Spatially discrete wave maps on
(1+2)-dimensional space-time

by

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SPATIALLY DISCRETE WAVE MAPS ON (1 + 2)-DIMENSIONAL SPACE-TIME

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Dedicated to Professor Jürgen Moser on the occasion of his 70th birthday

1. Introduction

Let $N$ be a smooth, compact manifold without boundary of dimension $k$. By Nash’s embedding theorem we may assume $N \subset \mathbb{R}^n$ isometrically for some $n$. A wave map $u = (u^1, \ldots, u^n) \colon \mathbb{R} \times \mathbb{R}^2 \to N \hookrightarrow \mathbb{R}^n$ by definition is a stationary point for the action integral

$$A(u; Q) = \int_Q \mathcal{L}(u, dz, Q \subset \subset \mathbb{R} \times \mathbb{R}^2,$$

with Lagrangian

$$\mathcal{L}(u) = \frac{1}{2}(|\nabla u|^2 - |u_t|^2)$$

with respect to compactly supported variations $u_\varepsilon$ satisfying the “target constraint” $u_\varepsilon(\mathbb{R} \times \mathbb{R}^2) \subset N$. Equivalently, a wave map is a solution to the equation

$$\Box u = u_{tt} - \Delta u = A(u)(D_u, D_u) \perp T_u N,$$

where $A$ is the second fundamental form of $N$, $T_p N \subset T_p \mathbb{R}^n$ is the tangent space to $N$ at a point $p \in N$, and “$\perp$” means orthogonal with respect to the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$.

We denote points on Minkowski space as $z = (t, x) = (x^a)_{0 \leq a \leq 2} \in \mathbb{R} \times \mathbb{R}^2$ and let $Du = (u_t, \nabla u) = (\partial_{\alpha} u)_{0 \leq \alpha \leq 2}$ denote the vector of space-time derivatives. Moreover, we raise and lower indices with the Minkowski metric $\eta = (\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \text{diag}(-1, 1, 1)$. A summation convention is used; thus, $\Box u = -\partial_{\alpha} \partial^\alpha u$.

Finally, we abbreviate

$$A(u)(D_u, D_u) = A(u)(\partial^\alpha u, \partial_\alpha u).$$

Recall that locally, near any point $p_0 \in N$, letting $\nu_{k+1}, \ldots, \nu_n$ be a smooth orthonormal frame for the normal bundle $TN^\perp$ near $p_0$, that is, vector fields such that $(\nu_{k}(p))_{k < \ell \leq n}$ is an orthonormal basis for the normal space $T_p N^\perp$ at any $p$ near $p_0$, we have

$$A(p)(v, w) = A^f(p)(v, w) \nu_{\ell}(p)$$

at any such $p$, where

$$A^f(p)(v, w) = \langle v, d\nu(p), w \rangle$$

is the second fundamental form of $N$ with respect to $\nu$. 

Given $u_0 : \mathbb{R}^2 \to N, u_1 : \mathbb{R}^2 \to \mathbb{R}^n$ satisfying the condition $u_1(x) \in T_{u_0(x)}N$ for all $x \in \mathbb{R}^2$, that is, $(u_0, u_1) : \mathbb{R}^2 \to TN$, we consider the Cauchy problem for wave maps $u$ with initial data

$$(u, u_1)|_{t=0} = (u_0, u_1) : \mathbb{R}^2 \to TN$$

of finite energy

$$E_0 = \frac{1}{2} \int_{\mathbb{R}^2} (|u_1|^2 + |\nabla u_0|^2) \, dx.$$ 

Specifically, in the present paper we study the relation between solutions $u$ of (1), (2) on $\mathbb{R} \times \mathbb{R}^2$ and their spatially discrete counterparts $u^h : \mathbb{R} \times M_h \to N \to \mathbb{R}^n$, where $\mathbb{R}^2$ is replaced by a uniform square lattice $M_h = (h\mathbb{Z})^2$ of mesh-size $h \to 0$.

In a previous paper [12], jointly with Vladimir Sverák, we studied the time-independent case and showed that a weakly convergent family of harmonic maps $u^h \in H^1(T_h; N)$ on a periodic lattice $T_h = (h\mathbb{Z})^2/\mathbb{Z}^2$ as $h \to 0$ accumulates at a harmonic map $u$ on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Here we extend this result to the time dependent case; see our main result Theorem 4.1 below. Since the Cauchy problem for wave maps on a spatially discrete domain is equivalent to an initial value problem for a system of ordinary differential equations which can be solved globally for any mesh-size $h$ in view of the uniform energy bounds available, as a corollary we reobtain our existence result from [11] for global weak solutions to the Cauchy problem (1), (2) for wave maps on $(1+2)$-dimensional Minkowski space; see Theorem 5.1. The methods we use are similar to the methods of [12]. We essentially rely on our previous weak compactness results [7], [8] with Freire and exploit the equivalent formulation of (1) as a Hodge system as in [3] or [9] to which compensation techniques may be applied in a way similar to the work of Hélein [9], [10], Evans [5], and Bethuel [1] on weakly harmonic maps, that is, time independent solutions of (1). (See [7] for further references and a detailed comparison of the elliptic and hyperbolic cases.)

2. Technical framework

Whenever possible, we use the same notations as in [12] regarding difference calculus, discrete Hodge theory, interpolation and discretization. For the reader's convenience we recall the definition at each first appearance of a symbol.

2.1. Differential forms. For $h > 0$ with $h^{-1} \in \mathbb{N}$ let $M_h = (h\mathbb{Z})^2, T_h = (h\mathbb{Z})^2/\mathbb{Z}^2$ with generic point $x = x_h = (x^1_h, x^2_h)$, and let $S^1 = \mathbb{R}/\mathbb{Z}$ with generic point $t = t^0 = x^0_h$. Differential forms on $\mathbb{R} \times M_h$ or $S^1 \times T_h$ may be most conveniently expressed in terms of the standard basis $dx^\alpha, dx^\alpha \wedge dx^\beta, 0 \leq \alpha < \beta \leq 2$, and $dt \wedge dx^1 \wedge dx^2 = dz$. In particular, for a 1-form $\varphi^h$ we have $\varphi^h = \varphi^h_\alpha dx^\alpha$, and a 2-form $b^h$ may be written in the standard form

$$b^h = b^h_0 dx^1 \wedge dx^2 - b^h_1 dx^0 \wedge dx^2 + b^h_2 dx^0 \wedge dx^1 = b^h_{\alpha\beta} dx^\alpha \wedge dx^\beta$$

with real-valued functions $\varphi^h_\alpha, b^h_{\alpha\beta}.$
The Hodge $*_g$-operator with respect to either the Euclidean metric $g = eucl$ or the Minkowski metric $g = \eta$ in terms of this basis is defined as

$$
*_{g} 1 = dz, *_{g} dz = 1
$$
$$
*_{g} \varphi^h = g^{00} \varphi_0^h dx^1 \wedge dx^2 - \varphi_1^h dx^0 \wedge dx^2 + \varphi_2^h dx^0 \wedge dx^1
$$
$$
*_{g} b^h = g^{00} b_0^h dx^0 + b_1^h dx^1 + b_2^h dx^2,
$$
where $(g^{\alpha\beta}) = g^{-1} = \text{diag}(\pm 1, 1, \ldots, 1)$ and $\varphi^h = \varphi^h_\alpha dx^\alpha$, etc., as above.

From this definition we immediately deduce that $*_{g} \circ *_{g} = id$ and, moreover,

$$
\varphi^h \wedge *_{g} \varphi^h = (*_{g} \varphi^h) \wedge \varphi^h = g^{\alpha\beta} \varphi^h_\alpha \varphi^h_\beta dx, \\
\varphi^h \wedge *_{g} b^h = (*_{g} \varphi^h) \wedge b^h = g^{\alpha\beta} b_\alpha^h b_\beta^h dx
$$
for any 1-form $\varphi^h$ or 2-form $b^h$ as above.

Finally, two forms $\varphi^h, \psi^h$ of the same degree may be contracted by letting

$$
\varphi^h \wedge \psi^h dz = \varphi^h \wedge *_{g} \psi^h = g^{\alpha\beta} \varphi^h_\alpha \psi^h_\beta dz.
$$

Spatially discrete differential and co-differential are defined as follows.

