

Max-Planck-Institut  
für Mathematik  
in den Naturwissenschaften  
Leipzig

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by

*Miroslav Chlebík*

Preprint-Nr.: 44

1997





# ON VARIATION OF SETS

MIROSLAV CHLEBÍK

## INTRODUCTION

A set  $A \subseteq \mathbb{R}^n$  is said to be of finite perimeter if it is Lebesgue measurable and the gradient in the sense of distributions  $D\chi^A = (\partial_1\chi^A, \partial_2\chi^A, \dots, \partial_n\chi^A)$  of its characteristic function  $\chi^A$  is an  $\mathbb{R}^n$  valued Borel measure on  $\mathbb{R}^n$  with finite total variation. The value of the perimeter of  $A$ , denoted by  $P(A)$ , is then the total variation  $\|D\chi^A\|$  of the vector valued measure  $D\chi^A$ . Otherwise, let the perimeter of  $A$  be equal to  $+\infty$ . (Another equivalent definition of perimeter was given in [DG1], see also [DG2] and [FH2].)

Given a direction  $\tau \in S^{n-1}$ , a set  $A \subseteq \mathbb{R}^n$  is said to have bounded variation at the direction  $\tau$  if it is Lebesgue measurable and the directional derivative  $\partial_\tau\chi^A$  of its characteristic function  $\chi^A$  is a signed Borel measure on  $\mathbb{R}^n$  with finite total variation. The value of the variation at direction  $\tau$  of  $A$ , denoted by  $\text{var}_\tau(A)$ , is then the total variation  $\|\partial_\tau\chi^A\|$  of the signed measure  $\partial_\tau\chi^A$ . Otherwise, let  $\text{var}_\tau(A) = +\infty$ .

It is well known that for a Lebesgue measurable set  $A$  and  $\tau = e_i$

$$\text{var}_i(A) = \int m_i^A(z) dz$$

where  $m_i^A(z)$  is the infimum of the variations in  $x_i$  of all functions defined on the line  $L_i(z)$  (parallel to the  $x_i$  axis and meeting  $z$ ) which are equivalent to  $\chi^A|_{L_i(z)}$  and the integration is over the  $(n-1)$  dimensional linear subspace of  $\mathbb{R}^n$  orthogonal to the  $x_i$  axis.

It is known that the perimeter of  $A$ , if this is finite, is equal to the  $(n-1)$  measure (Hausdorff or, equivalently, integralgeometric) of the set  $fr_r A$  that is called the reduced boundary (see [FH2]) or equivalently it is equal to  $(n-1)$  measure of the essential boundary  $fr_e A$  of  $A$  (see [VA] or [FH3, 4.5.6]). Specifically,  $x \in fr_r A$  iff there is an  $(n-1)$  plane  $\pi$  through  $x$  such that the symmetric difference of  $A$  and one of the halfspaces determined by  $\pi$  has density zero at  $x$ . Further,  $x \in fr_e A$  iff both  $A$  and complement of  $A$  have positive outer upper density at  $x$ .

Moreover, if  $(n-1)$  measure of  $fr_e A$  is finite then  $A$  is of finite perimeter ([FH3, 4.5.11]). Hence  $(n-1)$  measure of  $fr_e A$  is equal to the perimeter of  $A$  for a general set  $A \subseteq \mathbb{R}^n$ . (Our method offers also simple proof of this fact for an integralgeometric  $(n-1)$  measure.)

The main purpose of this paper is to show that the directional variation of a set  $A \subseteq \mathbb{R}^n$  (without any assumptions on regularity of  $A$ ) is equal to the measure of projection (with multiplicities taken into account) of the measure-theoretic boundary of  $A$ .

## 1. NOTATION AND TERMINOLOGY

Throughout the whole paper we deal with the sets in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . We tacitly assume that  $n \geq 2$  but results trivially hold in the case  $n = 1$ .

Let  $e_1, e_2, \dots, e_n$  stand for the orthonormal base in  $\mathbb{R}^n$ ,  $e_1 = (1, 0, 0, \dots, 0)$ ,  $e_2 = (0, 1, 0, \dots, 0)$ ,  $\dots$ , and  $e_n = (0, 0, 0, \dots, 1)$ .

For  $x, y \in \mathbb{R}^n$  we denote by  $|x|$  the euclidean norm of  $x$  and by  $x \circ y$  the inner product of  $x$  and  $y$ . The symbol  $[x, y]$  stands for the convex hull of the set  $\{x, y\}$  and  $|x, y[$  means  $[x, y] \setminus \{x, y\}$ .

Whenever  $x \in \mathbb{R}^n$  and  $r > 0$   $B(x, r)$  and  $U(x, r)$  stand for the closed and open balls, respectively, with center  $x$  and radius  $r$  and  $Q(x, r)$  stands for the cubic interval  $\{y \in \mathbb{R}^n: |y_i - x_i| \leq r, 1 \leq i \leq n\}$ . We put

$$S^{n-1} = \{x \in \mathbb{R}^n: |x| = 1\} \quad \text{and} \quad L_\tau(x) = \{x + t\tau: t \in \mathbb{R}\} \quad \text{for} \quad x \in \mathbb{R}^n \quad \text{and} \quad \tau \in S^{n-1}.$$

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The paper in this form was prepared while I visited Max-Planck Institute at Leipzig. I am indebted to directors of MPI and their colleagues for letting me visit the Institute during April-June 1997.

For  $\tau \in S^{n-1}$  we denote by  $\mathbb{R}^{n-1}(\tau)$  the orthogonal complement in  $\mathbb{R}^n$  to the one dimensional subspace  $\{t\tau : t \in \mathbb{R}\}$  and by  $p_\tau$  the orthogonal projection of  $\mathbb{R}^n$  onto  $\mathbb{R}^{n-1}(\tau)$ . We write briefly  $L_i(x)$ ,  $\mathbb{R}^{n-1}(i)$  and  $p_i$  in the case  $\tau = e_i$ .

For any  $A \subseteq \mathbb{R}^n$  we denote by  $A^c$  the complement of  $A$  and by  $\chi^A$  the characteristic function of  $A$ .

For an open set  $\Omega \subseteq \mathbb{R}^n$  we will denote by  $C_0^\infty(\Omega)$  (and  $C_0^\infty(\Omega, \mathbb{R}^n)$ ) the space of all infinitely differentiable real valued functions with compact support in  $\Omega$  (and the space of all infinitely differentiable  $\mathbb{R}^n$  valued vector functions with compact support in  $\Omega$ , respectively). These spaces are considered to be equipped with the “sup norm”.

