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from the $N=2$ String

by

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EXTENDED SELF-DUAL YANG-MILLS FROM THE $N=2$ STRING *

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Abstract

We show that the physical degrees of freedom of the critical open string with $N=2$ superconformal symmetry on the worldsheet are described by a self-dual Yang-Mills field on a *hyperspace* parametrised by the coordinates of the target space $\mathbb{R}^{2,2}$ together with a *commuting* chiral spinor. A prepotential for the self-dual connection in the hyperspace generates the infinite tower of physical fields corresponding to the inequivalent pictures or spinor ghost vacua of this string. An action is presented for this tower, which describes consistent interactions amongst fields of arbitrarily high spin. An interesting truncation to a theory of five fields is seen to have no graphs of two or more loops.

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1 Introduction

The $N=2$ string¹ has many unique features descending from various remarkable properties of the (1+1)-dimensional $N=2$ superconformal algebra, the gauge symmetry on the worldsheet. In particular, these superconformal gauge symmetries kill all oscillatory modes of the string, yielding a peculiar string without any massive modes. The $N=2$ worldsheet supersymmetry is the maximal one for which the perturbative (e.g. BRST) quantisation yields a positive critical dimension. This turns out to be four [2]. However, the target space coordinates carry a complex structure which implies a euclidean (4, 0) or kleinian (2, 2) signature. In the target space, the (naive) single degree of freedom corresponds to a scalar field, which in the open (resp. closed) sector is the dynamical degree of freedom of a self-dual gauge (resp. graviton) field [3]. The absence of physical oscillator excitations of the string also implies that all scattering amplitudes beyond the three-point function vanish [3, 4]. The relation to self-duality provides this string with its particularly rich geometric structure, and we shall describe a further novel feature in this paper. We restrict ourselves to the open string sector, though analogous arguments hold for the closed sector as well; and we adopt the more interesting case of indefinite signature metric and specialise to a flat target space $\mathbb{R}^{2,2}$ [5].

Perturbative open string Hilbert spaces are defined using the *relative* cohomology of the BRST operator Q_{BRST} on the Fock space of open string excitations. In other words, *physical* open string states correspond to elements of the coset $im Q_{BRST}/ker Q_{BRST}$, after imposing the subsidiary conditions $b_0 = 0 = \tilde{b}_0$. Here, b_0 and \tilde{b}_0 are the anticommuting antighost zero modes of the open $N=2$ string; and we do *not* demand further restrictions involving the spinor antighost zero modes β_0^\pm . Chiral bosonisation of the spinor ghosts [6] enlarges the open $N=2$ string Fock space. It is then no longer graded by only the mass level and the *total ghost number* $u \in \mathbb{Z}$, but acquires *two* additional gradings: the *picture numbers* π_\pm labelling inequivalent spinor ghost vacua called *pictures*. These are related by *spectral flow* (\mathcal{S}) and *picture-raising* (\mathcal{P}^\pm) transformations which commute (up to BRST-exact terms) with one another as well as with Q_{BRST} [7, 8, 9, 10, 11, 12, 13]. These maps, however, do not afford a complete equivalence between the *relative* BRST cohomologies in the different pictures², although on states with non-zero momentum it is possible to establish an equivalence [15, 16]. Further analysis proves that each picture contains exactly *one* (absolute as well as relative) BRST cohomology class for a given non-zero lightlike momentum [16].

The physical state in each picture, however, changes under target space $SO(2, 2)$ action in a fashion characteristic of the particular picture. Specifically, it transforms as a highest weight state of a spin j representation of the non-manifest $SL(2, \mathbb{R})$ subgroup of $SO(2, 2)$. The spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ may be used as a convenient picture label instead of the *total picture number* $\pi_+ + \pi_- \in \mathbb{Z}$ [11, 12, 13]. A fully $SO(2, 2)$ -covariant formulation of the $N=2$ string cannot be

¹For a review of the subject until 1992, see ref. [1].

²A local picture-lowering operator inverting \mathcal{P}^\pm does not exist in the $N=2$ string [7, 8]. It occurs in the $N=1$ string, where, because it does not commute with b_0 , it guarantees picture-equivalence [14] only for the *absolute* BRST cohomology on the unrestricted Fock space obtained by dropping the subsidiary conditions.

expected to identify unequal-multiplicity-states in different pictures. This reinforces the notion that the pictures include physically distinct states. We adopt the attitude that the total picture number is a physical quantum number for $N=2$ string excitations. In contrast, there is no physics in the difference $\pi_+ - \pi_-$. We stress that the situation is completely different from the $N=1$ case. There, only *one* picture grading by $\pi \in \frac{1}{2}\mathbb{Z}$ appears. Most importantly, physical states in pictures with π differing by an integer have identical target space transformation properties and are therefore rightly identified, yielding only a single NS ($\pi \in \mathbb{Z}$) and a single R ($\pi \in \mathbb{Z} + \frac{1}{2}$) sector.

$N=2$ supermoduli transformations include twists of the $U(1)$ gauge bundle generated by *spectral flow* [17] relating the one-parameter family of twists of the fermionic boundary conditions interpolating between periodic (NS) and antiperiodic (R). It also changes $\pi_+ - \pi_-$ but not the total picture number. Since the quantum theory involves a functional integral over the supermoduli, there is no physical distinction between sectors having different boundary conditions, and all sectors are equivalent to the NS sector ($\pi_+ - \pi_- \in \mathbb{Z}$). This property implies the absence of target space fermions, and therefore this string fails to reflect its worldsheet supersymmetry in a target space supersymmetry. However, as we shall demonstrate, it realises an alternative extension of the Poincaré algebra, obtainable from the $N=1$ super-Poincaré algebra by changing the statistics of the Grassmann-odd (fermionic) generators. The algebra thus obtained is a *Lie* (rather than super) extension of the Poincaré algebra by Grassmann-even (bosonic) spin $\frac{1}{2}$ generators.

This extended Lie algebra is a genuine symmetry algebra on the space of physical states, with picture-raising being interpreted as an *even* variant of a supersymmetry transformation. The earlier viewpoint [3] of a one-dimensional physical state space is thus revealed to be a (consistent) truncation of an infinite tower of physical states of increasing spin. We present an effective action for this infinite tower of fields and demonstrate it to be a component version of an extension of self-dual Yang-Mills to a *hyperspace* with standard vectorial coordinates $x^{\alpha\dot{\alpha}}$ supplemented by an even (commuting) chiral spinor η^α . A prepotential for the self-dual connection in the hyperspace is shown to generate the entire tower of physical fields.

The plan of this paper is as follows. In section 2 we review some relevant features of $N=2$ string theory, describing in particular the infinite set of superconformally inequivalent physical states. In section 3, by considering the tree-level scattering amplitudes for these physical states, we deduce an effective action, S_∞ , for the corresponding infinite set of target space fields. This action is seen to consistently truncate to a two-field action previously considered in [18], as well as a novel five-field action. In section four we rederive S_∞ from a consideration of generalised self-dual Yang-Mills on an even-spinorial extension of $\mathbb{R}^{2,2}$, mimicking the construction of superspace using spinorial coordinates of the ‘wrong’ statistics. On a chiral subspace of this hyperspace, the well known Leznov functional [19, 20] is then seen to be a hyperspace-covariant version of S_∞ . Finally, we present an $SO(2, 2)$ -invariant action for the five-field model, reminiscent of the action for $N=4$ *supersymmetric* self-dual Yang-Mills [21].

2 N=2 Open String Worldsheets

Strings with two world-sheet supersymmetries in the NSR formulation are built from an $N=2$ world-sheet supergravity multiplet containing the metric h_{mn} , a $U(1)$ gauge field a_m , and two charged Majorana gravitini χ_m^\pm . Quantum consistency demands $c_{\text{matter}} = 6$, corresponding to a target space of real dimension four. Analysis of $N=(2,2)$ non-linear σ -models produces three distinct possibilities for worldsheet matter fields [22]:

- (a) two chiral superfields
- (b) one chiral and one twisted-chiral superfield
- (c) one semi-chiral superfield.

