Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Some Remarks on the Article "SU(N) Quantum Yang-Mills Theory in Two Dimensions: A Complete Solution" by Ashtekar, Lewandowski, Marolf, Mourão and Thiemann

by

Christian Fleischhack

Preprint-Nr.: 26 1998

Some Remarks on the Article "$SU(N)$ Quantum Yang-Mills Theory in Two Dimensions: A Complete Solution" by Ashtekar, Lewandowski, Marolf, Mourão and Thiemann

Christian Fleischhack*

Institut für Theoretische Physik
Fakultät für Physik und Geowissenschaften
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig, Germany

Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstraße 22-26
04103 Leipzig, Germany

June 23, 1998

Abstract

The expectation values of Wilson loop products for the pure Euclidean Yang-Mills theory on $\mathbb{R} \times \mathbb{R}$ given by Ashtekar et al. [ALM$^+$96] are determined directly for all piecewise analytic loops. For this a new kind of hoop independence is introduced and their regularization scheme is slightly modified.

1 Introduction

For quite a long time the quantization of Yang-Mills theories has been investigated. One of the main emphases is the approach via functional integration. The crucial point is the definition of an appropriate measure $d\mu$ on the space $\mathcal{A}/\mathcal{G}$ of all connections modulo gauge transformations. Heuristically one sets simply $d\mu := e^{-S(A)}\mathcal{D}A$, where $S(A)$ is the Yang-Mills action and $\mathcal{D}A$ is a kinematical measure on $\mathcal{A}/\mathcal{G}$, but the resulting mathematical problems are enormous. Some years ago, Ashtekar and Isham [AI92] developed an interesting idea to overcome these difficulties. They considered a certain completion of $\mathcal{A}/\mathcal{G}$, the compact Hausdorff space $\overline{\mathcal{A}/\mathcal{G}}$. Now, Ashtekar and Lewandowski [AL93] were able to construct a natural kinematical measure $d\mu_0$ corresponding to $\mathcal{D}A$, but the extension of S onto the whole

*e-mail: Christian.Fleischhack@itp.uni-leipzig.de *or* Christian.Fleischhack@mis.mpg.de

$\overline{\mathcal{A}/\mathcal{G}}$ remained to be difficult. This problem was circumvented using the duality between measures on $\overline{\mathcal{A}/\mathcal{G}}$ and positive linear functionals on the space of all Wilson loop products. Using the lattice regularization, Thiemann [Thi95] and Ashtekar et al. [ALM$^+$96] defined these expectation values and received the measure $d\mu$.

Nevertheless, some technical problems remained open. The authors of [ALM$^+$96] did not specify the type of hoop independence used for the projection $\overline{\mathcal{A}/\mathcal{G}} \longrightarrow \mathbf{G}^n$. Both the strong independence and the weak independence [AL93] are not applicable – the first one because obviously the lattice loops $\beta_{x,y} = \rho_{x,y}\Box_{x,y}\rho_{x,y}^{-1}$ cannot be strongly independent for lattices with more than two rows and columns, and the second one because then the integral would become ill-defined [AL93]. Furthermore, the authors of [ALM$^+$96] used the completeness of the plaquette loops $\beta_{x,y}$, i.e. that the subgroup of the hoop group generated by the $\beta_{x,y}$ coincides with the subgroup generated by all loops in the lattice. But, in general, the completeness is not guaranteed if one chooses arbitrary paths $\rho_{x,y}$ from the base point to the plaquette (x, y). So we will prove that there *exists* a choice for the $\rho_{x,y}$ such that the plaquette loops $\beta_{x,y}$ are complete. For the same reasons, the proof of the decomposition lemma, which ensures that any loop α without self-intersections can be expressed by a product of the loops corresponding to the plaquettes in the interior of α, has to be modified.

The present article is intended to provide these missing mathematical details. Moreover, we drop the restriction on quadratic lattices. We admit now any finite connected graph – a "floating" lattice – for the regularization. For this we slightly modify the regularization of the Yang-Mills action simply replacing a^2 ($a \ldots$ lattice spacing) by the area $|G|$ of the plaquette (see also [AK98]) and adapting the regularization to the given loops and not, as usual, vice versa. Thus, the use of floating lattices allows us to calculate the Wilson loop expectation values for all sets of hoops directly, i.e. without approximating them in a certain sense by loops in a quadratic lattice and without a subsequent (naive) limit. On the other hand we need a little bit more sophisticated – and, unfortunately, more technical – analysis, even if we would consider only quadratic lattices. At the beginning we define a new type of independence – the so-called moderate independence, which stands between the strong and the weak independence and is well-suited to make the calculations mathematically rigorous. We prove that it is strong enough to make the integration calculus still applicable. Then we generalize the propositions in [ALM$^+$96] to the case of floating lattices. The loops $\beta_{x,y}$ correspond now to the so-called flags f_G, i.e. loops that run from the base point m to the interior domain G – the generalized plaquette –, traverse G once and return to m. Choosing a flag to each interior domain we get a flag world. The crucial point is now the proof that there is a (moderately) independent and complete flag world for any graph. Moreover, the generalized decomposition lemma yields that, if one refines the underlying graph, any flag world can be naturally refined to a new (again moderately independent and complete) flag world and each flag f of the old flag world is a product of exactly the flags of the new one that correspond to domains in the interior of f.

By means of these propositions we can finally compute the Wilson loop expectation values reusing the calculations of Thiemann and Ashtekar et al.

2 Preliminaries

In this section we summarize the basic facts about the space $\overline{\mathcal{A}/\mathcal{G}}$ of generalized connections modulo gauge transformations following [AI92, AL93, ALM$^+$96].

Let P be a fixed principal fibre bundle over the base manifold M with structure group $\mathbf{G}$ and m any fixed point in M. Furthermore, let $\{U_i\}$ be a covering of M, $\{\chi_i\}$ a trivialization of P over $\{U_i\}$ and j a fixed index with $m \in U_j$. In the following we suppose $\mathbf{G}$ to be either $SU(N)$, $N \geq 2$, or $U(1)$. Connections on P are described by their connection 1-form A on P or, equivalently, their localized forms A_i on U_i. Similarly, we describe a gauge transformation by its corresponding equivariant map $\rho : P \longrightarrow \mathbf{G}$ or its localized forms $\rho_i : U_i \longrightarrow \mathbf{G}$. We will only consider C^∞ connections and C^∞ gauge transformations. The spaces of all connections and all gauge transformations are denoted by $\mathcal{A}$ and $\mathcal{G}$, respectively, and their quotient w.r.t. the natural action of $\mathcal{G}$ on $\mathcal{A}$ is denoted by $\mathcal{A}/\mathcal{G}$.

Next, we define $\mathcal{L}_m$ to be the set of all piecewise analytic loops in M with base point m, i.e. all piecewise analytic maps $\alpha : [0,1] \longrightarrow M$, $\alpha(0) = \alpha(1) = m$. Two loops α_1 and α_2 are multiplied by $\alpha_1 \circ \alpha_2(t) := \begin{cases} \alpha_1(2t), & \text{for } t \in [0, \frac{1}{2}] \\ \alpha_2(2t-1), & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$. Note, that $\circ$ is not associative. For any $\alpha \in \mathcal{L}_m$ and $A \in \mathcal{A}$ we define the holonomy $h_\alpha(A) = h_A(\alpha) = h(\alpha, A) \in \mathbf{G}$ as the group element, which corresponds to the parallel transport w.r.t. A of $\chi_j^{-1}(m, e_{\mathbf{G}})$ along α. In the trivialization χ_j we have $h(\alpha, A) = \mathcal{P}e^{-\oint_\alpha A_j}$ if α is completely contained in U_j. A change of the trivialization yields a conjugation of $h(\alpha, A)$ independent of α. Moreover, we have $h_\alpha h_\beta = h_{\alpha\circ\beta}$ for all $\alpha, \beta \in \mathcal{L}_m$.

The fundamental idea of Ashtekar and Isham was to use the description of connections by the traces of their holonomies, the so-called Wilson loops. First, they defined an equivalence relation on $\mathcal{L}_m$. Two loops $\alpha_1, \alpha_2 \in \mathcal{L}_m$ are said to be holonomically equivalent $\alpha_1 \sim \alpha_2$ iff $h_{\alpha_1}(A) = h_{\alpha_2}(A)$ for any $A \in \mathcal{A}$. The equivalence classes $[\alpha]$ are called hoops[1]. The hoop group $\mathcal{HG}$ is the set of all hoops with the well-defined projected multiplication of $\mathcal{L}_m$: $[\alpha_1] \circ [\alpha_2] = [\alpha_1 \circ \alpha_2]$ and $[\alpha]^{-1} = [\beta]$ with $\beta(t) = \alpha(1-t)$. For instance, two loops are holonomically equivalent if they can be obtained from each other by reparametrization or insertion of retracings. Second, Ashtekar and Isham made use of the so-called Wilson loops $T_\alpha : \mathcal{A} \longrightarrow \mathbb{C}$ defined by $T(\alpha, A) = T_\alpha(A) = \frac{1}{N}\mathrm{tr}\, h_\alpha(A)$. Obviously, T factorizes over $\sim$ and $\mathcal{G}$, i.e. $T : \mathcal{HG} \times \mathcal{A}/\mathcal{G} \longrightarrow \mathbb{C}$. Next, they defined the algebra $\mathcal{HA} := \{f : \mathcal{A}/\mathcal{G} \longrightarrow \mathbb{C} \mid f = \sum_{i=1}^n c_i \prod_{j_i=1}^{n_i} T_{\alpha_{j_i}} \mid n, n_i \in \mathbb{N}, c_i \in \mathbb{C}\}$ of all finite linear combinations of finite products of Wilson loops and called its completion $\overline{\mathcal{HA}}$ with respect to the sup-norm on $\mathcal{A}/\mathcal{G}$ holonomy algebra. Clearly, $\overline{\mathcal{HA}}$ is a commutative C^* algebra. This allows to use the powerful tools provided by the theory of C^* algebras. Due to the Gelfand-Naimark theorem there exists a compact Hausdorff space $\mathcal{M}(\overline{\mathcal{HA}})$, the space of all characters of $\overline{\mathcal{HA}}$, i.e. all nontrivial, linear, multiplicative functionals on $\overline{\mathcal{HA}}$, such that $\overline{\mathcal{HA}} \cong C(\mathcal{M}(\overline{\mathcal{HA}}))$. Giles [Gil81] had proved that given all Wilson loops one can reconstruct the corresponding connection up to a gauge transformation. Rendall [Ren93] observed that, therefore, $\mathcal{A}/\mathcal{G}$ can be densely embedded into $\mathcal{M}(\overline{\mathcal{HA}})$. This justifies the Ashtekar-Isham definition $\overline{\mathcal{A}/\mathcal{G}} := \mathcal{M}(\overline{\mathcal{HA}})$ of the space of the generalized connections modulo gauge transformations. The elements of $\overline{\mathcal{A}/\mathcal{G}}$ are denoted by $\overline{A}$. The isomorphism between $\overline{\mathcal{HA}}$ and $C(\mathcal{M}(\overline{\mathcal{HA}}))$ is given by the Gelfand transformation

$$\sim : \begin{array}{ccc} \overline{\mathcal{HA}} & \longrightarrow & C(\overline{\mathcal{A}/\mathcal{G}}) \\ f & \longmapsto & \tilde{f} \end{array} \qquad \text{with} \qquad \tilde{f} : \begin{array}{ccc} \overline{\mathcal{A}/\mathcal{G}} & \longrightarrow & \mathbb{C}. \\ \overline{A} & \longmapsto & \overline{A}(f) \end{array}$$

The theory of C^* algebras yields also the measure theory and representation theory on $\overline{\mathcal{A}/\mathcal{G}}$. There is a one-to-one correspondence between Borel measures μ on $\overline{\mathcal{A}/\mathcal{G}}$, linear con-

[1] In the following we often drop the brackets. Then the symbol $=$ means equality of loops and the symbol $\sim$ means equality of hoops.

tinuous positive functionals F on $\overline{\mathcal{HA}}$ and continuous cyclic Hilbert space representations ϕ of $\overline{\mathcal{HA}}$. More precisely, any such functional F can be obtained by $F(f) = \int_{\overline{\mathcal{A}/\mathcal{G}}} \tilde{f}\, d\mu_F$ with a certain unique Borel measure μ_F and any such ϕ is unitary equivalent to the representation φ of $\overline{\mathcal{HA}}$ on $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_\phi)$ by multiplication operators $\varphi(f)\psi = \tilde{f}\cdot\psi$ with a certain measure μ_ϕ.

Ashtekar and Lewandowski [AL93] (in the following shortly denoted by A-L) discovered a second description of $\overline{\mathcal{A}/\mathcal{G}}$ via the hoop group $\mathcal{HG}$.[2] They defined two kinds of independence on $\mathcal{L}_m$. A finite subset $\boldsymbol{\beta} := \{\beta_i\}$ of $\mathcal{L}_m$ is called strongly independent iff each β_i contains an open segment which is traced once and only once by β_i and which is intersected by the remaining β_j at most in a finite set of points. $\boldsymbol{\beta}$ is weakly independent iff to any $(g_1, \ldots, g_n) \in \mathbf{G}^n$ there exists an $A \in \mathcal{A}$ such that $h_{\beta_i}(A) = g_i$ for all i. They proved that strong independence implies weak independence. Then they could give a bijection between $\overline{\mathcal{A}/\mathcal{G}}$ and the space $\mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad}$ of *all* homomorphisms from $\mathcal{HG}$ to $\mathbf{G}$ modulo a hoop independent conjugation. More precisely, any $h \in \mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad}$ yields an $\overline{A}_h \in \overline{\mathcal{A}/\mathcal{G}}$ via $\overline{A}_h(T_\alpha) := \frac{1}{N}\mathrm{tr}\, h(\alpha)$ and vice versa.

This graph-theoretical approach was used by A-L to define a natural integration measure, the so-called induced Haar measure [AL93]. They introduced an equivalence relation on $\overline{\mathcal{A}/\mathcal{G}}$ for finitely generated subgroups $\mathcal{HG}(\boldsymbol{\beta}) \subseteq \mathcal{HG}$: $\overline{A}_1 \sim \overline{A}_2$ with respect to $\mathcal{HG}(\boldsymbol{\beta})$ iff $h_{\overline{A}_1}(\gamma) = g^{-1}h_{\overline{A}_2}(\gamma)g$ for all $\gamma \in \mathcal{HG}(\boldsymbol{\beta})$ with a (hoop independent) $g \in \mathbf{G}$. $\pi_{\boldsymbol{\beta}} : \overline{\mathcal{A}/\mathcal{G}} \longrightarrow \overline{\mathcal{A}/\mathcal{G}}/\!\sim$ is the corresponding projection. Thus, there is a bijection $\overline{\mathcal{A}/\mathcal{G}}/\!\sim\, \longleftrightarrow \mathrm{Hom}(\mathcal{HG}(\boldsymbol{\beta}), \mathbf{G})/\mathrm{Ad}$ as for $\overline{\mathcal{A}/\mathcal{G}}$ and $\mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad}$. $\mathrm{Hom}(\mathcal{HG}(\boldsymbol{\beta}), \mathbf{G})/\mathrm{Ad}$ itself is isomorphic to $\mathbf{G}^{\#\boldsymbol{\beta}}/\mathrm{Ad}$ if $\boldsymbol{\beta}$ is weakly independent. Therefore A-L could reduce the integration over $\overline{\mathcal{A}/\mathcal{G}}$ under certain circumstances to the case of the integration over a finite dimensional Lie group. In detail, they defined cylindrical functions, i.e. functions f being pullbacks $\pi_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}}$ of continuous functions $f_{\boldsymbol{\beta}}$ on $\mathrm{Hom}(\mathcal{HG}(\boldsymbol{\beta}), \mathbf{G})/\mathrm{Ad} = \mathbf{G}^{\#\boldsymbol{\beta}}/\mathrm{Ad}$ with strongly independent β and showed that the set $\mathcal{C}$ of all such functions is dense in $\overline{\mathcal{HA}} = C(\overline{\mathcal{A}/\mathcal{G}})$. Now, they defined $\int_{\overline{\mathcal{A}/\mathcal{G}}} f\, d\mu_0 := \int_{\mathbf{G}^{\#\boldsymbol{\beta}}/\mathrm{Ad}} f_{\boldsymbol{\beta}}\, d\mu_{\boldsymbol{\beta}}$ and chose $d\mu_{\boldsymbol{\beta}}$ to be the Haar measure for each $\boldsymbol{\beta}$. Thus they got a well-defined, regular and positive measure μ_0 on $\overline{\mathcal{A}/\mathcal{G}}$, the so-called induced Haar measure.

Ashtekar and Lewandowski realized that μ_0 could serve as a kinematical measure of physical theories in the functional integral approach. Since the elements of $\mathcal{A}/\mathcal{G}$ are classical potential configurations, the completion $\overline{\mathcal{A}/\mathcal{G}}$ seems to be a candidate for the space of histories in the quantum regime and the physical measure is built from $d\mu_0$ by multiplication with e^{-S}, where S is the physical action of the theory. The crucial point was to choose such an S defined not only on $\mathcal{A}/\mathcal{G}$ but on $\overline{\mathcal{A}/\mathcal{G}}$. Neglecting that fact, one could compute via $\langle f\rangle = \int e^{-S} f\, d\mu_0$ any expectation value of the theory supposed f to be a function on $\overline{\mathcal{A}/\mathcal{G}}$. Thiemann [Thi95] and Ashtekar et al. [ALM+96] (in the following shortly denoted by T-A+) proposed a solution of that problem in the case of the 2-dimensional quantum Yang-Mills theory using lattice regularization. The main problem was the replacement of the Yang-Mills action $S_{YM} = \frac{1}{4}\int_M F_{\mu\nu}F^{\mu\nu} dx$ by an expression whose domain is $\overline{\mathcal{A}/\mathcal{G}}$. The only a priori available quantities are the generalized holonomies. This indicates the use of Wilson's lattice regularization. For this one places a finite quadratic lattice with spacing a and length R on the 2-plane and defines $S^{reg}_{YM} = \frac{N}{\mathrm{g}^2 a^2}\sum_\Box(1 - \frac{1}{N}\mathrm{Re\ tr}\, h_\Box)$ where the sum goes over all plaquettes of the lattice. $h_\Box$ denotes the holonomy around the plaquette $\Box$. In the limit $a \to 0$ and

[2] Marolf and Mourão [MM95] obtained a third description of $\overline{\mathcal{A}/\mathcal{G}}$ via projective limits. However, this approach is unimportant for our purpose and we only mention it for completeness.

$R \to \infty$ one can show naively the regularized action to converge to S_{YM}. The advantage of S_{YM}^{reg} is its natural extendability to $\overline{\mathcal{A}/\mathcal{G}}$. Now, T-A$^+$ could compute the expectation values of the Wilson loops expected to determine the whole pure quantum YM$_2$ theory:

$$\begin{aligned}\langle T_{\alpha_1} \cdots T_{\alpha_n} \rangle &= \frac{1}{Z} \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0 \; e^{-\lim_{a\to 0, R\to\infty} S_{YM}^{reg}} T_{\alpha_1} \cdots T_{\alpha_n} \\ &= \lim_{a\to 0, R\to\infty} \frac{1}{Z_{a,R}} \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0 \; e^{-S_{YM}^{reg}} T_{\alpha_1} \cdots T_{\alpha_n} \end{aligned} \tag{1}$$

after exchanging limit and integral.[3] Afterwards they expressed each loop $\alpha_1, \ldots, \alpha_n$ and each plaquette loop $\square$ by a product of "simple" loops (i.e. loops traversing exactly one plaquette and connecting it with the base point m by conjugation), provided, however, $\alpha_1, \ldots, \alpha_n$ are contained in the lattice. Under the assumption that these loops are independent they could reduce the integration over $\overline{\mathcal{A}/\mathcal{G}}$ to the integration over G^n, n finite. Finally, they computed the integrals explicitely and got an algebraic expression depending only on the areas enclosed by the loops. For general $\alpha_1, \ldots, \alpha_n$ they suggested to approximate these loops naively by lattice loops and to consider the limit of the expectation values, but this is simply given by the limit of the enclosed areas.

3 Moderate Independence

In this section we will introduce a new type of independence being crucial for the considerations below – the so-called moderate independence.

3.1 Why a New Type of Independence?

We consider a quadratic lattice with spacing a and length $R = la$, $l \in \mathbb{N}^+$, i.e. with l^2 plaquettes, see e.g. figure 1. Now we assign (see [ALM+96]) a loop $\beta_{x,y} := \rho_{x,y} \circ f_{x,y} \circ \rho_{x,y}^{-1}$ to each plaquette $\square_{x,y}$. x, y indicates the position of the plaquette, as follows: First, choose a path $\rho_{x,y}$ from the base point m to the bottom left corner (x, y) and then define $\beta_{x,y} := \rho_{x,y} \circ f_{x,y} \circ \rho_{x,y}^{-1}$ where $f_{x,y}$ is a path traversing $\square_{x,y}$ counterclockwise. For our example, we choose $\rho_{x,y}$ to consist of a horizontal and a subsequent vertical path as in figure 1.

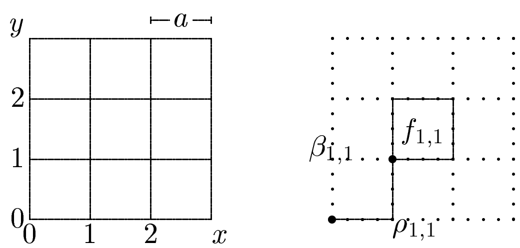


Figure 1: Example of a lattice ($l = 3$) and the loop $\beta_{1,1}$

Obviously, the set $\boldsymbol{\beta}$ of all these loops $\beta_{x,y}$ is not strongly independent (for the exact definition see next subsection) because, e.g., $\beta_{1,1}$ does not have a segment which is intersected by any other $\beta_{x,y}$ at most in a finite number of points. Of course, one can prove that $\boldsymbol{\beta}$

[3] The factor $\frac{1}{Z}$ guarantees $\langle T_1 \rangle = 1$.

is weakly independent, but this is not sufficient to allow the application of the integration calculus. Therefore we need a third type of independence between these two ones; this will be the moderate independence.

3.2 Moderate Independence: Definition and Position among the Independences

In the following, $\boldsymbol{\beta}$ denotes any finite subset $\{\beta_i\}$ of $\mathcal{L}_m$ (or $\mathcal{HG}$) and $\mathcal{HG}(\boldsymbol{\beta})$ the subgroup of $\mathcal{HG}$ generated by $\boldsymbol{\beta}$.[4] First, we recall the definition of the strong independence [AL93].

Definition 3.1 Strong Independence in $\mathcal{L}_m$
$\boldsymbol{\beta} \subseteq \mathcal{L}_m$ is *strongly independent* iff any $\beta_i \in \boldsymbol{\beta}$ contains an open segment e_i, the so-called *free segment*, traced exactly once by β_i and intersected by any β_j, $j \neq i$, in at most a finite number of points.[5]

Our definition of the moderate independence differs very little from the previous one. We only replace $j \neq i$ by $j < i$:

Definition 3.2 Moderate Independence in $\mathcal{L}_m$
$\boldsymbol{\beta} \subseteq \mathcal{L}_m$ is *moderately independent* iff any $\beta_i \in \boldsymbol{\beta}$ contains an open segment e_i, the so-called *free segment*, traced exactly once by β_i and intersected by any β_j, $j < i$, in at most a finite number of points.[6]

We have simply replaced the rigid condition of a simultaneous freeness of segments by the flexible condition of an iterative freeness. We will see that this keeps the integration calculus valid and makes the set of all plaquette loops (cf. fig. 1) independent.