For any function  $f$ , any set  $A$  and any value  $y$ , the multiplicity  $N(f, A, y)$  is defined as the number of elements (possibly  $+\infty$ ) of the set  $\{x \in A : f(x) = y\}$ .

**1.1. Hausdorff measures.** For an integer  $k$  such that  $0 \leq k \leq n$  let  $H_k$  stand for the  $k$ -dimensional Hausdorff outer measure on  $\mathbb{R}^n$ , which is normalized in such a way that

$$H_k \{ x \in \mathbb{R}^n : 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq k \text{ and } x_i = 0 \text{ for } k < i \leq n \} = 1 .$$

In particular,  $H_0$  is a counting measure and  $H_n$  coincides with the Lebesgue outer measure on  $\mathbb{R}^n$ .

The constant  $V(n-1)$  stands for the volume of the unit ball in  $\mathbb{R}^{n-1}$  (with  $V(0) = 1$ ) and the constant  $A(n)$  means the area of  $S^{n-1}$ .

We define the equivalence  $\sim$  for subsets of  $\mathbb{R}^n$  by the prescription

$$A \sim B \text{ iff } H_n [(A \setminus B) \cup (B \setminus A)] = 0 .$$

**1.2. Projection measures  $\mu_\tau$ .** For  $\tau \in S^{n-1}$  the result of Caratheodory’s construction (see e.g. [FH3,2.10.1]) from the set function

$$B \longmapsto H_{n-1} [ p_\tau(B) ]$$

which is defined on the covering family of all Borel sets in  $\mathbb{R}^n$  will be called the projection measure at the direction  $\tau$  and denoted by  $\mu_\tau$ . Then  $\mu_\tau$  is a Borel regular outer measure on  $\mathbb{R}^n$  and  $\mu_\tau \leq H_{n-1}$ .

From Fubini’s theorem it follows that  $H_n(C) = 0$  whenever  $C \subseteq \mathbb{R}^n$  is such that  $\mu_\tau(C) < \infty$ .

**1.3. Integralgeometric measure  $\mathfrak{S}_1^{n-1}$ .** The result of Caratheodory’s construction from the set function

$$B \longrightarrow \frac{1}{2V(n-1)} \int_{S^{n-1}} H_{n-1} [ p_\tau(B) ] dH_{n-1}(\tau)$$

which is defined on the covering family of all Borel sets in  $\mathbb{R}^n$  is usually termed  $(n-1)$  dimensional integralgeometric measure with exponent 1 and denoted by  $\mathfrak{S}_1^{n-1}$ . (For the existence of the above integral see e.g. [FH3, 2.10.5].)

$\mathfrak{S}_1^{n-1}$  is a Borel regular outer measure on  $\mathbb{R}^n$  and obviously  $2V(n-1)\mathfrak{S}_1^{n-1} \leq A(n)H_{n-1}$ . Moreover  $\mathfrak{S}_1^{n-1} \leq H_{n-1}$  by [FH3, 3.2.26, 3.3.14 and 3.3.16].

**1.4. Densities.** For every set  $A \subseteq \mathbb{R}^n$  and each  $x \in \mathbb{R}^n$  we define the upper outer density  $\bar{d}(x, A)$  and the lower outer density  $\underline{d}(x, A)$  of  $A$  at  $x$  by the formulas

$$\begin{aligned} \bar{d}(x, A) &= \overline{\lim}_{r \rightarrow 0+} \frac{H_n[A \cap B(x, r)]}{H_n[B(x, r)]} \\ \underline{d}(x, A) &= \underline{\lim}_{r \rightarrow 0+} \frac{H_n[A \cap B(x, r)]}{H_n[B(x, r)]} . \end{aligned}$$

In the case  $\bar{d}(x, A) = \underline{d}(x, A)$  this common value is denoted by  $d(x, A)$ . A point  $x$  for which  $d(x, A) = 1$  is termed the outer density point of  $A$ . (We will drop the adjective “outer” in our terminology whenever  $A$  is  $H_n$  measurable.)

**1.5. Essential and preponderant interior and boundary.** We define the essential interior  $int_e A$ , the essential boundary  $fr_e A$ , the preponderant interior  $int_{pr} A$  and the preponderant boundary  $fr_{pr} A$  of  $A \subseteq \mathbb{R}^n$  by the formulas

$$\begin{aligned} int_e A &= \{ x \in \mathbb{R}^n : d(x, A^c) = 0 \}, \\ fr_e A &= \{ x \in \mathbb{R}^n : \bar{d}(x, A) > 0 \text{ and } \bar{d}(x, A^c) > 0 \}, \\ int_{pr} A &= \left\{ x \in \mathbb{R}^n : \bar{d}(x, A^c) < \frac{1}{2} \right\}, \\ fr_{pr} A &= \left\{ x \in \mathbb{R}^n : \bar{d}(x, A) \geq \frac{1}{2} \text{ and } \bar{d}(x, A^c) \geq \frac{1}{2} \right\}. \end{aligned}$$

It is easy to see that  $int_e A \cap int_e(A^c) = \emptyset$ ,  $fr_e A = [int_e A \cup int_e(A^c)]^c$ ,  $int_{pr} A \cap int_{pr}(A^c) = \emptyset$ ,  $fr_{pr} A = [int_{pr} A \cup int_{pr}(A^c)]^c$ ,  $int_e A$  is of type  $F_{\sigma\delta}$ ,  $fr_e A$  is of type  $G_{\sigma\delta}$ ,  $int_{pr} A$  is of type  $F_\sigma$  and  $fr_{pr} A$  is of type  $G_\delta$ . In particular, all these sets are Borel.

**1.6. BV functions.** For a nonempty open set  $\Omega \subseteq \mathbb{R}^n$  and for any  $\tau \in S^{n-1}$  we define the space  $BV(\Omega, \tau)$  of all locally (in  $\Omega$ )  $H_n$  summable functions  $g$  for which there exists a finite signed Borel measure  $\Phi_{\Omega, \tau}^g$  on  $\Omega$  with the equality

$$\int_{\Omega} g(x) \tau \circ \text{grad } \varphi(x) dx = - \int_{\Omega} \varphi(x) d\Phi_{\Omega, \tau}^g(x)$$

whenever  $\varphi \in C_0^\infty(\Omega)$ .

$BV(\Omega)$  is defined as the space of all locally (in  $\Omega$ )  $H_n$  summable functions  $g$  such that there exist the finite signed Borel measures  $\Phi_{\Omega, 1}^g, \Phi_{\Omega, 2}^g, \dots, \Phi_{\Omega, n}^g$  with the equality

$$\int_{\Omega} g(x) \text{div } \psi(x) dx = - \sum_{i=1}^n \int_{\Omega} \psi_i(x) d\Phi_{\Omega, i}^g(x)$$

whenever  $\psi = (\psi_1, \psi_2, \dots, \psi_n) \in C_0^\infty(\Omega, \mathbb{R}^n)$ .

