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**Lipschitz minimizers
of the 3–well problem having
gradients of bounded variation**

by

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Abstract

We consider solutions of the three well problem. These are maps u satisfying $\nabla u \in SO(3)A^1 \cup SO(3)A^2 \cup SO(3)A^3$ a.e. in $\Omega \subset \mathbb{R}^3$ where the A^i are the diagonal matrices representing the cubic to tetragonal transformation. It is known that such maps can behave rather irregular. Under the additional assumption that the phase sets $E^i = \{x\,; \nabla u(x) \in SO(3)A^i\}$ are of finite perimeter we conclude that u is locally a function of one variable only.

1 Introduction

We consider the following kind of problem

$$\begin{array}{c}\text{Let } K \text{ be a compact set of } m \times n \text{ matrices. What are properties of} \\ \text{Lipschitz mappings } u : \Omega \to \mathbb{R}^m \text{ satisfying } \nabla u(x) \in K \text{ a.e. in } \Omega. \\ \text{In particular, which regularity can nontrivial solutions posses?}\end{array} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ and nontriviality of a solution u means its nonaffinity. The latter is usually enforced by suitable boundary conditions

$$u(x) = \phi(x) \text{ on } \partial\Omega.$$

Existence and further properties of solutions of this problem were systematically studied and well understood essentially in the scalar case $m = 1$ only. These results are based on the methods of viscosity solutions, see e.g. [1], [2] or [11], which has no satisfactory generalization to the vectorial case being addressed here. For this case existence results have been obtained only recently in [15] and [6], but one should also recall the older existence results for the C^1-case [12].

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The most interesting questions arise, of course, if K is not convex (and even not quasi-convex, see [5] or [6] for the definition). A prototype of such problems is given by choices like

$$K = \bigcup_{i=1}^{k} SO(n) \cdot A^i, \quad A^i \text{ being } n \times n \text{ symmetric, positive.} \tag{1.2}$$

This particular form of K is also motivated by its physical relevance. In fact, the zero energy deformation gradients in the phase transition of shape memory alloy crystals form precisely sets K of this type (with $n = 3$). See [3] for a mathematical description of the situation. The problem (1.1) given by a set K like in (1.2) is called the k-well problem. Because of the fine scale oscillating patterns observed in the shape memory crystals, the notion microstructure refers sometimes in the literature to nontrivial solutions of (1.1) and (1.2) and the structures the induce in their domain and target.

The most simple set K to start with are of course those consisting of two matrices X and Y. Just for simplicity (compare with [3]) we assume moreover that the gradient takes these values on two sets $M_X, M_Y \subset \Omega$ which touch along a sufficiently regular (e.g. C^1) boundary ∂M. Then continuity of u along the $(n-1)$-dimensional tangent spaces of ∂M implies that

$$X - Y = a \otimes n$$

where $a \in \mathbb{R}^n$ and n is the normal of an hyperplane locally containing ∂M. In this situation we say that X and Y are rank-1 connected (across an interface with normal n).

Such rank-1 connections exists for instance if

$$K = SO(2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cup SO(2) \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \text{ where } 0 < \lambda < 1 < \mu.$$

Been given this set K it is in fact possible to construct solutions to (1.1) which are so called local laminates. These are mappings u which have locally a very simple structure. Around each point $x \in \Omega$ there is a sufficiently small ball B_x on which the gradient ∇u takes only two values and is a function of one variable only (in a suitably chosen coordinate system). However, since any $X \in K$ is rank-one connected to two matrices in K the gradient can take altogether more than two values.

But since these laminates locally not really differ from affine maps, the question arises if it is possible to construct for such K mappings u which have a substantially more complicated local structure. Simultaneously one would like to know if this nonlaminatial solutions can keep at least some regularity or if they have to be very wild.

The first question was answered by [15] in the affirmative using Gromov's ideas on convex integration, see [12]. The solutions constructed in [15] are obtained as limits of sequences u_n of approximate solutions with rapidly increasing variations of ∇u_n. Therefore, the exact solutions are Lipschitz without any known additional regularity properties. In [6] solutions to (1.1),(1.2) are obtained using the Baire category principle. Even if this shows that such solutions prevail in a certain sense, category arguments do not allow us to conclude additional regularity.

The second question reminds problems in geometry were also partial differential relations like in (1.1) play an important role. There are already classical examples that regularity conditions have to be added in order to exclude unexpected solutions. One of the most famous concerns isometric immersions $\mathbb{S}^2 \to \mathbb{R}^3$. Due to work by Hilbert such maps are trivial provided they are of class C^2 but C^1-immersions with images of arbitrarily small diameters can be found, see [16],[14]. We refer to Gromov's book [12] for a systematic exposition.

First progress in our question was obtained in [8] where it was shown that in the 2-well case already mild additional regularity is sufficient to rule out solutions which are not local laminates. The regularity assumption required is essentially that $\nabla u \in BV(\Omega)$. This condition arises naturally from physical situations where not only the bulk energy like $\int_\Omega W(\nabla u(x))\, dx$ but also surface energy is taken into account. It reflects the fact that we require some kind of finite interface between the phases, i.e. between the two sets where the gradients belongs to one fixed well. For $\Omega \subset \mathbb{R}^2$ open and convex, $K = \bigcup_j^2 SO(2) \cdot A^j$ the authors of [8] proved the following.

Theorem 1.1 *Let $u : \Omega \to \mathbb{R}^2$ be a Lipschitz solution of (1.1) such that the sets $\{x \in \Omega\ ;\ \nabla u(x) \in SO(2) \cdot A^j\}$ have finite perimeter, then $\nabla u \in BV(\Omega)$ and u is a local laminate in Ω.*

In [8] the authors obtained their result in two steps. First, they generalized Liouville's theorem about the affinity of infinitesimal rotations on euclidean domains to the BV-context. More precisely, they showed the following. Let $E \subset \Omega$ be BV-indecomposable which essentially means that E has a boundary of finite area and can not be decomposed into two nontrivial parts without creating new boundary of positive area. (So this obviously implies connectedness of E). If $u : \Omega \to \mathbb{R}^n$ is Lipschitz , $\det(\nabla u) \geq c > 0$ on Ω and $\nabla u(x) \in SO(n)$ a.e. in E then $\chi_E \cdot \nabla u \in BV(\Omega)$ and u is affine on E (see Definition 2.3 and Theorem 2.4 for the details). In fact, the first conclusion holds even if one only knows about E that is it of finite perimeter, indecomposability can be omitted.

And indeed, in [8] only the first conclusion was used for the final second step. Here the authors managed to express the gradient of u in the form $\nabla u(x) = \exp(i\theta(x)) \cdot ((\nabla u(x))^T \nabla u(x))^{1/2}$, $\theta \in BV(\Omega, \mathbb{R})$ and to lift the problem essentially to the wave equation. The possibility to linearize the problem in this way is (among other things) essentially based on the commutativity of the involved gauge group $SO(2)$. Hence, in the moment there is no hope to attack the problems of three or more wells in a similar way.

In the present paper we investigate the 3 well problem with the noncommutative gauge group $SO(3)$ by means of a new approach. It exploits geometric properties of the BV-indecomposable parts of the phases of u in more detail.

Our final result is the following.

Theorem 1.2 *Let $A^1 = \operatorname{diag}(\lambda^{-2}, \lambda, \lambda)$, $A^2 = \operatorname{diag}(\lambda, \lambda^{-2}, \lambda)$, $A^3 = \operatorname{diag}(\lambda, \lambda, \lambda^{-2})$, let λ be positive, $\Omega \subset \mathbb{R}^3$ be open $u : \Omega \to \mathbb{R}^3$ be Lipschitz . If the sets*

$$E^j = \{x \in \Omega\ ;\ \nabla u(x) \in SO(3) \cdot A^j\}, j = 1, 2, 3$$

are of locally finite perimeter in Ω then u is a locally laminate.

The three wells considered in this statement are often called the tetragonal wells, since they consist precisely of the zero energy gradients in case of a cubic to tetragonal phase transition. For $\lambda = 1$ is the statement just a particular case of Liouville's theorem, so we will assume $\lambda \neq 1$. Moreover, to keep the computations in Section 5 as short as possible we replace the matrices A_1, A_2, A_3 by their rescalings

$$D^1 = \mathrm{diag}(a,1,1),\, D^2 = \mathrm{diag}(1,a,1) \text{ and } D^3 = \mathrm{diag}(1,1,a),\, a \in (0,\infty) \setminus \{1\}. \tag{1.3}$$

In the proof we consider a fixed lipschitz solution u and proceed as follows. By the already mentioned Liouville type result we know that the map u behaves particular simple, i.e. affine, on the BV-indecomposable subsets of the individual phases E^j. Motivated by this observation we study in more detail the geometry of BV-components which are the maximal sets open in the measure-theoretical sense and connected in the BV-topology. They define in a canonical way a countable partition of Ω whose (BV-)boundary consists in principle only of the original phase boundaries. A basic notion is that of a regular point, i.e. a point belonging to some of the BV-components. For a better imagination of this concept, one can think of the regular points as those where u is differentiable. But our main tool is the the notion of regularity of a line g. It essentially means that u behaves along this line g like a laminate. To be more precise, the following holds on regular lines.

1) There are only finitely many exceptional points on g and any interval I on g between two such points is entirely contained in a single BV-component, in particular ∇u is constant on I.

2) In the neighbourhood of an exceptional $x \in g$ the gradient $(\nabla u)_{|g}$ behaves like the gradient of a laminate with a single interface transversally intersected by g in x.

Therefore, if x, y are regular points on a regular line g then $\nabla u(y)$ can be reached from $\nabla u(x)$ via a sequence of rank-one connections across surfaces transversally to g. The further parts of the proof essentially rely on the fact (proven in Lemma 3.6) that for any given direction d almost each line g in this direction is regular. In particular, the path of rank-one connections between $\nabla u(x)$ and $\nabla u(y)$ exists whenever x and y are regular points.

Beside this observation about the fine geometric properties of BV-decompositions, we have also to use the special structure of rank-one connections between $SO(n)$ wells, for more details see [3]. First note that there are no such connections inside individual wells. Next, for two fixed wells given by A^i, A^j all rank-one connections between them are of one of the following two kinds.

$$Q \cdot A^i - Q \cdot Q^{i,j,1} A^j = (Q a^{i,j,1}) \otimes n^{i,j,1}, Q \in SO(n),$$
$$Q \cdot A^i - Q \cdot Q^{i,j,2} A^j = (Q a^{i,j,2}) \otimes n^{i,j,2}, Q \in SO(n).$$

Due to our transversality result, along a regular line orthogonal to some of the $n^{i,j,l}$ the corresponding rank-one connection can not occur between neighbouring points, i.e. points separated by one exceptional point only.

We assume now that the finite subgroup of $SO(n)$ generated by $\{Q^{i,j,l} \ ; \ i,j \leq k, l = 1,2\}$ is free. This assumption is in the tetragonal and similar cases certainly not too special since Baire category and other kind of arguments, see [17], show that it is fulfilled if we choose the parameter a in (1.3) generically[1)]. Under this assumption our strategy is rather straightforward. In any nonlaminatial microstructure we can find two nonparallel interfaces S_1 and S_2 sufficiently close to each other and far away from the boundary of Ω. Then we can choose a parallelogram $x_1x_2x_3x_4$ in Ω such that $x_1 - x_2 \| x_3 - x_4 \| S_1$ and $x_2 - x_3 \| x_4 - x_1 \| S_2$. Moreover, we can arrange that all x_i are regular points and that $[x_1, x_2]$ intersects S_2 and interfaces of type S_2 precisely once, similar for $[x_2, x_3]$. Now it can be very easily seen that the compatibility conditions between neighbouring points lead to a nontrivial expression of the identity matrix as a product of some $Q^{i,j,l}$'s. This contradicts of course to the nonexistence of any proper cancellation rule in free groups.

In the paper, however, we will treat all possible choices of the parameter a and, therefore, we can not use this kind of arguments. For this purpose, a more careful analysis based on special properties of the tetragonal wells has to be developed. Before we go into all the technical efforts necessary for this, we try to make the ideas just explained more transparent. We show how they give a very simple proof of the 2-well result from [8], see Theorem 1.1 above. This approach avoids also the technicalities in the calculus for BV-functions which had to be handled in [8]. However, we heavily build on the Liouville theorem derived there.

We identify the members of $SO(2)$ with the unit complex numbers by understanding complex multiplication as a linear operator on $\mathbb{R}^2$. Using suitable affine changes of coordinates both in the domain and the target we can ensure that all rank-one connections between the two well $SO(2) \cdot A^1$ and $SO(2) \cdot A^2$ are of the form

$$Q \cdot \exp(i\theta_j) \cdot A^2 - Q \cdot A^1 = (Qa_j) \otimes e_j, \ Q \in SO(2) \text{ and } j = 1,2$$

where $\theta_1 + \theta_2 = 0$, $0 < |\theta_1| < \pi/2$. (The computations can be found in the proof of Theorem 5.3 in [8]). Consequently, only rank-one connections across vertical or horizontal interfaces appear. From this and the transversality condition we see the following. Suppose x, y are regular points, the segment $[x, y] \subset \Omega$ is parallel to e_j and $\nabla u(x) = Q \cdot A^1$. Then

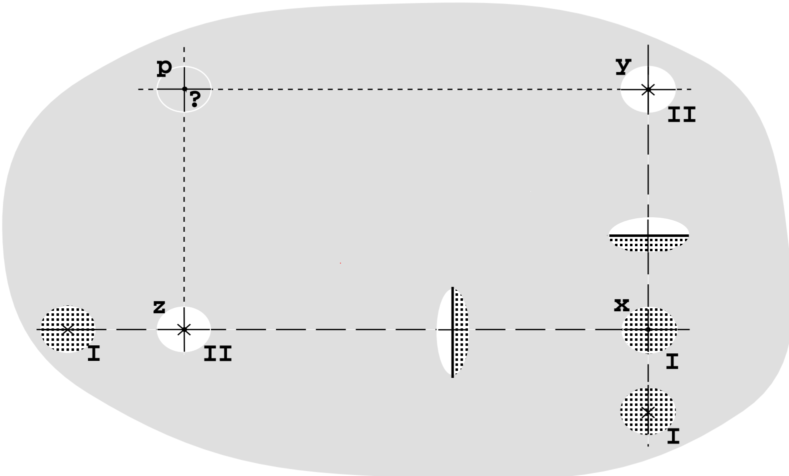
$$\nabla u(y) = \nabla u(x) \text{ if } \nabla u(y) \in SO(2) \cdot A, \text{ else} \nabla u(y) = Q \cdot \exp(i\theta_j) \cdot B. \tag{1.4}$$

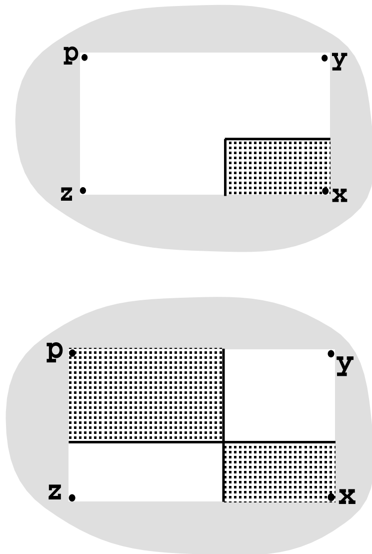
Now, assume $B(0, \sqrt{2}) \subset \Omega$ but $\nabla u_{|B(0,1)}$ is not a function of x_1 nor x_2. Therefore, we find a vertical and a horizontal line (i.e. the dashed lines in the picture below) and on each of them a pair of points (positions indicated by a $\times$) which belong to different phase sets. We consider the intersection x of these lines and use what was said before. So we find, interchanging the role of A^1 and A^2 if necessary, the points $y, z \in B(0,1)$ with $x_1 = y_1$, $x_2 = z_2$ and such that

$$\nabla u(x) = Q \cdot A^1, \nabla u(y) = Q \cdot \exp(i\theta_2) \cdot A^2 \text{ and } \nabla u(z) = Q \cdot \exp(i\theta_1) \cdot A^2.$$

[1)] However, there are situations where our assumption does not hold generically, see e.g. [4]. This phenomenon basically depends on the degree of linear dependence of the normals $n^{i,j,l}$. It remains a challenge to derive a more explicit description of the microstructures appearing in this situation

Obviously, $p = y + (z - x) = (z_1, y_2)$ is in $B(0, \sqrt{2})$ and since almost every point is regular we can assume the same for p. But if $\nabla u(p) \in SO(2) \cdot A^2$ then $\nabla u(y) = \nabla u(p) = \nabla u(z)$ and hence $\theta_1 = \theta_2$, contradiction. Hence, $\nabla u(p) \in SO(2) \cdot A^1$. Since $y - p \| e_1$ we have $\nabla u(p) = Q \cdot \exp(i(\theta_2 - \theta_1)) \cdot A^1$ and $z - p \| e_2$ yields $\nabla u(p) = Q \cdot \exp(i(\theta_1 - \theta_2)) \cdot A^1$. Consequently, $1 = \exp(2i(\theta_1 - \theta_2)) = \exp(4i\theta_1)$ which obviously contradicts to $0 < |\theta_1| < {}^{\pi}/_{2}$. The two possibilities we just ruled out are illustrated on the right hand side.