Since case (c) has not yet been much studied, and case (b) leads to free strings only [23], we will concentrate on the standard case (a), in which the component content of the worldsheet matter is given by the four string coordinates X^μ and their $U(1)$ charged NSR partners ψ^μ .

$N=2$ worldsheet supersymmetry implies a target space complex structure, so the (real) spacetime metric must have signature $(2,2)$ if we require light-like directions. Specialising to a flat target space, $\mathbb{R}^{2,2}$, we write

$$X^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} X^\mu = \begin{pmatrix} X^0+X^3 & X^1+X^2 \\ X^1-X^2 & X^0-X^3 \end{pmatrix}, \quad \alpha \in \{+, -\} \quad \dot{\alpha} \in \{\dot{+}, \dot{-}\}, \quad (2.1)$$

with a set of chiral gamma matrices σ_μ , $\mu = 0, 1, 2, 3$, appropriate for a spacetime metric $\eta_{\mu\nu} = \text{diag}(- + - +)$. We use the van der Waerden index notation, splitting $SO(2,2)$ vector indices μ into two $SL(2, \mathbb{R})$ spinor indices, α and $\dot{\alpha}$, which are raised and lowered using the $SL(2, \mathbb{R})$ -invariant skew-symmetric tensor, e.g., $\kappa \cdot \lambda = \kappa^\alpha \lambda_\alpha = \epsilon_{\alpha\beta} \kappa^\alpha \lambda^\beta = \epsilon^{\alpha\beta} \kappa_\beta \lambda_\alpha$; and vectors have an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})'$ invariant length-squared

$$\eta_{\mu\nu} X^\mu X^\nu = -\frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} X^{\alpha\dot{\alpha}} X^{\beta\dot{\beta}} = -\det X^{\alpha\dot{\alpha}}. \quad (2.2)$$

The NSR formulation [24] requires a specific choice of complex structure, $\mathbb{R}^{2,2} \rightarrow \mathbb{C}^{1,1}$, thus breaking this $SO(2,2)$ transformation group to a subgroup leaving the complex structure invariant, namely,

$$SL(2, \mathbb{R}) \times SL(2, \mathbb{R})' \rightarrow GL(1, \mathbb{R}) \times SL(2, \mathbb{R})'. \quad (2.3)$$

The residual $SL(2, \mathbb{R})$ transformations change the complex structure. Indeed, they generate the complex structure moduli space, $SL(2, \mathbb{R})/GL(1, \mathbb{R})$, where $GL(1, \mathbb{R})$ is a parabolic subgroup. We choose a representation for the $sl(2, \mathbb{R})$ algebra $\{L_{\pm\pm}, L_{+-}\}$, with L_{+-} diagonalised as one of the *boost* generators, having eigenvalues m called *boost charges*. The rotation and the second boost are generated by linear combinations of the nilpotent L_{++} and L_{--} . Then, following [13], we may choose the unbroken $gl(1, \mathbb{R})$ generator to be L_{++} .

As mentioned in the Introduction, the most elegant way to classify physical open string states is by solving the relative cohomology of the BRST operator Q_{BRST} . This yields a spectrum of only massless states distinguished by the total ghost number $u \in \mathbb{Z}$ and a pair of picture charges (π_+, π_-) labelling inequivalent spinor ghost vacua. For physical states these three quantum

numbers are related by $u = \pi_+ + \pi_- + 1$, so that in every picture there exists exactly one cohomology class for a given non-zero lightlike momentum [16]. It is often convenient to use the sum and difference,

$$\begin{aligned}\pi &\equiv \pi_+ + \pi_- \in \mathbb{Z} && \text{total picture} \\ \Delta &\equiv \pi_+ - \pi_- \in \mathbb{R} && \text{picture twist} \quad ,\end{aligned}\tag{2.4}$$

and denote states like $|\pi; \dots\rangle_\Delta$. Then, in the picture $(-1, -1)$, the BRST analysis [25] leads to a single physical state [3] with $u = -1$, namely,

$$|-2; k\rangle_0 = V(k)|0; 0\rangle_0\tag{2.5}$$

where $V(k)$ represents the string vertex operator for the center-of-mass mode with lightlike momentum k^μ . Nothing else appears. In particular, the massive states one would naively expect of a string do not materialise. Since we are dealing with open strings, a Chan-Paton adjoint gauge index, which we generally suppress, is to be assumed. It should be noted that null vectors like k^μ factorise into spinors,

$$\eta_{\mu\nu} k^\mu k^\nu = 0 \quad \Leftrightarrow \quad k^{\alpha\dot{\alpha}} = \kappa^\alpha \kappa^{\dot{\alpha}} \quad ,\tag{2.6}$$

which simplifies the massless dynamics.

The picture twist may be changed continuously, $\Delta \rightarrow \Delta + 2\rho$, by applying the spectral-flow operator $\mathcal{S}(\rho)$, $\rho \in \mathbb{R}$.³ The $\mathcal{S}(\rho)$ form an abelian algebra with zero ghost number, and they commute with Q_{BRST} , $\tilde{\mathcal{P}}^\pm$, and \tilde{b}_0 but not with b_0 . Nevertheless, since the $N=2$ string integrates over the parameter of spectral flow, it identifies states of differing picture twist Δ for fixed π . For convenience, we shall use the $\Delta=0$ representative and drop the Δ label from now on.

Can we also identify physical states having different π ? As in the $N=1$ case, picture-raising commutes (up to BRST-exact terms) with both Q_{BRST} and the antighost zero-modes b_0 and \tilde{b}_0 . As for the $N=1$ string, picture-lowering with the same properties can be defined on states with non-zero momentum [15]. Consequently, for $k \cdot k = 0$ but $k \neq 0$, an equivalence relation exists between the single relative cohomology classes appearing in any two pictures. From the worldsheet point of view, it may therefore seem reasonable to conjecture picture equivalence to identify all physical states, yielding a single massless scalar field's worth of physical degree of freedom [3]. However, things are not so simple [11, 12, 13], for there exist *two* such picture-raising operators, $\tilde{\mathcal{P}}^\alpha = \tilde{\mathcal{P}}^\pm$, as components of an $SL(2, \mathbb{R})$ *spinor*. They do not twist the pictures ($\Delta=0$) and commute modulo BRST exact terms.⁴ Of course, the difference $\tilde{\mathcal{P}}^+ - \tilde{\mathcal{P}}^-$ is BRST exact, but in a non-local way (involving division by momentum components). In the following, we shall argue against the identification $\tilde{\mathcal{P}}^+ \cong \tilde{\mathcal{P}}^-$ in view of target space properties.

Clearly, iterated picture-raising on $|-2; k\rangle$ creates states of $\pi = -2 + 2j$ thus:

$$\tilde{\mathcal{P}}^{(\alpha_1} \tilde{\mathcal{P}}^{\alpha_2} \dots \tilde{\mathcal{P}}^{\alpha_{2j})} |-2; k\rangle = |-2 + 2j; (\alpha_1 \alpha_2 \dots \alpha_{2j}), k\rangle \quad , \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots\tag{2.7}$$

³The argument of \mathcal{S} is not compact. Writing $\rho = c + \vartheta/2\pi$, ϑ is the angle of spectral flow and c is the change in $U(1)$ instanton number [9, 10].