**1.7. Directional variation and perimeter of sets.** For a nonempty open set  $\Omega \subseteq \mathbb{R}^n$  and for any  $\tau \in S^{n-1}$  the set functions  $\text{var}_{\Omega, \tau}$  and  $P_\Omega$  are defined for any subset  $A$  of  $\mathbb{R}^n$  as follows:

(1) If  $A \cap \Omega$  is  $H_n$  measurable then we put

$$\begin{aligned} \text{var}_{\Omega, \tau}(A) &= \sup \left\{ \int_{\Omega} \chi^A(x) \tau \circ \text{grad } \varphi(x) dx : \varphi \in C_0^\infty(\Omega) \text{ and } |\varphi| \leq 1 \right\}, \\ P_\Omega(A) &= \sup \left\{ \int_{\Omega} \chi^A(x) \text{div } \psi(x) dx : \psi \in C_0^\infty(\Omega, \mathbb{R}^n) \text{ and } |\psi| \leq 1 \right\}. \end{aligned}$$

(2) If  $A \cap \Omega$  is not  $H_n$  measurable then we put

$$\text{var}_{\Omega, \tau}(A) = P_\Omega(A) = \infty.$$

The value  $\text{var}_{\Omega, \tau}(A)$  is termed the variation at direction  $\tau$  of the set  $A$  in the open set  $\Omega$  and  $P_\Omega(A)$  is the perimeter of  $A$  in  $\Omega$ .

In the case  $\text{var}_{\Omega, \tau}(A) < \infty$  the symbol  $\Phi_{\Omega, \tau}^A$  stands for the (uniquely determined) signed Borel measure on  $\Omega$  such that

$$\int_{\Omega} \chi^A(x) \tau \circ \text{grad } \varphi(x) dx = - \int_{\Omega} \varphi(x) d\Phi_{\Omega, \tau}^A(x)$$

holds whenever  $\varphi \in C_0^\infty(\Omega)$ . We write briefly  $\Phi_{\Omega, i}^A$  in the case  $\tau = e_i$ .

## 2. AUXILIARY RESULTS

**2.1. Lebesgue outer density theorem.** For any set  $A \subseteq \mathbb{R}^n$   $H_n$  almost every point of  $A$  is an outer density point of  $A$ .

**2.2. Remark.** Let  $\Omega \subseteq \mathbb{R}^n$  be nonempty open and  $A \subseteq \mathbb{R}^n$  be arbitrary. Using Lebesgue density theorem and Borel regularity of Lebesgue outer measure one can easily show that the following statements are true:

(i) If  $A \cap \Omega$  is  $H_n$  measurable then

$$\Omega \cap \text{int}_e A \sim \Omega \cap \text{int}_{pr} A \sim \Omega \cap A \quad \text{and} \quad H_n(\Omega \cap \text{fr}_e A) = H_n(\Omega \cap \text{fr}_{pr} A) = 0 .$$

(ii) If  $A \cap \Omega$  is not  $H_n$  measurable then

$$H_n\{x \in \Omega: d(x, A) = 1 \quad \text{and} \quad d(x, A^c) = 1\} > 0$$

and especially

$$H_n(\Omega \cap \text{fr}_e A) \geq H_n(\Omega \cap \text{fr}_{pr} A) > 0 .$$

### 2.3. Observations.

- (1) If  $g \in BV(\Omega)$  then  $g \in BV(\Omega, \tau)$  for every  $\tau \in S^{n-1}$ .
- (2) If  $\tau_j \in S^{n-1}$ ,  $\alpha_j \in \mathbb{R}$ ,  $g \in BV(\Omega, \tau_j)$  ( $j = 1, 2, \dots, r$ ) and  $\tau = \sum_{i=1}^r \alpha_j \tau_j \in S^{n-1}$  then  $g \in BV(\Omega, \tau)$  and  $\Phi_{\Omega, \tau}^g = \sum_{i=1}^r \alpha_j \Phi_{\Omega, \tau_j}^g$ .
- (3) If  $\tau_1, \tau_2, \dots, \tau_n \in S^{n-1}$  are linearly independent and  $g \in BV(\Omega, \tau_j)$  ( $j = 1, 2, \dots, n$ ) then  $g \in BV(\Omega)$ .
- (4)  $\text{var}_{\Omega, \tau}(A) < \infty$  holds if and only if  $\chi_A|_{\Omega} \in BV(\Omega, \tau)$ . In this case  $\text{var}_{\Omega, \tau}(A) = |\Phi_{\Omega, \tau}^A|(\Omega)$  holds.
- (5)  $P_{\Omega}(A) < \infty$  holds if and only if  $\chi_A|_{\Omega} \in BV(\Omega)$ . In this case  $P_{\Omega}(A) = |\Phi_{\Omega}^A|(\Omega)$ , where  $\Phi_{\Omega}^A = (\Phi_{\Omega, 1}^A, \Phi_{\Omega, 2}^A, \dots, \Phi_{\Omega, n}^A)$ .
- (6) If  $P_{\Omega}(A) = +\infty$  then the set  $\{\tau \in S^{n-1}: \text{var}_{\Omega, \tau}(A) < \infty\}$  is contained in an  $(n-1)$ -dimensional linear subspace of  $\mathbb{R}^n$ .

**2.4. Lemma.** Let  $B \subseteq \mathbb{R}^n$  be a Borel set.

(i) For any  $\tau \in S^{n-1}$  the function  $z \mapsto N(p_{\tau}, B, z)$  defined on  $\mathbb{R}^{n-1}(\tau)$  is  $H_{n-1}$  measurable and

$$\mu_{\tau}(B) = \int_{\mathbb{R}^{n-1}(\tau)} N(p_{\tau}, B, z) dH_{n-1}(z) .$$

(ii) The function  $\tau \mapsto \mu_{\tau}(B)$  defined on  $S^{n-1}$  is  $H_{n-1}$  measurable and

$$\mathfrak{S}_1^{n-1}(B) = \frac{1}{2V(n-1)} \int_{S^{n-1}} \mu_{\tau}(B) dH_{n-1}(\tau) .$$

*Proof.* See [FH3, 2.10.10 and 2.10.15].