However, it has to be noted that we can not control the total number of interfaces the closed path $xzpy$ passes through (the gray zones in the picture should indicate this). In the 2-well problem this fact does not cause any trouble since we have perfect control of ∇u independent of the number of interfaces passed. Anyhow, precisely this fact is the main difference between the situation for lipschitz solutions piecewise affine on a finite decomposition of Ω and the situation in our class of lipschitz mappings with BV-gradients. This difference becomes a principal obstacle if one tries to apply our method to an arbitrary finite number of wells. This motivated us to present in Section 6 a construction of examples which show the following. The geometric information we have about the admissible normals of the BV-components and about the absolute measure of their boundaries does indeed not ensure the existence of a nontrivial closed path fulfilling an apriori bound for the number of interfaces passed.

The paper is organized as follows. Section 2 recalls basic facts about sets of finite perimeter and quotes results from [8] which we will use later on extensively. In Section 3 we give a rather selfcontained analysis of existence and properties of decompositions of sets of finite perimeter in their BV-components. However, who is not interested in the geometric measure theory techniques used there can skip the proofs. All we will need in the sequel is stated in the paragraphs 3.4 till 3.6. Section 4 presents the consequences of this analysis for an arbitrary finite number of wells and Section 5 focuses on the case of three tetragonal wells. First we introduce a kind of instructive notation for chains of rank-one connections. This allows us to show that there are distinguished directions along which we get better control about ∇u. Even if there is no simple rule like (1.4), we are still able to show that along such lines the gradient is essentially contained in some cyclic subgroup of $SO(3)$. Therefore, we are in a position to obtain a set of explicit compatibility conditions derived from suitably chosen closed paths. The proof is then finished by ruling out each of these compatibility conditions individually. Since the underlying equations consists of rational functions and polynomials up to order 12 (in the parameter a from (1.3)), the necessary computations were carried out using the symbolic calculation package maple. The Appendix 7 presents only the main arguments of these computations.

2 Generalities

In the remainder of this paper we write $B(x, r)$ for the closed ball around x of radius r. For a set $A \subset \mathbb{R}^n$ we use the notations $\mathscr{L}^n(A)$ or $|A|$ for its Lebesgue measure and $\mathscr{H}^{n-1}(A)$ stays for the $(n-1)$-dimensional Hausdorff measure - all sets appearing will be measurable in the appropriate sense.

Given a set A in n-space we denote by

$$\begin{aligned} A_I &= \{x \in \mathbb{R}^n \; ; \; \lim_{r \searrow 0} r^{-n} \mathscr{L}^n(B(x,r) \setminus A) = 0\} \\ A_O &= \{x \in \mathbb{R}^n \; ; \; \lim_{r \searrow 0} r^{-n} \mathscr{L}^n(B(x,r) \cap A) = 0\} \\ \partial_* A &= \mathbb{R}^n \setminus (A_I \cup A_O) \end{aligned}$$

the measure-theoretic interior, exterior and boundary of A. These are almost topological notions, except facts like that $A_I \subset A$ holds only if we neglect Lebesgue zero sets.

Let $\Omega \subset \mathbb{R}^n$ be open. A function $f : \Omega \to \mathbb{R}$ is said to be of bounded variation, $f \in BV(\Omega)$ for short, if there exists a measure μ_f on Ω and a function $\nu_f : \Omega \to \mathbb{S}^{n-1}$ such that

$$\int_\Omega f \,\mathrm{div} g \, dx = \int_\Omega \langle g(x), \nu_f(x) \rangle \, d\mu_f(x) \text{ for all } g \in C^1_0(\Omega, \mathbb{R}^n).$$

Frequently one writes ∇f for the vector measure $\nu_f d\mu_f$ and $\|\nabla f\|$ for its variation μ_f. For vector valued functions the definition applies coordinatewise. A set $A \subset \Omega$ is said to be of finite perimeter (in Ω) if $\chi_A \in BV(\Omega)$ and we denote by $\mathrm{Per}_\Omega(A) = \|\nabla \chi_A\|(\Omega)$ the perimeter of A relatively to Ω. A very nice introduction into the theory of BV-functions can be found in Chapter 5 of [9]. We will also need the concept of the reduced boundary $\partial^* A$ of a set of finite perimeter. It consist of those points x where $\nu_{\chi_A}(x) = \lim_{r \searrow 0} \nabla \chi_A(B(x,r))/\|\nabla \chi_A\|(B(x,r))$. For such A it can be shown that $\partial^* A \subset \partial_* A$, $\mathscr{H}^{n-1}(\partial_* A \setminus \partial^* A) = 0$ and moreover that each single point $x \in \partial^* A$ is a boundary point of A in a quite classical sense. In fact, defining $H_- = \{y \; ; \; \langle y, \nu_{\chi_A}(x) \rangle < 0\}$ and the blowups $A_{x,r}(y) = (A - x)/r$ one has $\chi_{A_{x,r}} \to \chi_{H_-}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \searrow 0$, see Theorem 5.7.1 in [9]. This picture is completed by the following statement about the geometric structure of the whole reduced boundary.

Theorem 2.1 (**Structure Theorem for Sets of finite Perimeter**)
Assume E has locally finite perimeter in $\mathbb{R}^n$. Then

(i) $\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N$, *where* $\|\partial E\|(N) = \mathscr{H}^{n-1}(N) = 0$ *and each* K_k *is a compact subset of a* C^1*-hypersurface* S_k.

(ii) Furthermore, $\nu_{E\,|K_k}$ *is normal to* S_k *for all* $k \geq 1$.

(iii) $\|\partial E\| = \mathscr{H}^{n-1} \llcorner \partial^* E$

For a proof we refer again to [9], Theorem 2 in 5.7. As an immediate consequence which will be used later we obtain the following.

Corollary 2.2 *Let* E *be a set of finite perimeter and* $d \in \mathbb{S}^{n-1}$. *Then the set* $M_d = \{x \in \partial^* E \,;\, \nu_E(x) \perp d\}$ *has under the orthogonal projection onto* $d^\perp$ *an* $(n-1)$*-zero image.*

Proof Obviously, the orthogonal projection proj_d onto $d^\perp$ does not increase distances, and hence not $\mathscr{H}^{n-1}$ measure. Hence, since $\lim_{r\searrow 0} \|\partial E\|(B(x,r)\cap E\setminus K_k)/r^{n-1} = 0$ $\mathscr{H}^{n-1}$-a.e. in K_k, it suffices to show, using the notation of Theorem 2.1 above, that for each k the set $\mathrm{proj}_d(\{x \in K_k \,;\, \nu_{K_k}(x) \perp d\})$ is of zero $(n-1)$-dimensional measure. But this is an direct consequence of the area formula, since $d \in \mathrm{proj}_d^{-1}(0) \cap \mathrm{Tan}_x S_k$ for all $x \in M_d \cap K_k$, therefore $\mathrm{rank}(\mathrm{proj}_{d|\mathrm{Tan}_x S_k}) \leq n-2$ whenever $x \in M_d \cap K_k$.

We introduce the concept of BV-indecomposability like in [8], Definition 2.11, but compare also with [10], 4.2.25.

Definition 2.3 *Let* $E \subset \mathbb{R}^n$ *be a set of finite perimeter. We say that* E *is (BV)-indecomposable if for any set* $A \subset E$ *of finite perimeter the identity*

$$Per(E) = Per(A) + Per(E \setminus A)$$

implies that

$$\mathscr{L}^n(A) = 0 \text{ or } \mathscr{L}^n(E \setminus A) = 0.$$

This notion is also in perfect agreement with the following fact, see e.g. Proposition 2.12 in [8]. The set E is BV-indecomposable iff any $f \in BV(\mathbb{R}^n)$ with $\|\nabla f\|(E) = 0$ is necessarily constant on E.

Finally we quote Liouville type Theorem 4.1 from [8].

Theorem 2.4 *Let* $\Omega \subset \mathbb{R}^n$ *be open and assume* $u : \Omega \to \mathbb{R}^n$ *is a Lipschitz map fulfilling* $\det \nabla u \geq c > 0$ *a.e. Suppose that* $E \subset \Omega$ *has finite perimeter and that*

$$\nabla u(x) \in SO(n) \text{ for a.e. } x \in E.$$

Then the function $f = \chi_E \nabla u$ *satisfies*

$$f \in BV(\Omega, \mathbb{R}^{n\times n}) \text{ and } Df \llcorner (\Omega \setminus \partial_* E) = 0.$$

If, moreover, E *is (BV)-indecomposable, then* u *is a rigid motion on the measure-theoretic interior* E_I, *i.e. there exist* $A \in SO(n)$, $b \in \mathbb{R}^n$ *such that*

$$u(x) = Ax + b \text{ for all } x \in \overline{E_I}.$$

3 Decompositions into BV-components

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be open and $\{E_k\}_{k=0}^\infty \subset \Omega$ measurable sets such that $E_0 \subset \bigcup_{k=1}^\infty E_k$ and $\sum_{k=1}^\infty Per_\Omega(E_k) < \infty$. Then*

$$\mathscr{H}^{n-1}([\Omega \cap (\partial_* E_0 \cup E_{0,I})] \setminus \bigcup_{k=1}^\infty \partial_* E_k \cup E_{k,I}) = 0.$$

Proof Denote by M the exceptional set $(\Omega \setminus E_{0,O}) \cap \bigcap_{k=1}^\infty E_{k,O}$. Clearly, it suffices to show that for any $\varepsilon \in (0, 1/2)$

$$M_\varepsilon = \{x \in M \; ; \; \limsup_{r \searrow 0} r^{-n} |B(x,r) \cap E_0| > \varepsilon\}$$

is of $\mathscr{H}^{n-1}$-measure zero.

For this purpose, let $K_0 \in \mathbb{N}$ be arbitrary but fixed. By definition of $E_{k,O}$ the inclusion

$$M_\varepsilon \subset \{x \in M \; ; \; \limsup_{r \searrow 0} r^{-n} |B(x,r) \cap \bigcup_{k=K_0}^\infty E_{k,O}| > \varepsilon\}$$

holds. Hence, for any $x \in M_\varepsilon$ there is a positive $R_x < \operatorname{dist}(x, \partial\Omega), 1/K_0$ and a $k_x \in \mathbb{N}$ such that $|B(x, R_x) \cap \bigcup_{k=0}^{k_x} E_{K_0+k}| > \varepsilon R_x^n$. Since $x \in \bigcap_{k=1}^\infty E_{k,O}$, there also exists an $r_x \in (0, R_x)$ with

$$|B(x, r_x) \cap \bigcup_{k=0}^{k_x} E_{K_0+k}| = \varepsilon r_x^n.$$

The isoperimetric inequality (see [9], 5.6.2) now implies that

$$\varepsilon r_x^n \le (\sigma_n \|\partial(\bigcup_{k=K_0}^{K_0+k_x} E_k)\|(B(x, r_x)))^{\frac{n}{n-1}}$$

i.e.

$$r_x^{n-1} \le \varepsilon^{\frac{1-n}{n}} \sigma_n \|\partial(\bigcup_{k=K_0}^{K_0+k_x} E_k)\|(B(x, r_x)) \le \varepsilon^{\frac{1-n}{n}} \sigma_n \sum_{k=K_0}^\infty \|\partial E_k\|(B(x, r_x))$$

By the usual covering arguments

$$\mathscr{H}^{n-1}(M_\varepsilon) \le \mathrm{const}_n \, \varepsilon^{\frac{1-n}{n}} \sum_{k=K_0}^\infty \mathrm{Per}_\Omega(E_k)$$

follows. Since K_0 was arbitrary, we are done.

Corollary 3.2 *If $E \subset \Omega$ is of finite perimeter in Ω, $\chi_E = \sum_{k=1}^\infty \chi_{E_k}$ $\mathscr{L}^n$-a.e. and $Per_\Omega(E) = \sum_{k=1}^\infty Per_\Omega(E_k)$, then* *a)* $\mathscr{H}^{n-1}(\partial_* E_k \cap \partial_* E_{k'} \cap \Omega) = 0$ *for all* $k \neq k'$.

b) $\mathscr{H}^{n-1}((\partial_* E \cap \Omega) \setminus \bigcup_{k=1}^{\infty} \partial^* E_k) = 0$.

c) $\mathscr{H}^{n-1}((\partial_* E_k \cap \Omega) \setminus \partial_* E) = 0$ *for all* k.

d) $\mathscr{H}^{n-1}((E_I \cap \Omega) \setminus \bigcup_{k=1}^{\infty} E_{k,I}) = 0$.

e) If $M \subset E$ *measurable, then* $Per_\Omega(M) = \sum_{k=1}^{\infty} Per_\Omega(M \cap E_k)$.

Proof The σ-subadditivity of the perimeter together with our assumptions imply that for all $\mathrm{Per}_\Omega(E_k \cup E_{k'}) \geq \mathrm{Per}_\Omega(E_k) + \mathrm{Per}_\Omega(E_{k'})$ for all $k \neq k'$, moreover $\partial_*(E_k \cup E_{k'}) \subset \partial_* E_k \cup \partial_* E_{k'}$. This demonstrates a). Due to the foregoing Lemma 3.1 $\mathscr{H}^{n-1}[(\partial_* E \cap \Omega) \setminus \bigcup_{k=1}^{\infty}(\partial_* E_k \cup E_{k,I})] = 0$, but $E_{k,I} \subset E_I$ for all k, hence b) follows. By a) and b)

$$\begin{aligned}\sum_{k=1}^{\infty} \mathscr{H}^{n-1}(\partial_* E_k \cap \Omega) &= \sum_{k=1}^{\infty} \mathrm{Per}_\Omega(E_k) = \mathrm{Per}_\Omega(E) = \mathscr{H}^{n-1}(\Omega \cap \partial_* E) \\ &= \sum_{k=1}^{\infty} \mathscr{H}^{n-1}(\partial E_k \cap \partial_* E \cap \Omega),\end{aligned}$$

therefore $\mathscr{H}^{n-1}(\partial_* E_k \cap \Omega) = \mathscr{H}^{n-1}(\partial E_k \cap \partial_* E \cap \Omega)$ for each k, which gives c). Next, we know from c) that $\mathscr{H}^{n-1}(E_I \cap \Omega \cap \partial_* E_k) = 0$ for all $k \geq 1$, and by Lemma 3.1 $\mathscr{H}^{n-1}(E_I \cap \Omega \setminus \bigcup_{k=1}^{\infty}(\partial_* E_k \cup E_{k,I})) = 0$, showing just d). Finally, if $\mathrm{Per}_\Omega(M) = \infty$, e) follows by σ-subadditivity. Else it suffices to verify that

$$\mathscr{H}^{n-1}(\Omega \cap \partial_*(E_k \cap M) \setminus (\partial_* M \cap \bigcap_{i \neq k} E_{i,O}) = 0,$$

which ensures $\mathscr{H}^{n-1}(\partial_*(E_k \cap M) \cap \partial(E_{k'} \cap M) \cap \Omega) = 0$ for all $k' \neq k$ and hence e). But indeed, $x \in \partial_*(E_k \cap M)$ implies $x \notin E_{k,O}$. If $x \in E_{k,I}$, then obviously $x \in \partial_* M$ and $x \in E_{i,O}$ for all $i \neq k$. Else $x \in \partial_* E_k$ which means due to a) and c) for $\mathscr{H}^{n-1}$-a.e. $x \in \Omega$ that $x \in \partial_* E \cap \bigcap_{i \neq k} E_{i,O}$. Since $M \subset E$ and $x \notin M_O$, again $x \in \partial_* M$ holds.