⁴These are not the naive picture-raising operators but contain a spectral flow factor.

where the extra label $(\alpha_1\alpha_2 \dots \alpha_{2j})$ denotes the tensorial transformation property under $SL(2, \mathbb{R})$; all states being singlets with respect to $SL(2, \mathbb{R})'$. We emphasise that states of both integer and half-integer spin j have the *same* (viz. bosonic) statistics. The appearance of a $(2j+1)$ -dimensional tensor representation would seem to contradict the earlier statement of unit multiplicity for each picture [16]. However, as we have already mentioned, any specific choice of complex structure breaks $SO(2, 2)$ as in (2.3); and the physical state in a given picture is indeed a singlet under the manifest transformation group $GL(1, \mathbb{R}) \times SL(2, \mathbb{R})'$. On the other hand, the transformations in the coset $SL(2, \mathbb{R})/GL(1, \mathbb{R})$ not only change the complex structure but also transform the physical states according to spin j $SL(2, \mathbb{R})$ -representations, where j depends on the picture [11, 12, 13]. The string thus supports $SL(2, \mathbb{R})$ multiplets of physical states of *any* spin, but the specific choice of complex structure is tantamount to a projection to a highest weight state of the $SL(2, \mathbb{R})$ multiplet. The components of the $SL(2, \mathbb{R})$ multiplets in (2.7) are therefore related to each other by changes of the complex structure. In fact, they can all be considered to be different components of a single physical state, the linear combination

$$|-2+2j; e, \theta, k\rangle := \sum_{(\alpha_1\alpha_2\dots\alpha_{2j})} v_{\alpha_1} v_{\alpha_2} \dots v_{\alpha_{2j}} |-2+2j; (\alpha_1\alpha_2 \dots \alpha_{2j}), k\rangle \quad , \quad (2.8)$$

where the components of the spinor v_α parametrise the two dimensional parabolic coset of complex structures, $SL(2, \mathbb{R})/GL(1, \mathbb{R})$. Conversely, it turns out that

$$|-2+2j; (\alpha_1\alpha_2 \dots \alpha_{2j}), k\rangle \propto \kappa^{\alpha_1} \kappa^{\alpha_2} \dots \kappa^{\alpha_{2j}} |-2+2j; e, \theta, k\rangle \quad , \quad (2.9)$$

i.e. the $SL(2, \mathbb{R})$ charges are carried exclusively by momentum spinors appearing in the normalisation of the state. We remark that the proportionality factor becomes singular when $v \cdot \kappa = 0$.

On the worldsheet the parameters of the space of complex structures correspond to the (open) string coupling e and the worldsheet instanton angle θ . For fixed values of e and θ , there exists therefore one physical state in each picture. Following [11, 13] we may put

$$\begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \sqrt{e} \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{pmatrix} \quad . \quad (2.10)$$

This amounts to prescribing non-trivial transformation behaviour for the string coupling “constants”: e is a boost velocity, and θ a rotation angle! If we further choose a $\theta=0$ complex structure, only the highest $SL(2, \mathbb{R})$ weights (all $\alpha_i=+$) survive. In particular $\tilde{\mathcal{P}}^-$ disappears.

What about pictures with $\pi < -2$? Contrary to previous belief [13], the BRST cohomology is not trivial there, but again yields a single massless state [16] in each picture. Indeed, for generic light-like momentum, with the factorisation (2.6) denoted $k = \kappa \bar{\kappa}$, the physical Fock space has a non-degenerate scalar product,

$$\langle -k, \theta, e; -2-2j \mid -2+2j; e, \theta, k \rangle \propto (v \cdot \kappa)^{2j} \quad \text{for } j \geq 0 \quad (2.11)$$

where the conjugate states in pictures $\pi = -2-2j$ are constructed in a fashion analogous to the $\pi = -2+2j$ states, but with conjugate spinors \bar{v} and $\bar{\kappa}$ satisfying

$$\bar{v} \cdot v = 1 \quad \text{and} \quad \bar{\kappa} \cdot \kappa = 1 \quad . \quad (2.12)$$

In the absence of a standard picture-lowering operator, we do not know a direct way of obtaining physical $\pi < -2$ states from $|-2; k\rangle$. Since the states $|-2+2j; (\alpha_1\alpha_2 \dots \alpha_{2j}), k\rangle$, for fixed j, k , are all proportional to one another, the metric in the spin j representation space has rank one. Nevertheless, the conjugate states

$$|-2-2j; (\alpha_1\alpha_2 \dots \alpha_{2j}), k\rangle \quad , \quad j > 0 \quad , \quad (2.13)$$

form an $SL(2, \mathbb{R})$ representation of the same spin, with highest and lowest weights interchanged. Hence, only $\alpha_i = -$ survives in the $|-2-2j; e, \theta=0, k\rangle$ representative. It is convenient to label *all* the states using $\pi = -2+2j$, allowing for negative spin j . The pattern which emerges is symmetric around $\pi = -2$.

Having determined the physical states, string amplitudes are computed by choosing a set of physical vectors as external states and integrating their product with the string measure over the $N=2$ supermoduli. This simplifies at tree-level to a sum over different world-sheet topologies in the form of $U(1)$ instanton sectors classified by the first Chern number $c \in \mathbb{Z}$ of the principal $U(1)$ gauge bundle. Inspection of the string path integral shows that only the range $|c| \leq J := n-2$ contributes to the n -point scattering amplitude [26]. Using only the representatives $|-2+2j; e, \theta, k\rangle$ in (2.8) as external states, with the selection rule

$$\sum_{s=1}^n j_s = J \quad , \quad (2.14)$$

the instanton sum is automatically generated, with the correct weights⁵

$$v_{\alpha_1} v_{\alpha_2} \dots v_{\alpha_{2J}} = v_+^{J+c} v_-^{J-c} = e^J \cos^{J+c} \frac{\theta}{2} \sin^{J-c} \frac{\theta}{2} \quad (2.15)$$

multiplying the individual contributions carrying boost charges $m = c = -J, \dots, +J$ [11]. All terms for a fixed value of c are identical, since one may picture-change the open spinor indices freely from one external state to another. It has been shown that all tree-level amplitudes vanish [4], except for the two- and three-point functions. The latter is given by

$$A(k_1, k_2, k_3; e, \theta) = f^{a_1 a_2 a_3} v_\alpha k^{\alpha\dot{\alpha}} v_\beta k^{\beta\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}} = f^{a_1 a_2 a_3} v \cdot \kappa_1 v \cdot \kappa_2 \dot{\kappa}_1 \cdot \dot{\kappa}_2 \quad (2.16)$$

where $f^{a_1 a_2 a_3}$ are the (antisymmetric) structure constants of the gauge algebra, a_i are Chan-Paton indices and the momenta $k_i = \kappa_i \dot{\kappa}_i$, ($i = 1, 2, 3$) satisfy

$$k_1 + k_2 + k_3 = 0 \quad \text{and} \quad \kappa_i \cdot \kappa_j = 0 \quad . \quad (2.17)$$

This amplitude is totally symmetric under interchange of the three legs. For the choice $\theta=0$, it reduces to

$$A(k_1, k_2, k_3; e, \theta=0) = e f^{a_1 a_2 a_3} \kappa_1^+ \kappa_2^+ \dot{\kappa}_1 \cdot \dot{\kappa}_2 \quad . \quad (2.18)$$

⁵Since $J > 0$, all j_s can be chosen to be non-negative and so no \bar{v} 's need appear.

3 Target Space Actions

Target space perturbations are induced by source terms in the worldsheet action of the form

$$\int d^2\xi V^r(X(\xi)) \varphi_r(X(\xi)) \quad (3.1)$$

coupling vertex operators V^r to target space excitations $\varphi_r(X)$, where X stands for generic worldsheet fields. Hence, there is a one-to-one correspondence between physical string states $|r; k\rangle$ and spacetime fields $\varphi_r(x)$. The on-shell dynamics of the φ_r is determined by tree-level string scattering amplitudes, $\langle V^r V^s \dots \rangle$, from which the coupling coefficients in the effective target space action can be obtained. In particular, string three-point functions afford immediate determination of the cubic terms of the effective target space action thus:

$$\int d^4k_1 d^4k_2 d^4k_3 \langle V^r V^s V^t \rangle(k_i) \tilde{\varphi}_r(k_1) \tilde{\varphi}_s(k_2) \tilde{\varphi}_t(k_3) \delta(k_1+k_2+k_3) \quad (3.2)$$

(here written in momentum space). Further, since the tree-level $N=2$ string amplitudes vanish for more than three external legs [4], the contribution of iterated cubic vertices to $(n>3)$ -point functions must cancel either automatically or in virtue of any necessary further terms in the effective target space action. For the Leznov action, it has been checked for $n \leq 6$, that such higher-order terms are not necessary [20]. Though a general proof does not seem to exist, we shall assume that our cubic target space actions are (tree-level) exact.