**2.5. Definition.** Let  $L_0$  be a line in  $\mathbb{R}^n$  and let  $L \subseteq L_0$  be relatively open in  $L_0$ . For any set  $A \subseteq \mathbb{R}^n$  the point  $x \in L$  is termed a hit of  $L$  on  $A$  provided both  $L \cap A \cap U(x, r)$  and  $(L \setminus A) \cap U(x, r)$  have a positive  $H_1$  measure for every  $r > 0$ .

For  $\Omega \subseteq \mathbb{R}^n$  nonempty open,  $A \subseteq \mathbb{R}^n$ ,  $z \in \mathbb{R}^n$  and  $\tau \in S^{n-1}$ , the symbol  $M_{\Omega, \tau}^A(z)$  stands for the set of all hits of  $L_{\tau}(z) \cap \Omega$  on  $A$ ,  $m_{\Omega, \tau}^A(z)$  stands for the number of elements (possibly  $+\infty$ ) of  $M_{\Omega, \tau}^A(z)$  and we put

$$M_{\Omega, \tau}^A = \cup \{ M_{\Omega, \tau}^A(z): z \in \mathbb{R}^n \} .$$

We write briefly  $M_{\Omega, i}^A(z)$ ,  $m_{\Omega, i}^A(z)$  and  $M_{\Omega, i}^A$  in the case  $\tau = e_i$ .

The starting point to our results is the following well known lemma.

**2.6. Lemma.** Let  $\Omega \subseteq \mathbb{R}^n$  be nonempty open and  $A \subseteq \mathbb{R}^n$  be such that  $A \cap \Omega$  is  $H_n$  measurable. Then for every  $\tau \in S^{n-1}$  the function  $z \mapsto m_{\Omega, \tau}^A(z)$  defined on  $\mathbb{R}^{n-1}(\tau)$  is  $H_{n-1}$  measurable and

$$\text{var}_{\Omega, \tau}(A) = \int_{\mathbb{R}^{n-1}(\tau)} m_{\Omega, \tau}^A(z) dH_{n-1}(z) .$$

*Proof.* See e.g. [MJ] and Chap. 7 of [KK].

### 3. CHARACTERIZATION OF DIRECTIONAL VARIATION OF SET

**3.1. Notation.** Let  $\Omega \subseteq \mathbb{R}^n$  be nonempty open and  $A \subseteq \mathbb{R}^n$  be such that  $A \cap \Omega$  is  $H_n$  measurable. Let us identify  $\mathbb{R}^n$  with  $\mathbb{R}^{n-1} \times \mathbb{R}$ . For any  $\alpha, \beta$  such that  $-\infty \leq \alpha < \beta \leq +\infty$  put

$$E_\Omega(\alpha, \beta; A) = \{z \in \mathbb{R}^{n-1} : \{z\} \times (\alpha, \beta) \subseteq \Omega \text{ and } H_1(\{z\} \times (\alpha, \beta) \setminus A) = 0\}.$$

It is easy to show using Fubini's theorem that these sets are  $H_{n-1}$  measurable.

**3.2. Lemma.** *Let  $\Omega \subseteq \mathbb{R}^n$  be nonempty open and  $A \subseteq \mathbb{R}^n$  be such that  $A \cap \Omega$  is  $H_n$  measurable. Then there is an  $H_{n-1}$  null set  $N \subseteq \mathbb{R}^{n-1}$  such that every  $z \in \mathbb{R}^{n-1} \setminus N$  has the following properties:*

- a) *If  $\alpha, \beta \in \mathbb{Q} \cup \{\pm\infty\}$  ( $\mathbb{Q}$  being the set of rationals) with  $-\infty \leq \alpha < \beta \leq +\infty$  are such that  $z \in E_\Omega(\alpha, \beta; A)$  ( $z \in E_\Omega(\alpha, \beta; A^c)$ , respectively) then  $z$  is a density point in  $\mathbb{R}^{n-1}$  of  $E_\Omega(\alpha, \beta; A)$  ( $E_\Omega(\alpha, \beta; A^c)$ ).*
- b) *If  $-\infty \leq \alpha < \beta \leq +\infty$  are such that  $\{z\} \times (\alpha, \beta) \subseteq \Omega$  and  $H_1(\{z\} \times (\alpha, \beta) \setminus A) = 0$  ( $H_1(\{z\} \times (\alpha, \beta) \setminus A^c) = 0$ , respectively) then  $\{z\} \times (\alpha, \beta) \subseteq \text{int}_e A$  ( $\{z\} \times (\alpha, \beta) \subseteq \text{int}_e(A^c)$ ).*
- c)  *$\{x \in \Omega \cap \text{fr}_e A : p_n(x) = z\} \subseteq M_{\Omega, n}^A$ .*

*Proof.* For any  $H_{n-1}$  measurable set  $B \subseteq \mathbb{R}^{n-1}$  put  $\tilde{B} = \{z \in B : z \text{ is not a density point of } B\}$ . Due to Lebesgue density theorem  $\tilde{B}$  is  $H_{n-1}$  null set. Hence the set

$$N = \bigcup \left\{ \tilde{E}(\alpha, \beta; A) \cup \tilde{E}(\alpha, \beta; A^c) : \alpha, \beta \in \mathbb{Q} \cup \{\pm\infty\}, -\infty \leq \alpha < \beta \leq +\infty \right\}$$

is  $H_{n-1}$  null set and each  $z \in \mathbb{R}^{n-1} \setminus N$  has the property a).

If  $z \in \mathbb{R}^{n-1} \setminus N$  and  $-\infty \leq \alpha < \beta \leq +\infty$  are such that  $\{z\} \times (\alpha, \beta) \subseteq \Omega$  and  $H_1(\{z\} \times (\alpha, \beta) \setminus A) = 0$  ( $H_1(\{z\} \times (\alpha, \beta) \setminus A^c) = 0$ , respectively), then  $z$  is a density point in  $\mathbb{R}^{n-1}$  of  $E_\Omega(\alpha_1, \beta_1; A)$  ( $E_\Omega(\alpha_1, \beta_1; A^c)$ ) whenever  $\alpha_1, \beta_1 \in \mathbb{Q} \cup \{\pm\infty\}$  with  $\alpha < \alpha_1 < \beta_1 < \beta$ . From Fubini's theorem it follows that  $\{z\} \times (\alpha, \beta) \subseteq \text{int}_e A$  ( $\{z\} \times (\alpha, \beta) \subseteq \text{int}_e(A^c)$ ), hence b) holds true.