Proposition 3.3 *Let* E *be a bounded set of finite perimeter. Then there is a countable family* $\{E_i\}_{i \in \mathcal{I}}$ *such that*

(1) For all $i \in \mathcal{I}$ *is* $E \cap E_{i,I} = E_i \neq \emptyset$ *(BV)-indecomposable.*

(2) $\mathscr{H}^{n-1}(E_I \cap E \setminus \bigcup_{i \in \mathcal{I}} E_i) = 0$.

(3) $Per(E) = \sum_{i \in \mathcal{I}} Per(E_k)$ *and* $E_i \cap E_j = \emptyset$ *if* $i \neq j$.

Moreover, this family $\{E_i\}_{i \in \mathcal{I}}$ *is unique in the following sense: whenever* $\{E'_i\}_{i \in \mathcal{I}'}$ *fulfills (1),(3) and the weaker assumption*

(2') $|E \setminus \bigcup_{i \in \mathcal{I}'} E'_i| = 0$

only, then we can already conclude the existence of bijection π from $\mathcal{I}'$ onto $\mathcal{I}'$ such that $E_{\pi(i')} = E'_{i'}$ for all $i' \in \mathcal{I}'$.

Therefore, we call the members of $\{E_k\}_{k=1}^{L}$ the indecomposable BV-components of E.

The concept of indecomposability in the BV-sense (and more generally for currents) and the existence part of this theorem already appears in [10], 4.2.25. Since the proof given there is in a rather condensed form, we present here all the necessary details.

Proof

First, we show the existence of at least one indecomposable set in E, then the usual exhausting argument ensures the existence of a countable family satisfying (1),(2') and (3).

For this purpose, we define for any set $M \subset E$ of finite perimeter

$$\mathscr{S}(M) = \{N \subset M \; ; \; \mathrm{Per}(N) + \mathrm{Per}(M \setminus N) = \mathrm{Per}(M)\}.$$

First, note that $\mathscr{S}(M)$ is a σ-algebra on M. Indeed, $N \in \mathscr{S}(M)$ obviously implies $M \setminus N \in \mathcal{S}(M)$. Moreover, for $N_1, N_2 \in \mathscr{S}(M)$ we infer by Corollary 3.2.e) that

$$\begin{aligned} \mathrm{Per}(N_2) &= \mathrm{Per}(N_1 \cap N_2) + \mathrm{Per}(N_2 \setminus N_1), \\ \mathrm{Per}(M \setminus N_2) &= \mathrm{Per}(N_1 \setminus N_2) + \mathrm{Per}(M \setminus (N_1 \cup N_2)). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathrm{Per}(M) &= \mathrm{Per}(N_1 \cap N_2) + \mathrm{Per}(N_2 \setminus N_1) + \mathrm{Per}(N_1 \setminus N_2) + \mathrm{Per}(M \setminus (N_1 \cup N_2)) \\ &\geq \mathrm{Per}(N_1 \cap N_2) + \mathrm{Per}(M \setminus (N_1 \cap N_2)) \geq \mathrm{Per}(M), \end{aligned}$$

hence $N_1 \cap N_2 \in \mathscr{S}(M)$. Last, if $N_1 \subset N_2 \subset \dots$ are in $\mathscr{S}(M)$, then for $N = \bigcup_{k=1}^{\infty} N_k$ and $N' = M \setminus N$ we obviously have $\chi_{N_k} \to \chi_N$ as well as $\chi_{M \setminus N_k} \to \chi_{N'}$ in L^1. Now, lowersemicontinuity implies

$$\begin{aligned} \mathrm{Per}(M) \leq \mathrm{Per}(N') + \mathrm{Per}(N) &\leq \liminf_k \mathrm{Per}(N_k) + \liminf_k \mathrm{Per}(M \setminus N_k) \\ &\leq \liminf_k (\mathrm{Per}(N_k) + \mathrm{Per}(M \setminus N_k)) = \mathrm{Per}(M); \end{aligned}$$

so $N \in \mathscr{S}(M)$ and $(\mathscr{L}^n \llcorner M, \mathscr{S}(M))$ is a complete measure space. Obviously, Corollary 3.2.e) gives also the relativization

$$N \in \mathscr{S}(M) \text{ implies } \mathscr{S}(N) = \{X \subset N \; ; \; X \in \mathscr{S}(M)\}.$$

Next, we note that

$$\left.\begin{array}{l} \mathscr{S}(M) \text{ either contains an atom (i.e. } N \in \mathscr{S}(M) \text{ such that } |N| > 0 \\ \text{and } |N'| \cdot |N \setminus N'| = 0 \text{ for any } N' \in \mathscr{S}(M) \text{ with } N' \subset N) \text{ or for any} \\ \varepsilon \in (0, |M|) \text{ there is an } N \in \mathscr{S}(M) \text{ with } |N| = \varepsilon. \end{array}\right\} \quad (3.1)$$

This is a very special case (sometimes stated as Alexandroff's theorem) of Ljapunoff's theorem about the convexity of the range of a vector measure, see [7] , but for the convenience of the reader we sketch a proof. First assume $\mathscr{S}(M)$ has no atoms and there is an $\varepsilon \in (0, |M|]$ such that for all $N \in \mathscr{S}(M)$ $|N| \notin (\varepsilon/2, \varepsilon]$. Then obviously $\mathcal{I} = \{N \in \mathscr{S}(M) \; ; \; |N| \leq \varepsilon/2\}$ is closed under (even countable) unions, and therefore $|N_0| = \sup\{|N| \; ; \; N \in \mathcal{I}\}$ for some $N_0 \in \mathcal{I}$. Since also $\mathscr{S}(M \setminus N_0)$ contains no atoms, $M \setminus N_0$ can be divided into arbitrary many sets of positive measure and belonging to $\mathscr{S}(M)$, hence there is an $N_1 \in \mathscr{S}(M \setminus N_0)$ with $0 < |N_1| < \varepsilon/2$. But then $N_1, N_0 \cup N_1 \in \mathcal{I}$ and $|N_0 \cup N_1| > |N_0|$, contradiction. Once knowing that for any sigma-algebra $\mathscr{S}(M)$ the nonexistence of atoms implies for any $\delta \in (0, |M|)$ the existence of $\tilde{N} \in \mathscr{S}(M)$ with $|\tilde{N}| \in (\delta/2, \delta)$, it is obvious how to construct the N required in (3.1).

The next step in our existence proof is to show that any nontrivial $\mathscr{S}(M)$ contains at least one atom. Indeed, otherwise (3.1) ensures that for any $K \in \mathbb{N}$ there are mutually disjoint $M_1, \ldots, M_K \in \mathscr{S}(M)$ such that $M = \bigcup_{k=1}^{K} M_k$ and $|M_k| = |M|/K$ for all k. One sees also (using induction with respect to K if necessary) that $\mathrm{Per}(M) = \sum_{k=1}^{K} \mathrm{Per}(M_k)$. But then the isoperimetric inequality yields

$$\mathrm{Per}(M) \geq \sum_{k=1}^{K} \mathrm{Per}(M_k) \geq \sum_{k=1}^{K} \mathrm{const}_n |M_k|^{\frac{n-1}{n}} \geq \mathrm{const}_n \sum_{k=1}^{K} \frac{M_k}{\sqrt[n]{|M|/K}} \geq \sqrt[n]{K} \frac{\mathrm{const}_n}{\sqrt[n]{|M|}} |M|,$$

which is a contradiction for K sufficiently large.

Finally, we define $\{E_i\}_{i \in \mathcal{I}}$ to be the set

$$\mathcal{A} = \{A_I \cap E \; ; \; A \text{ is an atom of } \mathscr{S}(E)\}.$$

Obviously, if A_1, A_2 are atoms of $\mathscr{S}(E)$ then either $|A_1 \Delta A_2| = 0$ or $|A_1 \cap A_2| = 0$. This gives $A_{1,I} = A_{2,I}$ or $A_{1,I} \cap A_{2,I} = \emptyset$, and since obviously any member of $\mathcal{A}$ is of positive measure, one sees that $\mathcal{A}$ is at most countable. Therefore, $E \setminus \bigcup \mathcal{A} \in \mathscr{S}(E)$ and contains no atom, hence $|E \setminus \bigcup \mathcal{A}| = 0$. Since atoms are obviously indecomposable, we see that (1) and (2') are valid. Moreover, for all $\mathcal{F} \subset \mathcal{I}$ finite one has $\mathrm{Per}(E) = \sum_{i \in \mathcal{F}} \mathrm{Per}(E_i) + \mathrm{Per}(E \setminus \bigcup_{i \in \mathcal{F}} E_i)$ which gives (3) as $\mathcal{F}$ exhausts the countable set $\mathcal{I}$. To check that even the original condition (2) is true we infer from Corollary 3.2.d) that $\mathscr{H}^{n-1}(E_I \setminus \bigcup \{A_I \; ; \; A \text{ atom of } \mathscr{S}(E)\}) = 0$. Hence $\mathscr{H}^{n-1}(E_I \cap E \setminus \bigcup_{i \in \mathcal{I}} E_i) = 0$ holds also.

This finished the existence proof, and all what remains to be done is to show the uniqueness. But from Corollary 3.2.e) we know that for all $M \subset E$

$$\mathrm{Per}(M) = \sum_{i \in \mathcal{I}} \mathrm{Per}(M \cap E_i),$$

in particular, $M \cap E_i \in \mathscr{S}(M)$ if $i \in \mathcal{I}$ and $\mathrm{Per}(M) < \infty$. Consequently, if M is BV-indecomposable then for each $i \in \mathcal{I}$ $|M \setminus E_k| \cdot |M \cap E_k| = 0$. Hence, there exists precisely one $i_0 = i_0(M, \{E_i\}_{i \in \mathcal{I}})$ with $|M \cap E_{i_0}| > 0$. So, if $\{E'_i\}_{i \in \mathcal{I}'}$ is a second sequence fulfilling (1),(2') and (3), then one easily verifies that the required permutation π is given by $\pi(i') = i_0(E'_{i'}, \{E_i\}_{i \in \mathcal{I}})$.

Notation 3.4 *Whenever $\mathcal{E} = \{E^1, \dots, E^m\}$ is a family of bounded pairwise disjoint subsets of $\mathbb{R}^n$ with $E^j = E^j_I$ and $Per(E) < \infty$ for all $j \leq m$, then we denote*

$$co_{\mathcal{E}}(x) = \begin{cases} E^j_{k,I} & \text{if } x \in E^j_{k,I} \text{ for a } BV\text{-indecomposable component } E^j_k \text{ of } E^j \\ \emptyset & \text{else} \end{cases}$$

Therefore, $co_{\mathcal{E}}(x)$ is the BV-indecomposable component determined by the partition $\mathcal{E}$ which contains x if such one exists. We also set

$$w(x) = w_{\mathcal{E}}(x) = \begin{cases} j & \text{if } \emptyset \neq co_{\mathcal{E}}(x) \subset E^j, \\ 0 & \text{else.} \end{cases}$$

Note that one for $\mathscr{H}^{n-1}$-a.e. point $x \in (\bigcup_{j=1}^m E^j)_I$ the following alternative holds. Either $co_{\mathcal{E}}(x) \neq \emptyset$ or there are two $j_1 \neq j_2$ and x^1, x^2 with $x^l \in co_{\mathcal{E}}(x^l) \subset E^{j_l}$ and $x \in \partial^* co_{\mathcal{E}}(x^1) \cap \partial^* co_{\mathcal{E}}(x^2)$. Next we will see that the situation is the same, if we consider slices of the partitioning sets $\{E^1, \dots, E^m\}$ by almost any line.

Definition 3.5 *Let $\mathcal{E} = \{E^1, \dots, E^m\}$ be a family of bounded subsets of $\mathbb{R}^n$ with $E^j = E^j_I$ and $Per(E) < \infty$ for all $j \leq m$. We say that a line $g = g(x, d) = \{x + \lambda d \ ; \ \lambda \in \mathbb{R}\}$ $(x, d \in \mathbb{R}^n, d \neq 0)$ is regular w.r.t. $\mathcal{E}$ provided there is a (unique) finite set $Exc(g) \subset g$ such that for any $\lambda_0 \in \mathbb{R}$ with $x + \lambda_0 d \in (\bigcup_{j=1}^m E^j)_I$ the following statements hold.*

(a) If $x + \lambda_0 d \notin Exc(g)$ then $x + \lambda d \in co_{\mathcal{E}}(x + \lambda_0 d)$ for all λ sufficiently close to λ_0.

(b) If $x + \lambda_0 d \in Exc(g)$ then there is an $\varepsilon > 0$ such that for the points $x^+ = x + (\lambda_0 + \varepsilon)d$ and $x^- = x + (\lambda_0 - \varepsilon)d$

(i) $x + \lambda_0 d \in \partial^* co_{\mathcal{E}}(x^+) \cap \partial^* co_{\mathcal{E}}(x^-)$.

(ii) $x + \lambda d \in co_{\mathcal{E}}(x^+)$ for $0 < \lambda - \lambda_0 < 2\varepsilon$.

(iii) $x + \lambda d \in co_{\mathcal{E}}(x^-)$ for $0 < \lambda_0 - \lambda < 2\varepsilon$.

(iv) $w_{\mathcal{E}}(x^+) \neq w_{\mathcal{E}}(x^-)$, but both positive.

(v) $\langle \nu_{co_{\mathcal{E}}(x^+)}(x + \lambda_0 d), d \rangle < 0$.

Lemma 3.6 *Let $\mathcal{E}$ be any "admissible" (in the sense of Definition 3.5 above) family. Then for each $d \in \mathbb{R}^n \setminus \{0\}$ and $\mathscr{H}^{n-1}$ a.e. $x \in d^\perp$ the line $g(x, d)$ is regular w.r.t. $\mathcal{E}$.*

Proof We can of course suppose $d \in \mathbb{S}^{n-1}$, and for $j \leq m$ we denote by $\{E^j_k\}_{k=1}^\infty$ a sequence containing all indecomposable BV-components of E^j, i.e. the family from Lemma 3.3 eventually filled up with empty sets.