We mention that the simple specification of the elements of a moderately independent set $\boldsymbol{\beta}$ is not sufficient. If we say "$\boldsymbol{\beta}$ is moderately independent" then there is an order of the elements $\beta_i \in \boldsymbol{\beta}$, such that the above criterion is valid. Analogously, the specification "$\{\beta_1, \beta_2\}$ or $\{\beta_2, \beta_1\}$, resp., are moderately independent" should be clear.

Finally, we recall the definition of weak independence [AL93].

Definition 3.3 Weak Independence in $\mathcal{L}_m$
$\boldsymbol{\beta} \subseteq \mathcal{L}_m$ is *weakly independent* iff for any $(g_1, \ldots, g_n) \in \mathbf{G}^n$, $n = \#\boldsymbol{\beta}$, there is an $A \in \mathcal{A}$, such that $h_{\beta_i}(A) = g_i$ for all $i = 1, \ldots, n$.

Obviously, this kind of independence can be extended from $\mathcal{L}_m$ to $\mathcal{HG}$.

Instead of the previous two definitions being graph-theoretical we have here an algebraic condition. Weak independence of $\boldsymbol{\beta}$ means no relations between the holonomies h_{β_i} and so it ensures the freeness of the corresponding subgroup $\mathcal{HG}(\boldsymbol{\beta}) \subseteq \mathcal{HG}$, see subsection 3.3.

The position of the moderate independence clarifies the next

Proposition 3.1 $\boldsymbol{\beta}$ strongly independent $\Longrightarrow$ $\boldsymbol{\beta}$ moderately independent $\Longrightarrow$ $\boldsymbol{\beta}$ weakly independent.

Proof
- The first implication is obvious.
- The proof of the second implication is technical and can be found in appendix A.

qed

[4] To avoid technical complications we set $\mathcal{HG}(\emptyset) = \{[1]\}$.

[5] The intersection condition can be replaced by "$e_i \cap \beta_j = \emptyset \quad \forall j \neq i$". However, this yields to an equivalent definition.

[6] Footnote 5 holds analogously in the case of moderate independence: "$e_i \cap \beta_j = \emptyset \quad \forall j < i$".

3.3 Algebraic Consequences of the Weak Independence

Proposition 3.2 Let $\boldsymbol{\beta} \subseteq \mathcal{HG}$ be weakly independent. Then the following holds:

1. $\mathcal{HG}(\boldsymbol{\beta})$ is freely[7] generated by $\boldsymbol{\beta}$.
2. Let there be given a $\boldsymbol{\gamma} \subseteq \mathcal{HG}$, such that $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\boldsymbol{\gamma})$. Then we have:
   $\boldsymbol{\gamma}$ is weakly independent $\Longleftrightarrow$ $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ have the same cardinality.

Proof 1. See [AL93].

2. $\Longleftarrow$
   - For $\mathcal{HG}(\boldsymbol{\gamma}) = \mathcal{HG}(\boldsymbol{\beta})$ there are expressions
     $$\gamma_i \sim \prod_{k_i=1}^{K_i} \beta_{j(i,k_i)}^{\epsilon(i,k_i)} \text{ and } \beta_j \sim \prod_{l_j=1}^{L_j} \gamma_{i(j,l_j)}^{\eta(j,l_j)}$$
     for any $i, j \in [1, n]$, $n := \#\boldsymbol{\beta} = \#\boldsymbol{\gamma}$, and thus
     $$\gamma_i \sim \prod_{k_i=1}^{K_i} \left(\prod_{l_{j(i,k_i)}=1}^{L_{j(i,k_i)}} \gamma_{i(j(i,k_i),l_{j(i,k_i)})}^{\eta(j(i,k_i),l_{j(i,k_i)})} \right)^{\epsilon(i,k_i)} \qquad \forall i \in [1, n].$$
   - Due to the first point $\boldsymbol{\beta}$ is a free system of generators for $\mathcal{HG}(\boldsymbol{\beta})$. Since $\boldsymbol{\gamma}$ also generates $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\boldsymbol{\gamma})$ and $\#\boldsymbol{\gamma} = \#\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ is a free system of generators for $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\boldsymbol{\gamma})$ ([Kur70]).
   - Let there be given $(g_1, \ldots, g_n) \in \mathbf{G}^n$ and let $\mathbf{H}$ be the group generated by $\{g_1, \ldots, g_n\}$. Since $\mathcal{HG}(\boldsymbol{\gamma})$ has the free rank n there is [Kur70] an epimorphism $\pi : \mathcal{HG}(\boldsymbol{\gamma}) \longrightarrow \mathbf{H}$ with $\pi(\gamma_i) = g_i$.
   - Since $\boldsymbol{\beta}$ is weakly independent, there exists an $A \in \mathcal{A}$ with $h_{\beta_j}(A) = \prod_{l_j=1}^{L_j} g_{i(j,l_j)}^{\eta(j,l_j)} \quad \forall j$, i.e. we have for all $i \in [1, n]$
     $$\begin{aligned} h_{\gamma_i}(A) &= h_{\prod_{k_i=1}^{K_i} \beta_{j(i,k_i)}^{\epsilon(i,k_i)}}(A) = \prod_{k_i=1}^{K_i} \left(\prod_{l_{j(i,k_i)}=1}^{L_{j(i,k_i)}} g_{i(j(i,k_i),l_{j(i,k_i)})}^{\eta(j(i,k_i),l_{j(i,k_i)})} \right)^{\epsilon(i,k_i)} \\ &= \pi \left(\prod_{k_i=1}^{K_i} \left(\prod_{l_{j(i,k_i)}=1}^{L_{j(i,k_i)}} \gamma_{i(j(i,k_i),l_{j(i,k_i)})}^{\eta(j(i,k_i),l_{j(i,k_i)})} \right)^{\epsilon(i,k_i)} \right) = \pi(\gamma_i) = g_i. \end{aligned}$$

     Thus, $\boldsymbol{\gamma}$ is weakly independent.

   $\Longrightarrow$

   Let $\boldsymbol{\gamma}$ be weakly independent, i.e. $\mathcal{HG}(\boldsymbol{\gamma}) = \mathcal{HG}(\boldsymbol{\beta})$ is free. Consequently, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ have the same cardinality [Kur70]. **qed**

3.4 Graphs and Loops

We recall some fundamental facts about graphs (see e.g. [Mas89]).

A graph (X, X_0) consists of a Hausdorff space X and a discrete subspace X_0, the space of the so-called vertices. $X \setminus X_0$ is a disjoint union of edges, i.e. open subsets e_i isomorphic to the interval $(0, 1)$. e_i can connect one or two vertices. In the first case e_i is called sling. Two

[7] In the case $\mathbf{G} = U(1)$ we understand by "free" anytime "abelian free".

vertices are connected by a multiple edge iff there are at least two different edges connecting these vertices. Iff a graph has neither slings nor multiple edges, it is called ordinary. Furthermore, (X, X_0) is finite iff both the set of edges and the set of vertices are finite. A graph (X', X'_0) is called subgraph (or refinement) of a graph (X, X_0) iff $X' \subseteq X$ and $X'_0 \subseteq X_0$. Obviously, any (finite) graph is subgraph of an ordinary (finite) graph. In the following we will briefly denote a graph by X instead of (X, X_0). Additionally, $X \leq X'$ means that X is a subgraph of X'.

In a natural way one can choose an orientation to any edge. The initial (terminal) vertex of an edge e is denoted by v_e^- (v_e^+). A path f in a graph is a finite sequence of (oriented) edges $(e_1, \ldots, e_n)$, $n \leq 0$, such that the terminal vertex of e_i coincides with the initial vertex of e_{i+1} $(1 \leq i < n)$ w.r.t. the chosen orientation. Iff $n = 0$, f is called trivial. Iff the initial vertex v_f^- and the terminal vertex v_f^+ of f are equal, f is called closed path or loop with base point $v_f = v_f^{\pm}$. f is called reduced iff no edge is retraced immediately and is called genuine iff no vertex is traced twice (exception: initial and terminal vertex can be equal). Finally, a tree T is a graph without any non-trivial genuine closed path.

Obviously, any graph contains trees. If we partially order the set of all trees in a graph using the inclusion, i.e. subgraph relation, we get

Lemma 3.3 Any tree in a graph X is contained in a maximal tree in X. If X is connected, then a tree T in X is maximal if and only if T containes all vertices of X.

Using this lemma one can construct explicitely the fundamental group of a connected graph. First choose a vertex v_0 and a maximal tree. Let $\{e_\lambda \mid \lambda \in \Lambda\}$ be the set of all edges of X not contained in T and choose an orientation for each e_λ. Now denote by t_λ^- and t_λ^+ the (unique) reduced path along T from v_0 to the initial vertex of e_λ and resp. from the terminal vertex of e_λ to v_0. Finally, define α_λ to be the product of t_λ^-, e_λ and t_λ^+. We have

Proposition 3.4 The fundamental group $\pi(X, v_0)$ is the free group generated by $\{\alpha_\lambda \mid \lambda \in \Lambda\}$, where α_λ denotes here not the loop itself, but its homotopy class.

The Euler-Poincaré characteristic $\chi(X)$ of a finite graph is per def. the difference of the number of vertices and the number of edges.

Proposition 3.5 Let X be finite and connected. Then $\pi(X, v_0)$ is a free group with $1-\chi(X)$ generators and X is a tree iff $\chi(X) = 1$.

Let there be given now a finite set of loops $\boldsymbol{\beta} = \{\beta_i\} \subseteq \mathcal{L}_m$ in a manifold M. Note that $\mathcal{L}_m$ contains only piecewise analytic loops. The image of $\boldsymbol{\beta}$ in M defines naturally a finite connected graph $\Gamma_{\boldsymbol{\beta}}$ via the following (see also [AL93])

Construction 3.4
1. Mark all end-points of overlapping intervals of two loops and all intersection points outside those overlapping intervals. These points become the vertices of $\Gamma_{\boldsymbol{\beta}}$. Due to the piecewise analyticity the number of vertices is finite.
2. Divide any β_i into paths between "neighbouring" vertices and call these paths edges of $\Gamma_{\boldsymbol{\beta}}$. Again due to the piecewise analyticity the set of edges is finite.
3. Since any β_i is a loop with base point m, $\Gamma_{\boldsymbol{\beta}}$ is connected.

3.5 Relations between the Fundamental Group and the Hoop Group of a Graph

In this subsection Γ is a finite connected graph and m an arbitrary, but fixed vertex of Γ. Furthermore, we denote by $\mathcal{HG}(\Gamma)$ the subgroup of $\mathcal{HG}$ generated by all loops in Γ.

It was an important observation of Ashtekar and Lewandowski [AL93] that there is a close relation between the representation of a loop as a hoop and as an equivalence class w.r.t. the homotopy in a graph. In detail, they got

Lemma 3.6 Two homotopically equivalent loops are holonomically equivalent, i.e. there is an epimorphism $\phi : \pi(\Gamma, m) \longrightarrow \mathcal{HG}(\Gamma)$. ϕ is an isomorphism if $\mathbf{G} = SU(N)$. For $\mathbf{G} = U(1)$ we have $\ker\phi = [\pi(\Gamma, m), \pi(\Gamma, m)]$.

Consequently, in the case $\mathbf{G} = SU(N)$ two loops are holonomically equivalent if and only if they can be obtained from each other by reparametrizations or (if necessary successively) cancelling retracings. Obviously, we have

Lemma 3.7 Let T be a maximal tree and $\{\alpha_\lambda\}$ the set of the corresponding generators of $\pi(\Gamma, m)$ as in Proposition 3.4. Then $\{\alpha_\lambda\}$ is strongly independent and complete in Γ, i.e. we have $\mathcal{HG}(\{\alpha_\lambda\}) = \mathcal{HG}(\Gamma)$.

The free segments are the edges e_λ not contained in T. Additionally, one can express any finite set of hoops by a finite set of strongly independent hoops [AL93].

Lemma 3.8 For any finite set $[\boldsymbol{\beta}]$ of hoops there is a set $\boldsymbol{\alpha} \subseteq \mathcal{L}_m$, such that

1. $\mathcal{HG}(\boldsymbol{\beta}) \subseteq \mathcal{HG}(\boldsymbol{\alpha})$,
2. $\boldsymbol{\alpha}$ is strongly independent and
3. $\#\boldsymbol{\alpha} = \operatorname{rank} \pi(\Gamma_{\boldsymbol{\beta}}, m)$.

For this choose the natural graph $\Gamma_{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$. Choose now some generating set $\boldsymbol{\alpha}$ of the fundamental group $\pi(\Gamma_{\boldsymbol{\beta}}, m)$. Obviously, $\boldsymbol{\alpha}$ fulfills the required conditions.

Now we want to investigate the independence of loops.

Lemma 3.9 Let n be the rank of $\pi(\Gamma, m)$. Then any $\boldsymbol{\beta} \subseteq \mathcal{L}_m$ with $\#\boldsymbol{\beta} = n$ and $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\Gamma)$ is weakly independent.

Proof Choose any maximal tree T in Γ and a corresponding system $\{\alpha_\lambda\}$ of generators of $\pi(\Gamma, m)$. $\{\alpha_\lambda\}$ has n elements and is a free generating system. Due to Lemma 3.7 $\{\alpha_\lambda\}$ is strongly independent and thus weakly independent. Proposition 3.2 finishes the proof. **qed**

Generally, one can not conclude that $\boldsymbol{\beta}$ is even moderately independent. To see this let $\pi(\Gamma, m)$ be generated by two loops α_1, α_2 as in Proposition 3.4. Set $\beta_1 := \alpha_1\alpha_2\alpha_1^{-1}$ and $\beta_2 := \alpha_1\alpha_2$.

- We have $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\Gamma) = \mathcal{HG}(\{\alpha_1, \alpha_2\})$, because $\alpha_1 = \beta_1^{-1}\beta_2$ and $\alpha_2 = \beta_2^{-1}\beta_1\beta_2$.
- Suppose $\{\beta_1, \beta_2\}$ are moderately independent. Any segment of β_2 is already traced by β_1. This is a contradiction to the assumption β_2 has a free segment.
  The case "$\{\beta_2, \beta_1\}$ are moderately independent" yields an analogous contradiction.

Thus $\boldsymbol{\beta}$ is not moderately independent.

We finish this section with a criterion for the completeness of loops in a given graph.

Proposition 3.10 Let Γ be a finite connected graph and $\boldsymbol{\beta}$ a set of moderately independent loops in Γ. Then is $\boldsymbol{\beta}$ complete w.r.t. Γ if and only if the cardinality of $\boldsymbol{\beta}$ equals the rank of $\pi(\Gamma, m)$.

Proof The $\Longrightarrow$ direction is simple. Due to Lemma 3.8 there is a set $\boldsymbol{\alpha}$ with $\mathcal{HG}(\boldsymbol{\alpha}) = \mathcal{HG}(\Gamma) = \mathcal{HG}(\boldsymbol{\beta})$, whose cardinality is just equal to the rank of $\pi(\Gamma, m)$. Proposition 3.2 yields that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have the same cardinality.
The $\Longleftarrow$ direction is a little bit technical.
The free segments of the β_i are as usual denoted by e_i and the cardinality of $\boldsymbol{\beta}$ by n. Suppose first that no β_i has a retracing interval.

1. W.l.o.g. the free segments e_i of β_i are edges of Γ. Otherwise, if necessary, restrict any e_i, such that it is still contained in only one edge k_i. Since β_i has no retracing intervals, the whole k_i is a free segment of β_i. Thus one can set $e_i := k_i$.
2. The graph $T := \Gamma \setminus \bigcup_{i=1}^n \{e_i\}$ created by removing all free segments is again a connected graph.
   Set $\Gamma_j := \Gamma \setminus \bigcup_{i=j}^n \{e_i\}$. Then $\Gamma_{n+1} = \Gamma$, $\Gamma_1 = T$. Due to the moderate independence of the β_i we have $\beta_i \cap e_{i'} = \emptyset \quad \forall i' > i$, i.e., β_i is a loop in Γ_{i+1}. Suppose T is not connected. Then there would exist a $j \in [1, n]$, such that all Γ_i with $i > j$ are connected, but Γ_j is not connected. Since β_j is a loop in Γ_{j+1} and β_j passes e_j, β_j has to pass vertices of both connected components of $\Gamma_j = \Gamma_{j+1} \setminus \{e_j\}$. Thus e_j must be passed at least once in each direction by β_j, i.e. we have a contradiction to the assumption that e_j is a free segment. Thus, Γ_i is connected for all $i \in [1, n+1]$.
3. T is a maximal tree in Γ.
   Due to Proposition 3.5 we have $n = \operatorname{rank} \pi(\Gamma, m) = 1 - \chi(\Gamma) = 1 - \epsilon_\Gamma + \kappa_\Gamma$, where ϵ_Γ and κ_Γ are the numbers of vertices and edges of Γ, respectively. Since $T = \Gamma \setminus \bigcup_{i=1}^n \{e_i\}$ we have $\kappa_T = \kappa_\Gamma - n$ and obviously $\epsilon_T = \epsilon_\Gamma$. For T connected, we have $\chi(T) = \epsilon_T - \kappa_T = \epsilon_\Gamma - \kappa_\Gamma + n = \chi(\Gamma) + n = 1$. Thus T is a tree in Γ due to Proposition 3.5. T is even maximal because T contains all vertices of Γ.
4. Let $\boldsymbol{\alpha} := \{\alpha_i\}$ be a free system of generators of $\pi(\Gamma, m)$ due to Proposition 3.4 for the just constructed maximal tree T and the edges $\{e_i\}$. Thus, $\boldsymbol{\alpha}$ fulfills $\mathcal{HG}(\boldsymbol{\alpha}) = \mathcal{HG}(\Gamma)$. W.l.o.g. α_i traces the edge e_i in the same direction as β_i. We show that $\boldsymbol{\beta}$ is complete in Γ.
   a) β_1 is a loop in $T \cup \{e_1\} = \Gamma_{1+1}$, where e_1 is traced once and in the same direction as α_1 is. Thus $\beta_1 = t_+ e_1 t_- \sim \alpha_1$ with certain paths $t_\pm$ in T, i.e. $\mathcal{HG}(\{\beta_1\}) = \mathcal{HG}(\{\alpha_1\})$, i.e., $\{\beta_1\}$ is complete in Γ_{1+1}.
   b) Let $\mathcal{HG}(\{\beta_1, \ldots, \beta_i\}) = \mathcal{HG}(\Gamma_{i+1}) = \mathcal{HG}(\{\alpha_1, \ldots, \alpha_i\})$ hold for all $i < j$. We have now $\beta_j = k_{j,+} e_j k_{j,-}$, where $k_{j,\pm}$ are some paths in $\Gamma_{j+1} \setminus \{e_j\} = \Gamma_j$. Furthermore, we have $\alpha_j = t_{j,+} e_j t_{j,-}$ with $t_{j,\pm} \subseteq T \subseteq \Gamma_j$. Thus $\beta_j \sim k_{j,+} t_{j,+}^{-1} \ \alpha_j \ t_{j,-}^{-1} k_{j,-}$. Since $k_{j,+} t_{j,+}^{-1}$ and $t_{j,-}^{-1} k_{j,-}$ are loops in Γ_j, we have $[k_{j,+} t_{j,+}^{-1}], [t_{j,-}^{-1} k_{j,-}] \in \mathcal{HG}(\Gamma_j) = \mathcal{HG}(\{\alpha_1, \ldots, \alpha_{j-1}\}) = \mathcal{HG}(\{\beta_1, \ldots, \beta_{j-1}\})$. Due to $\alpha_j \sim t_{j,+} k_{j,+}^{-1} \beta_j k_{j,-}^{-1} t_{j,-} \in \mathcal{HG}(\{\beta_1, \ldots, \beta_j\})$ we have $\mathcal{HG}(\{\alpha_1, \ldots, \alpha_{j-1}\} \cup \{\alpha_j\}) \subseteq \mathcal{HG}(\{\beta_1, \ldots, \beta_j\})$. Since β_j is a loop in Γ_{j+1}, we get immediately the $\supseteq$ relation, i.e. $\mathcal{HG}(\Gamma_{j+1}) = \mathcal{HG}(\{\alpha_1, \ldots, \alpha_j\}) = \mathcal{HG}(\{\beta_1, \ldots, \beta_j\})$. Thus $\{\beta_1, \ldots, \beta_j\}$ is complete in Γ_{j+1}.

The induction yields also $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\boldsymbol{\alpha}) = \mathcal{HG}(\Gamma_{n+1})$, i.e., $\boldsymbol{\beta}$ is complete in $\Gamma_{n+1} = \Gamma$.

We allow now the β_i to have retracing intervals. Denote by β_i' the loop that remains after cancelling all these intervals in β_i. Obviously, β_i' lies in the same hoop class as β_i, i.e. $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\boldsymbol{\beta}')$. Thus, since we have already proven the proposition for the retracing-free $\boldsymbol{\beta}'$, we get immediately the claim for arbitrary $\boldsymbol{\beta}$. **qed**

4 Flag Worlds

This section provides some facts about the hoop group of a graph ("lattice") Γ in the two-dimensional manifold $M = \mathbb{R}^2$. For this we can specialize the facts of subsection 3.5 to the case of planar graphs (see e.g. [Hal89]). These have a crucial advantage: one can define domains enclosed by the graph edges. The set of all these domains induces a basis of the corresponding hoop group $\mathcal{HG}(\Gamma)$. Finally, we will investigate the behaviour of that set under refinement of the graph Γ generalizing the results of T-A$^+$.

4.1 Planar Graphs

This subsection collects some basic and simple facts about planar graphs and is intended to clarify the notations. We call a graph X *planar* iff there exists a homomorphism $\iota : X \longrightarrow \Gamma \subseteq \mathbb{R}^2$. We identify X and Γ in the sequel. Furthermore, in the following any graph is supposed to be planar, finite and connected.

Any graph is the complement of a disjoint union of domains. Exactly one of them if unbounded – the so-called exterior domain G_{ext}. The set of the remaining domains, the so-called interior domains, is denoted by $L_{\text{int}}(\Gamma)$ and we set $L(\Gamma) := L_{\text{int}}(\Gamma) \cup \{G_{\text{ext}}\}$. We say that a domain G is contained in Γ iff its boundary ∂G is in Γ and $G \cap G_{\text{ext}} = \emptyset$.

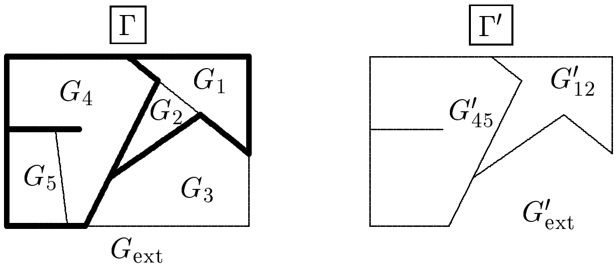
One easily proves Euler's polyhedron formula $\epsilon - \kappa + \lambda = 2$, where ϵ, κ and λ are the numbers of vertices, edges and domains, resp., of the graph. Since $\lambda - 1 = 1 - (\epsilon - \kappa) = 1 - \chi(\Gamma)$, we have using Proposition 3.5

Lemma 4.1 The number of interior domains of a graph Γ is equal to the rank of $\pi(\Gamma, m)$.

We are now interested in the behaviour of $L(\Gamma')$ under refinement of Γ'. Clearly, if we refine a graph Γ' to a graph Γ, then any domain of Γ' is refined into a certain set of domains in Γ (see e.g. figure 2). We have in detail the simple

Proposition 4.2 Let $\Gamma' \leq \Gamma$. Then the following holds:

1. For any $G \in L(\Gamma), G' \in L(\Gamma')$ we have $G \cap G' \neq \emptyset \Longrightarrow G \subseteq G'$. Especially, two interior domains of one and the same graph are disjoint or equal.
2. For any $G \in L(\Gamma)$ there exists exactly one $G' \in L(\Gamma')$ with $G \cap G' \neq \emptyset$.
3. For any $G' \in L(\Gamma')$ there exists exactly one $L_{G'} \subseteq L(\Gamma)$, such that $G \cap G' \neq \emptyset \Longleftrightarrow G \in L_{G'}$ and $\overline{\bigcup_{G \in L_{G'}} G} \supseteq G'$.
4. Let now G' be any domain in Γ, not necessarily an interior domain. There is exactly one set $L_{G'}(\Gamma) \subseteq L_{\text{int}}(\Gamma)$, such that for all interior domains G holds: $G \in L_{G'}(\Gamma) \Longleftrightarrow G \cap G' \neq \emptyset$ and $\overline{\bigcup_{G \in L_{G'}} G} \supseteq G'$.