To prove c) let us fix  $z \in \mathbb{R}^{n-1} \setminus N$  and assume that we can find  $x \in \Omega \setminus M_{\Omega, n}^A$  such that  $p_n(x) = z$ . Our aim is to prove that then necessarily  $x \notin \text{fr}_e A$ . As  $x \in \Omega \setminus M_{\Omega, n}^A$  we can find real numbers  $\alpha < \beta$  such that  $x \in \{z\} \times (\alpha, \beta) \subseteq \Omega$  and either  $H_1(\{z\} \times (\alpha, \beta) \setminus A) = 0$  or  $H_1(\{z\} \times (\alpha, \beta) \setminus A^c) = 0$ . From b) it follows that  $x \in \text{int}_e A$  or  $x \in \text{int}_e(A^c)$ , hence  $x \notin \text{fr}_e A$ . This completes the proof.

**3.3. Lemma.** *Let  $X, Y \subseteq \mathbb{R}$  be two disjoint sets of type  $F_\sigma$  such that every  $x \in X$  is a bilateral accumulation point of  $\mathbb{R} \setminus Y$  and every  $y \in Y$  is a bilateral accumulation point of  $\mathbb{R} \setminus X$ . Then  $]a, c[ \setminus (X \cup Y)$  is nonempty whenever  $a \in X$  and  $c \in Y$ .*

*Proof.* We have  $X = \bigcup_{k=1}^\infty X_k$  and  $Y = \bigcup_{k=1}^\infty Y_k$  where  $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$  and  $Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \dots$  are closed. Let  $a \in X$  and  $c \in Y$  be arbitrarily chosen. Suppose, if possible, that  $X \cup Y \supseteq ]a, c[$ . Our assumptions imply that  $X \cap ]a, c[ \neq \emptyset$ ,  $Y \cap ]a, c[ \neq \emptyset$ , every  $x \in X \cap ]a, c[$  is a bilateral accumulation point of  $X$ , and every  $y \in Y \cap ]a, c[$  is a bilateral accumulation point of  $Y$ . We can construct by induction an infinite sequence of nonnegative integers  $\{k_r\}_{r=0}^\infty$  and the sequences  $\{a_r\}_{r=0}^\infty$  and  $\{c_r\}_{r=0}^\infty$  of real numbers such that  $a_0 = a$ ,  $c_0 = c$ ,  $k_0 = 0$  and, for every positive integer  $r$ ,

$$a_r \in X_{k_r} \cap ]a_{r-1}, c_{r-1}[ , \quad c_r \in Y_{k_r} \cap ]a_{r-1}, c_{r-1}[ \text{ and } (X_{k_r} \cup Y_{k_r}) \cap ]a_r, c_r[ = \emptyset,$$

as follows:

Put  $a_0 = a$ ,  $c_0 = c$  and  $k_0 = 0$ . If  $a_{r-1}, c_{r-1}$  and  $k_{r-1}$  have been constructed choose  $\tilde{a}_r \in X \cap ]a_{r-1}, c_{r-1}[$  and  $\tilde{c}_r \in Y \cap ]a_{r-1}, c_{r-1}[$  arbitrarily, and an integer  $k_r$  so large that  $k_r > k_{r-1}$ ,  $\tilde{a}_r \in X_{k_r}$  and  $\tilde{c}_r \in Y_{k_r}$ . As  $[\tilde{a}_r, \tilde{c}_r] \cap X_{k_r}$  and  $[\tilde{a}_r, \tilde{c}_r] \cap Y_{k_r}$  are two disjoint nonempty compact sets, we can choose their points  $a_r$  and  $c_r$ , respectively, such that they realize the distance of these sets. Then we have  $a_r \in X_{k_r} \cap ]a_{r-1}, c_{r-1}[$ ,  $c_r \in Y_{k_r} \cap ]a_{r-1}, c_{r-1}[$  and  $(X_{k_r} \cup Y_{k_r}) \cap ]a_r, c_r[ = \emptyset$ .

Now it is easy to see that for the sequence  $\{[a_r, c_r]\}_{r=1}^\infty$  of intervals with the above properties we have

$$\emptyset \neq \bigcap_{r=1}^\infty [a_r, c_r] = \bigcap_{r=1}^\infty ]a_r, c_r[ \subseteq ]a, c[ \setminus (X \cup Y).$$

That completes the proof.

**3.4. Definition.** As the density of the ball  $B(0, 1) \subseteq \mathbb{R}^n$  is equal to  $\frac{1}{2}$  at every point of its boundary, we can fix for any positive integer  $k$  the constant  $\delta(k)$  (depending only on  $k$  and on dimension  $n$ ) such that  $0 < \delta(k) \leq 1$  and

$$H_n[B(e_1, \delta(k)) \cap B(0, 1)] \geq \frac{V(n)}{2} \left(1 - \frac{1}{8k}\right) [\delta(k)]^n .$$

As the function

$$y \longmapsto H_n[B(y, \delta(k)) \cap B(0, 1)]$$

is continuous on  $\mathbb{R}^n$  we can fix for  $k$  and  $\delta(k)$  as above the constant  $\varepsilon(k) > 0$  such that

$$H_n[B(y, \delta(k)) \cap B(0, 1)] \geq \frac{V(n)}{2} \left(1 - \frac{1}{4k}\right) [\delta(k)]^n$$

whenever  $y \in [e_1, (1 + \varepsilon(k))e_1]$ .

According to the homogeneity and the invariance under Euclidean isometries of  $H_n$  we see that

$$H_n[B(y, \delta(k)r) \cap B(x, r)] \geq \frac{V(n)}{2} \left(1 - \frac{1}{4k}\right) [\delta(k)r]^n$$

whenever  $k$  is a positive integer,  $0 < r < \infty$ ,  $x \in \mathbb{R}^n$  and  $y \in B(x, (1 + \varepsilon(k))r) \setminus U(x, r)$ .

**3.5. Theorem.** Let  $\Omega \subset \mathbb{R}^n$  be nonempty open,  $A \subseteq \mathbb{R}^n$  be arbitrary and  $\tau \in S^{n-1}$ . Then

$$\text{var}_{\Omega, \tau}(A) = \mu_\tau(\Omega \cap fr_e A) = \mu_\tau(\Omega \cap fr_{pr} A) .$$

*Proof.* We may assume that  $\tau = e_n$ . Since  $fr_{pr} A \subseteq fr_e A$ , it is sufficient to prove the inequalities

$$\mu_n(\Omega \cap fr_e A) \leq \text{var}_{\Omega, n}(A) \leq \mu_n(\Omega \cap fr_{pr} A) .$$

- (i) To prove the first inequality we may assume  $\text{var}_{\Omega, n}(A) < \infty$  and then  $A \cap \Omega$  is  $H_n$  measurable. According to Lemma 3.2 we have,

$$N(p_n, \Omega \cap fr_e A, z) \leq m_{\Omega, n}^A \quad \text{for } H_{n-1} \text{ a.e. } z \in \mathbb{R}^{n-1}(n).$$