Taking seriously the higher-spin states of the previous section, we associate

$$\begin{aligned} |-2-2j; (\alpha_1 \alpha_2 \dots \alpha_{2j}), k \rangle &\Leftrightarrow \varphi_{(\alpha_1 \alpha_2 \dots \alpha_{2j})}(x) \\ |-2-2j; e, \theta, k \rangle &\Leftrightarrow v^{\alpha_1} v^{\alpha_2} \dots v^{\alpha_{2j}} \varphi_{(\alpha_1 \alpha_2 \dots \alpha_{2j})}(x) \equiv \varphi_{(j)}(x) \end{aligned} \quad (3.3)$$

for $j \geq 0$ and analogously for $j < 0$, yielding a spectrum of target space fields taking values in the Lie algebra of the (Chan-Paton) gauge group. Choosing $\theta=0$, we associate to these fields their highest $GL(1, \mathbb{R})$ -boost eigenstates thus:

π	\dots	-4	-3	-2	-1	0	\dots	
$\varphi_{(j)}$	\dots	$\varphi_{(-1)}$	$\varphi_{(-\frac{1}{2})}$	$\varphi_{(0)}$	$\varphi_{(+\frac{1}{2})}$	$\varphi_{(+1)}$	\dots	.
φ^{\dots}	\dots	$e^{-1}\varphi^{++}$	$e^{-\frac{1}{2}}\varphi^+$	φ	$e^{+\frac{1}{2}}\varphi^-$	$e^{+1}\varphi^{--}$	\dots	

The effective action for this infinite tower of fields is surprisingly simple:

$$S_\infty = \int d^4x \text{Tr} \left\{ -\frac{1}{2} \sum_{j \in \mathbb{Z}/2} \varphi_{(-j)} \square \varphi_{(+j)} + \frac{1}{3} \sum_{j_1+j_2+j_3=1} \varphi_{(j_1)} \left[\partial^{+\dot{\alpha}} \varphi_{(j_2)}, \partial_{\dot{\alpha}}^+ \varphi_{(j_3)} \right] \right\} \quad (3.5)$$

produces all tree-level string amplitudes correctly. Interestingly, this action can be truncated to a finite number of fields in three ways. First, we may restrict ourselves to $|j| = 1$, which is closed under the interactions. Indeed,

$$S_2 = \int d^4x \text{Tr} \left\{ -\varphi^{++} \square \varphi^{--} + e \varphi^{++} \left[\partial^{+\dot{\alpha}} \varphi^{--}, \partial_{\dot{\alpha}}^+ \varphi^{--} \right] \right\} \quad (3.6)$$

is the two-field action of Chalmers and Siegel [18] for self-dual Yang-Mills in the Leznov gauge [19]. Concretely, the Leznov field φ^{--} coming from $\pi=0$ and a multiplier field φ^{++} from $\pi=-4$ interact via a $(j_1, j_2, j_3) = (-1, +1, +1)$ vertex [13]. Second, adding $\varphi=\varphi_{(0)}$ from $\pi=-2$ yields a three-field action S_3 containing a $(0, 0, +1)$ coupling as well. Third, allowing also the fields $\varphi_{(\pm\frac{1}{2})}$ yields a five-field action (boosting $e \rightarrow 1$)

$$S_5 = \int d^4x \operatorname{Tr} \left\{ \begin{aligned} & \frac{1}{2} \partial^{+\dot{\alpha}} \varphi \partial_{\dot{\alpha}}^- \varphi + \partial^{+\dot{\alpha}} \varphi^+ \partial_{\dot{\alpha}}^- \varphi^- + \partial^{+\dot{\alpha}} \varphi^{++} \partial_{\dot{\alpha}}^- \varphi^{--} \\ & + \frac{1}{2} \varphi [\partial^{+\dot{\alpha}} \varphi^-, \partial_{\dot{\alpha}}^+ \varphi^-] + \frac{1}{2} \varphi^{--} [\partial^{+\dot{\alpha}} \varphi, \partial_{\dot{\alpha}}^+ \varphi] \\ & + \varphi^{--} [\partial^{+\dot{\alpha}} \varphi^+, \partial_{\dot{\alpha}}^+ \varphi^-] + \frac{1}{2} \varphi^{++} [\partial^{+\dot{\alpha}} \varphi^{--}, \partial_{\dot{\alpha}}^+ \varphi^{--}] \end{aligned} \right\} \quad (3.7)$$

for the range $j = -1, \dots, +1$. Note that the fields φ^{++} and φ^+ effectively play the rôle of (propagating) Lagrange multipliers for the fields φ^{--} and φ^- respectively. Although the above actions merely serve to generate the (classical) background equations of motion, it is remarkable that the theories based on S_2 , S_3 , and S_5 are *one-loop exact*; their Feynman rules do not support higher-loop diagrams. Any attempt to include further fields beyond $|j| \leq 1$ requires the infinite set and no longer forbids two-loop diagrams.

We have seen that the physical states in different pictures are connected by acting with $\tilde{\mathcal{P}}^+$ (for $\theta=0$). Consequently, picture-raising induces a dual operation Q^+ on the set of spacetime fields, which lowers the spin by $\frac{1}{2}$. Interestingly, for $j \leq +1$,

$$Q^+ \varphi_{(j)} = (3-2j) \varphi_{(j-\frac{1}{2})} \quad (3.8)$$

turns out to be a derivation. By this we mean that the equations of motion for S_5 ,

$$\begin{aligned} \partial^{+\dot{\alpha}} \partial_{\dot{\alpha}}^- \varphi^{--} &= \frac{1}{2} [\partial^{+\dot{\alpha}} \varphi^{--}, \partial_{\dot{\alpha}}^+ \varphi^{--}] \\ \partial^{+\dot{\alpha}} \partial_{\dot{\alpha}}^- \varphi^- &= [\partial^{+\dot{\alpha}} \varphi^-, \partial_{\dot{\alpha}}^+ \varphi^{--}] \\ \partial^{+\dot{\alpha}} \partial_{\dot{\alpha}}^- \varphi &= [\partial^{+\dot{\alpha}} \varphi, \partial_{\dot{\alpha}}^+ \varphi^{--}] + \frac{1}{2} [\partial^{+\dot{\alpha}} \varphi^-, \partial_{\dot{\alpha}}^+ \varphi^-] \\ \partial^{+\dot{\alpha}} \partial_{\dot{\alpha}}^- \varphi^+ &= [\partial^{+\dot{\alpha}} \varphi^+, \partial_{\dot{\alpha}}^+ \varphi^{--}] + [\partial^{+\dot{\alpha}} \varphi, \partial_{\dot{\alpha}}^+ \varphi^-] \\ \partial^{+\dot{\alpha}} \partial_{\dot{\alpha}}^- \varphi^{++} &= [\partial^{+\dot{\alpha}} \varphi^{++}, \partial_{\dot{\alpha}}^+ \varphi^{--}] + [\partial^{+\dot{\alpha}} \varphi^+, \partial_{\dot{\alpha}}^+ \varphi^-] + \frac{1}{2} [\partial^{+\dot{\alpha}} \varphi, \partial_{\dot{\alpha}}^+ \varphi] \quad , \end{aligned} \quad (3.9)$$

all follow from the top one by applying

$$Q^+ : \varphi^{--} \longrightarrow \varphi^- \longrightarrow 2 \varphi \longrightarrow 2 \cdot 3 \varphi^+ \longrightarrow 2 \cdot 3 \cdot 4 \varphi^{++} \quad (3.10)$$

with the Leibniz rule. The reason for these curious facts will become clear in the following section.