For $u^h : \mathbb{R} \times M_h \to \mathbb{R}$, $h \neq 0$, we let $d^h u^h = \partial^h_{\alpha} u^h dx^\alpha$ with components

$$
\partial^h_{\alpha} u^h = \partial_{\alpha} u^h = \delta^h_{\alpha} u(z) = \frac{u(z + h \xi_{\alpha}) - u(z)}{h}, \quad \alpha = 1, 2,
$$
where $(\xi_{\alpha})_{1 \leq \alpha \leq 2}$ is the standard basis for $\mathbb{R}^2$. For a 1-form $\varphi^h = \varphi^h_\alpha dx^\alpha$ then

$$
d^h \varphi^h = \partial^h_{\beta} \varphi^h_\alpha dx^\alpha \wedge dx^\beta
$$
and for a 2-form $b^h$ as above,

$$
d^h b^h = \partial^h_{\alpha} b_\alpha^h dz.
$$

The co-differential (with respect to $g$) is

$$
\delta^h_{g} = - *_{g} \circ d^h \circ *_{g}.
$$

Explicitly, for $\varphi^h = \varphi^h_\alpha dx^\alpha$, $h \neq 0$, we have

$$
\delta^h_{g} \varphi^h = - g^{\alpha\beta} \partial^h_{\alpha} \varphi^h_\beta = - \partial^h_{2} \varphi^h_2 - \partial^h_{1} \varphi^h_1 - g^{00} \partial^h_{0} \varphi^h_0,
$$
and similarly for forms of higher degree. Clearly, we have $d^h \circ d^h = 0, \delta^h \circ \delta^h = 0$ for all $h \neq 0$.

Finally, for $h > 0$, we let

$$
\square^h = \Box^h = d^h \delta^h + \delta^h d^h = d^h \delta^h + \delta^h d^h
$$
denote the spatially discrete wave operator, acting on forms on $\mathbb{R} \times M_h$. Explicitly, we have

$$
\square^h u^h = \delta^h_{g} d^h u^h = (\partial^2_{\alpha} - \Delta^h) u^h, \\
\square^h (\varphi^h_\alpha dx^\alpha) = (\square^h \varphi^h_\alpha) dx^\alpha , \\
\square^h (b_\alpha^h dx^\alpha \wedge dx^\beta) = (\square^h b_\alpha^h) dx^\alpha \wedge dx^\beta, \\
\square^h (f^h dz) = (\square^h f^h) dz,
$$
where $\Delta^h = \Delta_{g}^h$ is the discrete (5-point) Laplace operator on $T_h$; that is, $\square^h$ acts as a diagonal operator with respect to the standard basis of forms.
Also note the product rule
\[
\partial^h_\alpha(u^h v^h) = \partial^h_\alpha u^h v^h + \tau^h_\alpha u^h \partial^h_\alpha v^h
\]
\[
= \partial^h_\alpha u^h \tau^h_\alpha v^h + u^h \partial^h_\alpha v^h.
\]
\[
(3)
\]
and
\[
\delta^h_g(\varphi^h f^h) = -g^{\alpha \beta} \partial^h_\alpha (\varphi^h f^h) = -g^{\alpha \beta} \left[ (\partial^h_\alpha \varphi_\beta^h) f^h + \tau^h_\alpha \varphi^h_\beta \partial^h_\alpha f^h \right].
\]
in particular, we have
\[
\delta^h_g(\tau^h_\alpha \varphi^h_\alpha dx^\alpha \cdot f^h) = -g^{\alpha \beta} \left[ (\partial^h_\alpha \varphi^h_\beta) f^h + \varphi^h_\beta \partial^h_\alpha f^h \right] = (\delta^h_g \varphi^h_\beta) f^h - \varphi^h \cdot g d^h f.
\]
Here and in the following we denote
\[
\tau^h_\alpha u^h = m^h_0 u^h = u^h, \tau^h_\alpha u^h = u^h(\pm h \mathbb{E}_\alpha), m^h_\alpha u^h = \frac{1}{2}(u^h + \tau^h_\alpha u^h), \alpha = 1, 2.
\]

2.2. Dirichlet’s integral. For \(u^h: \mathbb{R} \times M_h \to \mathbb{R}\) we let
\[
e_h(u^h) = \frac{1}{4} \sum_{0 \leq \alpha \leq 2} \left\{ |\partial^h_\alpha u^h|^2 + |\partial^h_{-\alpha} u^h|^2 \right\}
\]
be the energy density and let
\[
E_h(u^h(t)) = \int_{M_h} e_h(u^h(t)) = h^2 \sum_{x_h \in M_h} e_h(u^h(t, x_h))
\]
be the energy of \(u^h\) at any time \(t\). If \(h^{-1} \in \mathbb{N}\) and if \(u^h\) has period one in each variable, we regard \(u^h\) as a map \(u^h: S^1 \times T_h \to \mathbb{R}\). Then we define
\[
D_h(u^h) = \int_{S^1 \times T_h} e_h(u^h) = \int_0^1 h^2 \sum_{x_h \in T_h} e_h(u^h(t, x_h)) dt,
\]
and similarly for forms of degree \(\geq 1\).

Note that the first variation of \(D_h\) at \(u^h\) in direction \(v^h\) is given by
\[
\langle dD_h(u^h), v^h \rangle = \frac{d}{d\varepsilon} D_h(u^h + \varepsilon v^h)\bigg|_{\varepsilon = 0}
\]
\[
= \frac{1}{2} \sum_{\alpha} \int_{S^1 \times T_h} \left\{ \partial^h_\alpha u^h \partial^h_\alpha v^h + \partial^h_{-\alpha} u^h \partial^h_{-\alpha} v^h \right\}
\]
\[
= \sum_{\alpha} \int_{S^1 \times T_h} \partial^h_\alpha u^h \partial^h_\alpha v^h = - \int_{S^1 \times T_h} \Delta^h u^h v^h,
\]
where \(\Delta^h = \delta^h_{\alpha \alpha} d^h + d^h \delta^h_{\alpha \alpha} = -\partial^2_x - \Delta^h\) is the spatially discrete Laplace operator, acting on forms on \(S^1 \times T_h\).

Similarly, for \(u^h: \mathbb{R} \times M_h \to \mathbb{R}^n\) the spatially discrete Lagrangian of \(u^h\) is
\[
L_h(u^h) = \frac{1}{4} \sum_{\alpha} \left\{ (\partial^h_\alpha u^h, \partial^h_\alpha v^h) + (\partial^h_{-\alpha} u^h, \partial^h_{-\alpha} v^h) \right\}.
\]
The action integral over any spatially discrete domain \(Q \subset \subset \mathbb{R} \times M_h\) then is
\[
A_h(u^h; Q) = \int_Q L_h(u^h),
\]
and \( u^h \) is stationary for \( A_h \) with respect to compactly supported variations if and only if
\[
\langle dA_h(u^h), v^h \rangle = \frac{d}{dz} A_h(u^h + \varepsilon v^h)|_{\varepsilon=0}
\]
\[
= \int_{\mathbb{R} \times M_h} \eta^{a\beta} \langle \partial^a u^h, \partial^\beta v^h \rangle = \int_{\mathbb{R} \times M_h} \Box^h u^h v^h = 0
\]
for any \( v^h \in C_0^\infty(\mathbb{R} \times M_h) \); that is, if and only if \( \Box^h u^h = 0 \).

2.3. Hodge decomposition. Analogous to the continuous case or the case of a planar lattice, we have the following result on Hodge decomposition of forms on \( S^1 \times T_h \).

**Proposition 2.1.** Any 1-form \( \varphi^h = \varphi^h dx^\alpha \) on \( S^1 \times T_h \) may be decomposed uniquely as
\[
\varphi^h = d^h a^h + \delta^h_{\text{curl}} b^h + c^h
\]
where \( a^h \) and \( b^h \) are normalized to satisfy
\[
\int_{S^1 \times T_h} a^h = \int_{S^1 \times T_h} b^h = 0 \quad \text{for} \quad 0 \leq \alpha < \beta \leq 2, \quad d^h b^h = 0,
\]
and \( d^h c^h = 0, \delta^h_{\text{curl}} c^h = 0 \).

**Proof.** Let \( a^h, b^h \) be the unique solutions to the equations
\[
-\Delta^h a^h = \delta^h_{\text{curl}} \varphi^h, \quad -\Delta^h b^h = d^h \varphi^h,
\]
normalized by (7), obtained, for instance, by minimizing the integral
\[
F_h(a^h) = \int_{S^1 \times T_h} \{ e_h(a^h) - a^h \delta^h_{\text{curl}} \varphi^h \}
\]
among functions \( a^h \colon S^1 \times T_h \to \mathbb{R} \) satisfying (7), and similarly for \( b^h \). The remainder \( c^h = \varphi^h - d^h a^h - \delta^h_{\text{curl}} b^h \) then satisfies
\[
d^h c^h = d^h \varphi^h + \Delta^h b^h = 0, \delta^h_{\text{curl}} c^h = \delta^h_{\text{curl}} \varphi^h + \Delta^h a^h = 0,
\]
as desired. \( \square \)

Via the Euclidean Hodge \( s \)-operator, we obtain an analogous decomposition of 2-forms. Observe that the decomposition (6) is \( L^2 \)-orthogonal and hence we have
\[
\int_{S^1 \times T_h} |\varphi^h|^2 = \int_{S^1 \times T_h} (|d^h a^h|^2 + |\delta^h_{\text{curl}} b^h|^2 + |c^h|^2).
\]