Using Lemma 2.4 and 2.6 and integrating the above inequality we get

$$\mu_n(\Omega \cap fr_e A) = \int_{\mathbb{R}^{n-1}(n)} N(p_n, \Omega \cap fr_e A, z) dH_{n-1}(z) \leq \int_{\mathbb{R}^{n-1}(n)} m_{\Omega, n}^A(z) dH_{n-1}(z) = \text{var}_{\Omega, n}(A).$$

- (ii) To prove the inequality

$$\text{var}_{\Omega, n}(A) \leq \mu_n(\Omega \cap fr_{pr} A)$$

we may assume that

$$\mu_n(\Omega \cap fr_{pr} A) < \infty .$$

Then obviously  $H_n(\Omega \cap fr_{pr} A) = 0$ , the set  $\Omega \cap A$  is  $H_n$  measurable according to the Remark 2.2 and  $A \cap \Omega \sim \Omega \cap \text{int}_{pr} A$ . Hence

$$\text{var}_{\Omega, n}(A) = \text{var}_{\Omega, n}(\text{int}_{pr} A)$$

and due to the Lemma 2.6 and 2.4 it is sufficient to prove that

$$(1) \quad m_{\Omega, n}^{\text{int}_{pr} A}(z) \leq N(p_n, \Omega \cap fr_{pr} A, z) \quad \text{for } H_{n-1} \text{ a.e. } z \in \mathbb{R}^{n-1}(n).$$

For any positive integer  $k$  we put

$$A(k) = \left\{ x \in \mathbb{R}^n : H_n(B(x, r) \setminus A) \leq \frac{V(n)}{2} \left(1 - \frac{1}{k}\right) r^n \quad \text{if } r \in \left(0, \frac{1}{k}\right) \right\},$$

$$C(k) = \left\{ x \in \mathbb{R}^n : H_n(B(x, r) \cap A) \leq \frac{V(n)}{2} \left(1 - \frac{1}{k}\right) r^n \quad \text{if } r \in \left(0, \frac{1}{k}\right) \right\}.$$

Obviously  $A(k)$  and  $C(k)$  are closed and  $A(k) \uparrow \text{int}_{pr} A, C(k) \uparrow \text{int}_{pr}(A^c)$  with  $k \uparrow +\infty$ . For any pair of positive integers  $(k, m)$  we put

$$A^+(k, m) = \left\{ x \in A(k) : Q\left(x, \frac{8}{m}\right) \subseteq \Omega, \left[x, x + \frac{8}{m}e_n\right] \subseteq \text{int}_{pr}(A^c) \right\},$$

$$A^-(k, m) = \left\{ x \in A(k) : Q\left(x, \frac{8}{m}\right) \subseteq \Omega, \left[x, x - \frac{8}{m}e_n\right] \subseteq \text{int}_{pr}(A^c) \right\},$$

$$C^+(k, m) = \left\{ x \in C(k) : Q\left(x, \frac{8}{m}\right) \subseteq \Omega, \left[x, x + \frac{8}{m}e_n\right] \subseteq \text{int}_{pr} A \right\},$$

$$C^-(k, m) = \left\{ x \in C(k) : Q\left(x, \frac{8}{m}\right) \subseteq \Omega, \left[x, x - \frac{8}{m}e_n\right] \subseteq \text{int}_{pr} A \right\},$$

$$B = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} [A^+(k, m) \cup A^-(k, m) \cup C^+(k, m) \cup C^-(k, m)].$$

To prove (1) it is sufficient to prove that

$$(2) \quad m_{\Omega, n}^{\text{int}_{pr} A}(z) \leq N(p_n, \Omega \cap fr_{pr} A, z) \quad \text{if } z \in \mathbb{R}^{n-1}(n) \setminus p_n(B), \text{ and}$$

$$(3) \quad H_{n-1}[p_n(B)] = 0.$$

First we make the following observation :

If  $z \in \mathbb{R}^{n-1}(n) \setminus p_n(B)$  then the assumptions of lemma 3.3 are fulfilled with  $L_n(z), L_n(z) \cap \Omega \cap \text{int}_{pr} A$  and  $L_n(z) \cap \Omega \cap \text{int}_{pr}(A^c)$  instead of  $\mathbb{R}, X$  and  $Y$ , respectively. Therefore for such  $z$  there exists  $b \in ]a, c[ \cap fr_{pr} A$  whenever  $a \in L_n(z) \cap \text{int}_{pr} A$  and  $c \in L_n(z) \cap \text{int}_{pr}(A^c)$  are such that  $[a, c] \subseteq \Omega$ .

To prove(2) we fix  $z \in \mathbb{R}^{n-1}(n) \setminus p_n(B)$ . We may assume that

$$(4) \quad N(p_n, \Omega \cap fr_{pr} A, z) < \infty.$$

Then even the inclusion

$$(5) \quad M_{\Omega, n}^{\text{int}_{pr} A}(z) \subseteq \Omega \cap fr_{pr} A$$

holds. To prove it we fix a point  $x \in (L_n(z) \cap \Omega) \setminus fr_{pr} A$ . According to (4) we may fix  $\varepsilon > 0$  such that

$$[x - \varepsilon e_n, x + \varepsilon e_n] \subseteq \Omega \quad \text{and} \quad [x - \varepsilon e_n, x + \varepsilon e_n] \cap fr_{pr} A = \emptyset.$$

According to the observation made above, we get either

$$[x - \varepsilon e_n, x + \varepsilon e_n] \subseteq \text{int}_{pr} A \quad \text{or} \quad [x - \varepsilon e_n, x + \varepsilon e_n] \subseteq \text{int}_{pr}(A^c) \subseteq (\text{int}_{pr} A)^c.$$

Both cases imply that  $x$  does not belong to  $M_{\Omega, n}^{\text{int}_{pr} A}(z)$ . This completes the proof of (5) and (2).

To prove (3) we fix the positive integers  $k, m$  and we will prove that

$$(6) \quad H_{n-1}\{p_n[A^+(k, m)]\} = 0.$$

(In the same way one can prove that  $p_n[A^-(k, m)], p_n[C^+(k, m)]$  and  $p_n[C^-(k, m)]$  are  $H_{n-1}$  null sets.)

To prove (6) we put for any integer  $s$

$$A^+(k, m, s) = \left\{ x \in A^+(k, m) : \frac{s-1}{m} < x_n \leq \frac{s}{m} \right\}$$

and assume, on the contrary, that for some fixed  $s$  we have

$$(7) \quad H_{n-1}\{p_n[A^+(k, m, s)]\} > 0.$$

Due to Lebesgue outer density theorem we can fix  $z_0 \in p_n[A^+(k, m, s)]$  which is an outer density point (in the space  $\mathbb{R}^{n-1}(n)$ ) of  $p_n[A^+(k, m, s)]$ .