Consider the following sets

$$S1 = \{x \in d^\perp \ ; \ \operatorname{card}(g(x, d) \cap \bigcup_{j=1}^m \bigcup_{k=1}^\infty \partial_* E^j_k) = \infty\}$$

From ([10], 2.10.2) we infer that

$$\int_{d^\perp} \operatorname{card}(g(x,d) \cap \bigcup_{\substack{j\le m\\ k\ge 1}} (\partial_* E_k^j))\, d\mathscr{H}^{n-1}(x) \le \sum_{\substack{j\le m\\ k\ge 1}} \mathscr{H}^{n-1}(\partial_* E_k^j) = \sum_{j=1}^m \operatorname{Per}(E^j)$$

is finite, hence $\mathscr{H}^{n-1}(S1) = 0$.

$$S2 = \bigcup_{j=1}^m (E_I^j \setminus \bigcup_{k=1}^\infty E_{k,I}^j) \cup (\partial_* E^j \setminus (\bigcup_{k=1}^\infty \partial^* E_k^j \cap \partial^* E^j)),$$

by Corollary 3.2.(a),(b),(d) we have $\mathscr{H}^{n-1}(S2) = 0$.

$$S3 = \bigcup_{j=1}^m \bigcup_{k=1}^\infty \{x \in d^\perp\ ;\ \text{ there are } y_1 \in g(x,d) \cap E_{k,I}^j, y_2 \in g(x,d) \setminus E_{k,I}^j,$$
$$\text{but } \operatorname{conv}\{y_1,y_2\} \cap \partial_* E_k^j = \emptyset\}.$$

Due to ([9], Th.5.11(1)) we know that $\mathscr{H}^{n-1}(S3) = 0$. Finally,

$$S4(j,k) = \{y \in \partial^* E_k^j;\ \text{there is } \varepsilon > 0 : y - \langle \nu_{E_k^j}(y), d\rangle \cdot \lambda d \notin E_{k,I}^j \text{ if } 0 < \lambda < \varepsilon\}.$$

Then $\mathscr{H}^{n-1}(\operatorname{proj}_d(S4(j,k)) = 0$ for all $j \le m$,$k \ge 1$. Although this statement is in the spirit of [9],Th.5.11.2, claim #1, the proof given there requires some minor modifications to give our result. Again for convenience, we present the details. First of all, using Corollary 2.2 and considering both d and $-d$, we can replace $S4(j,k)$ by

$$\tilde{S}4(j,k) = \{y \in \partial^* E_k^j; \langle \nu_{E_k^j}(y), d\rangle > 0 \text{ and } y - \lambda d \notin E_{k,I}^j \text{ for all } \lambda \in (0, \varepsilon_y)\}.$$

Moreover, it is obviously sufficient to prove that for each $i,l \in \mathbb{N}$ the set $\tilde{S}_{i,l} \subset \tilde{S}4(j,k)$ of those $y \in \partial^* E_k^j$ fulfilling

- $\mathscr{L}^n(\{z \in B(y,R)\ ;\ \langle \nu_{E_k^j}(y), z-y\rangle \le 0 \text{ and } z \notin E_k^j\}) < \dfrac{\alpha(n-1)R^n}{(12i)^n}$ for $0 < R < 1/l$
- $\langle \nu_{E_k^j}(y), d\rangle \ge 1/i$,
- $y - \lambda d \notin E_{k,I}^j$ for all $\lambda \in (0, 1/i)$,

has an $\mathscr{H}^{n-1}$-zero orthogonal projection onto $d^\perp$. Additionally, dividing $\tilde{S}_{i,l}$ into (finitely many) parts if necessary, we can also assume $|\langle y - y', d\rangle| < 1/2i$ for all $y, y' \in \tilde{S}_{i,l}$ and that $l \ge i$. Next, we observe that

$$\mathscr{H}^1(g(y',d) \cap (B(y_0,R) \cap \{z\ ;\ \langle z - y_0, \nu_{E_k^j}(y_0) \langle \le 0\} \setminus E_{k,I}^j) \ge R/6 \tag{3.2}$$

provided

$$y', y_0 \in \tilde{S}_{i,l}\ ,\ 3\sqrt{1+i^2}|\mathrm{proj}_d(y'-y_0)| \le R < \frac{1}{5l} \text{ and } \langle y_0 - y', d\rangle \le \frac{R}{6}.$$

Indeed, fix $\nu = \nu_{E^j_k}(y_0)$, and the unique point $s(y') \in g(y', d) \cap \{z\ ;\ z - y_0 \perp \nu\}$. Then

$$\begin{aligned} |y - s(y')|^2 &\le |\mathrm{proj}_d(y_0 - s(y'))|^2 + |\langle y_0 - s(y'), d\rangle|^2 \\ &\le |\mathrm{proj}_d(y_0 - y')| \left(1 + \frac{1}{\langle \nu, d\rangle^2}\right) \le \left(\frac{R}{3}\right)^2. \end{aligned}$$

Hence, there is also a unique point

$$t(y') \in g(y', d) \cap \{z\ ;\ \langle z - y_0, \nu\rangle < 0 \text{ and } |z - y_0| = R\},$$

moreover,

$$\langle s(y') - t(y'), d\rangle = |s(y') - t(y')| \ge 2R/3 \tag{3.3}$$

Consequently, $\langle y_0 - t(y'), d\rangle \ge R/3$ and $\langle y' - t(y')\rangle \ge R/6$.

Summarizing, we obtain

$$\begin{aligned} \langle t(y'), d\rangle + \frac{R}{6} &\le \langle y', d\rangle \le \langle s(y'), d\rangle + \frac{R}{3} + \frac{1}{2k} \\ &\le \langle s(y'), d\rangle + \frac{7R}{3} + \frac{1}{2i} \le \langle s(y'), d\rangle + \frac{1}{i}. \end{aligned} \tag{3.4}$$

Obviously, our observation (3.2) follows from (3.3) and (3.4).

Now, choose an arbitrary $x \in \mathrm{proj}_d(y)$, $y \in \tilde{S}_{i,l}$ and $r \in (0, 1/30l\sqrt{1+i^2})$. Denote $M = \tilde{S}_{k,l} \cap \{\tilde{y}\ ;\ |\mathrm{proj}_d(y) - x| \le r\}$ and $R = 6\sqrt{1+i^2}r$. We pick $y_0 \in M$ such that $\langle y_0 - y', d\rangle \le R/6$ for all $y' \in M$. In virtue of the observation (3.2)

$$\mathscr{H}^1(g(y', d) \cap B(y_0, r) \cap \{z, \langle z - y_0, \nu_{E^j_k}(y_0)\rangle \le 0\} \setminus E^j_{k,I}\}) \le \frac{R}{6}$$

for all $y' \in M$, therefore

$$\frac{R}{6}\mathscr{H}^{n-1}(\mathrm{proj}_d M| \le \alpha(n-1)\frac{R^n}{(12i)^n}.$$

In other words,

$$\mathscr{H}^{n-1}(\mathrm{proj}_d(\tilde{S}_{i,l} \cap B(x,r)) \le \frac{6(6\sqrt{1+i^2})^{n-1}\alpha(n-1)r^{n-1}}{(12i)^n} \le \frac{\alpha(n-1)}{2i}r^{n-1}.$$

Hence, Lebesgue's density theorem applied in $d^\perp$ gives for the set $S4 = \bigcup_{(j,k)} S4(j,k)$ that $\mathscr{H}^{n-1}(\mathrm{proj}_d(\tilde{S}_{i,l})) = 0 = \mathscr{H}^{n-1}(\mathrm{proj}_d(S4))$.

We finish the proof of this lemma by showing that the line $g = g(x,d)$ is regular whenever

$$x \in d^{\perp} \setminus (S1 \cup S3 \cup \mathrm{proj}_d(S2 \cup S4),$$

and that we have

$$Exc(g) = g \cap \bigcup_{j=1}^{m} \bigcup_{k=1}^{\infty} \partial^* E_k^j$$

in this case. Indeed, $g \cap \bigcup_{(j,k)} \partial E_k^j$ is finite by definition of S1. Moreover, since $g \cap S2 = \emptyset$, we have $g \setminus (\bigcup_{j=1}^m E^j)_O \subset \bigcup_{(j,k)} E_{k,I}^j \cup Exc(g)$. Hence, if $x + \lambda_0 d \notin (\bigcup_{j=1}^m E^j)_O \cup Exc(g)$ then there are (j_0, k_0) and $\varepsilon_0 > 0$ such that $x + \lambda_0 d \in E_{k_0,I}^{j_0}$ and $x + \lambda d \notin Exc(g)$ for all $\lambda \in (l_0 - \varepsilon_0, \lambda_0 + \varepsilon_0)$. Because $x \notin S3$, we infer that $x + \lambda d \in E_{k_0,I}^{j_0}$ whenever $|\lambda - \lambda_0| < \varepsilon_0$. So condition (a) of Definition 3.5 is satisfied. To verify also (b), we assume that $y_0 = x + \lambda_0 d \in (\bigcup_{j=1}^m E^j)_I \cap Exc(g)$. Because $g \cap S2 = \emptyset$ there are precisely two $j_1 < j_2$ with $y_0 \in \partial_* E^{j_1} \cap \partial_* E^{j_2}$ and also unique k_1, k_2 such that $y_0 \in \partial^* E_{k_1}^{j_1} \cap \partial^* E_{k_2}^{j_2}$. Moreover, we can find $\varepsilon > 0$ with $g \cap B(y_0, 2\varepsilon) \cap Exc(g) = \{y_0\}$ and define $x^{\pm} = y_0 \pm \varepsilon d$ and $\nu = \nu_{E_{k_1}^{j_1}}(y_0) = -\nu_{E_{k_2}^{j_2}}(y_0) \in \mathbb{S}^{n-1}$. Obviously, $y_0 \notin S4$ implies $s = \langle d, \nu \rangle \neq 0$. There are also $\Delta_1, \Delta_2 \in (0, 2\varepsilon/|s|)$ such that $y_0 + (-1)^l \Delta_1 d \in E_{k_l,I}^{j_l}$ for $l = 1, 2$. Due to our choice of ε and the already shown (a) we know that

$$y_0 + (-1)^l s \Delta_l d \in E_{k_l,I}^{j_l} \text{ for all } \Delta \in (0, e/|s|), j = 1, 2,$$

consequently (b)(i)—(iv) follow. Last, if $s > 0$ then $x^- \in E_{k_1,I}^{j_1} = co_{\mathcal{E}}(x^-)$, hence $\langle \nu_{co_{\mathcal{E}}(x^+)}, d \rangle = \langle -\nu, d \rangle = -s < 0$ and if $s < 0$ then $x^+ \in E_{k_1,I}^{j_1} = co_{\mathcal{E}}(x^+)$, therefore $\langle \nu_{co_{\mathcal{E}}(x^+)(y_0)}, d \rangle = \langle \nu, d \rangle = s < 0$. So (b)(v) is also shown and the proof finished.

4 Compatibility conditions for BV-gradients of pointwise minimizers of the m-well problem

Notation 4.1 In this section we will consider an open bounded and convex set $\Omega \in \mathbb{R}^n$, symmetric and positive matrices $A^j \in M^{n \times n}$, $j = 1, \ldots, m$ with $A^j \notin SO(n)A^{j'}$ for $j \neq j'$ and a Lipschitz function $u : \Omega \to \mathbb{R}^n$ such that the mutually disjoint sets

$$E^j = \{x \; ; \; \nabla u(x) \in SO(n) \cdot A^j\} j = 1, \ldots, m$$

(the so called phases) fulfill

(a) $|\Omega \setminus \bigcup_{j=1}^m E^j| = 0$,

(b) $\mathrm{Per}(E^j) < \infty$ for each $j \leq m$.

So, if we denote $\mathcal{E} = \mathcal{E}(u) = \{E^1, \ldots, E^m\}$, the notations introduced in the last section, in particular in 3.4, are well defined.

Definition 4.2 *We say that $x \in \Omega$ is a regular point if $x \in co_{\mathcal{E}}(x)$ and that $[x,y] = \{x + \lambda(y-x)\ ;\ 0 \le \lambda \le 1\} \subset \Omega$ is a segment regular with respect to u provided:*

(i) $g = g(x, y-x)$ *is a line regular with respect to* $\mathcal{E}(u)$.

(ii) x, y *are different regular points.*

(iii) *if* $z_1, z_2 \in [x,y]$ *and* $[z_1, z_2] \cap Exc(g) = \emptyset$, *then* $\nabla u(z_1) = \nabla u(z_2) \in SO(n) \cdot A^{w(z_1)}$.

(iv) *if* $z_1, z_2 \in [x,y] \setminus Exc(g)$ *and* $[z_1, z_2] \cap Exc(g) = \{z\}$ *then there are* $e, n \in \mathbb{R}^n \setminus \{0\}$ *with* $\nabla u(z_1) - \nabla u(z_2) = e \otimes n$ *and* $\langle y-x, n\rangle \neq 0$.

Theorem 4.3 *In the setting of Notation 4.1 we have*

(a) *if* $x, y \in \Omega$ *are different regular points and* $g(x, y-x)$ *is a line regular with respect to* $\mathcal{E}(u)$, *then* $[x,y]$ *is a segment regular with respect to* u.

(b) *If* $x, y \in \Omega$ *are different, then there is an* $\varepsilon > 0$ *such that for almost each* $z \in B(x,\varepsilon)$ $[z, y + (z-x)]$ *is a segment regular with respect to* u *for almost each* $z \in B(x,\varepsilon)$.

Proof Due to the remark in Notation 4.1, we have $\mathscr{H}^{n-1}(\Omega \setminus \bigcup\{co_{\mathcal{E}}(x)\ ;\ x \in \Omega\} < \infty$, hence (b) is an immediate consequence of (a) and Lemma 3.6. So we focus on (a) and have only to verify (i)—(iv) of Definition 4.2. But (i) and (ii) are obvious by the choice of x, y and the definition of $Exc(g)$. Condition (iii) is a direct consequence of Theorem 2.4, and also (iv) follows easily from this Theorem and the definition of regular lines. Indeed, we can suppose that $\langle d, z_2 - z_1\rangle > 0$ for the direction $s = (y-x)/|y-x|$. Then by Definition 3.5(b) $y \in \partial^* co_{\mathcal{E}}(z_1) \cap co_{\mathcal{E}}(z_2)$ and $\nu = \nu_{co_{\mathcal{E}}(z_2)}(y) \notin d^{\perp}$, hence it suffices to show that $(\nabla f(z_2) - \nabla f(z_1))(v) = 0$ for all $v \perp \nu$, i.e. we could choose $n = \nu$.

For this purpose, fix any $\varepsilon > 0$ and by definition of the reduced boundary ∂^* we see that

$$\begin{aligned}&\lim_{r\searrow 0} r^{-n}|B(y + r\cdot n, \varepsilon\cdot r) \cap \{z\ ;\ \langle z-y, d\ \rangle > 0\} \setminus co_{\mathcal{E}}(z_1)\}|\\ &\quad = \lim_{r\searrow 0}|B(y + r\cdot n, \varepsilon\cdot r) \cap \{z\ ;\ \langle z-y, d\ \rangle > 0\} \setminus co_{\mathcal{E}}(z_2)\}| = 0,\end{aligned}$$

hence for $r > 0$ sufficiently small there are $x^+ \in B(y + r\cdot n, \varepsilon\cdot r) \cap co_{\mathcal{E}}(z_1)$ and $x^- \in B(y + r\cdot n, \varepsilon\cdot r) \cap co_{\mathcal{E}}(z_2)$. Since by Theorem 2.4

$$\begin{aligned}f(x^+) - f(y) &= \nabla f(z_2)\cdot(x^+ - y)\\ f(x^-) - f(y) &= \nabla f(z_1)\cdot(x^- - y),\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{lip}(f)\cdot 2\varepsilon\cdot r \ge |f(x^+) - f(x^-)| &\ge |\nabla f(z_2)(n) - \nabla f(z_1)(n)|\\ &\quad - |\nabla f(z_2)||x^+ - (y + n\cdot r)| - |\nabla f(z_1)||x^- - (y + n\cdot r)|\end{aligned}$$

therefore, $4\,\mathrm{lip}(f)\cdot\varepsilon\cdot r \ge r|\langle\nabla f(z_2) - \nabla f(z_1), n\rangle|$ for all $\varepsilon \in (0,1)$ and $0 < r < r_\varepsilon$, so we are done.

5 Pointwise BV-minimizers for three tetragonal wells are laminates

Notation 5.1

As already explained in the introduction we will now assume the set of admissible gradients to be the union of three tetragonal wells. So we suppose

$$\nabla u \in \bigcup_{i=1}^{3} SO(3)\cdot D^i, \text{ where } D^1 = \mathrm{diag}(a,1,1), D^2 = \mathrm{diag}(1,a,1), D^3 = \mathrm{diag}(1,1,a)$$

and a is a positive parameter of course different from one. For convenience, we also define $D^{i+3} = D^i$ for all integer i, i.e. we compute things modulus 3 if necessary. We choose also the auxillary matrix $Tra \in SO(3)$ given by $Tra(e^i) = e^{i+1}$ which describes the transition from one well to another. In fact, using that we obviously have $(x_1, x_2, x_3)\cdot Tra^T = (x_3, x_1, x_2)$ we conclude

$$D^{i+1} = Tra \cdot D^i \cdot Tra^T \text{ for any integer } i. \tag{5.1}$$

We will also use that $Tra^T = Tra^{-1} = Tra \cdot Tra$.