2.4. Discretization and interpolation. We discretize a map \( u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) by letting, for each \( t \in \mathbb{R} \),
\[
u^h(t, x_h) = h^{-2} \int_{Q^+_h(x_h)} u(t, x) \, dx, \quad x_h \in M_h,
\]
where for \( l \in \mathbb{N} \) the set
\[
Q^+_h(x_h) = \{ x = (x^1, x^2) \in \mathbb{R}^2; x^2_h < x^2 < x^2_h + lh, \alpha = 1, 2 \}
\]
is a square with lower left corner \( x^\alpha h \) of size \( lh \), and similarly for periodic maps
\( u : T^0 = S^1 \times T \to \mathbb{R} \), assuming \( h^{-1} \in \mathbb{N} \).
Conversely, we interpolate a map \( u^h : \mathbb{R} \times M_h \to \mathbb{R} \) either trivially, by letting
\[
u^h(t, x) = u^h(t, x_h) \quad \text{for } x \in Q^+_h(x_h), x_h \in M_h,
\]
or bilinearly, by letting
\[
u^b(t, x) = u^h(t, x_h) + \sum_{\alpha=1,2} \xi^\alpha \partial^\alpha u^h(t, x_h) + \xi^1 \xi^2 \partial_t^2 u^h(t, x_h)
\]
whenever \( x = x_h + \xi \in Q^+_h(x_h), x_h \in M_h \), and similarly for maps \( u^b : S^1 \times T_h \to \mathbb{R} \).

Observe that
\[
\partial^\alpha \nu^b(t, x) = \partial^\alpha u^b(t, x_h)
\]
for all \( t \in S^1, x \in Q^+_h(x_h), x_h \in T_h \); moreover,
\[
\partial_t \nu^b(t, x_h + h\xi) = (1 - \xi^2) \partial_t \nu^b(t, x_h) + \xi \partial_t \nu^b(t, x_h + h\xi)
\]
for \( t \in S^1, x_h \in T_h, \xi \in Q^+_1(0) \), and similarly with \( x^1 \) - and \( x^2 \)-directions exchanged.

From this identity the following result is immediate.

**Proposition 2.2.** For \( u^b : \mathbb{R} \times M_h \to \mathbb{R} \) with \( \sup_t E_h(u^b(t)) < \infty \) we have \( \nu^b \in L^\infty(\mathbb{R} H^1(\mathbb{R}^2)) \cap C^0(\mathbb{R} \times \mathbb{R}^2) \), and with a uniform constant \( C \) for all \( t \in \mathbb{R} \) there holds

i) \( \| (\nu^b - u^b)(t) \|_{L^\infty(Q^+_1(x_h))} \leq C \int_{Q_{2h}(x_h)} e_h(u^b(t)) \) for all \( x_h \in M_h \);

ii) \( \| (\nu^b - u^b)(t) \|_{L^2(\mathbb{R}^2)} \leq C h^2 E_h(u^b(t)) \);

iii) \( C^{-1} E_h(u^b(t)) \leq E(\nu^b(t)) \leq C E_h(u^b(t)) \).

Moreover, by comparing \( u^b \) and \( \nu^b \), using Proposition 2.2 i), it is clear that the Poincaré inequality
\[
\| (u^b - \overline{u}^b_{r, x_0}(t)) \|_{L^2(Q_r(x_0))} \leq C r^2 E_h(u^b(t); Q_{r+h}(x_0))
\]
holds for every \( (t, x_0) \in \mathbb{R} \times M_h \), any \( r = nh, k \in \mathbb{N} \), where
\[
Q_r(x_0) = \{ x = (x^1, x^2) ; \| x^\alpha - x^\alpha_0 \| < r, \alpha = 1, 2 \}
\]
and where
\[
\overline{u}^b_{r, x_0}(t) = \frac{1}{P_r(x_0)} \int_{Q_r(x_0)} u^b(t, x)
\]
is the mean value.

Similar results hold true if we also take time dependence into account.

For \( x_0 = (x^\alpha_0)_{0 \leq \alpha \leq 2}, r > 0 \), let
\[
P_r(x_0) = \prod_{\alpha=0}^2 [x^\alpha_0 - r, x^\alpha_0 + r]
\]
and let \( u^b : \mathbb{R} \times M_h \to \mathbb{R} \) with locally finite energy as above. For \( z \in \mathbb{R} \times M_h, r = nh, k \in \mathbb{N} \), we also let
\[
\overline{u}^b_{r, z} = \frac{1}{P_r(z)} \int_{P_r(z)} u^b
\]
denote the average of \( u^b \) on \( P_r(z) \).
**Proposition 2.3.** For any \( z = (t, x) \in \mathbb{R} \times M_h, 0 < h \leq r = kh, k \in \mathbb{N}, \alpha \in \{1, 2\}, \) with an absolute constant \( C \) there holds

\[
[(\tau_h^\alpha u^h - u^h)(z)]^2 \leq Ch^{-1} \int_{P_{2h}(z)} e_h(u^h); \\
\|u^h - u_{r,z}\|_{L^2(P_r(z))}^2 \leq Cr^2 \int_{P_{r+s}(z)} e_h(u^h).
\]

**Proof.**

i) Integrating in time, for any \( s \in [t - h, t + h[ \) we obtain

\[
|[(\tau_h^\alpha u^h - u^h)(t, x)| \leq \int_{t-h}^{t+h} |(\tau_h^\alpha u^h - u^h)(s, x)| + \int_{t-h}^{t+h} \left( |\partial_s(\tau_h^\alpha u^h)| + |\partial_s u^h| \right) \, ds.
\]

Squaring and averaging with respect to \( s \), in view of Proposition 2.2 i) we find

\[
|[(\tau_h^\alpha u^h - u^h)(z)]^2 \leq h^{-1} \int_{t-h}^{t+h} |(\tau_h^\alpha u^h - u^h)(s, x)|^2 \, ds \\
+ Ch \int_{t-h}^{t+h} \left( |\partial_s \tau_h^\alpha u^h|^2 + |\partial_s u^h|^2 \right) \, ds \leq Ch^{-1} \int_{P_{2h}(z)} e_h(u^h).
\]

ii) The asserted inequality is immediate from Proposition 2.2 i) and the usual Poincaré inequality, applied to the function \( \bar{\tau}^\alpha. \)

\[\square\]

If we consider the trivial extensions of a function \( u^h: \mathbb{R} \times M_h \to \mathbb{R} \) and its energy density \( e_h(u^h) \) to \( \mathbb{R} \times \mathbb{R}^2 \), Proposition 2.3 ii) remains valid for all \( z \in \mathbb{R} \times \mathbb{R}^2 \) and \( 0 < h \leq r. \)

Regarding a function \( u^h: S^1 \times T_h \to \mathbb{R} \) as a periodic function on \( \mathbb{R} \times M_h \), the above results also hold for \( u^h: S^1 \times T_h \to \mathbb{R} \). In addition, by integrating in time, from Proposition 2.2 iii) we obtain the following result.

**Proposition 2.4.** For \( u^h: S^1 \times T_h \to \mathbb{R} \) with \( D_h(u^h) < \infty \) we have \( \bar{\tau}^\alpha \in H^1(T^3) \) and with a uniform constant \( C \) there holds

\[
C^{-1} D_h(u^h) \leq D(\bar{\tau}^\alpha) = \frac{1}{2} \int_{T^3} (|u_t^h|^2 + |\nabla u^h|^2) \, dz \leq CD_h(u^h).
\]

In view of Proposition 2.4 we will say that \( u^h \to u \) weakly in \( H^1(T^3) \) as \( h \to 0 \), if \( \bar{\tau}^\alpha \to u \) weakly in \( H^1(T^3) \), or, equivalently, if \( u^h \to u \) and \( d_h u^h \to d u \) weakly in \( L^2(T^3) \), where \( u^h, d_h u^h \) denote the trivial extensions of \( u^h, d u^h \) to \( T^3 \), defined above.

# 3. Spatially Discrete Wave Maps

In analogy with the continuous case a map \( u^h: \mathbb{R} \times M_h \to N \hookrightarrow \mathbb{R}^m \) is a spatially discrete wave map if and only if \( u^h \) is stationary for \( \mathcal{A}_h \) among maps \( u^h: \mathbb{R} \times M_h \to N \) such that \( u^h = u^h \) at \( \varepsilon = 0 \) and outside some compact set \( Q \subset \mathbb{R} \times M_h \); in particular, then

\[
d \frac{d}{dz} \mathcal{A}_h(\pi_N(u^h + \varepsilon v^h)) = 0
\]

for all \( v^h \in C^\infty_0(\mathbb{R} \times M_h; \mathbb{R}^m) \), where \( \pi_N: U_0(N) \to N \) is the smooth map projecting a point \( p \) in a tubular neighborhood of \( N \) of sufficiently small width \( \delta > 0 \) to its nearest neighbor \( \pi_N(p) \in N \).
Computing the first variation using (5), we deduce that \( u^h \) satisfies the equation
\[
\frac{d}{dt} \pi_N(u^h) \Box^h u^h = 0;
\]
that is,
\[
\Box^h u^h \perp T_{u^h} N. \tag{9}
\]

Hence, letting \( \nu_{k+1}, \ldots, \nu_n \) be a local frame for \( TN^\perp \) as above, we have
\[
\Box^h u^h = \lambda' \nu_1 \circ u^h,
\]
where \( \lambda' \) may be computed as
\[
\lambda' = (\Box^h u^h, \nu_1 \circ u^h) = -\eta a^\beta \partial^\beta (\partial^\beta u^h, \nu_1 \circ u^h) + \eta a^\alpha \partial^\alpha u^h, \partial^\beta (\nu_1 \circ u^h)). \tag{10}
\]
Observe that for \( \alpha = 0, \beta = 0 \) the first term vanishes because \( (\partial^\beta u^h, \nu_1 \circ u^h) = 0 \).