$L(\Gamma) = \{G_1, G_2, G_3, G_4, G_5\} \cup \{G_{\text{ext}}\}$
$L(\Gamma') = \{G'_{12}, G'_{45}\} \cup \{G'_{\text{ext}}\}$
Examples:
$G_4 \cap G'_{45} \neq \emptyset \Longrightarrow G_4 \subseteq G'_{45}$
For G_1 there is exactly one domain (G'_{12}) in $L(\Gamma')$, such that $G_1 \cap G'_{12} \neq \emptyset$.
For G_3 there is exactly one domain (G'_{ext}) in $L(\Gamma')$, such that $G_3 \cap G'_{\text{ext}} \neq \emptyset$.
Exactly $\{G_4, G_5\}$ corresponds with G'_{45}.

Figure 2: Example for the decomposition of domains

We call $L(\Gamma)$ a refinement of $L(\Gamma')$ (and analogously for L_{int}) iff Γ is a refinement if Γ'.

Definition 4.1 A domain $G \subseteq \mathbb{R}^2$ is called *simple* iff it is the interior of a Jordan curve. A graph Γ is called *simple* iff each of its interior domains is simple.

Finally, we need

Proposition 4.3 Any ordinary graph Γ is subgraph of a simple, ordinary graph Γ' whose exterior domain coincides with that of Γ.

The proof is quite easy. First one eliminates the retracings, second the repetitions of edges, and finally the repetitions of vertices by inserting appropriate edges as demonstrated in figure 3.

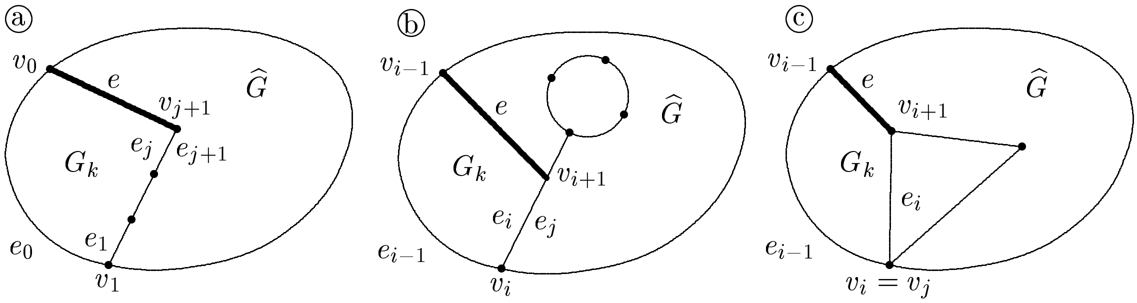


Figure 3: Cancelling (a) retracings, (b) repetition of edges, (c) repetition of vertices

4.2 Boundary Loops and Flags

We start with a simple

Lemma 4.4 For any simple domain $G \subseteq \Gamma$ and any $\widetilde{m} \in \Gamma_0 \cap \partial G$ there is exactly one genuine loop[8] $\alpha_{G,\widetilde{m}}$ in Γ with base point $\widetilde{m}$, such that

- $\alpha_{G,\widetilde{m}} = \partial G$ and
- $\alpha_{G,\widetilde{m}}$ traverses the domain G counterclockwise.

Vice versa, any such loop determines exactly one simple $G \subseteq \Gamma$.
We call $\alpha_{G,\widetilde{m}}$ *boundary loop* of G with base point $\widetilde{m}$.

[8]We recall that we do not distinguish between loops and hoops in the sequel. The symbol $=$ means equality of loops and the symbol $\sim$ means equality of hoops.

Analogously, for any $G \subseteq \Gamma$ and any $\widetilde{m} \in \Gamma_0 \cap \partial G$ there exists a loop $\alpha_{G,\widetilde{m}}$ in Γ with base point $\widetilde{m}$ and the properties above.

Now we are interested in loops with base point m, that traverse only one domain G in Γ. This is provided by

Definition 4.2 Flag
Let G be a simple domain in a graph Γ.
We call a loop $f_{G,m,\widetilde{m}}$ *flag* with base point m, flag point $\widetilde{m}$ and domain G iff

- $f = \rho_{m\widetilde{m}} \alpha_{G,\widetilde{m}} \rho_{m\widetilde{m}}^{-1}$,
- $\alpha_{G,\widetilde{m}}$ is a boundary loop of G with base point $\widetilde{m}$ and
- $\rho_{m\widetilde{m}}$ is a path from m to $\widetilde{m}$ in Γ;
- there is a $v \in \partial G$, such that
  - $\rho_{m\widetilde{m}} = \rho_{mv}\rho_{v\widetilde{m}}$,
  - $\rho_{mv} \cap \partial G = \{v\}$,
  - ρ_{mv} traces neither an edge nor a vertex twice and
  - $\rho_{v\widetilde{m}} \subseteq \partial G$ holds.

Then $\rho_{m\widetilde{m}}$ is called *flagpole.*
We call $f_{G,m,\widetilde{m}}$ *minimal* iff $v = \widetilde{m}$.

Since Γ is connected, we get from Lemma 4.4

Lemma 4.5 For any tripel $\{G, m, \widetilde{m}\}$ with the above properties there exists a corresponding flag $f_{G,m,\widetilde{m}}$.

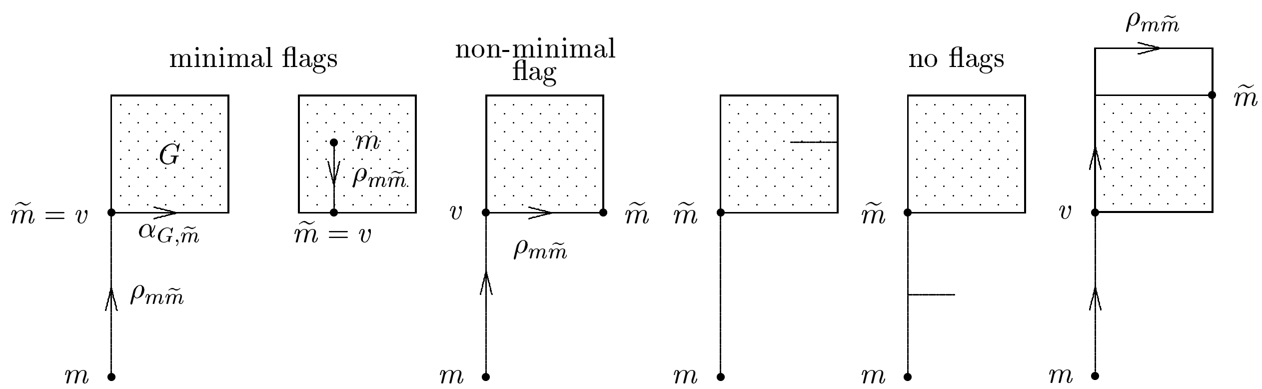


Figure 4: Flags: Examples and counterexamples

Remark 1. To any simple domain G and any $m \in \Gamma_0$ there exists a minimal flag. For this choose a maximal tree T and an $m' \in \partial G \cap \Gamma_0$. Furthermore, choose the shortest path ρ from m to m' along T. Let $\widetilde{m}$ be the nearest to m (w.r.t. the up to there traced edges of ρ) point in $\partial G \cap \rho$ and $\rho_{m\widetilde{m}}$ the corresponding initial path of ρ from m to $\widetilde{m}$. Obviously, $f_{G,m,\widetilde{m}} := \rho_{m\widetilde{m}} \alpha_{G,\widetilde{m}} \rho_{m\widetilde{m}}^{-1}$ with the boundary loop $\alpha_{G,\widetilde{m}}$ is a minimal flag for G.

2. All flags beginning with the same ρ_{mv} are equal modulo holonomy equivalence, esp. any flag is holonomically equivalent to a minimal flag.

3. For $\mathbf{G} = U(1)$ all flags to one and the same domain are holonomically equivalent.
   Let $f_i = \rho_{m\widetilde{m}_i}\alpha_{G,\widetilde{m}_i}\rho_{m\widetilde{m}_i}^{-1}$, $i = 1, 2$. We have
$$\begin{aligned} f_1 &= \rho_{m\widetilde{m}_1}\alpha_{G,\widetilde{m}_1}\rho_{m\widetilde{m}_1}^{-1} \sim \rho_{m\widetilde{m}_1}\rho_{\widetilde{m}_1\widetilde{m}_2}\alpha_{G,\widetilde{m}_2}\rho_{\widetilde{m}_1\widetilde{m}_2}^{-1}\rho_{m\widetilde{m}_1}^{-1} \\ &\sim \rho_{m\widetilde{m}_1}\rho_{\widetilde{m}_1\widetilde{m}_2}\rho_{m\widetilde{m}_2}^{-1}\rho_{m\widetilde{m}_2}\alpha_{G,\widetilde{m}_2}\rho_{m\widetilde{m}_2}^{-1}\rho_{m\widetilde{m}_2}\rho_{\widetilde{m}_1\widetilde{m}_2}^{-1}\rho_{m\widetilde{m}_1}^{-1} \\ &\sim \rho_{m\widetilde{m}_1}\rho_{\widetilde{m}_1\widetilde{m}_2}\rho_{m\widetilde{m}_2}^{-1}\rho_{m\widetilde{m}_2}\rho_{\widetilde{m}_1\widetilde{m}_2}^{-1}\rho_{m\widetilde{m}_1}^{-1}\rho_{m\widetilde{m}_2}\alpha_{G,\widetilde{m}_2}\rho_{m\widetilde{m}_2}^{-1} \\ &\sim f_2. \end{aligned}$$
   $\rho_{\widetilde{m}_1\widetilde{m}_2}$ is any path from $\widetilde{m}_1$ to $\widetilde{m}_2$ along ∂G. In the last but one step we used the commutativity of $\mathcal{HG} \supseteq \mathcal{HG}(\Gamma)$ induced by the commutativity of $U(1)$.
4. Two flags to disjoint domains are non-overlapping.

4.3 Flag Worlds: Definition and Existence

In this and the next subsection we only consider simple graphs, i.e. graphs with only simple interior domains, to avoid technical complications.

We are looking for a set $\boldsymbol{\beta}$ of hoops, such that any hoop in Γ can be expressed by a product of elements of $\boldsymbol{\beta}$, i.e. $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\Gamma)$ holds. Furthermore, we are interested in integrating cylindrical functions over $\mathcal{HG}(\boldsymbol{\beta})$. For this we need the moderate independence of $\boldsymbol{\beta}$, that means at least the weak independence. Due to Proposition 3.2 that is guaranteed only if the number of elements of $\boldsymbol{\beta}$ equals the number of generators of $\mathcal{HG}(\Gamma)$, i.e. equals the number of generators of the fundamental group $\pi(\Gamma, m)$. This in mind one could choose $\boldsymbol{\beta}$ to be a system of generators as in Proposition 3.4. But, because of our regularization we need loops enclosing an area being as tiny as possible, i.e. enclosing only one interior domain. For this the above defined flags are well-suited. We already know that the number of interior domains of Γ equals the rank of the fundamental group (cf. Euler's polyhedron formula in section 4.1). Thus the following definition is obvious.

Definition 4.3 **Flag World**
A set $\mathcal{F}$ of flags is called *flag world* to the simple graph Γ (with base point m) iff $\mathcal{F} = \{f_G \mid G \in L_{\text{int}}(\Gamma)\}$, where f_G is any flag to the domain G and to the base point m.
$\mathcal{F}$ is called *complete* iff $\mathcal{HG}(\mathcal{F}) = \mathcal{HG}(\Gamma)$.

Using Proposition 3.2 we have immediately

Corollary 4.6 The flags of a complete flag world are weakly independent.

Now we are interested in moderately independent flag worlds because they are necessary for the integration calculus and because of

Proposition 4.7 Let $\mathcal{F}$ be a moderately independent flag world in a simple graph Γ. Then $\mathcal{F}$ is complete.

Proof Γ is a finite, connected graph and $\mathcal{F}$ is a moderately independent set of loops in Γ, whose cardinality is equal to the rank of $\pi(\Gamma, m)$ due to Lemma 4.1. Proposition 3.10 finishes the proof. **qed**

We can construct naturally a flag world to any tree as follows.

Definition 4.4 Let T be a maximal tree in a simple graph Γ.
$\mathcal{F}$ is called T–flag world for Γ iff the following holds for all flags $f \in \mathcal{F}$:
1. f is a minimal flag.
2. The flagpole of f is a path in T.

Proposition 4.8 Let T be a maximal tree in a simple graph Γ.
1. There is a T–flag world for Γ.
2. Any T–flag world for Γ is moderately independent.

From this we get the crucial

Corollary 4.9 For any simple graph Γ there exists a moderately independent, i.e. also complete flag world.

Corollary 4.10 Any loop in Γ is holonomically equivalent to a product of mutually non-overlapping loops.

Proof (Proposition 4.8)

- First, let Γ be a tree, i.e. $\Gamma = T$. Then there is no interior domain and therefore no flag, too. We have $\mathcal{F} = \emptyset$ and $\mathcal{HG}(\mathcal{F}) = \{1\} = \mathcal{HG}(\Gamma)$.
- Now, Γ is not a tree. Let T be a maximal tree in Γ and $E := \{e_\lambda\}$ the corresponding set of edges of Γ not contained in T. Now we can construct Γ from T inserting successively edges e_λ. The intermediate graphs are denoted by Γ_λ. This allows to use induction on the number of interior domains increased exactly by 1 in each step. We can insert these edges, such that any new edge e_λ lies on the boundary of the corresponding graph Γ_λ.[9] Thus the interior domains of the intermediate graphs are simple due to $L_{\text{int}}(\Gamma_\lambda) \subseteq L_{\text{int}}(\Gamma)$. Obviously, any Γ_λ is finite, planar and connected.
- Suppose the proposition holds for any graph with $k - 1 \geq 0$ interior domains. Now, Γ has k interior domains, T and E are chosen as above and $e \in E$ is an edge in $\partial\Gamma$. We set $\Gamma' := \Gamma \setminus \{e\}$ and $E' := E \setminus \{e\}$. By inserting e in Γ' we get a new (simple) interior domain G, i.e. $L_{\text{int}}(\Gamma) = L_{\text{int}}(\Gamma') \cup \{G\}$. Obviously, T is also a maximal tree in Γ' and E' is the set of all edges of Γ' not contained in T. Γ' has exactly $k - 1$ interior domains and we have by induction:
  1. There exists a T–flag world for Γ'.
  2. Any T–flag world for Γ' is moderately independent.
- 1. Existence of a T–flag world for Γ
  We construct a flag for G. Since any vertex of Γ is contained in T, there is a path in T from m to a vertex of ∂G. We choose from among these paths a path ρ which is minimal w.r.t. to number of traced edges. The terminal vertex of ρ is denoted by $\widetilde{m}$, $\widetilde{m} \in \partial G$. Due to Lemma 4.4 we choose a boundary loop α of G with base point $\widetilde{m}$. $f := \rho\alpha\rho^{-1}$ is now a minimal flag for G and $\mathcal{F} := \mathcal{F}' \cup \{f\}$ is a T–flag world for Γ.

[9] Suppose there is a tree T' with $\partial\Gamma \subseteq T'$. Then $\partial\Gamma$ is a tree itself and $\partial\Gamma$ has no interior domain. Consequently, Γ has no interior domain, i.e. Γ is a tree. Thus, there is no tree T' with $\partial\Gamma \subseteq T'$ and so for any tree T in Γ there is an edge $e_\lambda \subseteq \partial\Gamma$ that is not contained in T.

2. Moderate independence of any T–flag world for Γ
   Γ', E' and G are still chosen as above. Set $\mathcal{F}' := \mathcal{F} \setminus \{f_G\}$, where $f_G \in \mathcal{F}$, $f_G = \rho\alpha\rho^{-1}$, is the flag for G with flagpole $\rho \subseteq T$. Obviously, $\mathcal{F}'$ is a T–flag world for Γ', and therefore moderately independent by induction.
   Since f_G is minimal, e is traced exactly once by f_G, and because $\mathcal{F}'$ is a flag world in $\Gamma' = \Gamma \setminus e$, not any $f_i \in \mathcal{F}'$ traces e. Therefore e is now a free segment of f_G.
   Finally, $\mathcal{F}'$ itself is moderately independent with the free segments e_i of the corresponding $f_i \in \mathcal{F}'$. Thus, $\mathcal{F} = \mathcal{F}' \cup \{f_G\}$ is moderately independent with the free segments $\{e_1, \ldots, e_{k-1}, e\}$. **qed**

Remark For $\mathbf{G} = U(1)$ even any flag world $\mathcal{F}$ is complete.
To prove this choose any complete flag world $\mathcal{F}'$ for Γ. Since (for $\mathbf{G} = U(1)$) all flags belonging to one and the same domain are equal up to holonomy equivalence, we have $\mathcal{HG}(\mathcal{F}) = \mathcal{HG}(\mathcal{F}') = \mathcal{HG}(\Gamma)$, i.e. $\mathcal{F}$ is complete.
In other words, for $U(1)$ all flag worlds to one and the same graph Γ are equal modulo holonomy equivalence.

The completeness of a flag world is not at all trivial for the $SU(N)$ because of

Proposition 4.11 Let $\mathbf{G} = SU(N)$. Then there exists a simple graph Γ, such that a non-complete (and so also not moderately independent) flag world exists to Γ.

Proof It is sufficient to give an example.
Due to $\mathbf{G} = SU(N)$ holonomy equivalence equals homotopy equivalence and we will indentify hoops and the corresponding elements of the fundamental group $\pi(\Gamma, m)$.
It is sufficient to construct a flag world $\mathcal{F}$, such that there is a loop $f \in \pi(\Gamma, m) = \mathcal{HG}(\Gamma)$ not contained in the subgroup $\mathcal{HG}(\mathcal{F})$ of the fundamental group generated by $\mathcal{F}$.

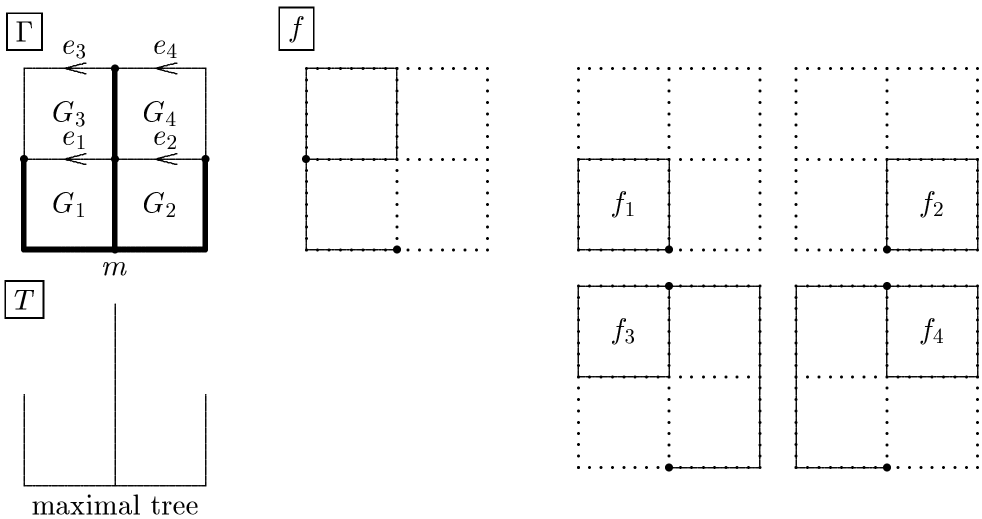


Figure 5: Example of a non-complete flag world

Let Γ be the graph in figure 5 with the flag world $\mathcal{F} := \{f_1, f_2, f_3, f_4\}$, the maximal tree T and the corresponding free edges e_1, e_2, e_3, e_4. We constuct from T and e_i the

free generators $\alpha_1, \dots, \alpha_4$ of $\pi(\Gamma, m)$ as in Proposition 3.4. We will prove, that $\mathcal{F}$ is not complete showing that $f \notin \mathcal{HG}(\mathcal{F})$, where f is the loop defined in figure 5.
A simple calculation shows:[10]

$$\begin{array}{lcl} f & \sim & \alpha_1^{-1}\alpha_3 \\ f_1 & \sim & \alpha_1 \\ f_2 & \sim & \alpha_2 \\ f_3 & \sim & \alpha_4\alpha_3\alpha_1^{-1}\alpha_4^{-1} \\ f_4 & \sim & \alpha_3^{-1}\alpha_2^{-1}\alpha_4\alpha_3. \end{array}$$

Suppose $f \in \mathcal{HG}(\mathcal{F})$, i.e. $f \sim \prod_{j=1}^{J} f_{i_j}^{\eta_j} \sim \prod_k \alpha_{\lambda_k}^{\epsilon_k}$ with $\eta_j \in \mathbb{Z}$ and $\epsilon_k \in \{-1, +1\}$. Choose this decomposition, such that the number J of used factors $f_{i_j}^{\eta_j}$ is minimal. Due to the freeness of $\mathcal{HG}(\Gamma) = \pi(\Gamma, m)$ there must exist a j' with $i_{j'} = 3$ and $\eta_{j'} \geq +1$, i.e.

$$\alpha_1^{-1}\alpha_3 \sim f \sim \prod_{j=1}^{j'-1} f_{i_j}^{\eta_j} \; f_3^{\eta_{j'}} \prod_{j=j'+1}^{J} f_{i_j}^{\eta_j} \sim \prod_k \alpha_{\lambda_k}^{\epsilon_k} \; \alpha_4 \; \prod^{\eta_{j'}} (\alpha_3\alpha_1^{-1}) \; \alpha_4^{-1} \prod_{k'} \alpha_{\lambda_{k'}}^{\epsilon_{k'}}.$$

In the last step f_j^{η} has been replaced by the corresponding reduced representation in the α_λ (see above), e.g. f_3^{η} by $\alpha_4(\alpha_3\alpha_1^{-1})^{\eta}\alpha_4^{-1}$ (i.e. not by $(\alpha_4\alpha_3\alpha_1^{-1}\alpha_4^{-1})^{\eta}$, since here (for $|\eta| > 1$) the $\alpha_4\alpha_4^{-1}$ terms are not reduced).
The right-hand decomposition of f in α_λ is (w.r.t. the number of used factors) longer than the left-hand one. Again by the freeness of $\mathcal{HG}(\Gamma)$ there must exist in the right-hand decomposition of f in α_λ a k with $\alpha_{\lambda_k}^{\epsilon_k} = \alpha_{\lambda_{k+1}}^{-\epsilon_{k+1}}$. This case does not occur in the decompositions of the f_i in α_λ above, thus this must occur during the multiplication $f_{i_j}^{\eta_j} f_{i_{j+1}}^{\eta_{j+1}}$ of two flags. From the decompositions above we see that such a collision of α_λ is only possible, if $i_j = i_{j+1}$. This is a contradiction to the minimality of the decomposition of f into a product of flags $f_i^{\eta} \in \mathcal{F}$.
Thus, $f \notin \mathcal{HG}(\mathcal{F})$, and $\mathcal{F}$ is not complete. **qed**

Remark 1. Up to now, we do not know, whether non-complete flag worlds can be constructed for graphs with less than 4 interior domains.