For every  $z \in p_n[A^+(k, m, s)]$  obviously one can choose a point  $x \in A^+(k, m, s)$  such that  $p_n(x) = z$ . Then

$$Q\left(x, \frac{8}{m}\right) \subseteq \Omega \quad \text{and} \quad \left]z + \frac{s}{m}e_n, z + \frac{s+7}{m}e_n\right[ \subseteq \left]x, z + \frac{s+7}{m}e_n\right[ \subseteq \text{int}_{pr}(A^c).$$

We put  $x_1 = z_0 + \frac{s+1}{m}e_n$ . According to the choice of  $z_0$  we can fix positive  $r_0$  such that  $r_0 \leq \frac{1}{m}, r_0 \leq \frac{1}{k}$  and

$$(8) \quad \frac{1}{V(n-1)r_0^{n-1}} H_{n-1}\{p_n[U(x_1, r_0)] \cap p_n[A^+(k, m, s)]\} \geq 1 - \frac{V(n)}{16kV(n-1)}[\delta(k)]^n$$

where  $\delta(k)$  is the constant from 3.4. Putting  $S = p_n[U(x_1, r_0)]$  from (8) we get

$$(9) \quad H_{n-1}\{S \cap p_n[A^+(k, m, s)]\} \geq H_{n-1}(S) - \frac{V(n)}{16k}[\delta(k)]^n r_0^{n-1}.$$

According to the choice of  $x_1$  and  $r_0$  we see that  $U(x_1, r_0) \cap A^+(k, m, s) = \emptyset$ . We can define the number  $t_0 \in ]\frac{s-1}{m}, \frac{s+1}{m}]$  by the formula

$$t_0 = \sup \left\{ t \in \left(\frac{s-1}{m}, \frac{s+1}{m}\right] : U(z_0 + te_n, r_0) \cap A^+(k, m, s) \neq \emptyset \right\}$$

and we put  $x_0 = z_0 + t_0 e_n$ . The ball  $U(x_0, r_0)$  has the following properties :

$$(10) \quad L_n(z) \cap U(x_0, r_0) \subseteq \text{int}_{pr}(A^c) \quad \text{whenever} \quad z \in p_n[A^+(k, m, s)]$$

$U(x_0, r_0) \subseteq \Omega$ , especially  $A \cap U(x_0, r_0)$  is  $H_n$  measurable,

$$A^+(k, m, s) \cap [B(x_0, (1+\varepsilon)r_0) \setminus U(x_0, r_0)] \neq \emptyset$$

whenever  $\varepsilon > 0$ .

We fix some  $y \in A^+(k, m, s) \cap [B(x_0, (1+\varepsilon(k))r_0) \setminus U(x_0, r_0)]$ , where  $\varepsilon(k)$  is as in 3.4. From (3.4) it follows that

$$(11) \quad H_n\{[B(y, \delta(k)r_0) \cap B(x_0, r_0)]\} \geq \frac{V(n)}{2} \left(1 - \frac{1}{4k}\right) [\delta(k)r_0]^n.$$

We define the function

$$g : \mathbb{R}^{n-1}(n) \longrightarrow [0, 2r_0]$$

by the formula

$$g(z) = H_1\{[L_n(z) \cap U(x_0, r_0)] \setminus \text{int}_{pr}(A^c)\}, \quad z \in \mathbb{R}^{n-1}(n).$$

According to remark 2.2 we have

$$[U(x_0, r_0) \setminus \text{int}_{pr}(A^c)] \sim [U(x_0, r_0) \cap A]$$

and using Fubini's theorem we get that  $g$  is  $H_{n-1}$  measurable and

$$(12) \quad H_n[U(x_0, r_0) \cap A] = \int_{\mathbb{R}^{n-1}(n)} g(z) dH_{n-1}(z).$$

From (10) we see that

$$\begin{aligned} g(z) &= 0 \quad \text{whenever } z \in p_n[A^+(k, m, s)], \quad \text{and obviously} \\ g(z) &= 0 \quad \text{whenever } z \in \mathbb{R}^{n-1}(n) \setminus S. \end{aligned}$$

Especially the set

$$\{z \in \mathbb{R}^{n-1}(n) : g(z) > 0\} = S \setminus \{z \in S : g(z) = 0\}$$

is  $H_{n-1}$  measurable and from (9) we get

$$\begin{aligned} (13) \quad H_{n-1}\{z \in \mathbb{R}^{n-1}(n) : g(z) > 0\} &= H_{n-1}(S) - H_{n-1}\{z \in S : g(z) = 0\} \leq \\ &\leq H_{n-1}(S) - H_{n-1}\{S \cap p_n[A^+(k, m, s)]\} \leq \frac{V(n)}{16k} [\delta(k)]^n r_0^{n-1} \end{aligned}$$

From (12) and (13) we see that

$$(14) \quad H_n[U(x_0, r_0) \cap A] \leq 2r_0 H_{n-1}\{z \in \mathbb{R}^{n-1}(n) : g(z) > 0\} \leq \frac{V(n)}{8k} [\delta(k)r_0]^n.$$

It is obvious that

$$(15) \quad H_n[B(y, \delta(k)r_0) \setminus A] \geq H_n[B(x_0, r_0) \cap B(y, \delta(k)r_0)] - H_n[A \cap U(x_0, r_0)].$$

According to (11), (14) and (15) we eventually get

$$(16) \quad H_n[B(y, \delta(k)r_0) \setminus A] \geq \frac{V(n)}{2} \left(1 - \frac{1}{2k}\right) [\delta(k)r_0]^n.$$

As  $y \in A^+(k, m, s) \subseteq A(k)$  and  $\delta(k)r_0 \leq r_0 \leq \frac{1}{k}$ , the inequality (16) contradicts with our definition of  $A(k)$ . Hence the assumption made in (7) leads to the contradiction and consequently (6) and (3) hold. This completes the proof.