In view of this representation of (9), for \( h > 0 \) equation (9) is equivalent to a system of ordinary differential equations of the form
\[
U^h = F(U^h, U^h_t)
\]
for \( U^h(t) = (u^h(t), x_h) \mid_{x_h \in M_h} \), with coupling involving only neighboring lattice sites.

Given \((u^h_0, u^h_t) : M_h \to TN\) with finite energy
\[
E_h(u^h(0)) = \frac{1}{2} \int_{M_h} (|u^h_1|^2 + |d^h u^h_0|^2), \tag{12}
\]
we therefore expect to obtain a unique global solution \( u^h \) of the initial value problem for (9) with initial data
\[
(u^h, u^h_t) |_{t = 0} = (u^h_0, u^h_t). \tag{13}
\]
In fact, we have the following result.

**Theorem 3.1.** For any \( h > 0 \), any \((u^h_0, u^h_t) : M_h \to N\) with \( E_h(u^h(0)) < \infty \) there exists a unique global solution \( u^h \) of the Cauchy problem (9), (13), and \( E_h(u^h(t)) = E_h(u^h(0)) \) for all \( t \).

The proof is achieved by combining the local existence and uniqueness results for systems of ordinary differential equations with the a priori bounds on solutions resulting from the following energy inequality.

### 3.1 Energy Inequality.

For \( u^h : \mathbb{R} \times M_h \to N \) let \( e_k(u^h) \) be the energy density defined in (4), and for \( \alpha = 1, 2 \) let
\[
g_k^\alpha(u^h) = (\partial^\alpha u_k^h, u_k^h)
\]
be the momentum of \( u^h \) in direction \( \alpha \).

For a solution of (9) then we have
\[
0 = \langle \Box^h u^h, u^h_t \rangle = \frac{d}{dt} e_k(u^h) - \frac{1}{2} \sum_{\alpha = 1, 2} (\partial^\alpha g^\alpha_k(u^h) + \partial^\alpha g^\alpha_k(u^h)). \tag{14}
\]
In particular, the total energy is conserved; that is,
\[
E_k(u^h(t)) = \int_{M_h} e_k(u^h(t)) = E_k(u^h(0)) \tag{15}
\]
for all \( t \).

For the proof of Theorem 3.1 and for our later purposes, we also need a local version of this result. Observe that in the discrete case (9) cannot exhibit finite propagation speed. However, as \( h \to 0 \) equation (9) approximates a system of wave equations. Therefore we expect the (essential) domains of influence and dependence of any given point to approach the light cone through that point; in particular, in the limit \( h \to 0 \), on any bounded region of space-time the discrete evolution should essentially be determined by the data on a finite rectangle of the hyperplane \( t = 0 \).

Below we verify this behavior in detail. Because in the discrete case we are working on a quadratic lattice, we prove the local energy inequality on squares, not on circles.

3.2. **Local energy inequality.** For any function \( \varphi \), upon multiplying (14) by the discretized function \( \varphi^h \) we obtain

\[
0 = \frac{d}{dt} (e^h(u^h)\varphi^h) - \frac{1}{2} \sum_{a=1,2} \left[ \partial_a^h (g_{a}^{-h}(u^h)\varphi^h) + \partial_a^{-h} (g_{a}^h(u^h)\varphi^h) \right]
\]

\[
- e^h(u^h)\partial \varphi^h + \frac{1}{2} \sum_{a=1,2} \left[ (g_{a}^{-h}(u^h)\partial_a^{-h} \varphi^h)(\cdot + he_a) + (g_{a}^h(u^h)\partial_a^{-h} \varphi^h)(\cdot - he_a) \right].
\]

Now let \( \psi : \mathbb{R} \to \mathbb{R} \) be given by

\[
\psi(s) = \begin{cases} 
  e^{-h^{-1/3}s}, & s \geq 0 \\
  2 - e^{-h^{-1/3}s}, & s < 0 
\end{cases}
\]

and choose

\[
\varphi(t, x) = \inf_{1 \leq a \leq 2} \psi(\|x^a\| + t) = \psi(\sup_{a} \|x^a\| + t),
\]

satisfying

\[
(\partial \varphi^h + \max_{a} \{ |\partial_a^h \varphi^h|, |\partial_a^{-h} \varphi^h| \})(t, x_h) \leq (\psi'(s) + \max_{a} \{ |\partial_a^h \psi(s)|, |\partial_a^{-h} \psi(s)| \}),
\]

for \( x_h \in M_h \), where \( s = \sup_{a} \|x^a\| + t \).

Integrating in spatial direction and shifting coordinates in the last two terms, we then find that

\[
\frac{d}{dt} \left( \int_{M_h} e^h(u^h)\varphi^h \right)
\]

\[
\leq \int_{M_h} \left( e^h(u^h)\partial \varphi^h + \frac{1}{2} \sum_{a=1,2} \left( |g_{a}^{-h}(u^h)||\partial_a^{-h} \varphi^h| + |g_{a}^h(u^h)||\partial_a^{-h} \varphi^h| \right) \right)
\]

\[
\leq \int_{M_h} e^h(u^h)\partial \varphi^h + \max_{1 \leq a \leq 2} \{ |\partial_a^h \varphi^h|, |\partial_a^{-h} \varphi^h| \}.
\]

Remark that at any point \((t, x_h)\) at most two of the terms \( \partial_a^h \varphi^h \neq 0 \); hence in the Cauchy–Schwarz inequality we may replace the Euclidean norm of \( \partial_a^h \varphi^h \) by the maximum norm.

Let

\[
\rho(s) = \psi'(s) + \max_{a} \{ |\partial_a^h \psi(s)|, |\partial_a^{-h} \psi(s)| \}.
\]

We distinguish the cases \( s \leq -h, s \geq h, -h \leq s \leq 0, \) and \( 0 \leq s \leq h. \)
If \( s \leq -h \), we have
\[
\rho(s) = \left( -h^{-1/3} + \max \left( \frac{e^{b_{2/3}} - 1}{h}, \frac{1}{h} \left( e^{b_{3/3}} - 1 \right) \right) \right) e^{h^{-1/3}s}
\]
\[
= h^{-1/3}e^{h^{-1/3}s} \left( \frac{e^{b_{2/3}} - 1}{h^{2/3}} - 1 \right).
\]
By Taylor’s formula
\[
\frac{e^{h^{2/3}} - 1}{h^{2/3}} - 1 = \frac{1}{2} h^{2/3} + O(h^{4/3}) \leq h^{2/3}
\]
for \( h \leq h_0 \). Hence for such \( h \) and \( s \) we conclude
\[
|\rho(s)| \leq h^{1/3} e^{h^{-1/3}s} \leq h^{1/3} \leq h^{1/3} \psi(s).
\]
Similarly, if \( s \geq h \), for \( h \leq h_0 \) we find
\[
\rho(s) = h^{-1/3}e^{-h^{-1/3}s} \left( \frac{e^{h^{2/3}} - 1}{h^{2/3}} - 1 \right) \leq h^{1/3} e^{-h^{-1/3}s} = h^{1/3} \psi(s).
\]
If \( -h \leq s \leq 0 \) we only need to check that
\[
\psi'(s) + |\partial^h \psi(s)| \leq -h^{-1/3}e^{h^{-1/3}s} + \frac{2 - e^{-h^{-1/3}s} - e^{-h^{-1/3}(s+h)}}{h}
\]
\[
\leq h^{-1/3}e^{h^{-1/3}s} \left( -1 + \frac{2e^{-h^{-1/3}s} - 1 - e^{b_{2/3}} e^{-2h^{-1/3}s}}{h^{2/3}} \right)
\]
\[
\leq C h^{1/3} e^{h^{-1/3}s} \leq C h^{1/3} \psi(s)
\]
with an absolute constant \( C \), if \( h \leq h_0 \). The estimate \( \psi'(s) + |\partial^h \psi(s)| \leq h^{1/3} \psi(s) \) for \( h \leq h_0 \) is obtained as in the case \( s \leq -h \).