2. Simultaneously, we have constructed an example for the fact that from $\mathcal{HG}(\boldsymbol{\beta}) \subseteq \mathcal{HG}(\boldsymbol{\alpha})$ and the equality of the cardinalities of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ not generally follows, that $\mathcal{HG}(\boldsymbol{\beta}) = \mathcal{HG}(\boldsymbol{\alpha})$.
But, obviously, $\mathcal{HG}(\mathcal{F})$ is freely generated by $\{f_1, f_2, f_3, f_4\}$. Thus, we have constructed a genuine (free) subgroup of $\mathcal{HG}(\Gamma)$ having the same rank as $\mathcal{HG}(\Gamma)$.

4.4 Refinement of Flag Worlds

Now we want to investigate the behaviour of flag worlds under refinement of the underlying graph. We need the following

Lemma 4.12 Let Γ be a simple graph and G a simple domain in Γ ($m \notin G$) with corresponding refinement $\{G_i \mid i \in I\} \subseteq L_{\mathrm{int}}(\Gamma)$. Let f be a minimal flag belonging to G with base point m. Furthermore, e is an arbitrary edge of Γ on ∂G.
Then, there exist minimal flags f_i with base point m, such that:

[10]Note, that in our convention the multiplication $\beta_1\beta_2$ of two paths means, that β_1 is traced first and β_2 second.

- f_i is a flag to domain G_i for all $i \in I$;
- f is holonomically equivalent to the product of all f_i in a certain order;
- $\{f_i\}$ is moderately independent and any of the free segments lies in $\mathrm{int}G \cup e$.

Proof Induction on the cardinality of I.[11]

1. $I = 1$ is trivial, i.e. $G = G_i$ is an interior domain itself.
2. First, we consider the case $I = \{1, 2\}$.
   We consider the case, that e and $\widetilde{m}$ do not lie on the boundary of one and the same interior domain. Topologically, we have the situation of figure 6; if necessary, one has to exchange the domains 1 and 2. Let $\rho_{m\widetilde{m}}$ be the flagpoles of f from m to

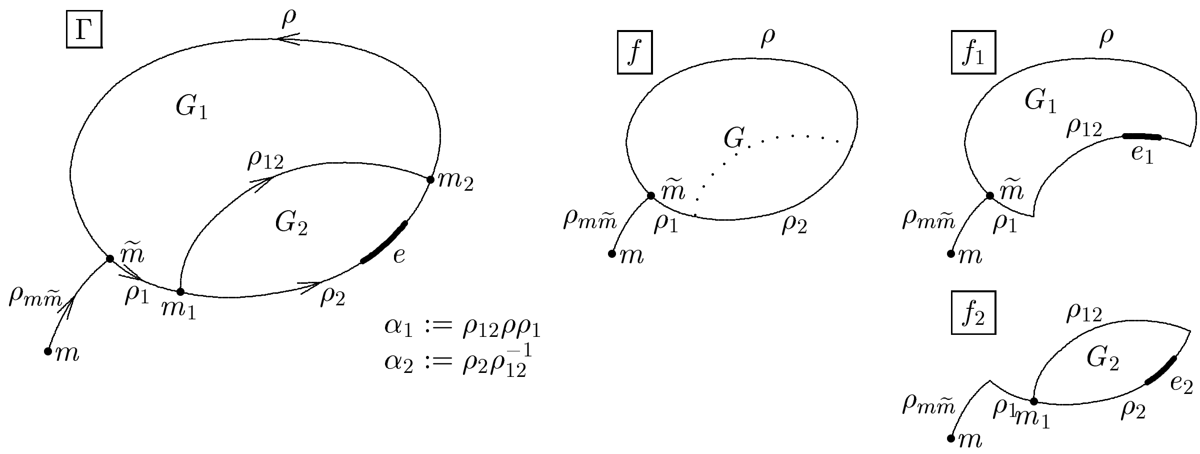


Figure 6: Refinement into two domains

   $\widetilde{m}$, $\rho, \rho_1, \rho_2, \rho_{12}$ as in figure 6 and α_i the corresponding boundary loop for G_i with base point m_1.
   We set $f_i := \rho_{m\widetilde{m}} \rho_1 \alpha_i {\rho_1}^{-1} \rho_{m\widetilde{m}}^{-1}$ for $i = 1, 2$, after cancelling possible retracings, i.e. we consider f_i to be minimal.
   - Obviously, f_i is a flag for G_i.
   - We have $f \sim f_2 f_1$.
   - Choose an edge $e_1 \subseteq \partial G_1 \cap \partial G_2$, i.e. $e_1 \subseteq \rho_{12}$, and set $e_2 := e$. Then $\{f_1, f_2\}$ is obviously moderately independent with the free segments $e_1, e_2 \subseteq \mathrm{int}\, G \cup e$.

   In the case that e and $\widetilde{m}$ lie on the boundary of one and the same domain, one has to exchange, if necessary, ρ_1 and ρ in construction above, such that $e \cap \rho_1 = \emptyset$. The rest of the proof is completely analogous.[12]
3. Suppose the lemma is proven for refinements by $k - 1 \geq 2$ domains und let now $\{G_i\}$ be a refinement of G by $k \geq 3$ domains.
   a) Choose any $i \in I$, such that $\overline{G_i} \cap \partial G$ contains at least one edge of Γ and the domain $\widetilde{G}$ built from the remaining G_j is again simple. (W.l.o.g. we set $i = k$ and j runs in the following from 1 to $k - 1$.) More precisely: ∂G,

[11] I is finite, since Γ is finite and thus $L_{\mathrm{int}}(\Gamma)$ is finite.

[12] Let e lie between $\widetilde{m}$ and m_2. Then f_i is constructed as above. Only, $\{f_2, f_1\}$ is now moderately independent if e_2 is any edge in ρ_{12} and $e_1 := e$.
Let e lie between $\widetilde{m}$ and m_1. Then m_1 and m_2 have to be exchanged. Thus, also ρ_1 is exchanged with ρ. By $f_i := \rho_{m\widetilde{m}} \rho_1 \alpha_i {\rho_1}^{-1} \rho_{m\widetilde{m}}^{-1}$ we get $f \sim f_1 f_2$. Furthermore, $\{f_2, f_1\}$ are moderately independent with e_2 to be an edge of $\rho_{12} = \partial G_1 \cap \partial G_2$ and $e_1 := e$.

∂G_k and $\rho_{m\widetilde{m}}$ span a finite and for $\overline{G_k} \cap \partial G \neq \emptyset$ again connected graph. We demand that the set of the interior domains in this graph is equal to $\{\widetilde{G}, G_k\}$ and that $\widetilde{G}$ is simple.
It remains the question, whether such an G_k exists. The first condition is trivial. To prove the second one it is sufficient to choose a domain G_k, such that $\partial G_k \cap \partial G$ is connected.
To see this let α be a boundary loop of G. One gets an $\widetilde{\alpha}$ from this, if one replaces the subpath α_k of α belonging to ∂G_k by the path $\widetilde{\alpha}_k$ corresponding to the boundary $\partial G_k \setminus \partial G$. Obviously, $\widetilde{\alpha}$ is a path in Γ. $\widetilde{\alpha}$ has neither repetitions of vertices nor of edges, because neither α nor α_k have the like and because $\widetilde{\alpha}_k$ touchs α only in its initial and terminal vertex (these are distinct). Otherwise, we would have a contradiction to the connectivity of $\partial G_k \cap \partial G$. Therefore $\widetilde{\alpha}$ is a Jordan path, i.e. a boundary of exactly one simple interior domain $\widetilde{G}$.
It remains now to ask for the existence of such a domain. Suppose not any $\partial G_i \cap \partial G$ is connected. Then there would exist a pair of indices (i_1, i_2), such that we have the situation in figure 7. Obviously, this is a contradiction to the connectivity of G_{i_1} and G_{i_2}.

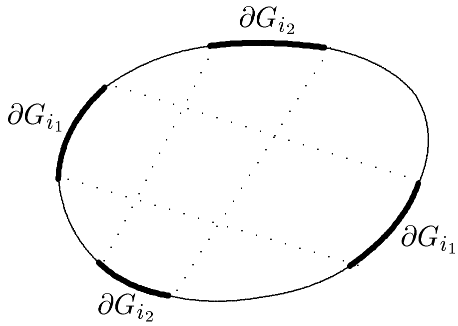


Figure 7: Existence of a G_k with connected $\partial G_k \cap \partial G$

Thus, there is a refinement of G into two simple domains $\{\widetilde{G}, G_k\}$, such that $\widetilde{G}$ itself has a refinement into $\{G_j\}$ in Γ.

b) Due to point 2. there are minimal flags $\widetilde{f}, f_k$ for $\widetilde{G}$ and G_k, resp., such that
   - $f \sim \widetilde{f} f_k$ (or $f_k \widetilde{f}$);
   - $\{\widetilde{f}, f_k\}$ or $\{f_k, \widetilde{f}\}$ is moderately independent, where the free segments $\widetilde{e}$ and e_k lie in $\operatorname{int} G \cup e$.
c) Let $\widetilde{e}$ be the free segment of $\widetilde{f}$. It is obviously an edge in $\Gamma \cap \partial\widetilde{G}$. Thus, by induction there are minimal flags f_j, such that:
   - $\widetilde{f}$ is a product of the f_j in a certain order;
   - $\{f_j\}$ is moderately independent, where any of the free segments e_j lies in $\operatorname{int} \widetilde{G} \cup \widetilde{e}$.
d) Thus, f can be represented as a hoop product of the f_i in a certain order.
e) The proof of the moderate independence of $\{f_j\} \cup \{f_k\}$ goes completely analogously to the case of two domains.
   i) Let $\{\widetilde{f}, f_k\}$ be moderately independent. Then e_k lies in $(\operatorname{int} G \cup e) \setminus \widetilde{f}$, otherwise e_k would already be traced by $\widetilde{f}$. Thus, $e_k = e$, since $(\operatorname{int} G \setminus$

$\widetilde{f}) \cap f_k = \emptyset$, and so $\widetilde{e} \subseteq \operatorname{int} G$. Due to $e_j \subseteq \operatorname{int} \widetilde{G} \cup \widetilde{e} \subseteq \operatorname{int} G$ we have $e_j, e_k \subseteq \operatorname{int} G \cup e$.
$\{f_1, \ldots, f_{k-1}, f_k\}$ is moderately independent because $e_1, \ldots, e_{k-1}$ are free segments of $f_1, \ldots, f_{k-1}$, and e_k is free, because $e_k \cap f_j \subseteq e_k \cap (\widetilde{f} \cup \operatorname{int} \widetilde{G}) = (e_k \cap \widetilde{f}) \cup (e_k \cap \operatorname{int} \widetilde{G}) = \emptyset$. The second intersection vanishes obviously and the first one does because $\{\widetilde{f}, f_k\}$ are moderately independent.

ii) Let $\{f_k, \widetilde{f}\}$ be moderately independent. The argumentation is analogous to the other case, however, here $\{f_k, f_1, \ldots, f_{k-1}\}$ is moderately independent. **qed**

We have now

Proposition 4.13 Let Γ, Γ' be simple graphs, Γ' a refinement of Γ and $m \in \Gamma$.
Then there exists for any moderately independent flag world $\mathcal{F}$ of Γ a moderately independent flag world $\mathcal{F}'$ of Γ', such that the following holds for all interior domains G_I of Γ: The flag $f_I \in \mathcal{F}$ to G_I is the hoop product of exactly these flags $f_{I,i_I} \in \mathcal{F}'$, that belong to the interior domains G_{I,i_I} with $G_{I,i_I} \subseteq G_I$, in a certain order.

Proof Obviously, we have $m \notin G_I$ for all $G_I \in L_{\text{int}}(\Gamma)$ because $m \in \Gamma$.
First, we define Γ'' to be the graph built from all interior domains of Γ' that are contained in the exterior domain of Γ and from all interior domains of Γ. Obviously, Γ'' is simple, $\Gamma \leq \Gamma'' \leq \Gamma'$ and the exterior domains of Γ'' and Γ' coincide. Let now $\mathcal{F} = \{f_I\} = \{f_1, \ldots, f_\Lambda\}$ be moderately independent with the free segments e_I. We can refine $\mathcal{F}$ to a moderately independent flag world $\mathcal{F}'' = \{f_1, \ldots, f_{\Lambda''}\} \supseteq \mathcal{F}$ of Γ'', where Λ'' is the number of interior domains of Γ'', analogously to the proof of Proposition 4.8. Next, we consider for any interior domain of Γ the corresponding refinement of G_I into the $G_{I,i_I} \in L_{\text{int}}(\Gamma')$. Due to Lemma 4.12 there exist minimal flags f_{I,i_I} with base point m, such that:

1. f_{I,i_I} is a flag to the domain G_{I,i_I}.
2. f_I is holonomically equivalent to the product of all f_{I,i_I} in a certain order.
3. $\{f_{I,i_I}\}$ is moderately independent and any free segment e_{I,i_I} is contained in $\operatorname{int} G_I \cup e_I$.[13]

The flags f_I in $\mathcal{F}'' \setminus \mathcal{F}$, i.e. those flags that belong to the interior domain of Γ', but are contained in the exterior domain of Γ, are left untouched. We only set $i_I := 1$ and $f_{I,i_I} := f_I$. Now[14] $\mathcal{F}' = \{f_{1,1}, \ldots, f_{1,\lambda_1}, f_{2,1}, \ldots, f_{2,\lambda_2}, \ldots\ldots, f_{\Lambda'',1}, \ldots, f_{\Lambda'',\lambda_{\Lambda''}}\}$ is a moderately independent flag world of Γ' because:

- e_{I,i_I} is traced exactly once by f_{I,i_I} per constructionem and is not traced by any $f_{I,j}$ with $j < i_I$ because due to the just stated point 3. $\{f_{I,j} \mid j \in [1, \lambda_I]\}$ is moderately independent with the free segments $e_{I,j}$ for a fixed I.
- But, e_{I,i_I} is also not traced by $f_{J,j}$ with $J < I$:
  $f_{J,j}$ traces only $f_J \cup \operatorname{int} G_J$ and we have $e_{I,i_I} \subseteq \operatorname{int} G_I \cup e_I$. Since the domains of Γ are disjoint, we have $\operatorname{int} G_J \cap \operatorname{int} G_I = \emptyset$, $f_J \cap \operatorname{int} G_I = \emptyset$ and $\operatorname{int} G_J \cap e_I = \emptyset$. Finally,

[13] W.l.o.g. e_I is an edge of Γ on ∂G_I.
[14] λ_I is the number of domains, that the G_I are refined into.

we have $f_J \cap e_I = \emptyset$ since $\mathcal{F}'' = \{f_1, \ldots, f_{\Lambda''}\}$ itself is moderately independent. Thus, $f_{J,j} \cap e_{I,i_I} = \emptyset$.

Thus, e_{I,i_I} fits all conditions for a free segment. Since $\mathcal{F}'$ is obviously a flag world of Γ', we get the proof. **qed**

4.5 Conclusions

We collect the most important facts with regard to the applications in section 6 neglecting sometimes mathematical details. For this see the cross-references. Any graph is finite, planar, connected and non-empty.

Let there be given an arbitrary graph Γ.

- There is a refinement of Γ to an ordinary graph Γ'.
- Any graph can be naturally associated with a finite set of connected interior domains and an exterior domain (section 4.1). By a refinement of Γ this set is refined.
- A graph is called simple iff its interior domains are simple, i.e. are bounded by Jordan loops.
- Any ordinary graph Γ' is subgraph of a simple, ordinary graph Γ''. The exterior domains of both graphs are the same (Proposition 4.3).
- Any simple domain G in a graph can be naturally associated with a flag, i.e. a loop running from a base point m to ∂G, traversing G exactly once and running back to m (Definition 4.2).
- By choosing a flag to each interior domain one gets a flag world (Definition 4.3). It is called complete iff it spans the full hoop group of Γ.
- We are looking for moderately independent and complete flag worlds. The completeness ensures that any loop in Γ can be expressed by elements of a flag world. The moderate independence is necessary for the integration of cylindrical functions. Fortunately, the moderate independence implies the completeness (Proposition 4.7).
- One can naturally construct flag worlds to any simple graph. For this one chooses a maximal tree in this graph and then for any interior domain a flag consisting of a path along the tree, a boundary loop of the corresponding domain and the inverse initial path. Any such flag world is moderately independent (Proposition 4.8).
- There is a moderately independent flag world for any simple graph (Corollary 4.9). Thus, any hoop can be represented as a hoop product of mutually non-overlapping loops.
- Under refinement of a simple domain G with a flag f one can choose flags f_i to the new domains G_i such that these generate all hoops "in G" and that f can be expressed as a hoop product of the f_i in a certain order (Lemma 4.12).
- In simple graphs Γ' any moderately independent flag world $\mathcal{F}$ of a simple subgraph Γ can be refined to a moderately independent flag world $\mathcal{F}'$ von Γ' such that any flag $f_G \in \mathcal{F}$ is a product of the flags $f_{G'} \in \mathcal{F}'$ to the interior domains $G' \subseteq G$ in a certain order (Proposition 4.13).

In section 6 we will see that especially the last point is crucial for the regularization of the Wilson loop functionals. We can now decompose the "banner" of a given flag in smaller "banners". But all small "banners" have "equal rights" since $f_I \sim f_{I,1} \cdots f_{I,\lambda_I}$. That is why they give identical contributions if we integrate cylindrical functions in f_I.

5 Integration on $\overline{\mathcal{A}/\mathcal{G}}$

In this section we slightly generalize the integration calculus on $\overline{\mathcal{A}/\mathcal{G}}$ which was in detail investigated by Ashtekar and Lewandowski [AL93]. Their key idea was to define first an equivalence relation on $\overline{\mathcal{A}/\mathcal{G}}$ which identifies two connections iff their holonomies on a certain finite set $\boldsymbol{\beta}$ of hoops are equal (up to conjugation), i.e. factorizing w.r.t. that relation they extracted the properties of a generalized connection on that finite set. But, if one knows these properties for all finite sets of hoops, one can reconstruct via $\overline{\mathcal{A}/\mathcal{G}} \sim \mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad}$ the generalized connection in $\overline{\mathcal{A}/\mathcal{G}}$. The main advantage of the factorization is the reduction of the infinite-dimensional problem to a finite-dimensional one, since $\overline{\mathcal{A}/\mathcal{G}}/\sim\; \cong \mathrm{Hom}(\mathcal{HG}(\boldsymbol{\beta}), \mathbf{G})/\mathrm{Ad} \cong \mathbf{G}^{\#\boldsymbol{\beta}}/\mathrm{Ad}$. Comparing that situation with the case of infinite-dimensional topological vector spaces, A-L defined first cylindrical functions as functions on $\overline{\mathcal{A}/\mathcal{G}}/\sim$ and second the integral of cylindrical functions $f = \pi_{\boldsymbol{\beta}}^* f_{\boldsymbol{\beta}}$ via $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \; d\mu = \int_{\overline{\mathcal{A}/\mathcal{G}}/\sim} f_{\boldsymbol{\beta}} \; d\mu_{\boldsymbol{\beta}}$, where $d\mu_{\boldsymbol{\beta}}$ is a measure on $\overline{\mathcal{A}/\mathcal{G}}/\sim\; \cong \mathbf{G}^{\#\boldsymbol{\beta}}/\mathrm{Ad}$. The main problem is to guarantee that this integral is well-defined. A-L could prove this for the choice that $d\mu_{\boldsymbol{\beta}}$ is the Haar measure on $\mathbf{G}^{\#\boldsymbol{\beta}}/\mathrm{Ad}$, and if only *strongly* independent $\boldsymbol{\beta}$ are allowed for calculating the integral above. The use of merely weakly independent $\boldsymbol{\beta}$ leads to contradictions. Our task is now to prove that the use of *moderately* independent $\boldsymbol{\beta}$ keeps instead the definition valid. This point is crucial for the calculation of the Wilson loop expectation values using the not strongly, but moderately independent flag worlds.

5.1 Equivalence of Connections

We recall [AL93] the following

Definition 5.1 Equivalence of Connections
Let $\mathcal{HG}(\boldsymbol{\beta}) \subseteq \mathcal{HG}$ be a finitely generated subgroup of the hoop group $\mathcal{HG}$ with weakly independent $\boldsymbol{\beta}$. Two (generalized) connections $\overline{A}_1$ and $\overline{A}_2$ are called *equivalent w.r.t.* $\mathcal{HG}(\boldsymbol{\beta})$ iff

$$h_{\overline{A}_1}(\gamma) = g^{-1} h_{\overline{A}_2}(\gamma) g \qquad \forall \gamma \in \mathcal{HG}(\boldsymbol{\beta})$$

with a fixed (hoop independent) $g \in \mathbf{G}$.
Furthermore, let $\pi_{\boldsymbol{\beta}} : \overline{\mathcal{A}/\mathcal{G}} \longrightarrow \overline{\mathcal{A}/\mathcal{G}}/\sim$ be the corresponding canonical projection.

Using the bijection $\overline{\mathcal{A}/\mathcal{G}} \longleftrightarrow \mathrm{Hom}(\mathcal{HG}, \mathbf{G})/\mathrm{Ad}$ Ashtekar and Lewandowski [AL93] could easily analyze the structure of $\overline{\mathcal{A}/\mathcal{G}}/\sim$.

Lemma 5.1
1. There is a bijection $\overline{\mathcal{A}/\mathcal{G}}/\sim \;\longrightarrow \mathrm{Hom}(\mathcal{HG}(\boldsymbol{\beta}), \mathbf{G})/\mathrm{Ad}$. That means, two generalized connections are equivalent if and only if they coincide mod Ad on $\mathcal{HG}(\boldsymbol{\beta})$.
2. Any choice of n weakly independent generators $\beta_i \in \mathcal{HG}(\boldsymbol{\beta})$ yields a bijection $\phi_{\boldsymbol{\beta}} : \overline{\mathcal{A}/\mathcal{G}}/\sim \;\longrightarrow \mathbf{G}^n/\mathrm{Ad}$.
3. Given $\mathcal{HG}(\boldsymbol{\beta}) \subseteq \mathcal{HG}$ the topology on $\overline{\mathcal{A}/\mathcal{G}}/\sim$ induced by the last point is independent of the choice of generators.

Furthermore, we have [AL93]

Corollary 5.2 Let $\mathcal{HG}(\boldsymbol{\beta})$ be a finitely generated subgroup of the hoop group and $\sim$ the induced equivalence relation on $\overline{\mathcal{A}/\mathcal{G}}$. Then any equivalence class $[\overline{A}] \in \overline{\mathcal{A}/\mathcal{G}}/\sim$ contains a regular connection.

5.2 Cylindrical Functions

In the following we set $\mathcal{B}_{\boldsymbol{\beta}} := \mathrm{Hom}(\mathcal{HG}(\boldsymbol{\beta}), \mathbf{G})/\mathrm{Ad} \cong \mathbf{G}^n/\mathrm{Ad}$ with $\boldsymbol{\beta}$ a weakly independent set of n hoops. Furthermore we usually do not distinguish between a function $f \in \overline{\mathcal{HA}}$ and its Gelfand transform $\widetilde{f} : \overline{\mathcal{A}/\mathcal{G}} \longrightarrow \mathbb{C}$.

We now provide a slightly modified version of the Ashtekar-Lewandowski definition of a cylindrical function.

Definition 5.2 **Cylindrical Function**
$f : \overline{\mathcal{A}/\mathcal{G}} \longrightarrow \mathbb{C}$ is called *cylindrical function* iff there is a finite set $\boldsymbol{\beta}$ of weakly independent hoops and a continuous $f_{\boldsymbol{\beta}} : \mathcal{B}_{\boldsymbol{\beta}} \longrightarrow \mathbb{C}$, such that $f = \pi^*_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}}$. Iff f can be obtained that way for a given $\boldsymbol{\beta}$, f is called *cylindrical* w.r.t. $\boldsymbol{\beta}$.
The set of all cylindrical functions is denoted by $\mathcal{C}$.