**3.6. Corollary.** *Let  $\Omega \subseteq \mathbb{R}^n$  be nonempty open and  $A \subseteq \mathbb{R}^n$  be arbitrary. Then the following are equivalent :*

- (i)  $P_\Omega(A) < \infty$ .
- (ii) *There exist linearly independent vectors  $\tau_1, \tau_2, \dots, \tau_n \in S^{n-1}$  such that  $\mu_{\tau_i}(\Omega \cap fr_{pr}A) < \infty$  for  $i = 1, 2, \dots, n$ .*

**3.7. Lemma.** *Let  $\Omega \subseteq \mathbb{R}^n$  be a nonempty open set and  $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_n)$  be an  $\mathbb{R}^n$  valued Borel measure on  $\Omega$  with finite total variation. For any  $\tau = (\tau_1, \tau_2, \dots, \tau_n) \in S^{n-1}$  let  $\Phi_\tau$  stand for the signed Borel measure  $\sum_{i=1}^n \tau_i \Phi_i$ . Then*

$$\|\Phi\| = \frac{1}{2V(n-1)} \int_{S^{n-1}} \|\Phi_\tau\| dH_{n-1}(\tau).$$

*Proof.* Let  $v : \Omega \rightarrow \mathbb{R}^n$  be the Radon-Nikodym derivative of  $\Phi$  with respect to its variational measure  $|\Phi|$ . Then  $v$  is a  $|\Phi|$  measurable  $\mathbb{R}^n$  valued function and  $|v(x)| = 1$  for  $|\Phi|$  a.e.  $x \in \Omega$  (see [FH3, 2.5.12]).

As  $\Phi_\tau(B) = \int_B \tau \circ v(x) d|\Phi|(x)$  for any Borel set  $B \subseteq \Omega$ , clearly

$$\|\Phi_\tau\| = \int_\Omega |\tau \circ v(x)| d|\Phi|(x).$$

Integrating over  $S^{n-1}$  and using Fubini's theorem we get

$$\int_{S^{n-1}} \|\Phi_\tau\| dH_{n-1}(\tau) = \int_\Omega \left( \int_{S^{n-1}} |\tau \circ v(x)| dH_{n-1}(\tau) \right) d|\Phi|(x).$$

As  $S^{n-1}$  and  $H_{n-1}$  are invariant under orthonormal transformations of  $\mathbb{R}^n$ ,

$$\int_{S^{n-1}} |\tau \circ w| dH_{n-1}(\tau) = |w| \int_{S^{n-1}} |\tau_1| dH_{n-1}(\tau) = 2V(n-1)|w| \quad \text{for any } w \in \mathbb{R}^n.$$

(See [FH3, 3.2.13] for an exact computation of constants  $V(n-1)$  and  $\int_{S^{n-1}} |\tau_1| dH_{n-1}(\tau)$ .) Hence

$$\int_{S^{n-1}} \|\Phi_\tau\| dH_{n-1}(\tau) = 2V(n-1) \int_\Omega |v(x)| d|\Phi|(x) = 2V(n-1)\|\Phi\|,$$

that completes the proof.

**3.8. Lemma.** *Let  $\Omega \subseteq \mathbb{R}^n$  be nonempty open and  $A \subseteq \mathbb{R}^n$  be arbitrary. Then*

$$P_{\Omega}(A) = \frac{1}{2V(n-1)} \int_{S^{n-1}} \text{var}_{\Omega, \tau}(A) dH_{n-1}(\tau).$$

*Proof.* If  $P_{\Omega}(A) = +\infty$  then clearly  $\text{var}_{\Omega, \tau}(A) = +\infty$  for  $H_{n-1}$  a.e.  $\tau \in S^{n-1}$  and the statement holds. If  $P_{\Omega}(A) < +\infty$  then  $D\chi^A$ , as the distribution over  $\Omega$ , is an  $\mathbb{R}^n$  valued Borel measure on  $\Omega$  with finite total variation. As  $P_{\Omega}(A) = \|D\chi^A\|$  and  $\text{var}_{\Omega, \tau}(A) = \|\tau \circ D\chi^A\|$ , the result follows from the lemma above.

**3.9. Theorem.** *Let  $\Omega \subseteq \mathbb{R}^n$  be nonempty open and  $A \subseteq \mathbb{R}^n$  be arbitrary. Then the following equalities hold :*

$$P_{\Omega}(A) = \frac{1}{2V(n-1)} \int_{S^{n-1}} \text{var}_{\Omega, \tau}(A) dH_{n-1}(\tau) = \mathfrak{S}_1^{n-1}(\Omega \cap fr_e A) = \mathfrak{S}_1^{n-1}(\Omega \cap fr_{pr} A).$$

*Proof.* Integrating equalities from Theorem 3.5 over  $S^{n-1}$  and using Lemma 2.4(ii) we get

$$\frac{1}{2V(n-1)} \int_{S^{n-1}} \text{var}_{\Omega, \tau}(A) dH_{n-1}(\tau) = \mathfrak{S}_1^{n-1}(\Omega \cap fr_e A) = \mathfrak{S}_1^{n-1}(\Omega \cap fr_{pr} A).$$

Due to Lemma 3.8 the first term is equal to  $P_{\Omega}(A)$ . That completes the proof.

**Remark.** The result  $P_{\Omega}(A) = \mathfrak{S}_1^{n-1}(\Omega \cap fr_e A)$  for an arbitrary set  $A \subseteq \mathbb{R}^n$  is known (see e.g. [FH3, 4.5.6 and 4.5.11]), but our simple proof does not use deep results of De Giorgi, Federer and Volpert on sets with finite perimeter. Combining our results with their we could replace integralgeometric measure by Hausdorff measure in the theorem above.

#### REFERENCES

- [FH1] H.FEDERER, *The Gauss-Green theorem*, Trans.Amer.Math.Soc. **58** (1945), 44-76.
- [FH2] ———, *A note on the Gauss-Green theorem*, Proc.Amer.Math.Soc. **9** (1958), 447-451.
- [FH3] ———, *Geometric measure theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1969.
- [FF] H.FEDERER, W.H.FLEMING, *Normal and integral currents*, Ann. of Math. **72** (1960), 458-520.
- [DG1] E.De GIORGI, *Su una teoria generale della misura  $(r-1)$ -dimensionale in uno spazio ad  $r$  dimensioni*, Annali di Mat. pura ed applicata Ser. 4, **36** (1954), 191-213.
- [DG2] ———, *Nuovi teoremi relativi alle misure  $(r-1)$ -dimensionale in uno spazio ad  $r$  dimensioni*, Ricerche di Mat. **4** (1955), 95-113.
- [KK] K.KRICKEBERG, *Distributionen, Funktionen beschränkter Variation und Lebesguescher Inhalt nichtparametrischer Flächen*, Annali di Mat. pura ed applicata **44** (1957), 105-133.
- [MJ] J. MAŘÍK, *The surface integral*, Czech.Math.Journal **81** (1958), 522-558.
- [VA] A. J.VOLPERT, *Prostranstva BV i kvazilinejnyje uravnenija*, Mat.sbornik **73** (1967), 255-302.