Similarly, for \( 0 \leq s \leq h \leq h_0 \) we have
\[
\psi'(s) + |\partial^h \psi(s)| \leq C h^{1/3} \psi(s).
\]
The remaining estimate
\[
\psi'(s) + |\partial^h \psi(s)| \leq h^{1/3} \psi(s), \quad h \leq h_0,
\]
is obtained as in the case \( s \geq h \).

Thus, we conclude that with the above choice of \( \varphi \) for \( h \leq h_0 \) there holds
\[
\partial_t \varphi^h + \max_a \{ |\partial^h \varphi^h|, |\partial^a \varphi^h| \} \leq C h^{1/3} \varphi^h
\]
with an absolute constant \( C \), and hence also
\[
\frac{d}{dt} \int_{M_h} e_h(u^h) \varphi^h \leq C h^{1/3} \int_{M_h} e_h(u^h) \varphi^h.
\]
We may shift the argument of \( \varphi \) by an arbitrary vector \((t_0, x_0)\) and integrate in time to obtain the following result.
Lemma 3.2. There exist constants $h_0 > 0, C$ such that for any $h \leq h_0$, any solution $u^h$ of (9), any $z_0 = (t_0, x_0) \in \mathbb{R} \times M_h$, if $0 \leq t \leq t_0$ there holds

$$
\int_{\{t\} \times M_h} e_h(u^h) |\varphi|^h_{z_0} \leq e^{C h^{1/2} t} \int_{\{0\} \times M_h} e_h(u^h) |\varphi|^h_{z_0},
$$

where $\varphi_{z_0}(t, x) = \varphi(t - t_0, x - x_0)$ is given by (16).

Proof of Theorem 3.1 We first consider initial data $(u_0^h, u_1^h) : M_h \to T N$ having compact support in the sense that $u_0^h \equiv \text{const., } u_1^h \equiv 0$ outside some compact set. Then for sufficiently large $K \in \mathbb{N}$ the support of $d^{\pm h} u_0^h, u_1^h$ is strictly contained in the square of edge-length $2Kh$ centered at $(0,0)$. Extending $u_0^h, u_1^h$ periodically with period $2Kh$ in the $x^1$- and $x^2$-directions, we may regard $u_0^h, u_1^h$ alternatively as maps $(u_0^h, u_1^h) : M_h/(2Kh\mathbb{Z})^2 =: M_{h,K} \to T N$ or as periodic maps on $M_h$.

The Cauchy problem for equation (9) now reduces to an initial value problem for a finite-dimensional system (11) of ordinary differential equations, which in view of the uniform a-priori bound on the energy

$$
E_{h,K}(u^h_K(t)) = \int_{M_{h,K}} e_h(u^h_K(t)) \equiv E_{h,K}(u^h_K(0)) = E_h(u^h(0))
$$

of a solution $u^h_K$, which results from integrating (14) over $M_{h,K}$, can be solved uniquely for all time.

Moreover, regarding $u^h_K : \mathbb{R} \times M_h \to N$ as spatially periodic solutions of (9), in view of these uniform energy bounds a subsequence $u^h_K \to u^h, \partial_t u^h_K \to \partial_t u^h$ locally uniformly on $\mathbb{R} \times M_h$ as $K \to \infty$, where $u^h$ satisfies (9). Combining (17), Lemma 3.2, and (15) we conclude that $E_h(u^h(t)) \equiv \text{const.}$ Indeed, given $t > 0, z_0 = (t_0, x_0)$, by exponential decay of $\varphi$ there are constants $K_0, C_1 = e^{C h^{1/2} t}$ such that for $L \geq K \geq K_0$ there holds

$$
2C_1 \int_{M_{h,K}} e_h(u^h(0)) |\varphi|^h_{z_0}(0) \geq C_1 \int_{M_h} e_h(u^h_0(0)) |\varphi|^h_{z_0}(0) \geq \int_{M_h} e_h(u^h_L(t)) |\varphi|^h_{z_0}(t) \geq \int_{\{x_n \in M_h, \|x_n\| \leq Kh\}} e_h(u^h_L(t)) |\varphi|^h_{z_0}(t).
$$

Fixing $K$ and letting $L \to \infty$, from locally uniform convergence $u^h_L \to u^h, d^h u^h_L \to d^h u^h$ we conclude that

$$
\int_{\{x_n \in M_h, \|x_n\| \leq Kh\}} e_h(u^h(t)) |\varphi|^h_{z_0}(t) \leq 4C_1 E_h(u^h(0)).
$$

Letting $K \to \infty$ and then $t_0 \to \infty$, we deduce that

$$
E_h(u^h(t)) \leq 2C_1 E_h(u^h(0)) < \infty
$$

locally uniformly in time and therefore, in fact, $E_h(u^h(t)) = E_h(u^h(0))$ for all $t$, by (15).

Uniqueness of $u^h$ is obtained as follows. Let $u^h, v^h : \mathbb{R} \times M_h \to N$ be solutions to (9) with $u^h(0, \cdot) = v^h(0, \cdot) = u_0^h$, $u^h(0, \cdot) = v^h(0, \cdot) = u_1^h$ and such that $E_h(u^h(t)) + E_h(v^h(t)) \leq C$, uniformly in $t$. Observe that this also implies that

$$
|u^h(t, x_h)|^2 + |v^h(t, x_h)|^2 \leq Ch^{-2},
$$

uniformly in $\mathbb{R} \times M_h$. 


Expanding (9) and (10), we deduce that \( w^h = u^h - v^h \) satisfies

\[
|\Omega^h w^h| \leq C \sum_{\alpha=1,2} \left( |\partial^\alpha u^h| + |\partial^\alpha v^h| + h^{-1} |\partial^\alpha w^h| + h^{-2} |w^h| \right) + C (|u^h_0| + |v^h_0|) |w^h| + C (\|u^h\|^2 + \|v^h\|^2) |w^h|
\]

\[
\leq C h^{-2} \sum_{\alpha=1,2} |w^h| + |v^h| + C h^{-1} |w^h|.
\]

Multiplying by \( w_t^h \) and integrating over \( M_h \), we obtain

\[
\frac{d}{dt} E_h(w^h(t)) \leq C (1 + h^{-2}) \int_{M_h} (|w^h(t)|^2 + |w_t^h(t)|^2)
\]

\[
\leq C (1 + h^{-2}) \int_{M_h} |w^h(t)|^2 + C (1 + h^{-2}) E_h(w^h(t)).
\]

Moreover, by Hölder’s inequality, for any \( t \geq 0 \), any \( x \in M_h \) we have

\[
|w^h(t,x)|^2 = \left( \int_0^t w_t^h(s,x) \, ds \right)^2 \leq t \int_0^t |w_t^h(s,x)|^2 \, ds.
\]

Hence for \( 0 \leq t \leq T \) we can estimate

\[
\int_{M_h} |w^h(t)|^2 \leq 2 t \int_0^t E_h(w^h(s)) \, ds \leq 2T^2 \sup_{0 \leq s \leq T} E_h(w^h(s)).
\]

Given \( T > 0 \), we fix \( t \in [0,T] \) such that

\[
E_h(w^h(t)) = \sup_{0 \leq s \leq T} E_h(w^h(s)).
\]

We may assume that \( T \leq 1 \). Integrating (18) from 0 to \( t \), it then follows that

\[
E_h(w^h(t)) = \sup_{0 \leq s \leq T} E_h(w^h(s)) \leq CT (1 + h^{-2}) \sup_{0 \leq s \leq T} E_h(w^h(s));
\]

Choosing \( T > 0 \) sufficiently small, we conclude that \( w^h \equiv 0 \) on \([0,T] \times M_h\). By iteration therefore \( w^h \equiv 0 \) on \( \mathbb{R} \times M_h \).

Finally, we may use (18) to remove the assumption that \( d^h u^h_{0,l}, u^h_{1,l} \) have compact support. Indeed, given data \((u^h_{0,l}, u^h_{1,l}) : M_h \rightarrow TN \) of finite energy we may approximate \((u^h_{0,l}, u^h_{1,l})\) by data \((u_{0,l}^h, u_{1,l}^h) : M_h \rightarrow TN, l \in \mathbb{N}\), such that \( d^h u^h_{0,l}, u^h_{1,l} \) have compact support for any \( l \) and such that

\[
\int_{M_h} (|d^h (u_{0,l}^h - u_{0}^h)|^2 + |u_{1,l}^h - u_{1}^h|^2) \rightarrow 0
\]

as \( l \rightarrow \infty \). (The proof of this density result is analogous to the proof that maps \( u \in H^1(\mathbb{R}^2) \) with \( \text{supp} \nabla u \subset \subset \mathbb{R}^2 \) are \( H^1 \)-dense in this space; see for instance [13].) Letting \((u_{0,l}^h)_{l \in \mathbb{N}}\) be the solutions to (9) with data \((u_{0,l}^h, \partial_t u_{1,l}^h)_{l \in \mathbb{N}} = (u_{0,l}^h, u_{1,l}^h)\), from (18), applied to \( w^h = u^h - u_{0,l}^h \) for large \( l,m \in \mathbb{N} \), we obtain convergence of \((u^h_{1,l})\) to the unique solution \( u \) of (9), (13). \( \square \)
4. Passing to the limit $h \to 0$

Our aim in this section is to prove the following weak convergence result.