We define the local coordinates of regular point $x \in \Omega$ to be $L(x) = \nabla u(x)\cdot (D^{w(x)})^{-1} \in SO(3)$. This is in fact a slight misuse of notation, since $L(x)$ rather describes the position of $\nabla u(x)$ in its well than the position of the point x itself, but this abbreviation should not cause any further misunderstandings.

5.2 A calculus for rank-1 connections We are going to introduce a sufficiently short and instructive notation for the compatibility conditions between two "neighbouring" local coordinates, i.e. we are giving a more compact way to write condition (iv) in Definition 4.2. First, we recall well-known results about the rank-1 connections between the 3 wells $SO(3)D^i$. The approach in [13], see also formula (3.19) in [3], together with the fact that for $e = (e^1 + e^3)/\sqrt{2}$ the transformation $R_0 = -Id + 2e \otimes e$ belongs to the point group $\mathcal{P}^{432}$ of the cube and fulfills $R_0 \cdot D^1 \cdot R_0^T = D^3$ gives the following. For both $c = -1$ and $c = 1$ the matrix

$$R(2,c) = \begin{pmatrix} 2a/(1+a^2) & 0 & c\cdot(1-a^2)/(1+a^2) \\ 0 & 1 & 0 \\ -c\cdot(1-a^2)/(1+a^2) & 0 & 2a/(1+a^2) \end{pmatrix} \tag{5.2}$$

and the vectors

$$v(2,c) = (1,0,c) \text{ and } \mathcal{J}(2,c) = \frac{1-a^2}{1+a^2}\cdot(a,0,-c)^T \tag{5.3}$$

satisfy

$$D^1 - R(2,c)\cdot D^3 = \mathcal{J}(2,c)\otimes v(2,c).$$

In fact $\mathrm{rank}(D^1 - R\cdot D^3) = 1$ and $R \in SO(3)$ implies $R = R(2,1)$ or $R = R(2,-1)$.

Consequently, for $i = 1,2,3$ and $Q \in SO(3)$ all solutions of $\mathrm{rank}(Q \cdot D^i - R \cdot D^{i+2}) = 1$ and $R \in SO(3)$ are given by

$$Q \cdot D^i - Q \cdot R(i+1,c) \cdot D^{i+2} = (Q \cdot \mathcal{J}(i+1,c)) \otimes v(i+1,c), \tag{5.4}$$

where $c = -1, 1$, and $R, \mathcal{J}, v$ satisfy the following transformation rules based on (5.1)

$$R(i+1,c) = Tra \cdot R(i,c) Tra^T$$
$$v(i+1,c) = v(i,c) \cdot Tra^T, \mathcal{J}(i+1,c) = Tra \cdot \mathcal{J}(i,c).$$

Note that

$$R(i,-c) = R(i,c)^T = R(i,c)^{-1} \text{ since } R(i,c) \in SO(3). \tag{5.5}$$

The next simple, but crucial observation is the following. Whenever $d \in \mathbb{R}^3$ fulfills $|d_1| = |d_2| = d_3 = 1$ then d is orthogonal to all three vectors $v(1,-d_2)$, $v(2,-d_1)$, $v(3,-d_1 \cdot d_2)$.

Putting this together with Definition 4.2.(iv) and (5.4), we infer that if

$$|d_1| = |d_2| = d_3 = 1,\ (x-y) \| d,\ [x,y] \text{ regular},\ \mathrm{card}([x,y] \cap Exc(g(x,d))) \le 1 \tag{5.6}$$

and $w(y) - w(x) \equiv 2_{(3)}$ then

$$L(y) = L(x) \cdot R(w(x)+1, c(w(x)+1,d)), \text{ where } c(j,d) = \begin{cases} d_2 & j \equiv 1_{(3)} \\ d_1 & j \equiv 2_{(3)} \\ d_1 d_2 & j \equiv 3_{(3)} \end{cases}$$

So it is well motivated to introduce the notation

$$[j \ \to \ j+2]_d = R(j+1, c(j+1,d)).$$

Obviously, (5.5) motivates to add the definition

$$[j+2 \ \to j]_d = R(j+1, -c(j+1,d)), \text{i.e. } [j \ \to \ j+1]_d = R(j+2, -c(j+2,d)).$$

Therefore it is finally possible to establish the following simple and suggestive rule describing the relation between the local coordinates in two neighbouring components

$$L(y) = L(x) \cdot [w(x) \to w(y)]_d \text{ provided only (5.6) holds.}$$

The next Proposition will show, that the relations between local coordinates keep quite simple, even if we allow more than one exceptional points on the joining segment. However, it should be noted, that it heavily relies on the fact, that we are dealing with not more than three wells.

Proposition 5.3 *Let $d = (d_1, d_2, 1)^T$, $d_1, d_2 \in \{-1, 1\}$, and let $x, y \in \Omega$ be regular points, such that $(x-y) \| d$. Then there is an integer k satisfying*

$$L(y) = L(x)[w(x) \to 1]_d([1 \to 3]_d[3 \to 2]_d[2 \to 1]_d)^k[1 \to w(y)]_d.$$

Proof Since the local coordinates L are constant on indecomposable components, Theorem 4.3 ensures that we can additionally assume $[x,y]$ to be a regular segment. Moreover, the considerations in 5.2 show that for our choice of d all matrices $[j \to j']_d$ are well-defined and that for $[x',y'] \subset [x,y]$ with $[x',y'] \cap Exc(g(x,y-x)) = \{z\}$ in fact

$$L(y') = L(x')[w(x') \to w(y')]_d$$

holds. Hence, we can choose $0 = t_0 < t_1 < \cdots < t_{l-1} < t_l = 1$ such that

$$L(x+t_{i+1}(y-x)) = L(x+t_i(y-x))[w(x+t_i(y-x)) \to w(x+t_{i+1}(y-x))]_d$$
$$\text{and } w(x+t_i(y-x)) \neq w(x+t_{i+1}(y-x)) \text{ for } i = 0,\ldots,l-1.$$

Now the statement of the Proposition can be shown by induction with respect to l, the induction step being a straightforward consequence of the following three "cancellation rules".

$$[1 \to 2]_d[2 \to 1]_d = [1 \to 3]_d[3 \to 1]_d = [1 \to 1]_d$$
$$[1 \to 2]_d[2 \to 3]_d = ([1 \to 3]_d[3 \to 2]_d[2 \to 1]_d)^{-1}[1 \to 3]_d$$
$$[1 \to 3]_d[3 \to 2]_d = ([1 \to 3]_d[3 \to 2]_d[2 \to 1]_d)[1 \to 2]_d.$$

The following Lemma presents all the algebraic constraints used in our approach. Because its proof requires completely elementary but rather lengthy computations, we postpone it to the Appendix

Lemma 5.4 *Let a be positive but different from one, $d^1 = (1,1,1)^T$ and $d^2 = (-1,1,1)^T$. We denote $M = [1 \to 3]_{d^1}[3 \to 2]_{d^1}[2 \to 1]_{d^1} \in SO(3)$,*

$$Q = \begin{pmatrix} \frac{1}{\sqrt{1+2a^2}} & \frac{a}{\sqrt{1+2a^2}} & \frac{a}{\sqrt{1+2a^2}} \\ 0 & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -2\frac{a}{\sqrt{2+4a^2}} & \frac{1}{\sqrt{2+4a^2}} & \frac{1}{\sqrt{2+4a^2}} \end{pmatrix} \in SO(3)$$

and $Di = Q \cdot M \cdot Q^T$.

Then $Di(1,1) = 1$, $Di(2,1) = Di(3,1) = Di(1,2) = Di(3,1) = 0$ and $\tilde{Di}(i,j) = Di(i+1,j+1)$, $1 \leq i,j \leq 2$ belongs to $SO(2)$.

Moreover,

a) for $(w_1,w_2) \in (\{2,3\} \times \{1,2,3\}) \cup \{(1,1)\}$ has the equation

$$Di^k = Q[1 \to w_1]_{d^1}[w_1 \to 1]_{d^2}[1 \to 2]_{d^1}[2 \to w_2]_{d^2}[w_2 \to 1]_{d^1}Q^T$$

no integer solution k.

b) for $w_1 \in \{1,2,3\}$ has the equation

$$Di^k = Q[1 \to w_1]_{d^1}[w_1 \to 1]_{d^2}[1 \to 2]_{d^1}[2 \to 3]_{d^2}[3 \to 1]_{d^2}Q^T$$

no integer solution k.

Lemma 5.5 *Let $U \subset \Omega$ be an open ball, $x^1, x^2 \in U$, $w(x^j) = j$ and $L(x^1) = L(x^2) \cdot R(3, +1)$. For both*

case 1) $(d^1, d^2, d^3) = ((1,1,1)^T, (-1,1,1)^T, (1,-1,1)^T)$ *and*

case 2) $(d^1, d^2, d^3) = ((-1,-1,1)^T, (1,-1,1)^T, (-1,1,1)^T)$

the following holds:
If $x^1 - x^2 \| d^1$ then

(a) $x^3 \in co_{\mathcal{E}}(x^1)$ provided $x^3 - x^1 \| d^2$ and both points $x^3, x^3 + (x^2 - x^1)$ belong to U.

(b) $x^4 \in co_{\mathcal{E}}(x^2)$ provided $x^4 - x^2 \| d^3$ and both points $x^4, x^4 + (x^1 - x^2)$ belong to U.

Proof First, we observe that the Lemma is proved once we verified statement (a) for case 1). Indeed, part (b) follows then directly considering instead of u the new function $\tilde{u}(x) = (u_2, u_1, u_3)(x_2, x_1, x_3)$, and case 2) is reduces to case 1) by the use of the substitution $\tilde{\tilde{u}}(x) = u(-x_1, -x_2, x_3)$ for u.

So, denote $x^4 = x^2 + (x^3 - x^1)$ and assume $x^3 \notin co_{\mathcal{E}}(x^1) = co_{\mathcal{E}}(x^1)_I$. Therefore, $0 \notin (\bigcup\{\{(U \cap co_{\mathcal{E}}(x)) - x^3 \ ; \ co_{\mathcal{E}}(x) \neq co_{\mathcal{E}}(x^1)\})_O$. Since obviously $0 \in ((co_{\mathcal{E}}(x^1) - x^1) \cap (co_{\mathcal{E}}(x^2) - x^2) \cap \{z \ ; \ z + x^4 \text{ is regular}\})_I$, we can, after renaming $x^j + z$ to x^j again, additionally suppose that all three segments $[x^1, x^3]$, $[x^3, x^4]$ and $[x^4, x^2]$ are regular and contained in U. For $j = 1, 2$ we also choose the parallel lines $g^j = g(x^j, d^2)$, the finite sets $T^j = \{t \in [0,1] \ ; \ x^j + t(x^3 - x^1) \in Exc(g^j)\} \cup \{2\}$ and $t^j = \min T^j > 0$. By our assumption, $t^1 < 1$. Now we have to distinguish three different cases.

First, let $t^1 < t^2$ and choose $\varepsilon > 0$ such that $(t^1 - 2\varepsilon, t^1 + 2\varepsilon) \cap (T^1 \cup T^2) = \{t^1\}$. Consequently, for $\tilde{x}^3 = x^1 + (t^1 + \varepsilon)(x^3 - x^1)$ and $\tilde{x}^4 = x^2 + (t^1 + \varepsilon)(x^3 - x^1)$ we know that $[\tilde{x}^3, \tilde{x}^4]$ is a regular segment and $w(\tilde{x}^3) \neq 1$, $co_{\mathcal{E}}(\tilde{x}^4) = co_{\mathcal{E}}(x^2)$. Hence, $L(\tilde{x}^3) = L(x^1) \cdot [1 \to w(\tilde{x}^3)]_{d^2}$, $L(\tilde{x}^4) = L(x^2)$ and due to Proposition 5.3 there exists an integer k such that $L(\tilde{x}^4) = L(\tilde{x}^3) \cdot [w(\tilde{x}^3) \to 1]_{d^1}([1 \to 3]_{d^1}[3 \to 2]_{d^1}[2 \to 1]_{d^1})^k \cdot [1 \to 2]_{d^1}$. Using the notation introduced in the foregoing Lemma 5.4, the just derived relations between the four local coordinates imply now

$$\begin{aligned} L(x^1) &= L(x^2) \cdot R(3,1) = L(\tilde{x}^4) \cdot R(3,1) \\ &= L(\tilde{x}^3) \cdot [w(\tilde{x}^3 \to 1]_{d^1} (M)^k \cdot [1 \to 2]_{d^1} \cdot R(3,1) \\ &= L(x^1) \cdot [1 \to w(\tilde{x}^3)]_{d^2} \cdot [w(\tilde{x}^3) \to 1]_{d^1} \cdot Q^T \cdot (Di)^k \cdot Q \cdot [1 \to 2]_{d^1} \cdot [2 \to 1]_{d^1}. \end{aligned}$$

Therefore,

$$Q \cdot [1 \to w(\tilde{x}^3)]_{d^1} \cdot [w(\tilde{x}^3) \to 1]_{d^2} \cdot [1 \to 2]_{d^1} \cdot [2 \to 1]_{d^1} \cdot Q^T = (Di)^k,$$

which is of course just the equation in part a) of Lemma 5.4 for $w_2 = 2$ and $w_1 = w(\tilde{x}^3)$, hence due to $w_1 \neq 1$ we know that this equation is unsolvable for integer exponents k.

So, we turn to the case $t^1 = t^2$, i.e. we consider the situation that for some $\tilde{x}^3 \in g^1$, $\tilde{x}^4 \in g^2$ with $\tilde{x}^3 - \tilde{x}^4 \| x^1 - x^2$ both $w(\tilde{x}^3) \neq w(x^1)$ and $w(\tilde{x}^4) \neq w(x^2)$ but still $L(\tilde{x}^3) =$

$L(x^1)[1 \to w(\tilde{x}^3)]_{d^2}$ and $L(\tilde{x}^4) = L(x^2)[2 \to w(\tilde{x}^4)]_{d^2}$. From this we infer as before that for $(w_1, w_2) = (w(\tilde{x}^3), w(\tilde{x}^4)) \in \{2,3\} \times \{1,3\}$ the matrix

$$S^{w_1,w_2} = Q \cdot [1 \to w_1]_{d^1} \cdot [w_1 \to 1]_{d^2} \cdot [1 \to 2]_{d^1} \cdot [2 \to w_2]_{d^2} \cdot [w_2 \to 1]_{d^1} \cdot Q^T \tag{5.7}$$

is a certain integer power of the matrix Di. Hence, also this case can be excluded using Lemma 5.4.a).

Consequently, we can focus on the remaining (and in fact slightly more complicated) case $0 < t^2 < t^1$. As before, we find a $t_0 \in (t^2, t^1)$ such that for $\tilde{x}^4 = x^2 + t_0(x^3 - x^1)$ $w(\tilde{x}^4) \in \{1,3\}$ and $L(\tilde{x}^4) = L(x^2)[2 \to w(\tilde{x}^4)]_{d_2}$. It is easy to see, that now (5.7) corresponds to $(w_1, w_2) \in \{1\} \times \{1,3\}$ in part a) of Lemma 5.4. However, the Lemma covers the choice $(w_1, w_2) = (1,1)$ only. Indeed, one easily computes

$$S^{1,3} = Q([1 \to 2]_{d^1}[2 \to 3]_{d^2}[3 \to 1]_{d^1})Q^T = Q([1 \to 2]_{d^1}[2 \to 3]_{d^1}[3 \to 1]_{d^1})Q^T = (Di)^{-1}.$$

Therefore, we can obviously not derive the usual contradictions. On the other hand, this is not at all surprising, since there could in fact be a $SO(3)D^2$–$SO(3)D^3$ interface normal to $(0,1,1)^T$ which first $[x^2, \tilde{x}^4]$ crosses and than $[\tilde{x}^4, \tilde{x}^3]$ returns across it.