4 Self-dual picture album

4.1 Extended Poincaré algebra

It is not often appreciated that the Poincaré algebra can have not only a \mathbb{Z}_2 -graded superalgebra extension, but a \mathbb{Z}_2 -graded *Lie* algebra extension as well. The super-Poincaré algebra of signature (p, q) is a \mathbb{Z}_2 -graded vector space $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ with a superskewsymmetric bilinear map (super commutator) $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that $[\mathcal{A}_\alpha, \mathcal{A}_\beta] \subset \mathcal{A}_{\alpha+\beta}$, with $\alpha, \beta \in \mathbb{Z}_2$. Here, \mathcal{A}_0 is the Poincaré algebra of signature (p, q) , and \mathcal{A}_1 is a spinor representation of $so(p, q)$ with $[\mathcal{A}_1, \mathcal{A}_1] \subset \mathbb{R}^{p, q}$. One can have a similar structure with the supercommutator replaced by a standard skewsymmetric Lie bracket. Thus giving elements of \mathcal{A}_1 the ‘wrong’ statistics yields an extension of the Poincaré algebra which remains a Lie algebra. Such Lie algebra extensions, as well as superalgebra extensions, of the Poincaré algebra have been classified for arbitrary dimension and signature only recently [27], where a correspondence is established (theorem 6.2) between superalgebra extensions of signature (p, q) and Lie algebra extensions of signature $(-q, -p) \bmod (4, \pm 4)$. For the $(2, 2)$ case, there clearly exists an even variant of the standard $N=1$ super-Poincaré algebra, with the *commutator* of two Grassmann-even spinorial generators squaring to an $\mathbb{R}^{2,2}$ translation, $[Q_{\dot{\alpha}}, Q_\alpha] = P_{\alpha\dot{\alpha}}$. Indeed, a representation by vector fields may easily be constructed:

$$\begin{aligned} Q_\alpha &\mapsto Q_\alpha = \frac{\partial}{\partial \eta^\alpha} + \frac{1}{2} \bar{\eta}^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \\ Q_{\dot{\alpha}} &\mapsto Q_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\eta}^{\dot{\alpha}}} - \frac{1}{2} \eta^\alpha \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \\ P_{\alpha\dot{\alpha}} &\mapsto [Q_{\dot{\alpha}}, Q_\alpha] = P_{\alpha\dot{\alpha}} = \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \end{aligned} \quad (4.1)$$

with the $so(2, 2)$ transformations being given in the usual fashion. Here $\eta, \bar{\eta}$ are *commuting* spinors. It is this Lie algebra which is relevant for a covariant description of the spectrum of $N=2$ string states described above.

4.2 Hyperspace self-duality

Define a *multi-picture hyperspace* $\widehat{\mathcal{M}}$ with coordinates $\{x^{\alpha\dot{\alpha}}, \bar{\eta}^{\dot{\alpha}}, \eta^\alpha\}$, where $\{\bar{\eta}^{\dot{\alpha}}, \eta^\alpha\}$ are *commuting* spinorial coordinates and $x^{\alpha\dot{\alpha}}$ are standard coordinates on $\mathbb{R}^{2,2}$, on which the *component fields* depend. The Lie algebra described in (4.1) clearly acts covariantly on this hyperspace. A *self-dual hyperconnection* is subject to the following constraints (c.f. [28, 29, 30, 31])

$$[\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}] = 0 \quad (4.2)$$

$$[\nabla_{\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] = 0 \quad (4.3)$$

$$[\nabla_{\dot{\beta}}, \nabla_\alpha] = \nabla_{\alpha\dot{\beta}} \quad (4.4)$$

$$[\nabla_\alpha, \nabla_\beta] = \epsilon_{\alpha\beta} \widehat{F} \quad (4.5)$$

$$[\nabla_\alpha, \nabla_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta} \widehat{F}_{\dot{\beta}} \quad (4.6)$$

$$[\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta} \widehat{F}_{\dot{\alpha}\dot{\beta}} \quad (4.7)$$

The first three conditions allow the choice of a *chiral basis* in which the covariant derivatives take the form

$$\nabla_{\dot{\alpha}} = \partial_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\eta}^{\dot{\alpha}}} \quad (4.8)$$

$$\nabla_{\alpha} = \mathcal{D}_{\alpha} + \bar{\eta}^{\dot{\alpha}} \mathcal{D}_{\alpha\dot{\alpha}} \quad (4.9)$$

$$\nabla_{\alpha\dot{\alpha}} = \mathcal{D}_{\alpha\dot{\alpha}} \quad , \quad (4.10)$$

where $(\mathcal{D}_{\alpha}, \mathcal{D}_{\alpha\dot{\alpha}})$ are gauge-covariant derivatives in the *chiral subspace*, $\widehat{\mathcal{M}}^+$, independent of the $\bar{\eta}^{\dot{\alpha}}$ coordinates. In this basis the single constraint (4.5) encapsulates the content of all the other constraints. The spinorial component of the gauge potential

$$\widehat{A}_{\alpha}(x, \bar{\eta}, \eta) = A_{\alpha}(x, \eta) + \bar{\eta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x, \eta) \quad (4.11)$$

describes the entire self-dual multi-picture hypermultiplet in the form of the curvature component \widehat{F} , which has a quadratic $\bar{\eta}$ -expansion in terms of chiral hyperfields of the form

$$\widehat{F}(x, \bar{\eta}, \eta) = F(x, \eta) + 2\bar{\eta}^{\dot{\alpha}} F_{\dot{\alpha}}(x, \eta) + \bar{\eta}^{\dot{\alpha}} \bar{\eta}^{\dot{\beta}} F_{\dot{\alpha}\dot{\beta}}(x, \eta) \quad . \quad (4.12)$$

The η -expansion of F yields an infinite tower of higher spin fields $\chi_{\alpha}, g_{\alpha\beta}, \psi_{\alpha\beta\gamma}, C_{\alpha\beta\gamma\delta}, \dots$ etc.

4.3 Extended self-dual component multiplet

Gauge-covariant derivatives in the chiral hyperspace take the form

$$\mathcal{D}_{\alpha} = \partial_{\alpha} + A_{\alpha} \quad , \quad \mathcal{D}_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} + A_{\alpha\dot{\alpha}} \quad , \quad (4.13)$$

where the partial derivatives $\partial_{\alpha} \equiv \frac{\partial}{\partial \eta^{\alpha}}$, $\partial_{\alpha\dot{\alpha}} \equiv \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}$ provide a holonomic basis for the tangent space. The components of the gauge connection $(A_{\alpha}, A_{\alpha\dot{\alpha}})$ take values in the Lie algebra of the gauge group, their transformations being parametrised by Lie algebra-valued sections on $\widehat{\mathcal{M}}^+$

$$\delta A_{\alpha} = -\partial_{\alpha} \tau(x, \eta) - [A_{\alpha}, \tau(x, \eta)] \quad (4.14)$$

$$\delta A_{\alpha\dot{\alpha}} = -\partial_{\alpha\dot{\alpha}} \tau(x, \eta) - [A_{\alpha\dot{\alpha}}, \tau(x, \eta)] \quad . \quad (4.15)$$

On $\widehat{\mathcal{M}}^+$, the self-duality conditions take the form of the following curvature constraints

$$[\mathcal{D}_{(\alpha}, \mathcal{D}_{\beta)}] = 0 \quad (4.16)$$

$$[\mathcal{D}_{(\alpha}, \mathcal{D}_{\beta)\dot{\beta}}] = 0 \quad (4.17)$$

$$[\mathcal{D}_{(\alpha\dot{\alpha}}, \mathcal{D}_{\beta)\dot{\beta}}] = 0 \quad , \quad (4.18)$$

or equivalently

$$[\mathcal{D}_{\alpha}, \mathcal{D}_{\beta}] = \epsilon_{\alpha\beta} F$$

$$[\mathcal{D}_{\alpha}, \mathcal{D}_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta} F_{\dot{\beta}} \quad (4.19)$$

$$[\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = \epsilon_{\alpha\beta} F_{\dot{\alpha}\dot{\beta}} \quad .$$

Here, $F_{\dot{\alpha}\dot{\beta}} = F_{\dot{\alpha}\dot{\beta}}(x, \eta)$ is symmetric and has the corresponding $\mathbb{R}^{2,2}$ Yang-Mills field-strength $F_{\dot{\alpha}\dot{\beta}}(x)$ as its leading component in an η -expansion. Henceforth all fields are chiral hyperfields, depending on both $x^{\alpha\dot{\alpha}}$ and η^α .