Lemma 5.3 Let f be cylindrical w.r.t. $\boldsymbol{\beta}$. Then f is cylindrical w.r.t. $\boldsymbol{\alpha}$, if the following holds:

1. $\boldsymbol{\alpha}$ is weakly independent.
2. $\mathcal{HG}(\boldsymbol{\alpha}) \supseteq \mathcal{HG}(\boldsymbol{\beta})$.

Proof We set $m := \#\boldsymbol{\alpha}$ and $n := \#\boldsymbol{\beta}$ and we have

$$\begin{array}{rcccccc} \pi_{\boldsymbol{\beta}} : & \overline{\mathcal{A}/\mathcal{G}} & \longrightarrow & \mathrm{Hom}(\mathcal{HG}(\boldsymbol{\beta}), \mathbf{G})/\mathrm{Ad} & = & \mathbf{G}^n/\mathrm{Ad} \\ & \overline{A} & \longmapsto & h_{[\overline{A}]} \ \mathrm{mod\ Ad} & \longmapsto & \left[h_{[\overline{A}]}(\beta_1), \ldots, h_{[\overline{A}]}(\beta_n)\right]_{\mathrm{Ad}} \end{array}$$

and analogously for $\pi_{\boldsymbol{\alpha}}$. Due to $\mathcal{HG}(\boldsymbol{\beta}) \subseteq \mathcal{HG}(\boldsymbol{\alpha})$ there exists for all $i \in [1, n]$ a decomposition $\beta_i = \prod_{k_i}^{K_i} \alpha_{j(i,k_i)}^{\epsilon(i,k_i)}$. Set

$$\begin{array}{rccc} \pi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} : & \mathbf{G}^m/\mathrm{Ad} & \longrightarrow & \mathbf{G}^n/\mathrm{Ad}. \\ & [g_1, \ldots, g_m]_{\mathrm{Ad}} & \longmapsto & \left[\prod_{k_1=1}^{K_1} g_{j(1,k_1)}^{\epsilon(1,k_1)}, \ldots, \prod_{k_n=1}^{K_n} g_{j(n,k_n)}^{\epsilon(n,k_n)}\right]_{\mathrm{Ad}} \end{array}$$

Obviously, $\pi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}$ is continuous and $\pi_{\boldsymbol{\beta}} = \pi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \pi_{\boldsymbol{\alpha}}$. The function $f_{\boldsymbol{\alpha}} := f_{\boldsymbol{\beta}} \pi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}$ is again continuous and we have

$$f = \pi^*_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}} = (\pi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} \pi_{\boldsymbol{\alpha}})^* f_{\boldsymbol{\beta}} = (\pi_{\boldsymbol{\alpha}})^* f_{\boldsymbol{\alpha}}.$$

Thus, f is cylindrical w.r.t. $\boldsymbol{\alpha}$. **qed**

Remark In contrast to [AL93] we define cylindrical functions not only on strongly independent, but also on weakly independent $\boldsymbol{\beta}$. For the present the set of cylindrical functions seems to be enlarged. But, it is easy to see, that given an $f \in \mathcal{C}$ there is a set $\boldsymbol{\alpha}$ of strongly independent loops, such that f is cylindrical w.r.t. $\mathcal{B}_{\boldsymbol{\alpha}}$.
Let $f \in \mathcal{C}$, i.e. there is a finite set $\boldsymbol{\beta}$ of weakly independent hoops w.r.t. that f is cylindrical. Following Lemma 3.8 there is a set $\boldsymbol{\alpha}$ of strongly independent loops, such that $\mathcal{HG}(\boldsymbol{\beta}) \subseteq \mathcal{HG}(\boldsymbol{\alpha})$. Due to the just proven lemma f is cylindrical w.r.t. the strongly independent set $\boldsymbol{\alpha}$. Thus, our definition is equivalent to that one in [AL93].

Finally, we quote [AL93]

Proposition 5.4 $\mathcal{C}$ is a normed *-algebra and $\overline{\mathcal{C}}$ is isomorphic to $\overline{\mathcal{HA}}$.

5.3 The Induced Haar Measure on $\overline{\mathcal{A}/\mathcal{G}}$

Definition 5.3 Let be $f \in \mathcal{C}$ and $\boldsymbol{\beta} \subseteq \mathcal{HG}$ be a moderately independent set of n hoops, such that f is cylindrical w.r.t. $\boldsymbol{\beta}$, i.e. $f = \pi^*_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}}$ with a continuous function $f_{\boldsymbol{\beta}} : \mathcal{B}_{\boldsymbol{\beta}} \longrightarrow \mathbb{C}$. Furthermore, $d\mu_{\boldsymbol{\beta}}$ is an arbitrary measure on $\mathcal{B}_{\boldsymbol{\beta}}$.
Then we define $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \; d\mu := \int_{\mathcal{B}_{\boldsymbol{\beta}}} f_{\boldsymbol{\beta}} \; d\mu_{\boldsymbol{\beta}}$.

We have to guarantee that the measures on the distinct $\mathcal{B}_{\boldsymbol{\beta}}$ are compatible in order to make the integral in the definition above well-defined.

Ashtekar and Lewandowski suggested to choose the Haar measure on each $\mathcal{B}_{\boldsymbol{\beta}}$, $\boldsymbol{\beta}$ strongly independent, induced from $\mathbf{G}^n/\mathrm{Ad}$ with n the cardinality of $\boldsymbol{\beta}$. Indeed, they could prove that the definition above provides a well-defined integral on $\overline{\mathcal{A}/\mathcal{G}}$. We are only left with the proof that the integral is still well-defined if we allow $\boldsymbol{\beta}$ to be merely moderately independent instead of strongly independent. Fortunately, for this we can reuse the A-L proof with slight modifications. Thus, we have

Theorem 5.5 Let $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \; d\mu_0$ be defined as in Definition 5.3, where the measure on $\mathcal{B}_{\boldsymbol{\beta}}$ is in each case the Haar measure on $d^n\mu_{\text{Haar}}$.

1. The integral $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \; d\mu_0$ is well-defined.
2. The functional
$$F : \overline{\mathcal{HA}} \longrightarrow \mathbb{C}, \qquad f \longmapsto \int_{\overline{\mathcal{A}/\mathcal{G}}} f(\overline{A}) \, d\mu_0(\overline{A})$$
is linear, continuous, positive and $\mathrm{Diff}(M)$-invariant.
3. The cylindrical measure $d\mu_0$ is a regular, positive and $\mathrm{Diff}(M)$-invariant measure on $\overline{\mathcal{A}/\mathcal{G}}$.

Proof It remains to prove the integral to be well-defined. If it is, then our measure coincides with the A-L measure defined by the only use of strongly independent hoops, since the A-L measure is unique and we did not remove any of the conditions the integral has to fulfill – because any strongly independent $\boldsymbol{\beta}$ is moderately independent. Consequently, all the other assertions of the theorem already proven in [AL93] using the strong independence can be generalized to our problem.

- Let there be given an $f \in \mathcal{C}$ and two sets $\boldsymbol{\beta}', \boldsymbol{\beta}'' \subseteq \mathcal{L}_m$ of moderately independent loops, such that f is cylindrical w.r.t. $\mathcal{B}_{\boldsymbol{\beta}'}$ and $\mathcal{B}_{\boldsymbol{\beta}''}$.
- W.l.o.g. choose the free segments $e'_{i'}, e''_{i''}$ of $\beta'_{i'}, \beta''_{i''}$, such that they are in each case completely contained in an edge of $\Gamma_{\boldsymbol{\beta}' \cup \boldsymbol{\beta}''}$.[15] Connect now any vertex $v \neq m$ of $\Gamma_{\boldsymbol{\beta}' \cup \boldsymbol{\beta}''}$ with the base point m by a piecewise analytic Jordan path h_v, such that $h_v \cap h_{v'} \quad \forall v \neq v'$ and $h_v \cap \Gamma_{\boldsymbol{\beta}' \cup \boldsymbol{\beta}''}$ consist of at most a finite number of points. Construct all paths $\beta_i := h_{e_i^-} e_i h^{-1}_{e_i^+}$, where e_i runs over all egdes of $\Gamma_{\boldsymbol{\beta}' \cup \boldsymbol{\beta}''}$. Obviously, $\boldsymbol{\beta}', \boldsymbol{\beta}'' \subseteq \mathcal{HG}(\boldsymbol{\beta}), \boldsymbol{\beta} := \{\beta_i \mid i = 1, \ldots, n\}$, and also $\mathcal{HG}(\boldsymbol{\beta}'), \mathcal{HG}(\boldsymbol{\beta}'') \subseteq \mathcal{HG}(\boldsymbol{\beta})$. More precisely: Let $\beta'_j = \prod_{k_j=1}^{K_j} e_{i(j,k_j)}^{\epsilon(j,k_j)}$ be a (minimal) decomposition of β'_j into a sequence of edges, so $\beta'_j \sim \prod_{k_j=1}^{K_j} \beta_{i(j,k_j)}^{\epsilon(j,k_j)}$ is a (minimal) decomposition of β'_j in β_i. The same holds for β''_j.
  Next, $\boldsymbol{\beta}$ is strongly independent with the free segments e_i.

[15] If necessary, free $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}''$ from retracings and use then the argumentation of Proposition 3.10. For the definition of $\Gamma_{\boldsymbol{\beta}' \cup \boldsymbol{\beta}''}$ see Construction 3.4.

Since $\mathcal{HG}(\boldsymbol{\beta}'), \mathcal{HG}(\boldsymbol{\beta}'') \subseteq \mathcal{HG}(\boldsymbol{\beta})$, f is also cylindrical w.r.t. $\mathcal{B}_{\boldsymbol{\beta}}$. Thus, it is sufficient to prove $\int_{\mathcal{B}_{\boldsymbol{\beta}'}} f_{\boldsymbol{\beta}'}\, d\mu_{\mathcal{B}_{\boldsymbol{\beta}'}} = \int_{\mathcal{B}_{\boldsymbol{\beta}}} f_{\boldsymbol{\beta}}\, d\mu_{\mathcal{B}_{\boldsymbol{\beta}}}$.

- Since we fixed the generators of $\mathcal{HG}(\boldsymbol{\beta})$, we can interpret the integration on $\mathcal{B}_{\boldsymbol{\beta}}$ as an integration on $\mathbf{G}^n/\mathrm{Ad}$. Since the Haar measure is Ad-invariant, we can pull back any function of $\mathbf{G}^n/\mathrm{Ad}$ onto the whole $\mathbf{G}^n$ and integrate hereon. The analogon holds for $\mathcal{B}_{\boldsymbol{\beta}'}$.
- Now we can express any $\beta'_{i'} \in \boldsymbol{\beta}'$ by a product of $\beta_i \in \boldsymbol{\beta}$, such that for all $i \in [1, n']$ there exists a $K(i') \in [1, n]$ and that the following holds:
  1. $i' \neq j' \Longleftrightarrow K(i') \neq K(j')$;
  2. $\beta_{K(i')}$ is not used in any decomposition of the $\beta'_{j'}$, $j' < i'$, into elements of $\boldsymbol{\beta}$;
  3. $\beta_{K(i')}$ (or $\beta^{-1}_{K(i')}$) is used in any decomposition of $\beta_{i'}$ exactly once.

  To see this choose $K(i')$, such that $e_{K(i')}$ contains the free segment of $\beta'_{i'}$. Since there is a bijection $e_i \longleftrightarrow \beta_i$, these three conditions are only a reformulation of the criteria for the moderate independence of the $\beta'_{i'}$.
- Since f is as well cylindrical w.r.t. $\boldsymbol{\beta}$ as w.r.t. $\boldsymbol{\beta}'$, $f = \pi^*_{\boldsymbol{\beta}} f_{\boldsymbol{\beta}} = \pi^*_{\boldsymbol{\beta}'} f_{\boldsymbol{\beta}'}$. Analogously to Lemma 5.3 we have $\pi_{\boldsymbol{\beta}'} = \pi \pi_{\boldsymbol{\beta}}$, where
$$\begin{array}{rccc} \pi : & \mathbf{G}^n/\mathrm{Ad} & \longrightarrow & \mathbf{G}^{n'}/\mathrm{Ad} \\ & [g_1, \ldots, g_n]_{\mathrm{Ad}} & \longmapsto & \left[\prod_{k_1=1}^{K_1} g_{i(1,k_1)}^{\epsilon(1,k_1)}, \ldots, \prod_{k_{n'}=1}^{K_{n'}} g_{i(n',k_{n'})}^{\epsilon(n',k_{n'})}\right]_{\mathrm{Ad}} \end{array}$$
  is defined due to the decompositions $\beta'_{i'} = \prod_{k'_i=1}^{K_{i'}} e_{i(i',k_{i'})}^{\epsilon(i',k_{i'})}$.
  Thus, we have $f_{\boldsymbol{\beta}} = \pi^* f_{\boldsymbol{\beta}'}$, i.e. $f_{\boldsymbol{\beta}}([g_1, \ldots, g_n]_{\mathrm{Ad}}) = (\pi^* f_{\boldsymbol{\beta}'})([g_1, \ldots, g_n]_{\mathrm{Ad}}) = f_{\boldsymbol{\beta}'}\left(\left[\prod_{k_1=1}^{K_1} g_{i(1,k_1)}^{\epsilon(1,k_1)}, \ldots, \prod_{k_{n'}=1}^{K_{n'}} g_{i(n',k_{n'})}^{\epsilon(n',k_{n'})}\right]_{\mathrm{Ad}}\right)$.
- Now we can integrate (considering $f_{\boldsymbol{\beta}}$ to be both a function on $\mathbf{G}^n$ and $\mathbf{G}^n/\mathrm{Ad}$):

$$\begin{aligned}
& \int_{\mathcal{B}_{\boldsymbol{\beta}}} f_{\boldsymbol{\beta}}\ d\mu_{\mathcal{B}_{\boldsymbol{\beta}}} \\
= & \int_{\mathbf{G}^n} \prod_{i=1}^n d\mu_i\ f_{\boldsymbol{\beta}}(g_1, \ldots, g_n) \\
= & \int_{\mathbf{G}^n} \prod_{i=1}^n d\mu_i\ f_{\boldsymbol{\beta}'}\left(\prod_{k_1=1}^{K_1} g_{i(1,k_1)}^{\epsilon(1,k_1)}, \ldots, \prod_{k_{n'}=1}^{K_{n'}} g_{i(n',k_{n'})}^{\epsilon(n',k_{n'})}\right) \\
= & \int_{\mathbf{G}^{n-n'}} \prod_{i=1, i\notin K([1,n'])}^n d\mu_i \int_{\mathbf{G}} d\mu_{K(1)} \cdots \int_{\mathbf{G}} d\mu_{K(n')} \\
& \qquad\qquad f_{\boldsymbol{\beta}'}\left(\cdots g_{K(1)} \cdots, \ldots, \cdots g_{K(n')} \cdots\right)
\end{aligned}$$

(Permutation of the order of integration. The three dots in $\cdots g_{K(i')} \cdots$ denote a product of g_i, which because of the construction above does not contain a $g_{K(j')}$ with $j' \geq i'$.)

$$\begin{aligned}
= & \int_{\mathbf{G}^{n-n'}} \prod_{i=1, i\notin K([1,n'])}^n d\mu_i \int_{\mathbf{G}} d\mu_{K(n')} \int_{\mathbf{G}} d\mu_{K(1)} \cdots \int_{\mathbf{G}} d\mu_{K(n'-1)} \\
& \qquad\qquad f_{\boldsymbol{\beta}'}\left(\cdots g_{K(1)} \cdots, \ldots, \cdots g_{K(n'-1)} \cdots, g_{K(n')}\right)
\end{aligned}$$

(Results from the translation invariance of the Haar measure, since for all $j' < n'$ $\cdots g_{K(j')} \cdots$ does not contain a factor $g_{K(n')}$ and since $g_{K(n')}$ appears in $\cdots g_{K(n')} \cdots$ exactly once.)

$$\vdots$$

$$= \int_{\mathbf{G}^{n-n'}} \prod_{i=1, i\notin K([1,n'])}^{n} d\mu_i \int_{\mathbf{G}} d\mu_{K(1)} \cdots \int_{\mathbf{G}} d\mu_{K(n')} \; f_{\boldsymbol{\beta}'}\left(g_{K(1)}, \ldots, g_{K(n')}\right)$$

(We used successively the translation invariance of the Haar measure in order to eliminate the $\cdots$-products as in the step above.)

$$= \int_{\mathbf{G}^{n'}} \prod_{i=1}^{n'} d\mu_i \;\; f_{\boldsymbol{\beta}'}\left(g_1, \ldots, g_{n'}\right)$$

(Normalization of the Haar measure and bijection $i' \longleftrightarrow K(i')$)

$$= \int_{\mathcal{B}_{\boldsymbol{\beta}'}} f_{\boldsymbol{\beta}'} \;\; d\mu_{\mathcal{B}_{\boldsymbol{\beta}'}}.$$

- Thus, $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \;\, d\mu_0$ is well-defined.[16] **qed**

Remark The proof that the integral is well-defined gives us the earlier mentioned importance of moderate independence. Though the flag worlds in section 4 are usually not strongly independent, they can be used for the integration calculus. If one instead demanded only the weak independence for the definition of the integral, the integral would become ill-defined. Let, e.g., $\mathbf{G} = SU(2)$ and β be a strongly or, equivalently, a moderately independent loop. $\gamma := \beta^2$ is no longer moderately independent, but, of course, still weakly independent, since extracting the square root is possible in $SU(2)$. Let now $f = \text{tr } h_\gamma = \text{tr } h_\beta^2$. f is cylindrical w.r.t. γ and w.r.t. β. We integrate f w.r.t. β and receive $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \, d\mu_0 = \int_{\mathbf{G}} \text{tr } g \, d\mu_{\text{Haar}} = 0$. But, w.r.t. γ we have $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \; d\mu_0 = \int_{\mathbf{G}} \text{tr } g^2 \; d\mu_{\text{Haar}} = -1$, i.e. the integral is ill-defined. Thus, the moderate independence is best-suited for the mathematically rigorous calculation of the Wilson loop expectation values in section 6.

6 Calculation of the Wilson Loop Expectation Values

In this section the expectation values of the Wilson loop products

$$\langle T_{\alpha_1} \cdots T_{\alpha_n} \rangle := \lim_{a\to 0, L_x, L_y \to \infty} \frac{1}{Z_{a,L_x,L_y}} \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0 \; e^{-S^W_{reg}} T_{\alpha_1} \cdots T_{\alpha_n}$$

of the pure Yang-Mills theory are computed. Thiemann [Thi95] and Ashtekar et al. [ALM+96] were the first who succeeded in calculating $\langle T_{\alpha_1} \cdots T_{\alpha_n} \rangle$ – at least for loops α_i that lie in a certain quadratic lattice – in the Ashtekar framework. Our goal is now to generalize their results for arbitrary α_i.

It is well-known [AL93] that given the expectation values $\langle T_{\alpha_1} \cdots T_{\alpha_n} \rangle$ for all α_i one can reconstruct the measure $d\mu_{YM}$ of the theory and vice versa. A direct definition $d\mu_{YM} := e^{-S[\overline{A}]} d\mu_0$ is difficult since one has to define the action S not only on $\mathcal{A}/\mathcal{G}$ but on the whole $\overline{\mathcal{A}/\mathcal{G}}$. The first step to overcome this problem is an appropriate regularization $S^W_{reg}(A)$ of $S_{YM}(A) = \int_M \frac{1}{4} \text{tr } F_{\mu\nu} F^{\mu\nu} \; dx$. Since the only variables used a priori in the Ashtekar approach

[16] We assumed in our calculation, that in any case $\cdots g_{K(i')} \cdots$ appears and $\cdots g^{-1}_{K(i')} \cdots$ does not. Otherwise in the last but three step we get a function partially in $g^{-1}_{K(i')}$. The claim remains valid since the Haar measure is invariant under inversions, i.e. we have $\int_{\mathbf{G}} d\mu_{\text{Haar}} \; f(g) = \int_{\mathbf{G}} d\mu_{\text{Haar}} \; f(g^{-1})$.

are the Wilson loops, it seems very likely to use the lattice regularization. Strictly speaking, T-A$^+$ set

$$S^W_{reg}(A) \quad := \quad \frac{N}{g^2 a^2} \sum_{\Box} (1 - \frac{1}{N} \mathrm{Re\ tr}\ h_{\Box}(A)), \tag{2}$$

where a denotes the lattice spacing, $\Box$ runs over all plaquettes of the lattice and $h_\Box(A)$ is the holonomy around the boundary of $\Box$. On the one hand, S^W_{reg} converges naively to S_{YM}, when the lattice grows ad infinitum and a goes to zero, and on the other hand, S^W_{reg} is a function of Wilson loops, i.e. it can be naturally extended from $\mathcal{A}/\mathcal{G}$ onto the whole $\overline{\mathcal{A}/\mathcal{G}}$. The second step of T-A$^+$ was now the definition of $\langle T_{\alpha_1} \cdots T_{\alpha_n} \rangle$ exchanging limit and integration (L is the length of the lattice):

$$\begin{aligned}
\langle T_{\alpha_1} \cdots T_{\alpha_n} \rangle \quad &:= \quad \lim_{a\to 0, L\to\infty} \frac{1}{Z_{a,L}} \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0 \ e^{-S^W_{reg}} T_{\alpha_1} \cdots T_{\alpha_n} \\
\text{"} = \text{"} \quad & \frac{1}{Z} \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0 \ e^{-\lim_{a\to 0, L\to\infty} S^W_{reg}} T_{\alpha_1} \cdots T_{\alpha_n}.
\end{aligned}$$

Now they were able to calculate explicitely the expectation values for all $\alpha_1, \ldots, \alpha_n$ contained in a quadratic lattice. Finally, they suggested to compute these values for general α_i by approximating them by certain lattice loops.

We aviod this problem using a slightly modified regularization. The idea is to adapt the regularization to the given loops and not vice versa. We consider any finite lattice with certain interior domains G generalizing the quadratic plaquettes $\Box$. Then we replace in (2) $\Box$ by G and also a^2 by $|G|$, the area of the interior domain G, in the denominator. Following the calculations of T-A$^+$ we get an explicite formula for $\langle T_{\alpha_1} \cdots T_{\alpha_n} \rangle$ with arbitrary $\alpha_1, \ldots, \alpha_n$ that coincides with the naive limit of T-A$^+$.

6.1 Regularization of the Wilson Loop Functionals

In this subsection we want to introduce and discuss our regularization.

Definition 6.1 Regularized Yang-Mills Action

Let G be a simple domain in $\mathbb{R}^2$, $|G|$ its area, α_G a boundary loop of G and $[A] \in \mathcal{A}/\mathcal{G}$. Then we set[17]

$$S_G([A]) := \frac{N}{g^2} \frac{1}{|G|} \left(1 - \frac{1}{N} \mathrm{Re\ tr}\ h_{\alpha_G}(A)\right) = \frac{N}{g^2} \frac{1}{|G|} \left(1 - \mathrm{Re}\ T_{\alpha_G}(A)\right).$$

Let now $\{G\}$ be a finite set of mutually disjoint simple domains in $\mathbb{R}^2$, such that int $(\cup \overline{G})$ is again a simple domain. $\{G\}$ denotes not only the set of domains G, but also the supremum $\sup_G \mathrm{diam}\, G$ of their diameters. Finally, R denotes the supremum of the diameters of all circles with center[18] in m, which are completely contained in $\cup \overline{G}$.

We set the *regularized Yang-Mills action* to be $S_{\{G\}}([A]) := \sum_G S_G([A])$ and define $S(A) \equiv S([A]) := \lim_{R\to\infty, \{G\}\to 0} S_{\{G\}}([A])$.

[17] This definition is obviously independent of the choice of the boundary loop and the chosen $A \in [A]$.

[18] The choice m is arbitrary. One can choose any point in $M = \mathbb{R}^2$, but one has to fix that point once for all.