To handle this difficulty, we have to examine also situations of more than one jump along g^2, so first we choose $\varepsilon > 0$ such that $(t^1, t^1 + \varepsilon) \cap (T^1 \cup T^2) = \emptyset$. Since for all $t \in (0, t^1 + \varepsilon)$ $L(x^1 + t(x^3 - x^1)) = L(x^1) \cdot [1 \to w(x^1 + t(x^3 - x^1))]_{d^2}$, we infer from the already made considerations (corresponding to $(w_1, w_2) \in \{1,2,3\} \times \{1\}$) that necessarily $L(x^2 + t(x^3 - x^1)) \neq L(x^2) \cdot [2 \to 1]_{d^2}$ for all $t \in (0, t^1 + \varepsilon)$. But this implies that either $L(x^2)^{-1} \cdot L(x^2 + t(x^3 - x^1)) \in \{\mathrm{Id}, [2 \to 3]_{d^2}\}$ for all these t, which leads for $t > t^1$ to one of the already occurred contradictions (since Lemma 5.4 covers all situations where $w_1 \neq 1$), or there is an $t^0 \in (0, t^1 + \varepsilon)$ such that for $\tilde{x}^4 = x^2 + t^0(x^3 - x^1)$

$$w(\tilde{x}^4) = 1 \text{ and } L(\tilde{x}^4) = L(x^2) \cdot [2 \to 3]_{d^2} \cdot [3 \to 1]_{d^2}.$$

We can of course also assume $t^0 \neq t^1$, hence $L(x^1 + t^0(x^3 - x^1)) = L(x^1) \cdot [1 \to w(x^1 + t^0(x^3 - x^1))]_{d^2}$ with $w_1 = w(x^1 + t^0(x^3 - x^1)) \in \{1,2,3\}$. Similarly as before, we conclude that

$$S^{w_1} = Q[1 \to w_1]_{d^1}[w_1 \to 1]_{d^2}[1 \to 2]_{d^1}[2 \to 3]_{d^2}[3 \to 1]_{d^2}Q^T$$

is an integer power of Di. Therefore, applying Lemma 5.4.b) a last time, we are done.

Corollary 5.6 *Suppose $U = U(0,R) \subset \Omega$ and $x, x^0, x^1, x^2, x^3 \in U$ fulfill*

a) $x - x^1 \perp (1,1,0)^T$,

b) $x^1 - x^0 \| (1,1,1)^T$ *and* $x^1 - x^2 \| (-1,-1,1)^T$,

c) $\max_{i=0,1,2} |x^i| + \sqrt{3}|x - x^1| < R$ *and* $x + (x^2 - x^1), x + (x^0 - x^1) \in U$,

d) either $x^0 \in co_{\mathcal{E}}(x^2) \subset E^2$, $w(x^1) = 1$ *and* $L(x^1) = L(x^0) \cdot R(3,1)$ *or* $x^0 \in co_{\mathcal{E}}(x^2) \subset E^1$, $w(x^1) = 2$ *and* $L(x^0) = L(x^1) \cdot R(3,1)$.

Then $x \in co_{\mathcal{E}}(x^1)$.

Proof Again it suffices to consider in the assumption d) the first case only since the second one then follows if we replace the map u by $\tilde{u}(x) = (u_2, u_1, u_3)(x_2, x_1, x_3)$.

For $\Delta = x - x^1$, $\lambda = \langle \Delta, (-1/4, 1/4, 1/2)^T \rangle$ and $\mu = \langle \Delta, (1/4, -1/4, 1/2)^T \rangle$ we have due to a) that $\Delta = \lambda(-1,1,1)^T + \mu(1,-1,1)^T$. Since obviously $|\lambda(-1,1,1)^T| \leq \sqrt{3})|\Delta|$, the assumption c) ensures that all three point $x^3 = x^0 + \lambda(-1,1,1)^T$, $x^4 = x^1 + \lambda(-1,1,1)^T$ and $x^5 = x^2 + \lambda(-1,1,1)^T$ belong to U. Consequently, $x^4 \in co_{\mathcal{E}}(x^1)$ by Lemma 5.5, case 1.a), and $x^5 \in co_{\mathcal{E}}(x^2)$ by the same Lemma case 2.b). Since $x^4 - x^5 = x^1 - x^2 \| (-1,-1,1)^T$ and because $x = x^4 + \mu(1,-1,1)^T$ as well as $x + (x^5 - x^4) = x + (x^2 - x^1)$ are in U, we infer from Lemma 5.5, case 2.a), that $co_{\mathcal{E}}(x) = co_{\mathcal{E}}(x^4) = co_{\mathcal{E}}(x^1)$ as required.

Proposition 5.7 *Suppose the origin belongs to the domain* Ω *and that* $j \in \{1,2,3\}$, $c \in \{-1,1\}$, $0 \in \partial^* E^j \cap \partial^* E^{j+2}$ *and* $g = g(0, v(j+1,c))$ *is a regular line. Then we can find two points* $x^i \in g \cap co_{\mathcal{E}}(x^i) \subset E^i$ *for* $i = j, j+2$ *with* $x^j + x^{j+2} = 0$ *and that the following is true. Whenever a point* x *satisfies* $\langle x, v(j+1,c) \rangle = 0$ *and* $t \cdot x \in \Omega$ *for all* $t \in [0,1]$ *then there is an* ε *positive such that*

$$U(x,\varepsilon) \cap H^i(x) \subset co_{\mathcal{E}}(x^i) \text{ for both } i = j, j+2,$$

where

$$H^i(x) = \{y \in \mathbb{R}^3 \; ; \; \langle y - x, v(j+1,c) \rangle \langle x^i, v(j+1,c) \rangle > 0\}.$$

Proof Performing a cyclic change of coordinates we can ensure that $j = 2$ (hence $j + 2 \equiv 1$) and applying the substitution $\mathrm{diag}(-1,1,-1) \in SO(3)$ if necessary we assume also that $c = +1$. So $g = g(0, (1,1,0)^T)$.

Since g is regular (and rotating $\mathbb{R}^3$ about the angle π around e^3 if needed), we find R positive such that

i) $U(0, 13R) \subset \Omega$.

ii) for $x^1 = R(1,1,0)^T$ and $x^2 = -R(1,1,0)^T$ both $w(x^i) = i$ and $0 \in \partial^* co_{\mathcal{E}}(x^i)$, $i = 1,2$ hold.

iii) $t(1,1,0)^T \in co_{\mathcal{E}}(x^1)$ and $-t(1,1,0)^T \in co_{\mathcal{E}}(x^2)$ for all $t \in (0, \sqrt{2}R)$.

iv)
$$\mathscr{L}^n(U(0,2R) \cap \{y \; ; \; \langle y, (1,1,0)^T \rangle > 0\} \setminus co_{\mathcal{E}}(x^1)) < \alpha(n) R^n / 2^{n+2}$$

and

$$\mathscr{L}^n(U(0,2R) \cap \{y \; ; \; \langle y, (1,1,0)^T \rangle < 0\} \setminus co_{\mathcal{E}}(x^2)) < \alpha(n) R^n / 2^{n+2}.$$

v) $L(x^1) = L(x^2) R(3,1)$,

since $\nabla u(x^1) - \nabla u(x^2) = e \otimes v(3,c)$ with $\langle v(3,c),(1,1,0)^T\rangle \neq 0$ implies $c = +1$.

We notice first that we are obviously done if we prove from i), ... ,v) only that $U(x,2R)\cap H^i(x) \subset co_{\mathcal{E}}(x^i)$ for all $x \in U(0,R)$ orthogonal to $(1,1,0)^T$. For this purpose, we fix $\tilde{x}^1 = x^1/2 = (R/2,R/2,0)^T$, $\tilde{x}^0 = (-R/2,-R/2,-R)^T$ and $\tilde{x}^2 = (-R/2,-R/2,R)^T$. Assumption iii) now ensures that the set of all $z \in B(0,R/2)$ for which $\tilde{x}^1 + z \notin co_{\mathcal{E}}(x^1)$ or $\tilde{x}^0 + z \notin co_{\mathcal{E}}(x^2)$ or $\tilde{x}^2 + z \notin co_{\mathcal{E}}(x^2)$ has measure less than $3\cdot\alpha(n)R^n/2^{n+1} = \mathscr{L}^n(B(0,R/2))$. In particular, there is an $z \in B(0,R/2)$ such that

$$\bar{x}^1 = \tilde{x}^1 + z \in co_{\mathcal{E}}(x^1) \cap H^1(0) \cap B(0,(\sqrt{2}+1)R/2)$$

and

$$\tilde{x}^i + z \in co_{\mathcal{E}}(x^2) \cap H^2(0) \cap B(0,(\sqrt{6}+1)R/2) \text{ for } i = 0,2.$$

Of course, $\bar{x}^1 - (\tilde{x}^0 + z)\|(1,1,1)^T$ and $\bar{x}^1 - (\tilde{x}^2 + z)\|(-1,-1,1)^T$. Consequently, Corollary 5.6 shows that $x \in co_{\mathcal{E}}(x^1)$ whenever $x \in U(0,6R)$ and $x - \bar{x}^1 \perp (1,1,0)^T$. (Note that then $|x - \bar{x}^1| < 6R$.)

Similarly, we infer the existence of an $\bar{x}^2 \in U((-R/2,-R/2,0),R/2)$ such that $x \in U(0,6R)$ and $x - \bar{x}^2 \perp (1,1,0)$ implies $x \in co_{\mathcal{E}}(x^2)$. But then obviously for any $t \in (0,\sqrt{2}R)$ and

$$x^1(t) = t\cdot(1,1,0)^T, y^1(t) = -t\cdot(1,1,0)^T$$

we find

$$\begin{aligned}
x^0(t) &= (\langle\bar{x}^2,(1,1,0)^T\rangle/2, \langle\bar{x}^2,(1,1,0)^T\rangle/2, -t + \langle\bar{x}^2,(1,1,0)^T\rangle/2)^T\\
x^2(t) &= (\langle\bar{x}^2,(1,1,0)^T\rangle/2, \langle\bar{x}^2,(1,1,0)^T\rangle/2, t - \langle\bar{x}^2,(1,1,0)^T\rangle/2)^T\\
y^0(t) &= (\langle\bar{x}^1,(1,1,0)^T\rangle/2, \langle\bar{x}^1,(1,1,0)^T\rangle/2, -t + \langle\bar{x}^1,(1,1,0)^T\rangle/2)^T \text{ and}\\
y^2(t) &= (-\langle\bar{x}^1,(1,1,0)^T\rangle/2, -\langle\bar{x}^1,(1,1,0)^T\rangle/2, t - \langle\bar{x}^1,(1,1,0)^T\rangle/2)
\end{aligned}$$

fulfilling Corollary 5.6.b).

Since $|\langle\bar{x}^1,(1,1,0)^T\rangle|, |\langle\bar{x}^2,(1,1,0)^T\rangle| < 2R$, we easily check that $x^0(t), x^1(t), x^2(t), y^0(t), y^1(t), y^2(t)$ all belong to $U(0,4R)$. Therefore, due to the choices of $\bar{x}^1$ and $\bar{x}^2$ Corollary 5.6.d) is also satisfied. Moreover, whenever $x \in U(0,R)$, $x \perp (1,1,0)^T$ and $x' \in U(0,2R) \cap H^1(x)$, then for $t = \langle x',(1,1,0)^T\rangle/2 \in (0,\sqrt{2}R)$ we have $x' - x^1(t) \perp (1,1,0)^T$ and hence $|x' - x^1(t)| \leq 2R + |x| \leq 3R$. So also 5.6.a) and c) hold. We conclude $x' \in co_{\mathcal{E}}(x^1)$ as required. The same argument applies for $x' \in U(x,2R) \cap H^2(x)$ and $y^1(\langle x',(1,1,0)^T\rangle/2)$, so $U(x,2R) \cap H^i(x) \subset co_{\mathcal{E}}(x^i)$ whenever $x \perp (1,1,0)^T$ is in $U(0,R)$.

Theorem 5.8 *If $U(z,2R) \subset \Omega$, then u is a laminate on $U(z,R)$, i.e there is an $j \in \{1,2,3\}$, $c \in \{+1,-1\}$, $Q \in SO(3)$ and a finite sequence $t_0 < t_1 < \cdots < t_l$ such that $-t_0, t_l > \sqrt{2}R$ and for any $x \in U(z,R)$ and $k \geq 0$*

$$\nabla u(x) = \begin{cases} Q\cdot D^j & \text{if } \langle x-z, v(j+1,c)\rangle \in (t_{2k}, t_{2k+1})\\ Q\cdot R(j+1,c)D^{j+2} & \text{if } \langle x-z, v(j+1,c)\rangle \in (t_{2k+1}, t_{2k+2}) \end{cases}$$

Proof First, we note that the proof of Theorem 4.3 and paragraph 5.2 show that for $\mathscr{H}^2$-a.e. $x \in \partial^* E_i^j \cap \partial^* E_{i'}^{j+2}$, $j = 1, 2, 3$ and E_i^j, $E_{i'}^{j+2}$ being BV-components of E^j and E^{j+2} there is some $c \in \{-1, +1\}$ such that $\nu_{E_i^j}(x) \| v(j+1, c)$. Due to Corollary 3.2.b) this is true even for $\mathscr{H}^2$-a.e. $x \in \partial^* E^j \cap \partial^* E^{j+2}$. So, if we denote $M(j, c)$ to be the set of all $x \in \partial^* E^{j-1} \cap \partial^* E^{j+1}$ for which $\nu_{E^{j-1}}(x) \| v(j, c)$ then due to Theorem 2.4

$$\|D(\nabla u)\|(\Omega \setminus \bigcup_{j=1}^{3} \bigcup_{c=\pm 1} M(j, c)) = 0.$$

From this we obtain easily that u is a laminate on $U(z, R)$ whenever there is at most one pair (j, c) such with $\mathscr{H}^2(M(j, c) \cap U(z, R)) > 0$.

Consequently, we can suppose the existence of $(j, c) \neq (j', c')$ for which both $M(j, c)$ and $M(j', c')$ intersect $U(z, R)$ in a set of positive 2-dimensional measure. Since the measure theoretical normal ν_{E^j} is $\mathscr{H}^2$-a.e. on $\partial^* E^j$ in fact normal to the approximative tangent space of this rectifiable set, we see that $U(z, R) \cap M(j, c)$ projects orthogonally onto a set of positive area in $v(j, c)^\perp$. Consequently, there is an $x \in U(z, R) \cap M(j, c)$ such that $g(x, v(j, c))$ is a regular line. Now Proposition 5.7 implies that for any $y \in U(z, 2R)$ with $y - x \perp v(j, c)$ there is a neighbourhood in which ∇u is constant along all planes orthogonal to $v(j, c)$. Since the same is true for (j', c') as well and since $v(j, c)$ and $v(j', c')$ can not be colinear, we obtain a contradiction finishing the proof from the elementary observation that the two planes $y + v(j, c)$ and $y' + v(j, c')$ intersect necessarily inside $U(z, 2R)$ whenever $y, y' \in U(z, R)$.

6 A highly nonpolygonal BV-geometry

Proposition 6.1 *Given any $N \in \mathbb{N}$, we can find open intervals $\tilde{U}_0 \subset (-1, 1)^2$ and $\{U_i\}_{i=1}^\infty$ such that for $U_0 = \operatorname{int}(\mathbb{R}^2 \setminus \tilde{U}_0)$*

a) $U_i \cap U_j = \emptyset$ for all $0 \leq i < j$.

b) $\mathscr{H}^1(\tilde{U}_0 \setminus \bigcup_{i=1}^\infty \overline{U_i}) = 0$.

c) $\displaystyle\sum_{i=1}^{\infty} \mathscr{H}^1(\partial U_i) < 1/N$.

d) There is a map $w : \mathbb{N}_0 \to \{0, 1\}$ satisfying $\overline{U_i} \cap \overline{U_j} = \emptyset$ whenever $w(i) = w(j)$.

e) There exists at least one proper cycle, but any proper cycle is of length N at least. (Here, by a cycle of length l we refer to a sequence $(j_0, \ldots, j_l)$ of indices such that $j_0 = j_l$ and that $\overline{U_{j_k}} \cap \overline{U_{j_{k+1}}}$ is a nontrivial line segment for any $k = 0, \ldots, l-1$. A cycle is said to be proper if $j_k \neq j_{k'}$ whenever $k - k' \in \{2, l-2\}$.)