The non-zero curvature components are not independent; they are related by super-Jacobi identities. Firstly, the dimension -3 Jacobi identity implies, in virtue of the constraint (4.18), the Yang-Mills equation for the *hyperfield* $F_{\dot{\alpha}\dot{\beta}}$,

$$\mathcal{D}_\alpha^\dot{\alpha} F_{\dot{\alpha}\dot{\beta}} = 0 \quad . \quad (4.20)$$

Next, in virtue of the constraints (4.17) and (4.18), the dimension $-2\frac{1}{2}$ Jacobi identity yields a dynamical equation for the dimension $-\frac{3}{2}$ curvature,

$$\mathcal{D}_\alpha^\dot{\alpha} F_{\dot{\alpha}} = 0 \quad . \quad (4.21)$$

Finally, the dimension -2 Jacobi identity says that

$$\mathcal{D}_{\alpha\dot{\alpha}} F = \mathcal{D}_\alpha F_{\dot{\alpha}} \quad . \quad (4.22)$$

We therefore obtain the equation of motion

$$\square F = [F^\dot{\alpha}, F_{\dot{\alpha}}] \quad , \quad (4.23)$$

where the covariant d'Alembertian is defined by $\square \equiv \frac{1}{2} \mathcal{D}^{\alpha\dot{\beta}} \mathcal{D}_{\alpha\dot{\beta}}$.

Repeated application of \mathcal{D}_a on F successively yields an infinite tower of higher spin fields

$$\chi_\alpha = \mathcal{D}_\alpha F \quad , \quad g_{\alpha\beta} = \mathcal{D}_{(\alpha} \mathcal{D}_{\beta)} F \quad , \quad \psi_{\alpha\beta\gamma} = \mathcal{D}_{(\alpha} \mathcal{D}_\beta \mathcal{D}_{\gamma)} F \quad , \quad C_{\alpha\beta\gamma\delta} = \mathcal{D}_{(\alpha} \mathcal{D}_\beta \mathcal{D}_\gamma \mathcal{D}_{\delta)} F \quad , \dots \quad (4.24)$$

all having bosonic statistics. First-order equations of motion for all these fields may be obtained on action of $\mathcal{D}_\alpha^\dot{\alpha}$ and use of the constraints (4.19). For instance, χ_α and $g_{\alpha\beta}$ satisfy

$$\mathcal{D}_\alpha^\dot{\alpha} \chi_\alpha = 3 [F_{\dot{\alpha}}, F] \quad (4.25)$$

$$\mathcal{D}_\alpha^\dot{\alpha} g_{\alpha\beta} = \frac{8}{3} [F_{\dot{\alpha}}, \chi_\beta] + 2 [\mathcal{D}_{\beta\dot{\alpha}} F, F] \quad . \quad (4.26)$$

The entire tower of fields satisfies such gauge-covariant and interacting variants of the first-order Dirac-Fierz equations for zero rest-mass fields of arbitrary spin [32, 33, 34]

$$\mathcal{D}_{\dot{\alpha}}^{\alpha_n} \varphi_{\alpha_1 \dots \alpha_n} = J_{\alpha_1 \dots \alpha_{n-1} \dot{\alpha}} \quad , \quad n \geq 2 \quad . \quad (4.27)$$

The interaction current depends on all lower spin fields and is covariantly conserved,

$$\mathcal{D}^{\alpha_1 \dot{\alpha}} J_{\alpha_1 \dots \alpha_n \dot{\alpha}} = 0 \quad , \quad (4.28)$$

in virtue of lower spin field equations. This provides a sufficient condition for the consistency of the linear equations (4.27). In fact, the η -expansion of equation (4.26) for the hyperfield $g_{\alpha\beta}$ yields the leading components of all the higher spin equations. Now due to the self-duality of the

connection (4.19), $\mathcal{D}_\beta^{\dot{\alpha}} \mathcal{D}_{\dot{\alpha}}^\alpha = -\delta_\beta^\alpha \square$, the gauge covariant d'Alembertian. Therefore, covariant derivation of (4.27) yields wave equations for the entire tower of fields of the form

$$\square \varphi_{\alpha_1 \dots \alpha_n} = -\mathcal{D}_{(\alpha_n}^{\dot{\alpha}} J_{\alpha_1 \dots \alpha_{n-1}) \dot{\alpha}} \quad , \quad n \geq 2 \quad . \quad (4.29)$$

The structure of this system of increasingly higher spin interacting fields is very similar to the arbitrary N extended supersymmetric self-dual system presented in [31]. In fact, just as in that supersymmetric case [35], the covariantly conserved sources (4.28) provide an infinite set of local conserved currents for this theory,

$$j_{\alpha_1 \dots \alpha_{n-1} \dot{\alpha}} = J_{\alpha_1 \dots \alpha_{n-1} \dot{\alpha}} - [A_{\dot{\alpha}}^{\alpha_n} ; \varphi_{\alpha_1 \dots \alpha_n}] \quad , \quad n \geq 2 \quad , \quad (4.30)$$

satisfying

$$\partial^{\alpha_1 \dot{\alpha}} j_{\alpha_1 \dots \alpha_{n-1} \dot{\alpha}} = 0 \quad . \quad (4.31)$$

It are these conserved currents which provide consistent interactions of fields of arbitrary spin with fields of lower spin. Consistency of higher spin interactions as a consequence of lower spin field equations is part of the nested structure characteristic of self-dual systems [31]. In fact, the absence of conjugation in $\mathbb{R}^{2,2}$ between dotted and undotted spinor indices weakens the compatibility conditions which, in Minkowski space, make it almost impossible to construct consistent gauge-covariant higher spin field equations. Moreover, the features of having only the Yang-Mills coupling constant and only the associated spin one gauge invariance render inapplicable traditional theorems forbidding higher spin couplings. Just as in the supersymmetric systems discussed in [31], all our fields (4.24) take values in the Lie algebra of the gauge group and are linear in the (dimensionless) Yang-Mills coupling constant, which we absorb into the definition of the fields. The vector potential transforms in the usual inhomogeneous fashion (4.15), whereas all other fields transform covariantly under gauge transformations, $\delta \varphi_{\alpha_1 \dots \alpha_n} = [\tau, \varphi_{\alpha_1 \dots \alpha_n}]$. These are the only gauge transformations of these fields; there are no higher spin gauge invariances. The latter are not required since all fields apart from the vector potential transform according to irreducible representations of $SO(2,2)$. They therefore do not contain any redundant degrees of freedom, which would have required elimination in virtue of further (higher spin) gauge invariances.