**Theorem 4.1.** Let $u^h : \mathbb{R} \times M_h \to N \hookrightarrow \mathbb{R}^n, h > 0$, be spatially discrete wave maps such that

$$E_h(u^h(t)) \leq C \text{ uniformly in } h > 0, t \in \mathbb{R}. \quad (20)$$

Then a subsequence $u^h \to u$ locally in $L^2(\mathbb{R}^{1+2}), d^h u^h \to Du$ weakly-a.s. in $L^\infty(\mathbb{R}; L^2(\mathbb{R}^2))$ as $h \to 0$ where $u : \mathbb{R} \times \mathbb{R}^2 \to N \hookrightarrow \mathbb{R}^n$ is a weak solution of (1) with

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^2} |Du(t)|^2 \, dx \leq \limsup_{h \to 0} E_h(u^h(t)) \leq C$$

uniformly in $t \in \mathbb{R}$.

The proof of Theorem 4.1 uses certain compensation properties of Jacobians exhibited by the first order equations equivalent to (1), (9), respectively, as in [7], [8], [12].

To derive these equations we proceed as in [3] or [9]. First suppose that $TN$ is parallelizable and let $\bar{e}_1, \ldots, \bar{e}_k$ be a smooth orthonormal frame. For any $h > 0$ and any $R^h : \mathbb{R} \times M_h \to SO(k)$ then

$$e^h_i = R^h_{ij}(\bar{e}_j \circ u^h), 1 \leq i \leq k,$$

is a frame field for $(u^h)^{-1} TN$.

4.1. First order equations. Let

$$\theta^h_{i,0} = \langle \partial_i u^h, e^h_i \rangle dt, \quad \theta^h_{i,\alpha} = \langle \partial^h_{\alpha} u^h, e^h_i (\cdot + h \omega^h_{\alpha}) \rangle, \alpha = 1, 2;$$

observe that the shift is arranged so that the functions

$$\theta^{-h}_{i,\alpha} = \theta^h_{i,\alpha}(\cdot - h \omega^h_{\alpha}) = \langle \partial^h_{\alpha} u^h, e^h_i \rangle, \alpha = 1, 2,$$

are the coefficients of the representation of $d^{-h} u^h$ in terms of the frame $(e^h_i)$. Also let

$$\omega^h_{ij,0} = \langle \partial_i e^h_j, e^h_i \rangle, \omega^h_{ij,\alpha} = \langle \partial^h_{\alpha} e^h_i, e^h_j, m_{i,j}^h \rangle, \alpha = 1, 2.$$

Clearly, the $\omega^h_{ij}$ are a discrete approximation of the connection 1-forms $\omega_{ij} = \langle de_i, e_j \rangle$ of a frame $(e_i)$ in the continuum limit $h = 0$. The definition is made to insure anti-symmetry $\omega^h_{ij} = -\omega^h_{ji}$ also in the discrete case.

Letting $\partial^h = \partial_t \circ \partial_i = 0, m_{i,j}^h = id$, we have

$$\theta^h_{i,\alpha} = \langle \partial^h_{\alpha} u^h, e^h_i (\cdot + h \omega^h_{\alpha}) \rangle, \theta^{-h}_{i,\alpha} = \langle \partial^h_{\alpha} u^h, e^h_i \rangle, \omega^h_{ij,\alpha} = \langle \partial^h_{\alpha} e^h_i, m_{i,j}^h \rangle,$$

for all $\alpha$. Then

$$\delta^h_{\alpha} \theta^h_{i,\beta} = -\eta^{\alpha\beta} \partial^h_{\alpha} \theta^h_{i,\beta} = -\eta^{\alpha\beta} \partial^h_{\alpha} \theta^{-h}_{i,\beta} = -\eta^{\alpha\beta} \langle \partial^h_{\alpha} u^h, e^h_i \rangle - \eta^{\alpha\beta} \langle \partial^h_{\alpha} u^h, \partial^h_{\alpha} e^h_i \rangle.$$

That is, $u^h : \mathbb{R} \times M_h \to N$ solves (9) if and only if

$$\delta^h_{\alpha} \theta^h_{i,\beta} = -\eta^{\alpha\beta} \langle \partial^h_{\alpha} u^h, \partial^h_{\alpha} e^h_i \rangle = -\eta^{\alpha\beta} \theta^h_{j,\alpha}, \omega^h_{ij,\alpha} = \tau^h_{i,\alpha}, \quad (21)$$

where

$$\tau^h_{i,\alpha} = -\eta^{\alpha\beta} \left[ \delta^h_{\alpha} \left( \frac{e^h_i (\cdot + h \omega^h_{\alpha}) - e^h_i}{2} \right) + \langle \partial^h_{\alpha} u^h, \eta \circ u^h (\cdot + h \omega^h_{\alpha}) \rangle \langle \partial^h_{\alpha} u^h, \partial^h_{\alpha} e^h_i \rangle \right].$$
Observe that there exists a constant $C = C(N)$ such that for $p,q \in N$ there holds
$$\|p - q, v(t(p))\| \leq C p - q^2.$$ It follows that
$$\|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} u_h \| \leq C h^{-1} \|u_h - u^{h_0}\|^2 = C h \|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} u_h\|^2.$$ Moreover, remark that
$$\|q_{ij,a} \tilde{\vartheta}^{h_{ij,a}} q_{ij,a} (e^h_{ij} + h_{ij,a} - e^h_{ij}) \| \leq h \|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} e^h_{ij}\|^2 \leq \|u_h - u^{h_0}\|^2 \|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} e^h_{ij}\|^2.$$ Thus, we may estimate the error term
$$\|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} u_h - u^h\| \leq C \sum_{\alpha = 1, 2} \|u_h - u^{h_0}\| \left( \|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} u_h\|^2 + \sum_j \|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} e^h_{ij}\|^2 \right).$$

Our aim is to pass to the distributional limit in (9) or, equivalently (21) for a suitable sequence $h \rightarrow 0$. As in [7], [8] we may convert this convergence problem into a problem on a compact domain, as follows. Given $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^2)$, let $Q$ be a cube centered at $(0,0)$ containing the support of $\varphi$. Scaling the coordinates suitably, we may assume that $Q = [-\frac{1}{Q}, \frac{1}{Q}]^3$; moreover, we may suppose that $\frac{1}{Q^2} \in \mathbb{N}$. We then extend $u^h$ by even reflection in the faces of $Q$ to periodic functions $v^h$ on $\mathbb{R} \times M_h$ of period 1 in each variable, satisfying (9) on the support of $\varphi$.

Given a frame $(e_i)$ for $(v^h)^{-1} T N$, then also (21) will hold on the support of $\varphi$. Regarding $v^h$ as maps $v^h : S^1 \times T_h \rightarrow N$ on the compact spatially discrete 3-torus; moreover, following Hélène [9], we may choose a frame $(e_i)$ which is in minimal Coulomb gauge, defined as follows.

4.2. Gauge condition. Choose $R^h = (R^h_{ij}) \in H^1(S^1 \times T_h; SO(k))$ such that
$$D_h(R^h(\varphi \circ u^h)) = \frac{1}{4} \int_{S^1 \times T_h} \sum_{\alpha,i} \left( \|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} e^h_{ij}\|^2 + \|\tilde{\vartheta}^{-h_{ij,a}} q_{ij,a} e^h_{ij}\|^2 \right) = \inf_R D_h(R(\varphi \circ u^h)),$$ and let $e^h_{ij} = R^h_{ij}(\varphi \circ u^h)$, $1 \leq i \leq k$. Observe that
$$D_h(e^h_{ij}) \leq C \int_{S^1 \times T_h} e^h_{ij} u^h \leq C D_h(u^h).$$ (22)

Moreover, minimality implies
$$0 = \frac{d}{d \varepsilon} D_h((id + \varepsilon S) e^h)_{\varepsilon = 0}$$
$$= \frac{1}{2} \int_{S^1 \times T_h} \left\{ \|\tilde{\vartheta}^{h_{ij,a}} q_{ij,a} e^h_{ij}\|^2 + \|\tilde{\vartheta}^{-h_{ij,a}} q_{ij,a} e^h_{ij}\|^2 \right\} = \inf_S D_h(S),$$
for all $S_{ij} \in SO(k)$, where we also used anti-symmetry of $S$ and the discrete product rule (3) to derive the second identity.