Proof: During the proof we will denote by π_H and π_V the horizontal and vertical projections, i.e. $\pi_V((x,y)) = x$ and $\pi_H((x,y)) = y$.

So let us be given an $N \geq 3$, for purely technical reasons we suppose N to be odd, and fix $\tilde{U}_0 = (-1/2, 1/2)^2$, $U_0 = \mathbb{R}^2 \setminus [-1/2, 1/2]^2$, $i_0 = 0$ and $w(0) = 1$. Our construction will continue by induction, so assume for some $n \geq 0$ open intervals $\tilde{U}_0, \ldots, U_{i_n}$ and integers $w(0), \ldots, w(i_n) \in \{0, 1\}$ are already chosen in a way respecting a),d) and the minimal length condition in e).

We denote by $\mathcal{I}_n$ the system of all connected components of the remaining open set $\tilde{U}_0 \setminus \bigcup_{i=0}^{i_n} \overline{U_i}$ and assume that

$$\mathcal{I}_n \text{ is a finite system of open intervals of positive distance and equal size.} \tag{6.1}$$

For $I = (a,b) \times (c,d) \in \mathcal{I}_n$ let $S_H(I)$ be the set consisting of 0, 1 and all $t \in (0,1)$ such that $(a + t(b-a), c)$ or $(a + t(b-a), d)$ belongs to the closure of more than one of the already chosen U_i, $0 \leq i \leq i_n$. Similar, $S_V(I)$ contains $0, 1$ and all $t \in (0,1)$ such that $(a, c + t(d-c))$ or $(b, c + t(d-c))$ is in the closure of more than one U_i.

We make also the assumption that

$$S_V(I) \cup S_H(I) \subset \{\frac{k}{N+1} \; ; \; k = 0, \ldots, N+1\} \text{ for any } I \in \mathcal{I}_n. \tag{6.2}$$

First, let n be even and $I = (a,b) \times (c,d) \in \mathcal{I}_n$. For each $t \in S_V(I)$ we define the interval $\mathcal{J}_t \subset (c,d)$ by

$$\mathcal{J}_t = \begin{cases} (c, c + (d-c)/4N^2) & \text{for } t = 0 \\ (c + (t - \frac{1}{8N^2})(d-c), c + (t + \frac{1}{8N^2})(d-c)) & \text{for } t \in (0,1) \cap S_V(I) \\ (d - (d-c)/4N^2, d) & \text{for } t = 1. \end{cases}$$

Clearly, the $\overline{\mathcal{J}_t}$ are mutually disjoint. Hence, $(c,d) \setminus \{\overline{\mathcal{J}_t} \; ; \; t \in S_V(I)\}$ consists of precisely $\operatorname{card}(S_V(I)) - 1$ open intervals and we will refer to such an interval $\tilde{\mathcal{J}}$ as $\tilde{\mathcal{J}}_t$ where $t \in S_V(I) \setminus \{1\}$ is given by $\inf \tilde{\mathcal{J}}_t = \sup \mathcal{J}_t$. Due to the definition of $S_V(I)$, for each $t \in S_V(I) \setminus \{1\}$ there are unique indices $i^+ = i^+(I,t), i^- = i^-(I,t) \leq i_n$ such that

$$\overline{U_{i^-}} \cap ([a,b] \times \overline{\tilde{\mathcal{J}}_t}) = \{a\} \times \overline{\tilde{\mathcal{J}}_t}, \; \overline{U_{i^+}} \cap ([a,b] \times \overline{\tilde{\mathcal{J}}_t}) = \{b\} \times \overline{\tilde{\mathcal{J}}_t},$$

and

$$\overline{U_i} \cap ([a,b] \times \overline{\tilde{\mathcal{J}}_t}) = \emptyset \text{ for } i \in \{0, \ldots, i_n\} \setminus \{i^+, i^-\}.$$

We choose $L = L(I,t) \in \{N-1, N\}$ such that $L + w(i^+) - w(i^-)$ is even, define

$$U(I,t,k) = (a + \frac{k-1}{N+1}(b-a), a + \frac{k}{N+1}(b-a)) \times \tilde{\mathcal{J}}_t,$$
$$\tilde{w}(U(I,t,k))) \equiv w(i^-(I,t)) + k)_{(2)}$$

for $k = 1, \ldots, L$ and

$$U(I,t,L+1) = (a + \frac{L}{N+1}(b-a), b) \times \tilde{\mathcal{J}}_t, \tilde{w}(U(I,t,L+1))) \equiv w(i^+(I,t)) + 1)_{(2)}.$$

Since $\mathrm{card}(S_V(I)\setminus\{1\}) < N+2$ and $L(I,t)+1 \le N+2$ for all t, we obtain the estimate

$$\sum_{t\in S_V(I)\setminus\{1\}} \sum_{k=1}^{L(I,t)+1} \mathscr{H}^1(\partial U(I,t,k)) \le (N+2)\mathscr{H}^1(\partial I). \tag{6.3}$$

Repeating this procedure in all the $I \in \mathcal{I}_n$, we can then simply define $\{U_i\}_{i=i_n+1}^{i_{n+1}}$ to be an enumeration of the family of all $U(I,t,k)$ for $I \in \mathcal{I}_n$, $t \in S_V(I)\setminus\{1\}$, $k = 1,\dots,L(I,t)+1$ and set $w(i) = \tilde{w}(U_i)$ for $i = i_n+1,\dots,i_{n+1}$. Now it is straightforward to check that a) and d) hold again. Since $\mathcal{I}_{n+1} = \{\pi_H^{-1}(\mathcal{J}) \cap I\ ;\ I \in \mathcal{I}_n \text{ and } t \in S_V(I)\}$, we see that (6.1) is still valid and that moreover

$$|\pi_H(I')| = \frac{|\pi_H(I)|}{4N^2}, |\pi_V(I')| = |\pi_V(I)| \text{ whenever } I \in \mathcal{I}_n \text{ and } I' \in \mathcal{I}_{n+1}. \tag{6.4}$$

Using (6.3) we infer also

$$\mathrm{card}(\mathcal{I}_{n+1}) \le (N+2)\mathrm{card}(\mathcal{I}_n) \text{ and } \sum_{i=i_n+1}^{i_{n+1}} \mathscr{H}^1(\partial U_i) \le (N+2)\sum_{I\in\mathcal{I}_n} \mathscr{H}^1(\partial I). \tag{6.5}$$

To verify (6.2) let I' be an arbitrary interval in $\mathcal{I}_{n+1}$, hence $I = \pi_H(\mathcal{J}_t) \cap I$ for some $I \in \mathcal{I}_n$, $t \in S_V(I)$. If $t = 0$, we obviously have $S_H(I') \subset S_H(I) \cup \{k/(N+1)\ ;\ k = 1,\dots,L(I,0)\}$ and $S_V(I) \cap (0,1) = \emptyset$. For $t = 1$ the statement (6.2) follows similarly. If $t \in (0,1) \cap S_V(I)$ and $t' = \max(S_V(I) \cap [0,t))$, then $S_H(I') = \{0,1\} \cup \{k/(N+1)\ ;\ k = 1,\dots,\max(L(I,t'),L(I,t))\}$. Moreover, $S_V(I') = \{0,1/2,1\}$, so (6.2) is satisfied, because $N+1$ was supposed to be even.

Last, we show that there is no proper cycle in $\{0,\dots,i_{n+1}\}$ of length less than N. Indeed, due to our induction assumption, any such cycle must contain an index bigger than i_n. This means, the corresponding chain of intervals must enter one of the "bridges" $\{U(I,t,k)\}_{k=1}^{L(I,t)+1}$ and since the cycle is supposed to be proper, it can not leave the bridge on the side it entered. Hence, the cycle contains at least $L(I,t)+2 > N$ indices, contradiction. So, the minimal length condition in e) is still satisfied. Moreover, already for $n = 1$ also the existence part of e) is fulfilled since one easily checks that $(0,1,\dots,i_1,0)$ is a proper cycle.

Therefore, we are now in a position to repeat this construction for odd $n+1$ instead of even n but interchanging the roles the vertical and horizontal directions play and still ensuring a),b) and e).

In this way we build up the whole sequence $\{U\}_0^\infty$ and $w : \mathbb{N}_0 \to \{0,1\}$ fulfilling a),b) and e). We finish the proof by establishing b) and c). From (6.3),(6.4) and the analogous estimate (6.4') for odd n,i.e. π_H and π_V being exchanged, we derive that for all $n \ge 0$

$$\mathrm{card}(\mathcal{I}_{n+2}) \le (N+2)^2\mathrm{card}(I_n)$$

and

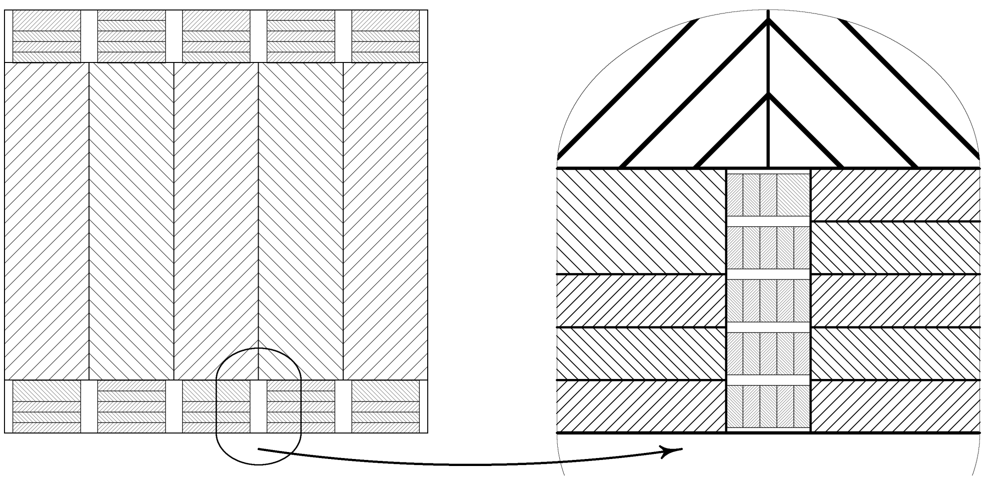
$$I'' \cong \frac{1}{4N^2} \cdot I \text{ provided } I'' \in \mathcal{I}_{n+2} \text{ and } I \in \mathcal{I}_n.$$

Consequently,

$$\sum_{I\in\mathcal{I}_{n+2}} \mathscr{H}^1(\partial I) \le \left(\frac{N+2}{2N}\right)^2 \sum_{I\in\mathcal{I}_n} \mathscr{H}^1(\partial I) \le \left(\frac{5}{6}\right)^2 \sum_{I\in\mathcal{I}_n} \mathscr{H}^1(\partial I)$$

which together with (6.3) implies $\sum_i \mathscr{H}^1(\partial U_i) < \infty$. Since $\tilde{U}_0 \setminus \bigcup_{i=1}^\infty \overline{U_i} \subset \bigcup \mathcal{I}_n$ for each n, also b) follows. Finally, to ensure also c) one can simply rescale the whole just constructed picture using a sufficiently shrinking homothety.

There is one more property of this construction which can easily be checked by the interested reader but which deserves at least to be mentioned. Namely, the construction is done in a way sufficiently selfsimilar in order to ensure the following. All possible blow-ups of this decomposition (which are due to compactness of the space BV again a union of intervals with finite total length of the boundary) fulfill the same lower estimate for the shortest nontrivial proper cycle. It is of course important to have this property since otherwise the consideration of blow-ups[2)] would lead to more simple compatibility conditions. Hopefully, the picture below illuminates the construction. The two different hatchings used correspond to the two different values of w.



7 Appendix

Here we will present the rather technical proof of Lemma 5.4. It has to be noted that the following calculations were carried out using Maple V. The correctness of our proof by contradiction is very sensitive to any kind of "random errors" (like typos etc.). In fact, the polynomial equations which represent the necessary conditions we derived before are generically unsolvable. Therefore, random errors lead almost surely to the desired contradiction in a seemingly perfect proof. On the other hand, writing down all computations and transformations involved would be possible but makes the paper much longer without being really

[2)]note that ∇u does not change in the process of blowing up illuminating nor ensuring that this kind of “random errors” will be detected. For these reasons the source code of the Maple session which contains the results of this appendix will be available on the internet (http://www.mis.mpg.de/preprints/98/preprint1298-addendum).

We return to the setting of Lemma 5.4. Using the results of paragraph 5.2, in particular (5.2), we obtain that

$$M = [1 \to 3]_{d^1}[3 \to 2]_{d^1}[2 \to 1]_{d^1} = R(2,1)R(1,1)R(3,1)$$

equals

$$\begin{pmatrix} -\frac{-7a^4-a^2+a^6-1}{(a^2+1)^3} & 4\frac{(a^2-1)a^3}{(a^2+1)^3} & -2\frac{(a^2-1)a}{(a^2+1)^2} \\ -2\frac{(a^2-1)a}{(a^2+1)^2} & 4\frac{a^2}{(a^2+1)^2} & \frac{a^2-1}{a^2+1} \\ 4\frac{(a^2-1)a^3}{(a^2+1)^3} & \frac{(a^2-1)(a^4-1-4a^2)}{(a^2+1)^3} & 4\frac{a^2}{(a^2+1)^2} \end{pmatrix}.$$

Since $(M - \mathrm{Id})x = 0$ has the normalized solution $x = (1,a,a)^\top/\sqrt{1+2a^2}$, we define $Q \in SO(3)$ using x as the first row. It is clear now that $Di = Q \cdot M \cdot Q^\top$ will fulfill $Di(1,1) = \langle x, x\rangle = 1$, $D(2,1) = D(3,1) = D(1,2) = D(1,3) = 0$ and $\tilde{Di}$ represents the action of the rotation M on the plane normal to the axis x of M. It remains to prove a) and b). For this purpose, let

$$S^{w_1,w_2} = Q[1 \to w_1]_{d^1}[w_1 \to 1]_{d^2}[1 \to 2]_{d^1}[2 \to w_2]_{d^2}[w_2 \to 1]_{d^1}Q^\top$$

and

$$S^{w} = Q[1 \to w_1]_{d^1}[w_1 \to 1]_{d^2}[1 \to 2]_{d^1}[2 \to 3]_{d^2}[3 \to 1]_{d^2}Q^\top$$

whenever $w_1, w_2, w \in \{1,2,3\}$. Since Di consists of two blocks, we infer that for $S = S^w$ or $S = S^{w_1,w_2}$ being a integer power of $Di \in SO(3)$ necessarily

$$S(2,1) = S(3,1) = S(1,2) = S(3,1) = 0. \tag{7.1}$$

After these introductionary observations the remainder of the proof consists in checking all different possible choice for (w_1, w_2) or w.

We start with part a), i.e. consider the pairs (w_1, w_2). For $(1,1)$ the matrix $S = S^{1,1}$ equals

$$\begin{pmatrix} \frac{7a^2-1}{(1+2a^2)(a^2+1)} & -\frac{\sqrt{2}(a^2-1)a}{\sqrt{1+2a^2}(a^2+1)} & -6\frac{(a^2-1)a}{\sqrt{1+2a^2}(a^2+1)\sqrt{2+4a^2}} \\ -\frac{\sqrt{2}a(-4a^2+3+a^4)}{(a^2+1)^2\sqrt{1+2a^2}} & 4\frac{a^2}{(a^2+1)^2} & -\frac{\sqrt{2}(5a^4-6a^2+1)}{\sqrt{2+4a^2}(a^2+1)^2} \\ 2\frac{a(5a^4-1-4a^2)}{\sqrt{2+4a^2}(a^2+1)^2\sqrt{1+2a^2}} & \frac{\sqrt{2}(3a^4-2a^2-1)}{\sqrt{2+4a^2}(a^2+1)^2} & -2\frac{a^2(a^4-1-6a^2)}{(1+2a^2)(a^2+1)^2} \end{pmatrix}.$$

and obviously $S^{1,1}(1,2) \neq 0$.