Our definition[19] reduces to that of Thiemann, Ashtekar et al. [Thi95, ALM+96, AL95] if all domains G are quadratic and congruent with area $|G| = a^2$ ($a \ldots$ lattice spacing). One can prove – at least in a naive limit – that $S(A)$ converges pointwise to $S_{YM} := \frac{1}{4}\int_M \mathrm{tr}\; F_{\mu\nu}F^{\mu\nu}dx$ with $F_{\mu\nu} = \partial_{[\mu}A_{\nu]} - \mathrm{ig}[A_\mu, A_\nu]$. Naive means that one expands $A_\mu(x)$ and consequently also $h_\alpha(A) = \mathcal{P}e^{-\mathrm{ig}\int_\alpha A}$ into a power series in x. Further calculations yield a series in $|G|$ and diam G. Comparing coefficients and applying the definition of the Riemann integral gives the limit. Obviously, this proof is only valid for analytical connections, for C^∞ connections the power series need not converge. But, using more sophisticated analysis one can surely get the prove in the general case.

The main advantage of Definition 6.1 is that it can be easily extended from $\mathcal{A}/\mathcal{G}$ to $\overline{\mathcal{A}/\mathcal{G}}$. One has only to replace the "standard" holonomy by the generalized holonomy $h_\alpha(\overline{A})$ via $\overline{A}(T_\alpha) = \frac{1}{N}\mathrm{Re}\;\mathrm{tr}\; h_\alpha(\overline{A})$. Thus, we have under the identical assumptions as in Definition 6.1

Definition 6.2 Generalized Yang-Mills Action
Let $\overline{A} \in \overline{\mathcal{A}/\mathcal{G}}$. We define $S(\overline{A}) := \lim_{R\to\infty,\{G\}\to 0}\sum_G S_G(\overline{A})$ to be the *generalized Yang-Mills action.*

In the rest of this subsection we want to focus on some properties of S on $\overline{\mathcal{A}/\mathcal{G}}$. First we investigate the existence of the limit for an important class of limiting processes – the refinements.

Lemma 6.1 If $(\{G\}_n) \to 0$ is an arbitrary, but fixed limiting process where each $\{G\}_{n+1}$ is a refinement of $\{G\}_n$, then for all $\overline{A} \in \overline{\mathcal{A}/\mathcal{G}}$ the limit $\lim_{\{G\}\to 0}\sum_G S_G(\overline{A}) \in [0, +\infty]$ exists.

Proof
- A short calculation shows that $1 - \frac{1}{N}\mathrm{Re}\;\mathrm{tr}\;\prod_{i=1}^n g_i \leq \sum_{i=1}^n (1 - \frac{1}{N}\mathrm{Re}\;\mathrm{tr}\; g_i)$ for all $n \in \mathbb{N}, g_1, \ldots, g_n \in \mathbf{G}$.
- Next, we prove that $\sum_{G\in\{G\}_n} S_G(\overline{A})$ increases with n.
  Let $\{G_I \mid I \in J\} = \{G\}_n$ and let $\{G\}_{n+1} = \{G_{I,i_I} \mid I \in J, i_I \in J_I\}$ be a refinement of $\{G\}_n$ for all n. Then we have $\sum_{i_I\in J_I} |G_{I,i_I}| = |G_I| \quad \forall I$, especially $|G_{I,i_I}| \leq |G_I| \quad \forall I, i_I$. Thus, we have ($c := \frac{N}{\mathrm{g}^2}$)

$$\begin{aligned}\sum_{G\in\{G\}_n} S_G(\overline{A}) &= c\sum_I \frac{1}{|G_I|}\left(1 - \frac{1}{N}\mathrm{Re}\;\mathrm{tr}\; h_{\alpha_I}(\overline{A})\right)\\ &= c\sum_I \frac{1}{|G_I|}\left(1 - \frac{1}{N}\mathrm{Re}\;\mathrm{tr}\;\prod_{i_I} h_{f_{I,i_I}}(\overline{A})\right)\\ &\leq c\sum_I \frac{1}{|G_I|}\sum_{i_I}\left(1 - \frac{1}{N}\mathrm{Re}\;\mathrm{tr}\; h_{\alpha_{I,i_I}}(\overline{A})\right)\end{aligned}$$

[19]While writing the present paper we found the article "Study of Wilson loop functionals in 2D Yang-Mills theories" of Aroca and Kubyshin [AK98]. They used an analogous regularization, i.e. they also permitted arbitrarily bounded domains instead of the usual quadratic plaquettes. They even considered a more general class of actions $S_{\{G\}}(A) := \sum_G S_1(h_{\alpha_G}(A))$, where G runs over all plaquettes which the lattice on the (compact) two-dimensional manifold is divided into and where S_1 has to fulfill the following axioms
1. $S_1(g) = S_1(g^{-1})$ for all $g \in \mathbf{G}$;
2. $S_1(g)$ has an absolute minimum in $g = e_{\mathbf{G}}$;
3. $\lim_{G\to\{x\}} \frac{1}{|G|}S_1(h_{\alpha_G}(A)) = \frac{1}{2}\mathrm{tr}\; F_{\mu\nu}(x)F^{\mu\nu}(x)$.

$$\leq c\sum_I\sum_{i_I}\frac{1}{|G_{I,i_I}|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ h_{\alpha_{I,i_I}}(\overline{A})\right)$$
$$=\sum_{G\in\{G\}_{n+1}} S_G(\overline{A}).$$

Here we have to explain the product $\prod_{i_I} h_{f_{I,i_I}}$. First the boundary loop α_I is expressed by a product $\alpha_I=\prod_{i_I} f_{I,i_I}$ of flags to the domains G_{I,i_I} due to Lemma 4.12. After pulling back the product on i_I we used tr $h_{f_{I,i_I}}$ = tr $h_{\alpha_{I,i_I}}$ for any boundary loop α_{I,i_I}.

- Since any monotonically increasing sequence in $\mathbb{R}$ has a limit, we get the proof with $S_G(\overline{A})\geq 0$ (which follows from $1-\frac{1}{N}\mathrm{Re\ tr}\ g\geq 0 \quad \forall g\in\mathbf{G}$). **qed**

We emphasize we did only prove that the limit $S(\overline{A})$ exists uniquely for any fixed sequence of refinements. But, nevertheless, the limit still depends on the concrete choice of such a sequence ($\{G\}_n$). To prove this we use generalized connections with one-point – say x – support defined by A-L [AL93].

For this they used that any connection $\overline{A}$ in $\overline{\mathcal{A}/\mathcal{G}}$ can equivalently be described (mod Ad) by a homomorphism $h_{\overline{A}}$ of the hoop group $\mathcal{HG}$ to the structure group $\mathbf{G}$ via $\overline{A}(T_\alpha)=\frac{1}{N}\mathrm{Re\ tr}\ h_{\overline{A}}(\alpha)$ and vice versa. So they could define an $\overline{A}\in\overline{\mathcal{A}/\mathcal{G}}$ in the following way. The corresponding $h_{\overline{A}}$ is equal to $e_{\mathbf{G}}$ if α does not pass x. If α does, then they set

$$h_{\overline{A}}(\alpha) \;:=\; \phi(-v_1^-)^{-1}\phi(v_1^+)\cdots\phi(-v_k^-)^{-1}\phi(v_k^+), \tag{3}$$

where v_i^- and v_i^+ is the direction of the incoming and outcoming tangent of α when it passes x for the ith time. Furthermore, $\phi(v)$ is any function from the space of directions in the tangent space of M in x, i.e. from $S^{\dim T_xM-1}$ to $\mathbf{G}$. Since $h_{\overline{A}}$ is obviously a homomorphism, it determines uniquely an $\overline{A}\in\overline{\mathcal{A}/\mathcal{G}}$.

Let us now consider two sequences $\{G\}_n$ and $\{G'\}_n$ of refinements. The first does not contain a domain G with $x\in\partial G$ for any n, but the second contains exactly two domains $G'_{n,-}$ and $G'_{n,+}$ for each n with $x\in\partial G'_{n,\pm}$. Furthermore, w.l.o.g. we demand that the boundaries $\partial G'_{n,\pm}$ pass the point x for all n in the same directions $\pm v$. We now specialize the connection $\overline{A}$ to be considered. For this we define $\phi(v')$ to be equal $e_{\mathbf{G}}$ if $v'\neq v$ and we subject the values in $v'=v$ only to the condition $\phi(-v)^{-1}\phi(+v)\neq e_{\mathbf{G}}$. The corresponding $h(\alpha)$ is defined by equation (3) and so we get $\overline{A}$.

Obviously, we have $S_{\{G\}_n}(\overline{A})=0$ for all n since no boundary of the considered domains contains x which is the support of $\overline{A}$ and we get $S(\overline{A})=0$ for the first limiting process. On the other hand, in the second sequence we have anytime contributions by the domains $G'_{n,\pm}$, namely $S_{G'_{n,\pm}}(\overline{A})=\frac{N}{\mathrm{g}^2}\frac{1}{|G'_{n,\pm}|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ \phi(\pm v)^{-1}\phi(\mp v)\right)=\frac{C}{|G'_{n,\pm}|}$ with C by construction a non-vanishing constant. Thus we get $S_{\{G'\}_n}(\overline{A})=S_{G'_{n,+}}(\overline{A})+S_{G'_{n,-}}(\overline{A})=\frac{C}{|G'_{n,+}|}+\frac{C}{|G'_{n,-}|}$. But, $\{G'\}\to 0$ implies $\left|G'_{n,\pm}\right|\to 0$ and thus $S(\overline{A})=\lim_{R\to\infty,\{G'\}\to 0} S_{\{G'\}}(\overline{A})=+\infty$. Consequently, the limit $S(\overline{A})$ depends, in general, on the limiting process.

Moreover, if we consider even limiting processes that are not successive refinements there is almost no chance at all to ensure the existence of the limit. For example, one can construct from the just treated sequences a third, new one, taking alternately a term of the first and of the second sequence. Obviously, $S(\overline{A})$ does not have a limit for this limiting process.

However, why should we need the existence or uniqueness of the limit $S(\overline{A})$. Actually, we only have to calculate terms like

$$\int_{\overline{\mathcal{A}/\mathcal{G}}} e^{-\lim_{R\to\infty,\{G\}\to 0}\sum_{\{G\}} S_G(\overline{A})} \widetilde{T}_{\alpha_1}(\overline{A})\cdots\widetilde{T}_{\alpha_n}(\overline{A})\ d\mu_0.$$

In order to use the integration calculus one has to exchange the limit and the integral. A priori we do not know, whether this is – at least mathematically – correct. Astonishingly, we will see that such an exchange makes the limit *of the integrals* independent of the limiting process. By now, we do not really know which effect is responsible for that behaviour and thus we stop the discussion here and will return very briefly to it after the explicit calculation of the Wilson loop expectation values.

6.2 What to calculate?

Given a finite set $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$ of loops. We have to calculate the following expressions

$$\chi(\alpha_1,\ldots,\alpha_n) := \lim_{R\to\infty,\{G\}\to 0} \underbrace{\frac{1}{Z}\int_{\overline{\mathcal{A}/\mathcal{G}}} e^{-\sum_{\{G\}} S_G(\overline{A})} \widetilde{T}_{\alpha_1}(\overline{A})\cdots\widetilde{T}_{\alpha_n}(\overline{A})\ d\mu_0}_{=:\chi_{\{G\}}(\alpha_1,\ldots,\alpha_n)}.$$

Z is chosen here so that we have $\chi(1) = 1$.[20]

Due to the analyticity of the loops the set $\boldsymbol{\alpha}$ generates a finite, non-empty, planar and connected graph $\Gamma_{\boldsymbol{\alpha}}$. We enlarge $\Gamma_{\boldsymbol{\alpha}}$ to an ordinary graph (subsection 3.4) and afterwards to a simple, ordinary graph (Proposition 4.3) denoted by Γ with the interior domains G_I. Furthermore we choose any moderately independent flag world $\mathcal{F} = \{f_I\}$ for Γ existing due to Proposition 4.9.

Now, due to Corollary 4.10 any hoop in Γ can be expressed by a hoop product of flags in $\mathcal{F}$, i.e. by a product of non-overlapping loops: $\alpha_i = \prod_{j=1}^{j_i} f_{I(i,j)}^{\epsilon(i,j)} \quad \forall i = 1,\ldots,n; \epsilon(i,j) = \pm 1$.

Thus, we get

$$\begin{aligned} N^n T_{\alpha_1}\cdots T_{\alpha_n} &= \operatorname{tr} h_{\prod_{j=1}^{j_1} f_{I(1,j)}^{\epsilon(1,j)}} \cdots \operatorname{tr} h_{\prod_{j=1}^{j_n} f_{I(n,j)}^{\epsilon(n,j)}} \\ &= \operatorname{tr} \prod_{j=1}^{j_1} h_{f_{I(1,j)}^{\epsilon(1,j)}} \cdots \operatorname{tr} \prod_{j=1}^{j_n} h_{f_{I(n,j)}^{\epsilon(n,j)}}. \end{aligned} \tag{4}$$

6.3 Calculation of the Expectation Values $\chi(\alpha_1, \ldots, \alpha_n)$ for $\mathbf{G} = SU(N)$

Throughout this subsection we reuse the calculations done by Ashtekar et al. in [ALM+96] and generalize them if necessary.

6.3.1 Tensorial form of the Wilson loop products

The expression (4) can be rewritten in a tensorial form. For this we use that any $g \in \mathbf{G}$ can be described by its matrix elements g_A^B. Thus, $\operatorname{tr} g = g_A^B \delta_B^A$ and $(gh)_A^B = g_A^C h_C^B = g_A^C h_D^B \delta_C^D = (g \otimes h)_{AD}^{CB} \delta_C^D$ are only contractions.

[20] Strictly speaking, Z actually depends on $\{G\}$, but we suppress this here and in the sequel.

We have (for simplicity we write T_α and h_α instead of $T_\alpha(\overline{A})$ and $h_\alpha(\overline{A})$, respectively):

$$N^n T_{\alpha_1} \cdots T_{\alpha_n} = \prod_{I=1}^{\lambda} (\otimes^{k_I^+} h_{f_I})^{\vec{B}_I^+}_{\vec{A}_I^+} (\otimes^{k_I^-} h_{f_I}^{-1})^{\vec{B}_I^-}_{\vec{A}_I^-} \mathcal{C}^{\vec{D}}_{\vec{C}}.$$

Here $\vec{A}_I^\pm, \vec{B}_I^\pm, \vec{C}, \vec{D}$ are certain multi-indices containing the tensor indices which are explicitely given in [ALM+96] or [Fle98]. The tensor $\mathcal{C}^{\vec{D}}_{\vec{C}}$ collects the δ-contractions.

Since $\det g = 1$, we can express $(g^{-1})^B_A$ as follows [ALM+96]:

$$(g^{-1})^B_A = \frac{1}{(N+1)!} \epsilon^{BE_1 \ldots E_{N-1}} \epsilon_{AF_1 \ldots F_{N-1}} g^{F_1}_{E_1} \cdots g^{F_{N-1}}_{E_{N-1}}$$

and we get with λ the number of flags in $\mathcal{F}$

$$N^n T_{\alpha_1} \cdots T_{\alpha_n} = \prod_{I=1}^{\lambda} (\otimes^{n_I} h_{f_I})^{\vec{B}_I}_{\vec{A}_I} \mathcal{C}^{\vec{D}}_{\vec{C}} \mathcal{E}^{\vec{F}}_{\vec{E}}.$$

Here $\mathcal{E}^{\vec{F}}_{\vec{E}}$ is the tensor that is built from the single ϵ-tensors. Furthermore $n_I = k_I^+ + (N-1)k_I^-$ with k_I^+ the number of the occurrence of f_I and k_I^- that of $(f_I)^{-1}$ in the decomposition of the α_i into flags $f_I \in \mathcal{F}$. Thus, we have

$$\begin{aligned}
\chi_{\{G_I\}}(\boldsymbol{\alpha}) &= \frac{1}{Z} \int_{\overline{\mathcal{A}/\mathcal{G}}} e^{-\sum_{\{G_I\}} S_{G_I}(\overline{A})} \widetilde{T}_{\alpha_1}(\overline{A}) \cdots \widetilde{T}_{\alpha_n}(\overline{A}) \; d\mu_0 \\
&= \frac{1}{N^n Z} \int_{\overline{\mathcal{A}/\mathcal{G}}} e^{-\sum_{\{G_I\}} \frac{N}{g^2} \frac{1}{|G_I|} \left(1 - \frac{1}{N} \mathrm{Re\ tr}\ h_{f_I}(\overline{A})\right)} \prod_{I=1}^{\lambda} (\otimes^{n_I} h_{f_I}(\overline{A}))^{\vec{B}_I}_{\vec{A}_I} \mathcal{C}^{\vec{D}}_{\vec{C}} \mathcal{E}^{\vec{F}}_{\vec{E}} \; d\mu_0 \\
&\qquad (\mathrm{tr}\ h_{\alpha_G} = \mathrm{tr}\ h_{f_G} \text{ if } \alpha_G \text{ is boundary loop and } f_G \text{ flag for the domain } G.) \\
&= \frac{1}{N^n Z} \int_{\mathbf{G}^\lambda} e^{-\sum_{\{G_I\}} \frac{N}{g^2} \frac{1}{|G_I|} \left(1 - \frac{1}{N} \mathrm{Re\ tr}\ g_{f_I}\right)} \prod_{I=1}^{\lambda} (\otimes^{n_I} g_{f_I})^{\vec{B}_I}_{\vec{A}_I} \mathcal{C}^{\vec{D}}_{\vec{C}} \mathcal{E}^{\vec{F}}_{\vec{E}} \; d\mu^\lambda_{\mathrm{Haar}} \\
&\qquad (\text{Replace } \overline{\mathcal{A}/\mathcal{G}} \longmapsto \mathbf{G}^\lambda \text{ and } h_{f_I}(\overline{A}) \longmapsto g_{f_I} \text{ as in section 5.}) \\
&= \frac{1}{N^n} \prod_{I=1}^{\lambda} \left(\frac{\int_{\mathbf{G}} e^{-\frac{N}{g^2} \frac{1}{|G_I|} \left(1 - \frac{1}{N} \mathrm{Re\ tr}\ g_{f_I}\right)} (\otimes^{n_I} g_{f_I}) \; d\mu_{\mathrm{Haar}}}{\int_{\mathbf{G}} e^{-\frac{N}{g^2} \frac{1}{|G_I|} \left(1 - \frac{1}{N} \mathrm{Re\ tr}\ g_{f_I}\right)} \; d\mu_{\mathrm{Haar}}} \right)^{\vec{B}_I}_{\vec{A}_I} \mathcal{C}^{\vec{D}}_{\vec{C}} \mathcal{E}^{\vec{F}}_{\vec{E}} \\
&\qquad (\text{Properties of the Haar measure and definition of } Z).
\end{aligned}$$

In the last but one step we used that $\mathcal{F}$ is moderately independent.

6.3.2 Refinement of the graph

Let Γ' be a refinement of Γ. Due to Proposition 4.13 there is a moderately independent and complete refinement $\mathcal{F}' = \{f_{I,i}\}$ of the flag world $\mathcal{F} = \{f_I\}$, such that $f_I = f_{I,1} \cdots f_{I,\lambda_I}$ and so $h_{f_I} = h_{f_{I,1}} \cdots h_{f_{I,\lambda_I}}$. λ'' is equal to λ, i.e. the number of interior domains in $\mathcal{F}$, plus the number of interior domains in Γ' which are in the exterior domain of Γ. λ_I is the number of domains which G_I is refined into, where $\lambda_I = 1$ if $I > \lambda$.

Completely analogously to the preceding paragraph we have for the refined graph

$$\begin{aligned}
\chi_{\{G_{I,i}\}}(\boldsymbol{\alpha}) &= \frac{1}{N^n}\prod_{I=1}^{\lambda}\left(\frac{\int_{\mathbf{G}^{\lambda_I}} e^{-\sum_{i=1}^{\lambda_I}\frac{N}{\mathrm{g}^2}\frac{1}{|G_{I,i}|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ g_{f_I}\right)}(\otimes^{n_I} g_{f_{I,1}}\cdots g_{f_{I,\lambda_I}})\ d\mu_{\mathrm{Haar}}^{\lambda_I}}{\int_{\mathbf{G}^{\lambda_I}} e^{-\sum_{i=1}^{\lambda_I}\frac{N}{\mathrm{g}^2}\frac{1}{|G_{I,i}|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ g_{f_I}\right)}\ d\mu_{\mathrm{Haar}}^{\lambda_I}}\right)_{\vec{A}_I}^{\vec{B}_I}\mathcal{C}_{\vec{C}}^{\vec{D}}\mathcal{E}_{\vec{E}}^{\vec{F}} \\
&\quad \prod_{I=\lambda+1}^{\lambda''}\frac{\int_{\mathbf{G}^{\lambda_I}} e^{-\sum_{i=1}^{\lambda_I}\frac{N}{\mathrm{g}^2}\frac{1}{|G_{I,i}|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ g_{f_I}\right)}\ d\mu_{\mathrm{Haar}}^{\lambda_I}}{\int_{\mathbf{G}^{\lambda_I}} e^{-\sum_{i=1}^{\lambda_I}\frac{N}{\mathrm{g}^2}\frac{1}{|G_{I,i}|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ g_{f_I}\right)}\ d\mu_{\mathrm{Haar}}^{\lambda_I}} \\
&= \frac{1}{N^n}\prod_{I=1}^{\lambda}\left(\prod_{i=1}^{\lambda_I}\frac{\int_{\mathbf{G}} e^{-\frac{N}{\mathrm{g}^2}\frac{1}{|G_{I,i}|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ g_{f_{I,i}}\right)}(\otimes^{n_I} g_{f_{I,i}})\ d\mu_{\mathrm{Haar}}}{\int_{\mathbf{G}} e^{-\frac{N}{\mathrm{g}^2}\frac{1}{|G_{I,i}|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ g_{f_{I,i}}\right)}\ d\mu_{\mathrm{Haar}}}\right)_{\vec{A}_I}^{\vec{B}_I}\mathcal{C}_{\vec{C}}^{\vec{D}}\mathcal{E}_{\vec{E}}^{\vec{F}} \\
&= \frac{1}{N^n}\prod_{I=1}^{\lambda}\left(\prod_{i=1}^{\lambda_I}\frac{\int_{\mathbf{G}}(\otimes^{n_I} g_{f_{I,i}})\ d\mu_{|G_{I,i}|}}{\int_{\mathbf{G}}\ d\mu_{|G_{I,i}|}}\right)_{\vec{A}_I}^{\vec{B}_I}\mathcal{C}_{\vec{C}}^{\vec{D}}\mathcal{E}_{\vec{E}}^{\vec{F}}.
\end{aligned}$$

In the first line we used that the flags $f_{I,i}$ for $I > \lambda$, i.e. for domains of Γ' that are outside Γ, do not occur in the decomposition of the α_i since these α_i only contain the flags f_I of $\mathcal{F}$ and thus only the flags f_{I_i} with $I \leq \lambda$. Thus, also the tensor indices for $I > \lambda$ are trivial. Consequently, the loops outside Γ do not make a contribution to $\chi(\boldsymbol{\alpha})$.

In the last step we set

$$d\mu_{|G|}(g) := e^{-\frac{N}{\mathrm{g}^2}\frac{1}{|G|}\left(1-\frac{1}{N}\mathrm{Re\ tr}\ g\right)}\ d\mu_{\mathrm{Haar}}(g).$$

Obviously, $d\mu_{|G|}$ is a conjugation invariant measure on $\mathbf{G}$.