Since $\omega_{ij,a} = -\omega_{ji,a}$ we conclude
$$\tilde{\vartheta}^{-h_{ij,a}} \omega_{ij,a} = \delta^{h_{ij,a}} \omega_{ij,a} - \delta^{-h_{ij,a}} \omega_{ij,a} = 0.$$
In view of (22) we may assume that, as $h \to 0$ suitably,
\[ e_i^h \to e_i \text{ weakly in } H^1(T^3), \]
\[ \theta_i^h \to \theta_i \text{ weakly in } L^2(T^3), \]
\[ \omega_{ij}^h \to \omega_{ij} \text{ weakly in } L^2(T^3), \]
where $e_i$ is a frame for $u^{-1}TN$ and $\theta_i = \langle du, e_i \rangle, \omega_{ij} = \langle de_i, e_j \rangle$.

Our aim is to show that
\[ \int_Q (\theta_i \cdot \eta d\varphi + \omega_{ij} \cdot \eta \theta_{ij} d\varphi) dz = 0, \]
where $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ with $\text{supp}(\varphi) \subset Q$ is the testing function that we chose above.

In fact, we will show that
\[ \delta_i \theta_i + \omega_{ij} \cdot \eta \theta_j = 0 \text{ in } D'(Q), \quad (23) \]
where we extend $u$ periodically as above and regard $Q$ as part of a fundamental domain for $T^3 = \mathbb{R}^3 / \mathbb{Z}^3$. In view of the equations (21), that is,\[ \delta_i \theta_i^h + \omega_{ij}^h \cdot \eta \theta_j^h = \tau_{ij}^h \text{ in } Q, \]
and distributional convergence $\delta_i^h \theta_i^h \to \delta_i \theta_i$ in $D'(T^3)$, it will suffice to show that
\[ \omega_{ij}^h \cdot \eta \theta_j^h - \tau_{ij}^h \to \omega_{ij} \cdot \eta \theta_i \text{ in } D'(T^3) \quad (24) \]
as $h \to 0$ suitably.

Let
\[ s_i \eta \theta_i^h = d^h \alpha_i^h + \delta_{euc}^h b_i^h + e_i^h \quad (25) \]
be the Hodge decomposition of $s_i \eta \theta_i^h$ on $S^1 \times T_h$ as determined in Proposition 2.1. We may assume that as $h \to 0$ suitably
\[ a_i^h \to a_i, b_i^h \to b_i \text{ weakly in } H^1(T^3), \]
and $c_i^h \to c_i$ smoothly. Observe that the harmonic forms $c_i^h, c_i$ are constant linear combinations of the basis $dx^\alpha \wedge dx^\beta, 0 \leq \alpha < \beta \leq 2$.

Using this decomposition, we may write
\[ \omega_{ij}^h \cdot \eta \theta_j^h dz = \omega_{ij}^h \wedge *\eta \theta_j^h = \omega_{ij}^h \wedge d^h \alpha_i^h + \omega_{ij}^h \wedge \delta_{euc}^h b_i^h + e_i^h. \]

Since $c_i^h \to c_i$ smoothly, passing to the desired limit in the last term is no problem. To show convergence of the second last term, for convenience denote $-\delta_{euc}^h b_j^h = \beta_j^h$. Observe that $\beta_j^h$ is a scalar function and $\beta_j^h \to \beta_j = -s_{euc} b_j$ weakly in $H^1(T^3)$, whence strongly in $L^2(T^3)$ by the Rellich-Kondrakov theorem. Then
\[ \omega_{ij}^h \wedge \delta_{euc}^h b_j^h = \omega_{ij}^h \wedge s_{euc} d^h \beta_j^h = \omega_{ij}^h \wedge \delta_{euc}^h d^h \beta_j^h dz = (s_{euc} \omega_{ij}^h) \wedge d^h \beta_j^h = d^h (s_{euc} \omega_{ij}^h \beta_j^h), \]
as $\delta_{euc}^h \omega_{ij}^h \beta_j^h = 0$ on account of the Coulomb gauge condition. (In coordinates, $\omega_{ij}^h \wedge \delta_{euc}^h d^h \beta_j^h = \omega_{ij}^h \beta_j^h \wedge \delta_{euc}^h \omega_{ij}^h \beta_j^h = \delta_{euc}^h \omega_{ij}^h \beta_j^h - (\delta_{euc}^h \omega_{ij}^h \beta_j^h)$.)
Since \( \omega_{ij}^h \to \omega_{ij} \) weakly in \( L^2 \), while \( \beta_{ij}^h \to \beta_j \) strongly in \( L^2 \), we conclude that
\[
\omega_{ij}^h \wedge \delta_{\text{cusp}} \beta_{ij}^h \to \omega_{ij} \wedge \delta_{\text{cusp}} \beta_j \text{ in } D'.
\]

For the remaining term by the discrete product rule we have
\[
\omega_{ij}^h \wedge a_{ij}^h = \omega_{ij}^h \partial_{\beta} a_{ij}^h \left( dx^\alpha \wedge dx^\beta \wedge dx^\gamma \right)
\]
\[
= \left[ \partial_{\beta} \left( \omega_{ij}^h \left( \cdot - h_{ij} \right) a_{ij}^h \right) - \partial_{\beta}^h \omega_{ij}^h \delta_{\beta} a_{ij}^h \right] dx^\alpha \wedge dx^\beta \wedge dx^\gamma,
\]
\[
= d^{-h} \omega_{ij}^h \wedge a_{ij}^h + \partial_{\beta}^h \left( \omega_{ij}^h \delta_{\beta} a_{ij}^h \right) dx^\alpha \wedge dx^\beta \wedge dx^\gamma.
\]

Since we also have that \( \tau_{ij}^h a_{ij}^h \to a_j \) weakly in \( H^1(T^2) \) and hence strongly in \( L^2 \), as \( h \to 0 \) the last term converges to \( \partial_{\beta} (\omega_{ij} a_{ij} \gamma) dx^\alpha \wedge dx^\beta \wedge dx^\gamma = d(\omega_{ij} a_j) \) in \( D' \).

Thus we have shown distributional convergence
\[
\omega_{ij}^h \eta \partial_{\beta}^h \left( dx^\alpha \wedge dx^\beta \wedge dx^\gamma \right) - d^{-h} \omega_{ij}^h \wedge a_{ij}^h \to \omega_{ij} \eta \partial_{\beta}^h \left( dx^\alpha \wedge dx^\beta \wedge dx^\gamma \right) - d \omega_{ij} \wedge a_j
\]
\[
(26)
\]
as \( h \to 0 \), and it remains to prove that
\[
d^{-h} \omega_{ij}^h \wedge a_{ij}^h - \tau_{ij}^h \to d \omega_{ij} \wedge a_j \text{ in } D'.
\]
(27)
The proof of (27) will be accomplished by adapting the ideas of [8] to the spatially discrete case.

Passing to a further subsequence, if necessary, we may assume that, as \( h \to 0 \),
\[
e_h(u^h) + e_h(D^h) \to \mu \text{ in } M(T^2)
\]
as Radon measures. Theorem 4.1 then will be a consequence of the following Proposition.

**Proposition 4.2.** There exists a Radon measure \( \nu \) such that, as \( h \to 0 \) suitably,
\[
d^{-h} \omega_{ij}^h \wedge a_{ij}^h - \tau_{ij}^h \to d \omega_{ij} \wedge a_j - \nu \text{ in } D'(Q),
\]
where
\[
\supp(\nu) \subset \Sigma = \{ z = (t,x) : \limsup_{R \to 0} \mu(P_R(z)) > 0 \}
\]
has finite 1-dimensional Hausdorff measure.

**Proof of Theorem 4.1** Combining Proposition 4.2 and (26), we conclude that, as \( h \to 0 \),
\[
0 = \delta_0^h \theta_i^h + \omega_{ij}^h \eta \theta_j^h - \tau_{ij}^h \to \delta_0 \theta_i + \omega_{ij} \eta \theta_j - \nu
\]
in \( D'(Q) \). Hence
\[
\nu = \delta_0 \theta_i + \omega_{ij} \eta \theta_j \in H^{-1} + L^1.
\]
But since the support of \( \nu \) is contained in a set of finite 1-dimensional Hausdorff measure, as in [8], Proof of Theorem 1.3, we conclude that, in fact, \( \nu = 0 \) and
\[
\delta_0 \theta_i + \omega_{ij} \eta \theta_j = 0 \text{ in } D'(Q),
\]
as claimed. \( \square \)
4.3. Proof of Proposition 4.2. We proceed as in [8]. The key ingredients in the proof are the duality between the Hardy space $H^1$ and $\text{BMO}$ (due to Fefferman and Stein [6]), the $H^1$ estimates for Jacobians of Coifman, Lions, Meyer and Semmes [4] (see Lemma 4.4 below for the discrete setting), and a characterization of concentration points in the spirit of concentration compactness for sequences of products whose factors are bounded in $H^1$ and $\text{BMO}$, respectively (see [8], Lemma 3.7). To obtain the $\text{BMO}$ estimate (see Lemma 4.3 below) we exploit the energy inequality and apply Campanato theory and Poincaré’s inequality. For elliptic problems similar arguments were used by Hélein [9], [10], Evans [5], Bethuel [1], and others.