For $S = S^{2,1}$ we obtain

$$\begin{pmatrix} \frac{73a^4-27a^2-25a^6+1+2a^8}{(1+2a^2)(a^2+1)^3} & -4\frac{\sqrt{2}a(3a^4-4a^2+1)}{\sqrt{1+2a^2}(a^2+1)^3} & 8\frac{a(-11a^4+10a^2+2a^6-1)}{\sqrt{1+2a^2}(a^2+1)^3\sqrt{2+4a^2}} \\ -4\frac{\sqrt{2}a(5a^6-15a^4+11a^2-1)}{(a^2+1)^4\sqrt{1+2a^2}} & \frac{-12a^6+a^8+38a^4-12a^2+1}{(a^2+1)^4} & 8\frac{\sqrt{2}a^2(-9a^4+a^6+11a^2-3)}{\sqrt{2+4a^2}(a^2+1)^4} \\ -8\frac{a(-17a^6+15a^4-1+2a^8+a^2)}{\sqrt{2+4a^2}(a^2+1)^4\sqrt{1+2a^2}} & -8\frac{\sqrt{2}a^2(a^6-5a^4+3a^2+1)}{\sqrt{2+4a^2}(a^2+1)^4} & \frac{-55a^8+128a^6-10a^2+2a^{10}-18a^4+1}{(1+2a^2)(a^2+1)^4} \end{pmatrix}.$$

In particular, we derive from $S^{2,1}(1,2) = S^{2,1}(1,3) = 0$ that $3a^4 - 4a^2 + 1 = 0$ and $-11\,a^4 + 10\,a^2 + 2\,a^6 - 1 = 0$, but the first equation gives then $a^2 = 1/3$ for which the second polynomial does not vanish.

Next,

$$S^{2,2} = \begin{pmatrix} \frac{7\,a^2-1}{(a^2+1)\,(1+2\,a^2)} & -\frac{(a^2-1)\,a\,\sqrt{2}}{\sqrt{1+2\,a^2}\,(a^2+1)} & -3\,\frac{a\,(a^2-1)\,\sqrt{2}}{(1+2\,a^2)\,(a^2+1)} \\ -\frac{a\,\sqrt{2}\,(-4\,a^2+3+a^4)}{\sqrt{1+2\,a^2}\,(a^2+1)^2} & 4\,\frac{a^2}{(a^2+1)^2} & -\frac{5\,a^4-6\,a^2+1}{\sqrt{1+2\,a^2}\,(a^2+1)^2} \\ \frac{a\,(5\,a^4-1-4\,a^2)\,\sqrt{2}}{(1+2\,a^2)\,(a^2+1)^2} & \frac{3\,a^4-2\,a^2-1}{\sqrt{1+2\,a^2}\,(a^2+1)^2} & -2\,\frac{(a^4-6\,a^2-1)\,a^2}{(1+2\,a^2)\,(a^2+1)^2} \end{pmatrix}.$$

So $S^{2,2}(2,1) = S^{2,2}(3,1) = 0$ implies $a^4 - 4a^2 + 3 = 5a^4 - 4a^2 - 1 = 0$, hence $4a^4 - 4 = 0$ which gives a contraction to $a \neq 1$.

If $(w_1, w_2) = (2,3)$, then S is equal to

$$\begin{pmatrix} \frac{7\,a^2-1}{(1+2\,a^2)\,(a^2+1)} & \frac{a\,\sqrt{2}\,(-16\,a^6+18\,a^4-3+a^8)}{(a^2+1)^4\,\sqrt{1+2\,a^2}} & 2\,\frac{a\,(-28\,a^6+6\,a^4+12\,a^2+9\,a^8+1)}{(a^2+1)^4\,\sqrt{1+2\,a^2}\,\sqrt{2+4\,a^2}} \\ -\frac{\sqrt{2}\,a\,(-4\,a^2+3+a^4)}{(a^2+1)^2\,\sqrt{1+2\,a^2}} & -\frac{19\,a^8-52\,a^6-2\,a^4+4\,a^2-1}{(a^2+1)^5} & \frac{\sqrt{2}\,a^2\,(-60\,a^6+26\,a^4+5\,a^8+1+28\,a^2)}{\sqrt{2+4\,a^2}\,(a^2+1)^5} \\ 2\,\frac{a\,(5\,a^4-1-4\,a^2)}{\sqrt{2+4\,a^2}\,(a^2+1)^2\,\sqrt{1+2\,a^2}} & -\frac{\sqrt{2}\,a^2\,(9\,a^8-60\,a^6+18\,a^4+28\,a^2+5)}{\sqrt{2+4\,a^2}\,(a^2+1)^5} & \frac{-42\,a^{10}+79\,a^8+64\,a^6+2\,a^{12}-6\,a^2-1}{(1+2\,a^2)\,(a^2+1)^5} \end{pmatrix}$$

and we can finish with the same argument as for $S^{2,2}$

So we turn to the three cases with $w_1 = 3$. First, $(3,1)$ gives for the matrix S

$$\begin{pmatrix} -\frac{-45\,a^4+15\,a^2+5\,a^6-1+2\,a^8}{(a^2+1)^3\,(1+2\,a^2)} & -4\,\frac{a\,\sqrt{2}\,(a^4-2\,a^2+1)}{(a^2+1)^3\,\sqrt{1+2\,a^2}} & -4\,\frac{a\,(11\,a^4-13\,a^2+a^6+1)}{(a^2+1)^3\,\sqrt{1+2\,a^2}\,\sqrt{2+4\,a^2}} \\ 4\,\frac{a\,\sqrt{2}\,(3\,a^6-7\,a^4+5\,a^2-1)}{(a^2+1)^4\,\sqrt{1+2\,a^2}} & -\frac{-12\,a^6+6\,a^4-12\,a^2+a^8+1}{(a^2+1)^4} & -8\,\frac{\sqrt{2}\,a^2\,(a^6-4\,a^4+5\,a^2-2)}{\sqrt{2+4\,a^2}\,(a^2+1)^4} \\ -4\,\frac{a\,(-20\,a^6+4\,a^2+14\,a^4+1+a^8)}{(a^2+1)^4\,\sqrt{1+2\,a^2}\,\sqrt{2+4\,a^2}} & -8\,\frac{\sqrt{2}\,a^4\,(a^4-2\,a^2+1)}{\sqrt{2+4\,a^2}\,(a^2+1)^4} & \frac{-25\,a^8+2\,a^4+72\,a^6-2\,a^2+2\,a^{10}-1}{(a^2+1)^4\,(1+2\,a^2)} \end{pmatrix}.$$

But here the one condition $S^{3,1}(1,2) = 0$ only already gives the impossible conclusion $(a^2-1)^2 = 0$.

Further, $S^{3,2}$ turns out to be

$$\begin{pmatrix} \frac{7\,a^2-1}{(1+2\,a^2)\,(a^2+1)} & \frac{a\,\sqrt{2}\,(a^2-1)}{\sqrt{1+2\,a^2}\,(a^2+1)} & -6\,\frac{(a^2-1)\,a}{\sqrt{1+2\,a^2}\,(a^2+1)\,\sqrt{2+4\,a^2}} \\ \frac{a\,\sqrt{2}\,(-4\,a^2+3+a^4)}{(a^2+1)^2\,\sqrt{1+2\,a^2}} & 4\,\frac{a^2}{(a^2+1)^2} & \frac{\sqrt{2}\,(5\,a^4-6\,a^2+1)}{\sqrt{2+4\,a^2}\,(a^2+1)^2} \\ 2\,\frac{a\,(5\,a^4-1-4\,a^2)}{\sqrt{2+4\,a^2}\,(a^2+1)^2\,\sqrt{1+2\,a^2}} & -\frac{\sqrt{2}\,(3\,a^4-2\,a^2-1)}{\sqrt{2+4\,a^2}\,(a^2+1)^2} & -2\,\frac{a^2\,(a^4-1-6\,a^2)}{(1+2\,a^2)\,(a^2+1)^2} \end{pmatrix}.$$

Obviously $S^{3,2}(1,2) \neq 0$.

Finally, $S^{3,3}$ equals

$$\begin{pmatrix} \frac{7\,a^2-1}{(1+2\,a^2)\,(a^2+1)} & -\frac{\sqrt{2}\,a\,(-12\,a^4+6\,a^2+3+a^8+2\,a^6)}{\sqrt{1+2\,a^2}\,(a^2+1)^4} & -2\,\frac{a\,(14\,a^6-12\,a^4-6\,a^2+3\,a^8+1)}{\sqrt{1+2\,a^2}\,(a^2+1)^4\,\sqrt{2+4\,a^2}} \\ \frac{a\,\sqrt{2}\,(-4\,a^2+3+a^4)}{(a^2+1)^2\,\sqrt{1+2\,a^2}} & \frac{11\,a^8-4\,a^6+22\,a^4+4\,a^2-1}{(a^2+1)^5} & -\frac{\sqrt{2}\,a^2\,(5\,a^8-12\,a^6+18\,a^4-4\,a^2-7)}{\sqrt{2+4\,a^2}\,(a^2+1)^5} \\ \frac{\sqrt{2}a\,(5\,a^4-1-4\,a^2)}{(a^2+1)^2\,(1+2\,a^2)} & -\frac{\sqrt{2}\,a^2\,(-20\,a^6+14\,a^4+3\,a^8+4\,a^2-1)}{\sqrt{2+4\,a^2}\,(a^2+1)^5} & \frac{-6\,a^{10}+49\,a^8+32\,a^6+12\,a^4+6\,a^2+2\,a^{12}+1}{(1+2\,a^2)\,(a^2+1)^5} \end{pmatrix}.$$

Since the conditions $S(2,1) = S(3,1) = 0$ lead to the same equations as for $S^{1,1}$, we find again a contradiction. Therefore, part a) of Lemma 5.4 is proved, and we have to consider S^1, S^2, S^3.

We obtain

$$S^1 = \begin{pmatrix} \frac{7\,a^2-1}{(1+2\,a^2)(a^2+1)} & \frac{\sqrt{2}\,(a^2-1)\,a}{\sqrt{1+2\,a^2}\,(a^2+1)} & -6\,\frac{(a^2-1)\,a}{\sqrt{1+2\,a^2}\,(a^2+1)\,\sqrt{2+4\,a^2}} \\ \frac{\sqrt{2}\,a\,(3\,a^4-5\,a^2+1+a^6)}{(a^2+1)^3\,\sqrt{1+2\,a^2}} & \frac{5\,a^4+2\,a^2+1}{(a^2+1)^3} & \frac{\sqrt{2}\,a^2\,(a^4-6\,a^2+5)}{\sqrt{2+4\,a^2}\,(a^2+1)^3} \\ 2\,\frac{a\,(-5\,a^2-3+a^6+7\,a^4)}{\sqrt{2+4\,a^2}\,(a^2+1)^3\,\sqrt{1+2\,a^2}} & \frac{\sqrt{2}\,a^2\,(3\,a^4-2\,a^2-1)}{\sqrt{2+4\,a^2}\,(a^2+1)^3} & \frac{17\,a^4+4\,a^2+2\,a^8+2\,a^6-1}{(1+2\,a^2)(a^2+1)^3} \end{pmatrix},$$

and $S^1(1,2) \neq 0$.

Next, we compute S^2 to be

$$\begin{pmatrix} \frac{86\,a^4-34\,a^2+a^8+1-8\,a^6+2\,a^{10}}{(1+2\,a^2)(a^2+1)^4} & -2\,\frac{a\,\sqrt{2}\,(3\,a^6-7\,a^4+5\,a^2-1)}{(a^2+1)^4\,\sqrt{1+2\,a^2}} & -4\,\frac{a\,(19\,a^4-25\,a^2+2\,a^8+3+a^6)}{(a^2+1)^4\,\sqrt{1+2\,a^2}\,\sqrt{2+4\,a^2}} \\ -2\,\frac{a\,\sqrt{2}\,(8\,a^6-34\,a^4+5\,a^8+24\,a^2-3)}{(a^2+1)^5\,\sqrt{1+2\,a^2}} & \frac{a^2\,(4\,a^6-38\,a^4+12\,a^2-11+a^8)}{-(a^2+1)^5} & \frac{3\,a^8+64\,a^6+2\,a^{10}-98\,a^4+30\,a^2-1}{-\sqrt{1+2\,a^2}\,(a^2+1)^5} \\ \frac{20\,a^9+160\,a^7-152\,a^5-24\,a^3+4\,a-8\,a^{11}}{\sqrt{2+4\,a^2}\,(a^2+1)^5\,\sqrt{1+2\,a^2}} & \frac{2\,a^{10}+35\,a^8-32\,a^6-2\,a^4-2\,a^2-1}{\sqrt{1+2\,a^2}\,(a^2+1)^5} & \frac{a^{10}+24\,a^8-158\,a^6+22\,a^4+13\,a^2+2\,a^{12}}{-(1+2\,a^2)(a^2+1)^5} \end{pmatrix}.$$

Now $S^2(1,2) = S^2(1,3) = S^2(2,1) = 0$ yield $p_1(b) = p_2(b) = p_3(b) = 0$, where $b = a^2$, $p_1(b) = -1 + 5\,b - 7\,b^2 + 3\,b^3$, $p_2(b) = 3 - 25\,b + 19\,b^2 + b^3 + 2\,b^4$, and $p_3(b) = -3 + 24\,b - 34\,b^2 + 8\,b^3 + 5\,b^4$. Since $(11 \cdot p_1 - 3(2 \cdot p_3 - 5 \cdot p_2))(b) = 52 - 464\,b + 412\,b^2$, we infer $b = 13/103$ but $p_1(b)(13/103) = -518400/1092727 \neq 0$.

Last of all, S^3 equals

$$\begin{pmatrix} \frac{24\,a^6+62\,a^4-30\,a^2-7\,a^8+1-2\,a^{10}}{(a^2+1)^4\,(1+2\,a^2)} & 4\,\frac{\sqrt{2}\,a^3\,(2\,a^2-3+a^4)}{\sqrt{1+2\,a^2}\,(a^2+1)^4} & -4\,\frac{a\,(8\,a^6+8\,a^4-20\,a^2+a^8+3)}{\sqrt{1+2\,a^2}\,(a^2+1)^4\,\sqrt{2+4\,a^2}} \\ \frac{\sqrt{8}\,a\,(-4\,a^6-14\,a^4+a^8+20\,a^2-3)}{(a^2+1)^5\,\sqrt{1+2\,a^2}} & \frac{a^2\,(4\,a^6+2\,a^4+28\,a^2-3+a^8)}{(a^2+1)^5} & \frac{\sqrt{2}\,(11\,a^8+16\,a^6+2\,a^{10}-58\,a^4+30\,a^2-1)}{\sqrt{2+4\,a^2}\,(a^2+1)^5} \\ 4\,\frac{a^3\,(12\,a^6+18\,a^4-28\,a^2-3+a^8)}{\sqrt{2+4\,a^2}\,(a^2+1)^5\,\sqrt{1+2\,a^2}} & \frac{5\,a^8-24\,a^6+2\,a^{10}+10\,a^4+6\,a^2+1}{\sqrt{1+2\,a^2}\,(a^2+1)^5} & \frac{a^2\,(16\,a^6-7\,a^8+106\,a^4-14\,a^2-2\,a^{10}-3)}{(1+2\,a^2)(a^2+1)^5} \end{pmatrix}$$

and $S^3(1,2) = 0$ forces $a^2 \in \{0, 1, -3\}$. This is the last contradiction needed to finish our proof.

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