4.4 Prepotential and action

As we have already mentioned, the NSR formulation of the $N=2$ string requires a specific choice of complex coordinates, leading to the breaking (2.3) of $SO(2,2)$. A convenient field-theoretical tool for describing complex structures is that of harmonic spaces [36]. This is a covariant description of the space of complex structures, the coset space $SL(2, \mathbb{R})/GL(1, \mathbb{R})$, given by equivalence classes under the parabolic $GL(1, \mathbb{R})$ subgroup. The quotient has homogeneous coordinates which may be organised into spinors $u^{\pm\alpha}$ satisfying $u^{+\alpha} u_{\bar{\alpha}}^- = 1$. These 'harmonics' provide a covariant version of the two-parameter description (2.10) of the space of complex structures. A particular choice of complex structure corresponds to choosing specific spinors $u^{\pm\alpha}$.

However, in the harmonic space method, these spinors are treated as independent variables, and this coset space is adopted as an auxiliary space with vector fields,

$$\begin{aligned}
\partial^{++} &= u^{+\alpha} \frac{\partial}{\partial u^{-\alpha}} \\
\partial^{--} &= u^{-\alpha} \frac{\partial}{\partial u^{+\alpha}} \\
\partial^{+-} &= u^{+\alpha} \frac{\partial}{\partial u^{+\alpha}} - u^{-\alpha} \frac{\partial}{\partial u^{-\alpha}} \quad ,
\end{aligned} \tag{4.32}$$

satisfying the $sl(2, \mathbb{R})$ algebra. Although this coset is non-compact and there certainly exist subtleties, we may apply formal rules of harmonic analysis on it (see e.g. [37]), understanding these in the sense of a Wick-rotated version of those applying to the compact case. The application of harmonic space methods to our hyperspace self-duality equations follows the treatment of other self-duality related systems, reviewed for instance in [38].

In analogy to the supersymmetric case [29] we enlarge $\widehat{\mathcal{M}}$ to a harmonic space with coordinates $\{x^{\dot{\alpha}\pm}, \bar{\eta}^{\dot{\alpha}}, \eta^{\pm}, u_{\alpha}^{\pm}\}$, where $x^{\dot{\alpha}\pm} = u_{\alpha}^{\pm} x^{\alpha\dot{\alpha}}$, and $\eta^{\pm} = u_{\alpha}^{\pm} \eta^{\alpha}$. The harmonic space gauge covariant derivatives are given by

$$\begin{aligned}
\mathcal{D}_{\dot{\alpha}}^{\pm} &= \frac{\partial}{\partial x^{\dot{\alpha}\pm}} + A_{\dot{\alpha}}^{\pm} \\
\nabla^{\pm} &= \frac{\partial}{\partial \eta^{\mp}} + A^{\pm} \\
\nabla_{\dot{\alpha}} &= \frac{\partial}{\partial \eta^{\dot{\alpha}}} + A_{\dot{\alpha}} \\
\mathcal{D}^{\pm\pm} &= \partial^{\pm\pm} + A^{\pm\pm} \quad .
\end{aligned} \tag{4.33}$$

The equations (4.2)-(4.7) are then equivalent to the following curvature constraints in harmonic space:

$$\begin{aligned}
[\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}] &= 0 \quad , \quad [\nabla_{\dot{\alpha}}, \mathcal{D}_{\dot{\beta}}^{\pm}] = 0 \quad , \quad [\nabla_{\dot{\beta}}, \nabla^{\pm}] = \mathcal{D}_{\dot{\beta}}^{\pm} \\
[\nabla^+, \mathcal{D}_{\dot{\beta}}^+] &= 0 \quad , \quad [\nabla^-, \mathcal{D}_{\dot{\beta}}^-] = 0 \\
[\mathcal{D}_{\dot{\alpha}}^+, \mathcal{D}_{\dot{\beta}}^+] &= 0 \quad , \quad [\mathcal{D}_{[\dot{\alpha}}^+, \mathcal{D}_{\dot{\beta}}^-] = 0 \quad , \quad [\mathcal{D}_{\dot{\alpha}}^-, \mathcal{D}_{\dot{\beta}}^-] = 0
\end{aligned} \tag{4.34}$$

together with the definitions of the non-zero curvatures

$$\begin{aligned}
[\nabla^+, \nabla^-] &= F \\
[\nabla^+, \mathcal{D}_{\dot{\beta}}^-] &= [\mathcal{D}_{\dot{\beta}}^+, \nabla^-] = F_{\dot{\beta}} \\
[\mathcal{D}_{\dot{\alpha}}^+, \mathcal{D}_{\dot{\beta}}^-] &= F_{\dot{\alpha}\dot{\beta}} \quad .
\end{aligned} \tag{4.35}$$

An equivalent set of curvature constraints is:

$$\begin{aligned}
[\nabla_{\dot{\alpha}}, \nabla_{\dot{\beta}}] &= 0 \quad , \quad [\nabla_{\dot{\alpha}}, \mathcal{D}_{\dot{\beta}}^{\pm}] = 0 \quad , \quad [\nabla_{\dot{\alpha}}, \mathcal{D}^{\pm\pm}] = 0 \\
[\nabla_{\dot{\beta}}, \nabla^{\pm}] &= \mathcal{D}_{\dot{\beta}}^{\pm} \quad , \quad [\nabla^+, \mathcal{D}_{\dot{\beta}}^+] = 0 \quad , \quad [\mathcal{D}_{\dot{\alpha}}^+, \mathcal{D}_{\dot{\beta}}^+] = 0 \\
[\mathcal{D}^{--}, \nabla^+] &= -\nabla^- \quad , \quad [\mathcal{D}^{--}, \mathcal{D}_{\dot{\alpha}}^+] = -\mathcal{D}_{\dot{\alpha}}^- \\
[\mathcal{D}^{--}, \nabla^-] &= 0 \quad , \quad [\mathcal{D}^{--}, \mathcal{D}_{\dot{\alpha}}^-] = 0 \quad .
\end{aligned} \tag{4.36}$$

The proof of equivalence is immediate in the *central frame* defined by $\mathcal{D}^{\pm\pm} = \partial^{\pm\pm}$, i.e. $A^{\pm\pm} = 0$, in which (4.36) has the partial solution $\mathcal{D}^{\pm} = u^{\pm\alpha} \mathcal{D}_{\alpha}$, $\mathcal{D}_{\dot{\alpha}}^{\pm} = u^{\pm\alpha} \mathcal{D}_{\alpha\dot{\alpha}}$.

The advantage of using harmonic space coordinates is that the existing flat subspaces thus become manifest, allowing the choice of a *Frobenius frame* in which $A_{\dot{\alpha}}$, A^+ and $A_{\dot{\beta}}^+$ are zero, i.e. $\nabla_{\dot{\alpha}}, \nabla^+$ and $\mathcal{D}_{\dot{\beta}}^+$ are partial derivatives $\partial_{\dot{\alpha}}, \partial^+$ and $\partial_{\dot{\beta}}^+$, respectively. In this frame the chiral (i.e. independent of $\bar{\eta}$) hyperfield $\Phi^{--} = A^{--}(x, \eta)$ becomes fundamental, determining all other fields occurring in the above constraints thus:

$$\begin{aligned} A^- &= \partial^+ \Phi^{--} \quad , & A_{\dot{\alpha}}^- &= \partial_{\dot{\alpha}}^+ \Phi^{--} \\ F &= \partial^+ \partial^+ \Phi^{--} \quad , & F_{\dot{\alpha}} &= \partial_{\dot{\alpha}}^+ \partial^+ \Phi^{--} \quad , & F_{\dot{\alpha}\dot{\beta}} &= \partial_{\dot{\alpha}}^+ \partial_{\dot{\beta}}^+ \Phi^{--} \quad . \end{aligned} \quad (4.37)$$