6.3.3 Calculation of the integrals

We have to calculate terms such as $\int_{\mathbf{G}}(\otimes^n g)\ d\mu_{|G|}$. First we decompose $\otimes^n g$ into irreducible representations using the projectors $p_{(m),j}^{(n)}$ [Fle98, ALM+96] built from the jth Young tableau (m). Since $\mathbf{1} = \oplus_{(m)} \oplus_{j=1}^{f_{(m)}^{(n)}} p_{(m),j}^{(n)}$ we get

$$\begin{aligned}
\int_{\mathbf{G}}(\otimes^n g)\ d\mu_{|G|} &= \left[\sum_{(m)}\sum_{j=1}^{f_{(m)}^{(n)}}\frac{1}{d_{(m)}}\left[\int_{\mathbf{G}}\mathrm{tr}\ (p_{(m),j}^{(n)}\otimes^n g)\ d\mu_{|G|}\right]p_{(m),j}^{(n)}\right]\otimes^n e_{\mathbf{G}} \\
&= \sum_{(m)}\sum_{j=1}^{f_{(m)}^{(n)}}\frac{1}{d_{(m)}}\left[\int_{\mathbf{G}}\chi_{(m)}(g)d\mu_{|G|}\right]p_{(m),j}^{(n)}\otimes^n e_{\mathbf{G}} \\
&= \sum_{(m)}\underbrace{\frac{1}{d_{(m)}}\left[\int_{\mathbf{G}}\chi_{(m)}(g)d\mu_{|G|}\right]}_{=:J_{(m)}(|G|,N)}p_{(m)}^{(n)}\otimes^n e_{\mathbf{G}}
\end{aligned}$$

where we used that the integral of a function ϕ w.r.t. a conjugation invariant measure on $\mathbf{G}$ is already determined by the integral of tr ϕ. Since all representations of one and the same tableau are mutually equivalent, the integral in the first line is independent of j. Finally, we set $d_{(m)}$ to be the dimension of the representation corresponding to the tableau (m) and $\chi_{(m)}(g) := \text{tr}\ (p^{(n)}_{(m),j} \otimes^n g)$ to be the character of (m).

6.3.4 Calculation of the limits $\chi(\boldsymbol{\alpha})$

We have (0 denotes the trivial representation)

$$\begin{aligned}
\chi_{\{G_{I,i}\}}(\boldsymbol{\alpha}) &= \frac{1}{N^n}\prod_{I=1}^{\lambda}\left(\prod_{i=1}^{\lambda_I}\sum_{(m)}\frac{J_{(m)}(|G_{I,i}|\,,N)}{J_0(|G_{I,i}|\,,N)}p^{(n)}_{(m)}\otimes^n 1\right)^{\vec{B}_I}_{\vec{A}_I}\mathcal{C}^{\vec{D}}_{\vec{C}}\mathcal{E}^{\vec{F}}_{\vec{E}} \\
&= \frac{1}{N^n}\prod_{I=1}^{\lambda}\left(\sum_{(m)}\left(\prod_{i=1}^{\lambda_I}\frac{J_{(m)}(|G_{I,i}|\,,N)}{J_0(|G_{I,i}|\,,N)}\right)p^{(n)}_{(m)}\otimes^n 1\right)^{\vec{B}_I}_{\vec{A}_I}\mathcal{C}^{\vec{D}}_{\vec{C}}\mathcal{E}^{\vec{F}}_{\vec{E}} \\
&\qquad (\text{since } p^{(n)}_{(m)}p^{(n)}_{(m')} = \delta_{(m)(m')}p^{(n)}_{(m)}).
\end{aligned}$$

We consider now an arbitrary limiting process $R \to \infty, \{G_{I,i}\} \to 0$, where all terms of the sequence $\{G_{I,i}\}$ are refinements of $\{G_I\}$, i.e., all the corresponding graphs are refinements of the simple graph Γ that corresponds to the graph $\Gamma_{\boldsymbol{\alpha}}$ spanned by $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\}$. We observe first that the limit $R \to \infty$ is trivial since the expression above depends only on the domains in the interior of Γ, but this graph is finite and all graphs in the limiting process are refinements of Γ. Consequently, we have only to deal with the limit $\{G_{I,i}\} \to 0$. Note that from $\{G_{I,i}\} \to 0$ follows $\lambda_I \to \infty \quad \forall I$. Thus, we get using the finiteness of the product over I and the sum over (m)

$$\begin{aligned}
\chi(\boldsymbol{\alpha}) &= \lim_{R\to\infty,\{G_{I,i}\}\to 0}\chi_{\{G_{I,i}\}}(\boldsymbol{\alpha}) \\
&= \lim_{\{G_{I,i}\}\to 0}\frac{1}{N^n}\prod_{I=1}^{\lambda}\left(\sum_{(m)}\left(\prod_{i=1}^{\lambda_I}\frac{J_{(m)}(|G_{I,i}|\,,N)}{J_0(|G_{I,i}|\,,N)}\right)p^{(n)}_{(m)}\otimes^n 1\right)^{\vec{B}_I}_{\vec{A}_I}\mathcal{C}^{\vec{D}}_{\vec{C}}\mathcal{E}^{\vec{F}}_{\vec{E}} \\
&= \frac{1}{N^n}\prod_{I=1}^{\lambda}\left(\sum_{(m)}\lim_{\{G_{I,i}\}\to 0}\left(\prod_{i=1}^{\lambda_I}\frac{J_{(m)}(|G_{I,i}|\,,N)}{J_0(|G_{I,i}|\,,N)}\right)p^{(n)}_{(m)}\otimes^n 1\right)^{\vec{B}_I}_{\vec{A}_I}\mathcal{C}^{\vec{D}}_{\vec{C}}\mathcal{E}^{\vec{F}}_{\vec{E}} \\
&= \frac{1}{N^n}\prod_{I=1}^{\lambda}\left(\sum_{(m)}e^{-\frac{1}{2}g^2c_{(m)}|G_I|}p^{(n)}_{(m)}\otimes^n 1\right)^{\vec{B}_I}_{\vec{A}_I}\mathcal{C}^{\vec{D}}_{\vec{C}}\mathcal{E}^{\vec{F}}_{\vec{E}} \\
&\qquad (\text{see appendix B}).
\end{aligned}$$

$c_{(m)}$ is the eigenvalue of the quadratic Casimir operator of the representation (m).

In conclusion we emphasize that obviously the limit above is completely independent of the limiting process $R \to \infty, \{G_{I,i}\} \to 0$ supposed Γ is a restriction of any graph in the limiting process. Thus, the existence and uniqueness of the limit is proven.

6.4 Calculation of the Expectation Values $\chi(\alpha_1, \ldots, \alpha_n)$ for $\mathbf{G} = U(1)$

The computation of $\chi(\boldsymbol{\alpha})$ is much easier in the case $\mathbf{G} = U(1)$ than for $\mathbf{G} = SU(N)$. The main reason for this is the commutativity of $U(1)$ which induces the commutativity of $\mathcal{HG}$. Furthermore, $U(1)$ is one-dimensional, i.e. we do not need tensor analysis, and the trace is trivial. Since the integration over $\overline{\mathcal{A}/\mathcal{G}}$ is completely analogous to that with $\mathbf{G} = SU(N)$, we can use the same strategy as in the last subsection.

Moreover, we have $T_{\alpha_1} \cdots T_{\alpha_n} = T_{\alpha_1 \cdots \alpha_n}$ for all $\{\alpha_i\} \subseteq \mathcal{HG}$. Therefore it is sufficient to compute the expectation values only for $T_\alpha, \alpha \in \mathcal{HG}$. We get with $N = 1$ from (4)

$$T_{\alpha_1} \cdots T_{\alpha_n} = T_{\alpha_1 \cdots \alpha_n} = \prod_{I=1}^{\lambda} h_{f_I^{n_I}}.$$

Here n_I is the "effective" winding number of $\alpha := \alpha_1 \cdots \alpha_n$ around the domain f_I. More precisely: n_I is equal to the difference between the occurrence of f_I and that of f_I^{-1} in the decomposition of α into the flags f_I. Due to the commutativity n_I is obviously independent of the choice of such a decomposition. In contrast to the $SU(N)$, n_I can become negative.

We have analogously to $SU(N)$

$$\begin{aligned}
\chi_{\{G_I\}}(\alpha) &= \frac{1}{Z} \int_{\overline{\mathcal{A}/\mathcal{G}}} e^{-\sum_{\{G_I\}} S_{G_I}(\overline{A})} \widetilde{T}_\alpha(\overline{A}) \; d\mu_0 \\
&= \frac{1}{Z} \int_{\overline{\mathcal{A}/\mathcal{G}}} e^{-\sum_{\{G_I\}} \frac{1}{g^2} \frac{1}{|G_I|} \left(1 - \mathrm{Re}\; h_{f_I}(\overline{A})\right)} \prod_{I=1}^{\lambda} h_{f_I}(\overline{A})^{n_I} \; d\mu_0 \\
&= \frac{1}{Z} \int_{\mathbf{G}^\lambda} e^{-\sum_{\{G_I\}} \frac{1}{g^2} \frac{1}{|G_I|} \left(1 - \mathrm{Re}\; g_{f_I}\right)} \prod_{I=1}^{\lambda} g_{f_I}^{n_I} \; d\mu_{Haar}^\lambda \\
&= \frac{1}{Z} \prod_{I=1}^{\lambda} \left(\int_{\mathbf{G}} e^{-\frac{1}{g^2} \frac{1}{|G_I|} \left(1 - \mathrm{Re}\; g_{f_I}\right)} g_{f_I}^{n_I} \; d\mu_{Haar} \right).
\end{aligned}$$

Let now Γ' be a refinement of Γ. Then we have due to Proposition 4.13 a refinement $\mathcal{F}' = \{f_{I,i}\}$ of the flag world $\mathcal{F} = \{f_I\}$, such that $f_I = f_{I,1} \cdots f_{I,\lambda_I}$ with λ_I the number of domains G_I is refined into. Using $d\mu_{|G|}(g) := e^{-\frac{1}{g^2} \frac{1}{|G|}(1 - \mathrm{Re}\; g)} \; d\mu_{Haar}(g)$ we get

$$\chi_{\{G_{I,i}\}}(\alpha) = \prod_{I=1}^{\lambda} \left(\prod_{i=1}^{\lambda_I} \frac{\int_{\mathbf{G}} g_{f_{I,i}}^{n_I} \; d\mu_{|G_{I,i}|}}{\int_{\mathbf{G}} \; d\mu_{|G_{I,i}|}} \right).$$

As for $SU(N)$ the domains of Γ' outside Γ make no contributions, such that the product over I runs only to λ and not to λ''.

Any $g \longmapsto g^m$, $m \in \mathbb{Z}$, yields a one-dimensional irreduzible representation of the $U(1)$ with charakter $\chi_m(g) = g^m$. We set $J_m(|G|, 1) := \int_{\mathbf{G}} \chi_m(g) \; d\mu_{|G|}$, where the 1 denotes the $U(1)$, and get

$$\chi_{\{G_{I,i}\}}(\alpha) = \prod_{I=1}^{\lambda} \left(\prod_{i=1}^{\lambda_I} \frac{J_{n_I}(|G_{I,i}|, 1)}{J_0(|G_{I,i}|, 1)} \right).$$

Using appendix B the limit $\{G_{I,i}\} \to 0$ and $R \to \infty$ is calculated as follows:

$$\begin{aligned}
\chi(\alpha) &= \lim_{R\to\infty,\{G_{I,i}\}\to 0} \chi_{R,\{G_{I,i}\}}(\alpha) \\
&= \lim_{\{G_{I,i}\}\to 0} \prod_{I=1}^{\lambda} \left(\prod_{i=1}^{\lambda_I} \frac{J_{n_I}(|G_{I,i}|,1)}{J_0(|G_{I,i}|,1)} \right) \\
&= \prod_{I=1}^{\lambda} \left(\lim_{\{G_{I,i}\}\to 0} \prod_{i=1}^{\lambda_I} \frac{J_{n_I}(|G_{I,i}|,1)}{J_0(|G_{I,i}|,1)} \right) \\
&= \prod_{I=1}^{\lambda} e^{-\frac{1}{2}\mathrm{g}^2 c_{n_I} |G_I|} \\
&= \prod_{I=1}^{\lambda} e^{-\frac{1}{2}\mathrm{g}^2 n_I^2 |G_I|}.
\end{aligned}$$

In the last step we used that the normalized Casimir operator (or, more precisely, its eigenvalue) of the representation $g \longmapsto g^{n_I}$ is equal to $\frac{(\mathrm{i}n_I)^2}{(\mathrm{i}\cdot 1)^2}$, i.e. $c_{n_I} = n_I^2$.

As for $\mathbf{G} = SU(N)$ the limit $\chi(\alpha)$ exists and is unique, i.e. is independent of the limiting process.

7 Discussion

In the last section we "calculated" the expectation values of the Wilson loop products. Actually, the word "calculated" is an exaggeration – de facto we *defined* the values even if the Yang-Mills action on $\mathcal{A}/\mathcal{G}$ influenced the definition of χ. But we did not deduce the values of χ from S_{YM} in a mathematically correct way. Formally we got χ by

$$\chi(\boldsymbol{\alpha}) = \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0 \; e^{-S_{YM}(\overline{A})} T_{\alpha_1} \cdots T_{\alpha_n} = \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0 \; e^{-\lim S_{reg}(\overline{A})} T_{\alpha_1} \cdots T_{\alpha_n},$$

i.e. by extending S_{YM} onto $\overline{\mathcal{A}/\mathcal{G}}$, and subsequently by exchanging the limiting process and the integration

$$\chi(\boldsymbol{\alpha}) := \lim \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu_0 \; e^{-S_{reg}(\overline{A})} T_{\alpha_1} \cdots T_{\alpha_n}.$$

Consequently, this definition is the actual start of our considerations. In principle, that approach is a kind of constructive quantum field theory that needs a physical justification only a posteriori.

In section 6.1 we already discussed that the regularization of S_{YM} by

$$S := \lim_{R\to\infty,\{G\}\to 0} \sum_{\{G\}} \frac{N}{\mathrm{g}^2} \frac{1}{|G|} \left(1 - \frac{1}{N} \mathrm{Re}\ \mathrm{tr}\ h_{\alpha_G} \right)$$

makes no problems on $\mathcal{A}/\mathcal{G}$, but breaks down on $\overline{\mathcal{A}/\mathcal{G}}$, because the limit does not exist in general. Thus, S cannot be in $\overline{\mathcal{HA}}$. But, surprisingly the exchange of limit and integral yields very regular results. We have even shown that the limit $\chi(\boldsymbol{\alpha})$ for our choice of regularization exists for all finite $\boldsymbol{\alpha} \subseteq \mathcal{L}_m$ and is independent of the limiting process. Is there a deeper reason behind that?

However, we know that the given expectation values define a unique Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ [AL93] because we can extend these values to a linear continuous positive functional on $\overline{\mathcal{HA}}$. Note that originally the expectation values are not mutually independent, but subjected to the so-called Mandelstam relations. Since we defined the expectation values using integrals on T_α, these relations are indeed implemented. What properties does μ have? Is μ *strictly* positive or is μ absolutely continuous w.r.t. the induced Haar measure μ_0? Is it even possible to define an action S on $\overline{\mathcal{A}/\mathcal{G}}$ directly, i.e. without regularization, and is it therefore possible to get the desired measure by $d\mu := e^{-S} d\mu_0$?

The choice of regularization is also worth being discussed. In the present case the regularization of S_{YM} depends crucially on the dimension 2. It cannot be extended to three or more dimensions because it uses – roughly speaking – the chance that for 2-dimensional manifolds a loop has both dimension and codimension 1. But the codimension is decisive. To avoid renormalization one has to regularize the d-dimensional Yang-Mills theory by

$$\lim_{\{G\}\to 0} \frac{N}{\mathrm{g}^2} \frac{1}{\mathrm{vol}\, G} \sum_{\{G\}} \left(1 - \frac{1}{N} \mathrm{Re\ tr}\ \mathcal{P} e^{-\mathrm{ig} \int_{\partial G} A}\right),$$

where $\{G\}$ is a decomposition of the base manifold into certain d-dimensional objects. How to connect $\mathcal{P}e^{-\mathrm{ig}\int_{\partial G} A}$ and $\mathcal{P}e^{-\mathrm{ig}\int_{\alpha} A}$? Moreover, the used propositions for planar graphs cannot be applied to higher dimensions. Thus, from dimension 3 on problems of knot theory will be important and so also methods of the topological quantum field theory. Perhaps using algebraic topology or invariant theory one can specify a class of constructible models.

Let us return finally to the concrete generalization of the two-dimensional Yang-Mills theory within the Ashtekar approach. In the last years some papers were published that calculated the expectation values of the Wilson loops in $\mathcal{A}/\mathcal{G}$ (e.g., [KK87, Dri89, GKS89]) and performed the continuum limit. They provided an area law, an indication for the confinement in the theory. All in all these papers delivered the same result as the Ashtekar approach does today. Thus, we get a little justification for our choice of the regularization. Perhaps it is possible to translate further models into the new approach and to confirm that way the results got on $\mathcal{A}/\mathcal{G}$. However, it seems to be unlikely that one gets – at least in the next time – general assertions for the equivalence of the "classical" and the Ashtekar approach. But, from the mathematical point of view this would be very interesting because some problems of the classical approach could be circumvented.

Acknowlegdements

I am very grateful to Gerd Rudolph for his great support while I wrote my diploma thesis and the present paper. I thank the Studienstiftung des deutschen Volkes for its long-time grant and the Max-Planck-Institut für Mathematik in den Naturwissenschaften for its generous promotion.

Appendix

A Proof of Proposition 3.1

Lemma A.1 Let γ be a Jordan path in M contained completely in a chart U_i with trivialization χ_i and p any point in the fibre over $\gamma(0)$. Furthermore, let $\mathbf{G}$ be compact and connected and $\epsilon \in]0, \frac{1}{2}[$ arbitrary.
Then $\tau_{\gamma,\mathcal{A}_{\epsilon,\gamma,i}}(p) = P_{\gamma(1)}$, where $\mathcal{A}_{\epsilon,\gamma,i}$ is defined by $\mathcal{A}_{\epsilon,\gamma,i} := \{A \in \mathcal{A} \mid A_i(\gamma(t)) \equiv 0 \text{ for } t \notin [\epsilon, 1-\epsilon]\}$, i.e. any point of $P_{\gamma(1)}$ can be reached by parallel transport starting in p w.r.t. connections in $\mathcal{A}_{\epsilon,\gamma,i}$.[21]

Proof
- Let $p \equiv p_i := s_i(\gamma(0)) := \chi_i^{-1}(\gamma(0), e_{\mathbf{G}})$. Then $\tau_{\gamma,p}(A) = \mathcal{P}e^{-\int_\gamma A_i(\dot\gamma)\,dt}$, where $\dot\gamma$ is the tangential vector field to γ and A_i is the connection A in the local trivialization χ_i.[22]
- Obviously, there is a 1-Form $a_i : TU_i \longrightarrow \mathbb{C}$ with $a_i(\dot\gamma) \mid_{\gamma([0,\epsilon]\cup[1-\epsilon,1])} \equiv 0$ and $-\int_\gamma a_i(\dot\gamma)\,dt = 1 \neq 0$.
- Set $A_{\lambda,i} := a_i \otimes \lambda$ for any $\lambda \in \mathbf{g}$ and extend $A_{\lambda,i}$ to a connection A_λ on TP.[23] Obviously, $A_\lambda \in \mathcal{A}_{\epsilon,\gamma,i}$ for any $\lambda \in \mathbf{g}$.
- For λ constant, we have $\tau_{\gamma,p}(A_\lambda) = \mathcal{P}e^{-(\int_\gamma a_i(\dot\gamma)\,dt)\lambda} = e^\lambda$.
- Since the image of the Lie algebra $\mathbf{g}$ under the exponential map is the connected component of unity of the Lie group $\mathbf{G}$, we have $\mathbf{G} \supseteq \tau_{\gamma,p}(\mathcal{A}_{\epsilon,\gamma,i}) \supseteq \{\tau_{\gamma,p}(A_\lambda) \mid \lambda \in \mathbf{g}\} = \{e^\lambda \mid \lambda \in \mathbf{g}\} = \mathbf{G}$, i.e. $\mathbf{G} = \tau_{\gamma,p}(\mathcal{A}_{\epsilon,\gamma,i})$ and thus $\tau_{\gamma,\mathcal{A}_{\epsilon,\gamma,i}}(p) = P_{\gamma(1)}$.
- Let now p be arbitrary. Since the parallel transport commutes with the right action, we have $\tau_{\gamma,\mathcal{A}_{\epsilon,\gamma,i}}(p) = (\mathrm{Ad}\psi_g)\tau_{\gamma,\mathcal{A}_{\epsilon,\gamma,i}}(p_i) = (\mathrm{Ad}\psi_g)P_{\gamma(1)} = P_{\gamma(1)}$ because $\mathbf{G}$ acts freely on P. We chose g, such that $p = p_i \cdot g$. **qed**

Proof (Proposition 3.1)
Let $\boldsymbol{\alpha} := \{\alpha_1, \dots, \alpha_n\}$ be a set of moderately independent loops. We have to show that for any n-tupel $(g_1, \dots, g_n) \in \mathbf{G}^n$ there is an $A \in \mathcal{A}$ with $h_{\alpha_i}(A) = g_i \quad \forall 1 \leq i \leq n$.[24]
Fix a covering $\{U_k\}$ of M. Choose a free segment e_i to any $\alpha_i \in \boldsymbol{\alpha}$ due to Definition 3.2, such that
- $\alpha_i = f_i^- e_i f_i^+$ with $f_i^\pm \cap e_i = \emptyset$ and $\alpha_j \cap e_i = \emptyset \quad \forall j < i$ and
- any free segment lies completely in a chart U_i.

Next, choose open neighbourhoods V_i of e_i, such that $V_i \subseteq U_i$ are mutually disjoint and that $\alpha_j \cap V_i = \emptyset$ for any $j < i$, and modify the covering of M in that way, that V_i lies in exactly one chart (denoted again by U_i). Furthermore, choose open $V_{i,\epsilon}$ and compact sets $V_i^c, V_{i,\epsilon}^c$ with some $\epsilon \in]0, \frac{1}{2}[$, $e_i \subseteq V_i^c \subseteq V_i$ and $V_{i,\epsilon}^c \subseteq V_{i,\epsilon} \subseteq V_i^c$, $V_{i,\epsilon} \cap f_i^\pm = \emptyset$ and $\gamma_i(t) \in V_{i,\epsilon}^c \iff t \in [\epsilon, 1-\epsilon]$, where $\gamma_i : [0,1] \longrightarrow e_i$ is a parametrization of e_i. (See fig. 8.)

[21] $\tau_{\gamma,A}(p) : P \longrightarrow P$ is the parallel transport of p along γ with respect to the connection A and $\tau_{\gamma,p}(A)$ the corresponding group element.
[22] We dropped the factor ig.
[23] This is possible, see e.g. [KN63](p. 67).
[24] Note, that we have fixed a trivialization χ and therefore a base point p in P_m from the very beginning.

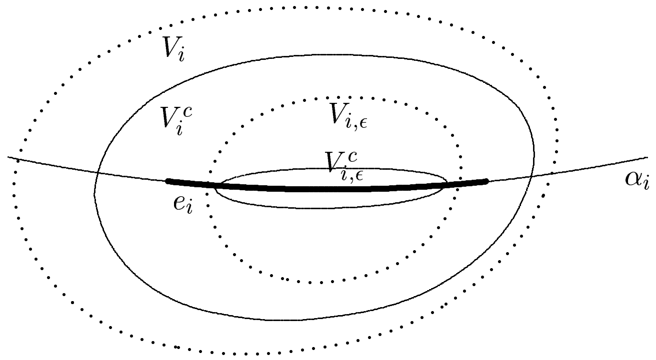


Figure 8: The domains V_i, V_i^c, $V_{i,\epsilon}$ and $V_{i,\epsilon}^c$

It is a well-known fact that there exists a $\phi \in C^\infty(M)$ with $\phi \equiv 1$ on $\cup V_i^c$ and $\phi \equiv 0$ on $M \setminus \cup V_i$ and analogously a $\phi_{i,\epsilon} \in C^\infty(M)$ with $\phi_{i,\epsilon} \equiv 1$ auf $V_{i,\epsilon}^c$ and $\phi_{i,\epsilon} \equiv 0$ on $M \setminus V_{i,\epsilon}$ for all i.
Let $B \in \mathcal{A}$ be some connection.

- $i = 0$.
  $A^{(0)} := B - \phi B$ is again a connection[25]. We have now $A_j^{(i)} \equiv 0$ on e_j for all $j > i = 0$ (and obviously $h_{\alpha_j}(A^{(i)}) = g_j$ for all $j \leq i = 0$).
- $i > 0$.
  Let $p_{i,-} := \tau_{f_i^-, A^{(i-1)}}(p) \in P_{\gamma_i(0)}$ be the parallel transport to $A^{(i-1)}$ of p along f_i^- and $p_{i,+} := \tau^{-1}_{f_i^+, A^{(i-1)}}(p \cdot g_i)$ the "inverse" parallel transport with respect to $A^{(i-1)}$ along f_i^+ leading from $P_{\gamma_i(1)}$ to $p \cdot g_i$. Due to the lemma above there is an $A' \in \mathcal{A}_{\epsilon,\gamma_i,i}$ with $p_{i,+} = \tau_{e_i,A'}(p_{i,-})$ and we have

$$\begin{aligned} p \cdot g_i &= \tau_{f_i^+, A^{(i-1)}}(\tau_{e_i,A'}(\tau_{f_i^-, A^{(i-1)}}(p))) \\ &= \tau_{f_i^+, A^{(i-1)}+\phi_{i,\epsilon}A'}(\tau_{e_i,A'}(\tau_{f_i^-, A^{(i-1)}+\phi_{i,\epsilon}A'}(p))) \quad \text{(due to } \phi_{i,\epsilon} \equiv 0 \text{ on } f_i^\pm) \\ &= \tau_{f_i^+, A^{(i-1)}+\phi_{i,\epsilon}A'}(\tau_{e_i, A^{(i-1)}+\phi_{i,\epsilon}A'}(\tau_{f_i^-, A^{(i-1)}+\phi_{i,\epsilon}A'}(p))) \\ &\qquad \text{(due to } A_i^{(i-1)} |_{e_i} \equiv 0 \text{ and } \phi_{i,\epsilon} |_{\text{supp } A_i' \cap e_i} \equiv 1) \\ &= \tau_{f_i^- e_i f_i^+, A^{(i-1)}+\phi_{i,\epsilon}A'}(p) \\ &= \tau_{\alpha_i, A^{(i)}}(p), \end{aligned}$$

  where we set $A^{(i)} := A^{(i-1)} + \phi_{i,\epsilon}A'$. Obviously, $A^{(i)}$ is a connection, and we get $h_{\alpha_i}(A^{(i)}) = g_i$.
  Since $A^{(i)} = A^{(i-1)}$ outside V_i and $V_i \cap \alpha_j = \emptyset \quad \forall j < i$, we have also $h_{\alpha_j}(A^{(i)}) = h_{\alpha_j}(A^{(i-1)}) = g_j \quad \forall j < i$ by induction. Furthermore, we have $A_j^{(i)} \equiv 0$ on e_j for all $j > i$.

The proof ends setting $A := A^{(n)}$. **qed**

[25]This simple notation means: There is an $A^{(0)}$ for that B, such that $A_i^{(0)} = (1-\phi)B_i$ on $\cup V_i$ and elsewhere $B = A^{(0)}$ since because of the special selection of V_i the compatibility conditions of chart changes are not touched.

B Calculation of $\lim_{\{G\}\to 0}\left(\prod_{i=1}^{\lambda_I}\frac{J_{(m)}(|G_{I,i}|,N)}{J_0(|G_{I,i}|,N)}\right)$

Our calculation is similar to that of T-A$^+$. Let $\mathbf{G}=SU(N)$ for $N>1$, $\mathbf{G}=U(1)$ for $N=1$, ϕ be a representation of $\mathbf{G}$ (and therefore also of the corresponding Lie algebra $\mathbf{g}$) and d_ϕ the dimension of ϕ. We set

$$J_{\phi,N}(\beta):=\int_{\mathbf{G}}d\mu(g)\ \frac{1}{d_\phi}\text{tr }\phi(g)=\int_{\mathbf{G}}d\mu_{\text{Haar}}(g)\ e^{-\left(1-\frac{1}{N}\text{Re tr }g\right)\beta}\frac{1}{d_\phi}\text{tr }\phi(g)\quad\text{with}\quad\beta=\frac{N}{\text{g}^2}\frac{1}{|G|}$$

and analogously

$$J_\phi(|G|)=\int_{\mathbf{G}}d\mu_{\text{Haar}}(g)\ e^{-\left(1-\frac{1}{N}\text{Re tr }g\right)\frac{N}{\text{g}^2}\frac{1}{|G|}}\frac{1}{d_\phi}\text{tr }\phi(g).$$

In the following we write J_ϕ instead of $J_{\phi,N}$. (We note that we use J_ϕ both with the argument β and $|G|$ without anytime explicitely distinguishing between them.) Furthermore, we define

$$J_0(\beta):=\int_{\mathbf{G}}d\mu_{\text{Haar}}(g)\ e^{-\left(1-\frac{1}{N}\text{Re tr }g\right)\beta}\qquad\text{and}\qquad F_\phi(\beta):=\frac{J_\phi}{J_0}$$

and analogously $J_0(|G|)$ and $F_\phi(|G|)$.

Obviously, F_ϕ is a real C^∞ function on $|G|\in(0,\infty)$ and $|F_\phi|\leq 1$. Furthermore, we know that $e^{-(1-\frac{1}{N}\text{Re tr }g)\beta}$ goes exponentially to zero for large β outside a small neighbourhood U_ϵ of $e_{\mathbf{G}}$. Thus, for $\beta\to\infty$ only U_ϵ makes contributions, but in U_ϵ the term $\frac{1}{d_\phi}\text{Re tr }\phi(g)$ is near 1 and thus we have $F_\phi(|G|)\to 1$ for $|G|\to 0$.

The key idea is now to expand g into a Taylor series [ALM$^+$96]. We have $g=e^A$ with $A=t^I\tau_I\in\mathbf{g}$, where τ_I are the generators of $\mathbf{g}$, such that $\text{tr }\tau_I\tau_J=-\tau N\delta_{IJ}$ with some constant $\tau\in\mathbb{R}^+$. Then we have

$$\begin{aligned}1-\frac{1}{N}\text{Re tr }g &= \frac{\tau}{2}\sum_K t^Kt^K+f_1(\vec{t}),\\ \text{and}\qquad 1-\frac{1}{d_\phi}\text{Re tr }\phi(g) &= -\frac{1}{2}t^It^J\frac{1}{d_\phi}\text{Re tr }\phi(\tau_I\tau_J)+f_2(\vec{t}),\end{aligned}$$

where the $f_i(\vec{t})$, $\vec{t}:=(t^1,\ldots,t^{\dim\mathbf{G}})$, collect the contributions of higher order in $\left|\vec{t}\right|$. The linear terms vanish since $SU(N)$ and $\phi(SU(N))$ are trace-free and since $U(1)$ is even in t.

We can reduce now the integration over $\mathbf{G}$ to the integration over $U\subseteq\mathbb{R}^{\dim\mathbf{G}}\cong\times_I(\mathbb{R}\tau_I)$ since there is a U with $0\in U=-U$, $e^U\subseteq\mathbf{G}\subseteq\overline{e^U}$ and injective $e:U\longrightarrow\mathbf{G}$. With the Baker-Campbell-Hausdorff-Theorem one has [ALM$^+$96]

$$\int_{\mathbf{G}}d\mu_{\text{Haar}}(g)\ f(g)=\kappa\int_U dt\left|\frac{\partial r(s,t)}{\partial s}\right|^{-1}f(g(t))$$

with a positive finite $\kappa\in\mathbb{R}$ and $\left|\frac{\partial r(s,t)}{\partial s}\right|^{-1}=1+f_3(\vec{t})$, $f_3(\vec{t})\to 0$ for $\left|\vec{t}\right|\to 0$, where $r(t,s)$ is defined by $e^{r(s,t)}=e^se^t$, $r(s,t)\in\overline{U}$.

Now we have to compute $\lim F_\phi'$ or, equivalently, $\lim(F_\phi-1)'$:

$$(F_\phi(|G|)-1)'=\left(\frac{J_\phi-J_0}{J_0}(|G|)\right)'=-\frac{(J_0-J_\phi)'}{J_0}(|G|)+\frac{J_0'(J_0-J_\phi)}{J_0^2}(|G|).$$

We treat only the first addend

$$\mathcal{J}:=\frac{(J_0-J_\phi)'}{J_0}(|G|)=\frac{\int_{\mathbf{G}}d\mu_{\text{Haar}}e^{-(1-\frac{1}{N}\text{Re tr }g)\beta}\frac{\text{g}^2}{N}\beta^2(1-\frac{1}{N}\text{Re tr }g)(1-\frac{1}{d_\phi}\text{Re tr }\phi(g))}{\int_{\mathbf{G}}d\mu_{\text{Haar}}e^{-(1-\frac{1}{N}\text{Re tr }g)\beta}}.$$

Again one easily proves that one can – up to corrections of the order of ϵ – replace the integration domain $\mathbf{G}$ for sufficiently large β by $U_{T(\epsilon)} := \{e^{\vec{t}} \in \mathbf{G} \mid \max |t^k| < T(\epsilon)\}$, $T(\epsilon) \to 0$ for $\epsilon \to 0$. We have now ($\approx$ means equality up to terms of order ϵ)

$$\begin{aligned}
\mathcal{J} &\approx \frac{\int_{U_{T(\epsilon)}} d\mu_{\text{Haar}} e^{-(1-\frac{1}{N}\text{Re tr } g)\beta} \frac{g^2}{N}\beta^2(1-\frac{1}{N}\text{Re tr } g)(1-\frac{1}{d_\phi}\text{Re tr } \phi(g))}{\int_{U_{T(\epsilon)}} d\mu_{\text{Haar}} e^{-(1-\frac{1}{N}\text{Re tr } g)\beta}} \\
&= \frac{g^2}{N}\frac{\int_{T_\epsilon} dt\ f_3(\vec{t})\, e^{-\frac{\tau}{2}\sum_K t^K t^K(1+\delta_1(\vec{t}))\beta}\beta^2(\frac{\tau}{2}\sum_L t^L t^L)(1+\delta_1(\vec{t}))(-\frac{1}{2}\sum_{I,J} t^I t^J \frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_J))(1+\delta_2(\vec{t}))}{\int_{T_\epsilon} dt\ f_3(\vec{t})\, e^{-\frac{\tau}{2}\sum_K t^K t^K(1+\delta_1(\vec{t}))\beta}} \\
&\approx \frac{g^2}{N}\frac{\int_{T_\epsilon} dt\ e^{-\frac{\tau}{2}\sum_K t^K t^K(1+\delta_1(\vec{t}))\beta}\beta^2(\frac{\tau}{2}\sum_L t^L t^L)(-\frac{1}{2}\sum_{I,J} t^I t^J \frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_J))}{\int_{T_\epsilon} dt\ e^{-\frac{\tau}{2}\sum_K t^K t^K(1+\delta_1(\vec{t}))\beta}} \\
&= \frac{g^2}{N}\frac{1}{\tau}\frac{\int_{\sqrt{\beta\tau}T_\epsilon} dt\ e^{-\frac{1}{2}\sum_K t^K t^K(1+\delta_1(\vec{t}/\sqrt{\beta\tau}))}(\frac{1}{2}\sum_L t^L t^L)(-\frac{1}{2}\sum_{I,J} t^I t^J \frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_J))}{\int_{\sqrt{\beta\tau}T_\epsilon} dt\ e^{-\frac{1}{2}\sum_K t^K t^K(1+\delta_1(\vec{t}/\sqrt{\beta\tau}))}}.
\end{aligned} \tag{5}$$

Here in the first step we performed a coordinate transform $\mathbf{G} \longrightarrow U$, i.e. $U_{T(\epsilon)} \longrightarrow T_\epsilon$; in the second step we used that δ_1, defined by $f_1(\vec{t}) = \delta_1(\vec{t})\ (\frac{\tau}{2}\sum_K t^K t^K)$, and, analogously, δ_2 go to 0 and that $f_3(\vec{t})$ goes to 1 for small $|\vec{t}|$; and in the third step we performed a second coordinate transform $\vec{t} \longrightarrow \vec{t}/\sqrt{\beta\tau}$.

If δ_1 were zero, then the sum on I, J would reduce to a sum on $I = J$ due to the symmetry of the integral under reflections $(t_1, \ldots, t_i, \ldots, t_{\dim \mathbf{G}}) \longmapsto (t_1, \ldots, -t_i, \ldots, t_{\dim \mathbf{G}})$. Indeed, since δ_1 is smaller than $C_1\epsilon$ for all $\vec{t}/\sqrt{\beta\tau} \in T_\epsilon$, those addends are neglectable up to corrections of order ϵ. Now we can throw out the δ_1-terms because the term $-\frac{1}{2}\sum_I t^I t^I \frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_I)$ is bounded below by $C\sum_I t^I t^I$, $C \in \mathbb{R}$. Finally, since $\sqrt{\beta\tau}T_\epsilon$ goes to $\mathbb{R}^{\dim \mathbf{G}}$ for large β (and fixed ϵ), we have

$$\begin{aligned}
\mathcal{J} &\approx \frac{g^2}{N}\frac{1}{\tau}\frac{\int_{\sqrt{\beta\tau}T_\epsilon} dt\ e^{-\frac{1}{2}\sum_K t^K t^K(1+\delta_1(\vec{t}/\sqrt{\beta\tau}))}(\frac{1}{2}\sum_L t^L t^L)(-\frac{1}{2}\sum_I t^I t^I \frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_I))}{\int_{\sqrt{\beta\tau}T_\epsilon} dt\ e^{-\frac{1}{2}\sum_K t^K t^K(1+\delta_1(\vec{t}/\sqrt{\beta\tau}))}} \\
&\approx \frac{g^2}{N}\frac{1}{\tau}\frac{\int_{\mathbb{R}^{\dim \mathbf{G}}} dt\ e^{-\frac{1}{2}\sum_K t^K t^K}(\frac{1}{2}\sum_L t^L t^L)(-\frac{1}{2}\sum_I t^I t^I \frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_I))}{\int_{\mathbb{R}^{\dim \mathbf{G}}} dt\ e^{-\frac{1}{2}\sum_K t^K t^K}} \\
&= -\frac{g^2}{4N}\frac{1}{\tau}\sum_{I,L}\frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_I)\frac{\int_{\mathbb{R}^{\dim \mathbf{G}}}\prod_K(dt^K\ e^{-\frac{1}{2}t^K t^K})t^I t^I t^L t^L}{\int_{\mathbb{R}^{\dim \mathbf{G}}}\prod_K(dt^K\ e^{-\frac{1}{2}t^K t^K})} \\
&= -\frac{g^2}{4N}\frac{1}{\tau}\sum_{I,L}\frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_I)(1+2\delta^{IL})\frac{\int_{\mathbb{R}^{\dim \mathbf{G}}} dt\ e^{-\frac{1}{2}\sum_K t^K t^K}}{\int_{\mathbb{R}^{\dim \mathbf{G}}} dt\ e^{-\frac{1}{2}\sum_K t^K t^K}} \\
&= -\frac{g^2}{4N}\frac{1}{\tau}\sum_I\frac{1}{d_\phi}\text{Re tr } \phi(\tau_I\tau_I)(\dim \mathbf{G}+2).
\end{aligned} \tag{6}$$

In the last but one step we used

$$\int_{\mathbb{R}} dt\ e^{-\frac{1}{2}x^2} = \int_{\mathbb{R}} dt\ e^{-\frac{1}{2}x^2}x^2 = \frac{1}{3}\int_{\mathbb{R}} dt\ e^{-\frac{1}{2}x^2}x^4. \tag{7}$$

Although we did not write down all details of the calculation, we have "shown" that there is a constant $C > 0$, such that for all (sufficiently small) $\epsilon > 0$ there is a β_0 with

$\left|\mathcal{J} - (-\frac{g^2}{4N}\frac{1}{\tau}\sum_I \frac{1}{d_\phi}\mathrm{Re}\ \mathrm{tr}\ \phi(\tau_I\tau_I)(\dim \mathbf{G}+2))\right| < C\epsilon$ for all $\beta > \beta_0$. Therefore we have

$$\lim_{|G|\to 0} -\frac{(J_0 - J_\phi)'}{J_0}(|G|) = \frac{g^2}{4N}\frac{1}{\tau}\sum_I \frac{1}{d_\phi}\mathrm{Re}\ \mathrm{tr}\ \phi(\tau_I\tau_I)(\dim \mathbf{G}+2). \tag{8}$$

Completely analogously we get for the second addend

$$\lim_{|G|\to 0} \frac{J_0'(J_0 - J_\phi)}{J_0^2}(|G|) = \lim_{|G|\to 0} \frac{J_0'}{J_0}\frac{(J_0 - J_\phi)}{J_0}(|G|) = \left(\frac{g^2}{4N}\frac{1}{\tau}\dim \mathbf{G}\right)\left(-\sum_I \frac{1}{d_\phi}\mathrm{Re}\ \mathrm{tr}\ \phi(\tau_I\tau_I)\right). \tag{9}$$

The only difference – out of the sign – between both addends is the missing of $+2$ in the $\dim \mathbf{G}$-term of the second addend that in the first one resulted from the integration of $e^{-\frac{1}{2}\vec{t}\cdot\vec{t}}t^It^It^It^I$ in (6). Such an integrand does not occur while computing (9). There we have both in $\frac{J_0'}{J_0}$ and in $\frac{(J_0-J_\phi)}{J_0}$ only terms like $e^{-\frac{1}{2}\vec{t}\cdot\vec{t}}t^It^I$. Due to (7) they yield only a factor 1 and not 3 unlike the $(t^I)^4$-Terms.

The result can be further simplified since $\sum_I \phi(\tau_I\tau_I)$ is the Casimir invariant, i.e., we have $\sum_I \phi(\tau_I\tau_I) = c_\phi\ \mathbf{1}_{d_\phi}$ with the eigenvalue $c_\phi = \sum_I \frac{1}{d_\phi}\mathrm{Re}\ \mathrm{tr}\ \phi(\tau_I\tau_I)$ depending on the normalization of the τ_I. We choose it so that the eigenvalue c_1 of the Casimir invariant of the fundamental representation is equal to 1. Then, since $1 = c_1 = \frac{1}{\dim \mathbf{G}}\sum_I \mathrm{Re}\ \mathrm{tr}\ \tau_I\tau_I = -\tau N$ (we had set $\mathrm{tr}\ \tau_I\tau_J = -\tau N\delta_{IJ}$), $\tau = -\frac{1}{N}$.

Thus, we have $\lim_{|G|\to 0} F_\phi'(|G|) = -\frac{1}{2}g^2 c_\phi$ and Taylor's theorem gives $F_\phi(|G|) = 1 - \frac{1}{2}g^2 c_\phi\,|G| + \frac{1}{2}F_\phi''(\vartheta(|G|)\,|G|)\,|G|^2$ with $\vartheta(|G|) \in (0,1)$.

The last step to analyze the power series is the proof that the second derivative of F_ϕ is bounded in a neighbourhood on the right of $|G| = 0$. We skip that here because this is very similar to the calculations above. Using the final lemma we have

$$\begin{aligned}\lim_{\{G\}\to 0}\prod_{i=1}^{\lambda_I}\frac{J_{(m)}(|G_{I,i}|, N)}{J_0(|G_{I,i}|, N)} &= \lim_{\{G\}\to 0}\prod_{i=1}^{\lambda_I}\left(1 - \frac{1}{2}g^2 c_{(m)}\,|G_{I,i}| + \frac{1}{2}F_{(m)}''(\vartheta(|G_{I,i}|)\ |G_{I,i}|)\,|G_{I,i}|^2\right)\\ &= e^{-\frac{1}{2}g^2 c_{(m)}|G_I|},\end{aligned}$$

since $\sum_{i=1}^{\lambda_I}|G_{I,i}| = |G_I|$ for all λ_I and all refinements $\{G_{I,i}\}$ of G_I.

Lemma B.1 Let $\sum_{i=1}^n c_{i,n} = c < \infty \quad \forall n$, where all $c_{i,n}$ are non-negative. Then we have $\lim_{n\to\infty}\prod_{i=1}^n(1 - c_{i,n} + f(c_{i,n})c_{i,n}^\alpha) = e^{-c}$ if:

- $\sup_i |c_{i,n}| \to 0$ for $n \to \infty$;
- $f : [0,\infty) \longrightarrow \mathbb{R}$ is bounded in a neighbourhood on the right of 0;
- $\alpha > 1$ is fixed.

The proof is straightforward.

References

[AI92] Abhay Ashtekar and C. J. Isham. Representations of the holonomy algebras of gravity and nonabelian gauge theories. *Class. Quant. Grav.*, 9:1433–1468, 1992. hep-th/9202053.

[AK98] J. M. Aroca and Yu. A. Kubyshin. Study of Wilson loop functionals in 2-D Yang-Mills theories. 1998. hep-th/9802048.

[AL93] Abhay Ashtekar and Jerzy Lewandowski. Representation theory of analytic holonomy C* algebras. 1993. In: *Knots and Quantum Gravity*, John C. Baez (ed.), Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 1994. gr-qc/9311010.

[AL95] Abhay Ashtekar and Jerzy Lewandowski. Projective techniques and functional integration for gauge theories. *J. Math. Phys.*, 36:2170–2191, 1995. gr-qc/9411046.

[ALM+96] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T. Thiemann. Closed formula for wilson loops for $SU(N)$ quantum Yang-Mills theory in two dimensions. 1996. hep-th/9605128. Published slightly enlarged in *J. Math. Phys.* 38:5453-5482,1997 with the title "$SU(N)$ Quantum Yang-Mills-Theory in Two Dimensions: A Complete Solution".

[Dri89] B. K. Driver. YM(2): Continuum expectations, lattice convergence, and lassos. *Commun. Math. Phys.*, 123:575, 1989.

[Fle98] Christian Fleischhack. Funktionalintegralzugang zu Eichtheorien und Gravitation im Rahmen des Ashtekarprogramms (Diplomarbeit), Universität Leipzig. February, 1998.

[Gil81] R. Giles. Reconstruction of gauge potentials from Wilson loops. *Phys. Rev.*, D24:2160–2168, 1981.

[GKS89] L. Gross, C. King, and A. Sengupta. Two-dimensional Yang-Mills theory via stochastic differential equations. *Ann. Phys.*, 194:65, 1989.

[Hal89] Rudolf Halin. *Graphentheorie.* Akademie-Verlag, Berlin, 1989.

[KK87] S. Klimek and W. Kondracki. A construction of two-dimensional quantum chromodynamics. *Commun. Math. Phys.*, 113:389, 1987.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, volume 1. Wiley, New York, 1963.

[Kur70] A. G. Kurosch. *Gruppentheorie 1.* Akademie-Verlag, Berlin, 1970.

[Mas89] William S. Massey. *Algebraic Topology: An Introduction.* Springer, New York, Heidelberg, Berlin, 1989.

[MM95] Donald Marolf and Jose M. Mourao. On the support of the Ashtekar-Lewandowski measure. *Commun. Math. Phys.*, 170:583–606, 1995. hep-th/9403112.

[Ren93] Alan D. Rendall. Comment on a paper of Ashtekar and Isham. *Class. Quant. Grav.*, 10:605–608, 1993.

[Thi95] T. Thiemann. An axiomatic approach to quantum gauge field theory. 1995. hep-th/9511122.