Higher spin fields then arise on iterative application of ∇^{\pm} according to (4.24). In this frame, most of the constraints in (4.34) are resolved and the only remaining dynamical equations for Φ^{--} are

$$\partial^{+\dot{\alpha}} \partial_{\dot{\alpha}}^- \Phi^{--} = \frac{1}{2} [\partial^{+\dot{\alpha}} \Phi^{--}, \partial_{\dot{\alpha}}^+ \Phi^{--}] \quad , \quad (4.38)$$

$$\partial^+ \partial_{\dot{\alpha}}^- \Phi^{--} - \partial^- \partial_{\dot{\alpha}}^+ \Phi^{--} = [\partial^+ \Phi^{--}, \partial_{\dot{\alpha}}^+ \Phi^{--}] \quad . \quad (4.39)$$

These equations are not independent; the former is obtained by acting on the latter by $\partial^{\dot{\alpha}}$. Moreover, they are also not independent of the equations for Φ^{--} following from the last two constraints in (4.36). Eq. (4.38) is the Euler-Lagrange equation for the generalised Leznov functional (c.f. [19, 39])

$$\begin{aligned} \mathcal{L}^{----} &= \text{Tr} \partial^{\dot{\alpha}} \left(\frac{1}{4} \partial^{[-} \Phi^{--} \partial_{\dot{\alpha}}^{+]} \Phi^{--} + \frac{1}{6} \Phi^{--} [\partial^+ \Phi^{--}, \partial_{\dot{\alpha}}^+ \Phi^{--}] \right) \\ &= \text{Tr} \left(\frac{1}{2} \partial^{+\dot{\alpha}} \Phi^{--} \partial_{\dot{\alpha}}^- \Phi^{--} + \frac{1}{6} \Phi^{--} [\partial^{+\dot{\alpha}} \Phi^{--}, \partial_{\dot{\alpha}}^+ \Phi^{--}] \right) \quad . \end{aligned} \quad (4.40)$$

One advantage of choosing (4.38) and (4.39) to be the equations determining the dynamics is that in this frame the harmonic variables u^{\pm} may be treated as parameters (this explicitly breaks the $SO(2, 2)$ invariance) and the explicit u -dependence may be ignored, treating $\{x^{\alpha\dot{\alpha}}\} \rightarrow \{x^{\pm\dot{\alpha}}\}$ as a fixed choice of complex structure. As we have seen, this is precisely the choice required for a comparison with string theory. Now, let us consider explicitly breaking the hyperspace-covariance of our theory by considering Φ^{--} to be independent of η^+ . It may then be Laurent-expanded in powers of η^- thus

$$\Phi^{--} = \dots + \frac{1}{\eta^-} \varphi^{---} + \varphi^{--} + \eta^- \varphi^{-} + (\eta^-)^2 \varphi^{+} + (\eta^-)^3 \varphi^{++} + (\eta^-)^4 \varphi^{+++} + (\eta^-)^5 \varphi^{++++} \dots \quad . \quad (4.41)$$

Inserting this expansion into (4.38) yields component equations of motion, which in general have infinitely many terms. Remarkably, inserting it into (4.40) and picking out the coefficient of $(\eta^-)^4$ yields precisely the charge-zero (homogeneous) lagrange functional in S_{∞} (3.5). So the chiral hyperfield prepotential $\Phi^{--}(x, \eta^-)$ is seen to be a generating function for the entire tower of physical states of the $N=2$ string, and the (dual) picture-lowering Q^+ (3.8) corresponds to a transformation from the coefficient at one order in an η^- -expansion to the coefficient at the next order.

Denoting the hyperfield having leading (i.e. η^- -independent) component $\varphi^{\dots}(x)$ by $\Phi^{\dots}(x, \eta^-)$, we remark that the leading term of the spinorial gauge potential is precisely the η^- -coefficient,

$$A^- = \Phi^- = \partial^+ \Phi^{--} \quad , \quad (4.42)$$

and the coefficient of $(\eta^-)^2$ is the leading term of the curvature component $F = \Phi = \partial^+ \partial^+ \Phi^{--}$. For η^+ -independent Φ^{--} , (4.39) takes the form

$$\partial_{\dot{\alpha}}^- \Phi^- = [\Phi^-, \partial_{\dot{\alpha}}^+ \Phi^{--}] \quad . \quad (4.43)$$

Acting on both sides by $\partial^{+\dot{\alpha}}$ yields the hyperfield version of the second equation in (3.9).

Now, if Φ^{--} is expandable as a *positive* power series in η^- ,

$$\Phi^{--} = \varphi^{--} + \eta^- \varphi^- + (\eta^-)^2 \varphi + (\eta^-)^3 \varphi^+ + (\eta^-)^4 \varphi^{++} + (\eta^-)^5 \varphi^{+++} \dots \quad , \quad (4.44)$$

the system of component equations are related by field transformations implied by (3.8), which amount to the action of ∂^+ on the corresponding hyperfields.

The restricted system of five fields $\varphi^{--}, \varphi^-, \varphi, \varphi^+, \varphi^{++}$ satisfies the rather distinguished set of equations (3.9) obtained from the action S_5 (3.7). An $SO(2, 2)$ -covariant action for this theory of five fields is given by

$$S = \int d^4x \operatorname{Tr} \left(\frac{1}{4} g^{\alpha\beta} F_{\alpha\beta} + \frac{1}{3} \chi^\alpha \mathcal{D}_{\alpha\dot{\alpha}} F^{\dot{\alpha}} + \frac{1}{8} \mathcal{D}^{\alpha\dot{\alpha}} F \mathcal{D}_{\alpha\dot{\alpha}} F + \frac{1}{2} F [F^{\dot{\alpha}}, F_{\dot{\alpha}}] \right) \quad , \quad (4.45)$$

where $g_{\alpha\beta}$ and χ_α play the rôle of (propagating) Lagrange multipliers for $A_{\alpha\dot{\alpha}}$ and $F_{\dot{\alpha}}$ respectively. This is in fact very reminiscent of the supersymmetric $N=4$ action of [21], with adjustments made for our different type of extension. Under this five-field truncation of the infinite system, only the first of the conserved currents in (4.30) survives, namely, $j_{\alpha\dot{\alpha}}$, the source current for the spin-one field $g_{\alpha\beta}$ (4.26), which is the Noether current corresponding to global gauge invariance of the action (4.45).

5 Concluding remarks

We have seen that a novel extension of self-dual Yang-Mills theory to a hyperspace with Grassmann-even spinorial auxiliary coordinates affords a covariant description of the physical degrees of freedom of the $N=2$ open string. It yields, moreover, a compact description of the infinite number of massless string degrees of freedom in terms of a scalar hyperspace prepotential, for which the generalised Leznov functional (4.40) yields the action S_∞ (3.5) describing the tree-level $N=2$ string amplitudes. The infinitely large multiplet of interacting massless higher spin fields is analogous to the $N=\infty$ supersymmetric self-dual multiplet presented in [31]. In fact the multiplets described by the three consistent truncations of S_∞ , namely, S_2, S_3 and S_5 , are remarkably reminiscent of the supersymmetric $N=1, 2, 4$ self-dual multiplets, respectively [29, 30, 21]. There appears to exist a correspondence between these pairs of theories.

Our infinite extension of the self-dual Yang-Mills system is amenable to solution by a twistor-type transform. In fact both the Ward splitting method and the ADHM construction yield themselves to modifications to accommodate our extension. Moreover, twistor theory makes intimate use of the sequence of zero-mass field equations of spin $\frac{m}{2}$ ($m \geq 0$) in a self-dual Yang-Mills background and of the associated space of solutions to the d'Alembert equation. These are

just the sets of equations (4.27) and (4.29), respectively, with the interaction currents $J_{\alpha_1 \dots \alpha_{n-1} \dot{\alpha}}$ set to zero. There is thus a tantalising similarity between the BRST-cohomological analysis yielding the tower of $N=2$ string states and the cohomological description of certain spaces used in twistor theory (see, for instance, [40]). We expect the interrelationship to be a fruitful direction for future research. The theories of $N=2$ closed as well as $N=(2,1)$ heterotic strings are also intimately related to self-dual geometry, and we expect our covariant description to generalise to both these cases.

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