Fix a function $\varphi \in C_0^\infty (B_1(0))$ with $\int_{\mathbb{R}^3} \varphi \, dz = 1$. For $f \in L^1(T^3)$ then let

$$(M_\varphi f)(z_0) = \sup_{0 < r < 1} \left| \int_{T^3} r^{-3} \varphi \left( \frac{z - z_0}{r} \right) f(z) \, dz \right|$$

be the regularized maximal function of $f$. The Hardy space on $T^3$ then is the space

$$(H^1(T^3) = \{ f \in L^1(T^3); \int_{T^3} f \, dz = 0, M_\varphi(f) \in L^1(T^3) \}$$

with norm

$$\| f \|_{H^1} = \| M_\varphi(f) \|_{L^1}.$$ 

Also let $\text{BMO}(T^3)$ be the space of functions $f \in L^1(T^3)$ such that

$$[f]_{\text{BMO}(T^3)} = \sup_{0 < r < \frac{1}{2}} \sup_{z_0 \in T^3} \int_{P_r(z_0)} |f - f_{r,z_0}| \, dz < \infty$$

with norm

$$\| f \|_{\text{BMO}(T^3)} = \int_{T^3} f \, dz + [f]_{\text{BMO}(T^3)},$$

where $P_r(z_0)$ and $f_{r,z_0}$ are defined as in Section 2.

By [6], $\text{BMO}(T^3)$ is the dual space of $H^1(T^3)$, and for $g \in H^1(T^3), f \in \text{BMO}(T^3)$ there holds

$$\langle f, g \rangle_{H^1 \times \text{BMO}} \leq C \| f \|_{\text{BMO}(T^3)} \| g \|_{H^1}.$$ 

Moreover, for any $\varphi \in C_c^\infty (T^3), f \in \text{BMO}(T^3)$ the function $f \varphi \in \text{BMO}(T^3)$ and

$$[f \varphi]_{\text{BMO}} \leq C \| f \|_{\text{BMO}} \| \varphi \|_{C^1}$$

see for instance [8], Proposition 3.8. In particular, for any $f \in \text{BMO}(T^3), g \in H^1(T^3)$ the product $T = fg$ is defined as a distribution in $T^3$ by letting

$$\langle T, \psi \rangle_{D' \times D} := \langle f \varphi, g \rangle_{H^1 \times \text{BMO}}$$

for any $\psi \in C_c^\infty (T^3)$.

Finally, for $0 \leq \lambda \leq 3, f \in L^2(T^3)$ let

$$[f]_{L^{2,\lambda}} = \sup_{0 < r \leq \frac{1}{2^\lambda}} \sup_{z_0 \in T^3} \int_{P_r(z_0)} |f - f_{r,z_0}|^2 \, dz$$

and for $0 \leq \lambda < 3$ denote

$$\| f \|_{L^{2,\lambda}} = \sup_{0 < r \leq \frac{1}{2^\lambda}} \sup_{z_0 \in T^3} \int_{P_r(z_0)} |f|^2 \, dz.$$ 

Define the Morrey-Companato spaces

$L^{2,\lambda}(T^3) = \{ f \in L^2(T^3); [f]_{L^{2,\lambda}} < \infty \}, L^{2,\lambda}(T^3) = \{ f \in L^2(T^3); \| f \|_{L^{2,\lambda}} < \infty \}$.
with norms \( \| \cdot \|_{L^{2,1}} \) and \( \| f \|_{C^{2,1}} = \| f \|_{L^2} + \| f \|_{C^{2,1}} \), respectively. Recall that \( L^{2,\lambda} = L^{2,\lambda} \) for \( 0 \leq \lambda < 3 \) and \( L^{2,3} = BMO \) with equivalent norms.

For an open set \( U \subset T^3 \) define the local \( BMO \)-seminorm by letting

\[
[f]_{BMO(U)} = \sup \{ \int_{B_r(z_0)} | f - f_{r,z_0} |^2 \, dz ; B_r(z_0) \subset U \}.
\]

**Lemma 4.3.** For any \( h > 0 \) we have \( a^h_j \in BMO(T^3) \) with \( d^h a^h_j \in L^{2,1}(T^3) \) and

\[
\| a^h_j \|_{BMO} \leq C \| d^h a^h_j \|^{2,1} \leq C E_h(u^h) \leq C
\]

independently of \( h \). Moreover, for any \( 0 < h \leq r < R < \frac{h}{2} \), any \( z_0 \in T^3 \) there holds

\[
[a^h_j]_{BMO(P_r(z_0))} + [d^h a^h_j]_{L^{2,1}(P_r(z_0))} \leq C \left( \frac{r}{R} \| a^h_j \|_{BMO(P_r(z_0))} + \| \theta^{-h} \|_{L^{2,1}(P_r(z_0))} \right).
\]

**Proof.** A global bound for \( a^h_j \) follows from (8). From (25) we obtain the equation

\[
-\Delta^h a^h_j = \delta^h \epsilon_{tu_d} d^h a^h_j = \delta^h \epsilon_{tu_d} \theta^{-h} \theta^h = D^h \theta^h,
\]

where \( D^h \) is a discrete first order differential operator with constant coefficients.

The proof now proceeds as the proof of [8], Lemma 3.11, in the case \( h = 0 \). Omitting the index \( j \) for brevity, given \( 0 < h \leq r < R = K h < \frac{h}{2} \), \( z_0 \in S^1 \times T_h \), we split \( a^h = a^h_1 + a^h_2 \) on \( P_R(z_0) \), where

\[
-\Delta^h a^h_1 = 0 \text{ in } P_R(z_0), a^h_1 = a^h \text{ on } \partial P_R(z_0),
\]

and

\[
-\Delta^h a^h_2 = D^h \theta^{-h} \text{ in } P_R(z_0), a^h_2 = 0 \text{ on } \partial P_R(z_0).
\]

Standard estimates yield that

\[
\| e^h(a^h_1) \|_{L^\infty(P_{R/2}(z_0))} \leq C R^{-2} \int_{P_R(z_0)} | a^h_1 - (a^h_1)_{R,z_0} |^2.
\]

Hence, from Proposition 2.3 ii), for any \( r = k h, z \in S^1 \times T_h \) such that \( P_{r+h}(z) \subset P_{R/2}(z_0) \) we conclude

\[
\int_{P_{r+h}(z)} | a^h_1 - (a^h_1)_{r,z_0} |^2 \leq C r^{-1} \int_{P_{r+h}(z)} e^h(a^h_1) \leq C r^2 \| e^h(a^h_1) \|_{L^\infty(P_{R/2}(z_0))} \leq C \left( \frac{r}{R} \right)^2 \| a^h_1 \|_{BMO(P_{R/2}(z_0))}^2.
\]

Clearly, these estimates remain valid for any \( r > h \) and any \( z \in T^3 \) with \( P_{r+h}(z) \subset P_{R/2}(z_0) \) if we extend \( a^h \) as the spatially piecewise constant function

\[
a^h(t,x) = a^h(t,x_h), \text{ for } x \in Q_h(x_h).
\]

Moreover, for \( 0 < r < h \), if we compare \( a^h_1 \) to its bilinearly interpolated function \( \pi^h, \) for any \( z_1 = (t_1, x_1) \in T^3 \) with \( P_{r}(z_1) \subset P_{2h}(z_h) \subset P_{R/2}(z_0) \) for some \( z_h = (t, x_h) \in S^1 \times T_h \), from Proposition 2.2 ii), iii) and the (standard) Poincaré inequality...
applied to \(a^k_1\) we obtain
\[
\int_{P_r(z_1)} \left| a^k_1 - (a^k_1)_{r,z_1}\right|^2 dz + r^{-1} \int_{P_r(z_1)} \left| d^k a^k_1\right|^2 dz \\
\leq C \int_{t_1-r}^{t_1+r} \left( (a^k_1 - a^k_1) (t) \right)^2_{L^2(\mathbb{R}^n)} dt + C \int_{P_r(z_1)} \left| (a^k_1 - (a^k_1))_{r,z_1}\right|^2 dz + r^{-1} \int_{P_r(z_1)} \left| d^k a^k_1\right|^2 dz \\
\leq C \int_{t_1-r}^{t_1+r} \int_{Q_{2r}(x_n)} \left( e_k(a^k_1(t)) + \left| d^k a^k_1(t) \right|^2 \right) dx dt \\
\leq C \sup_{t_1-r} \left( \left| a^k_1 \right|_{\text{BMO}(P_r(z_1))}\right) \leq C \left( \frac{\langle h \rangle^2}{R} \right) \left | a^k_1 \right|_{\text{BMO}(P_r(z_1))}.
\]

It follows that for \(r \geq h\) there holds
\[
\left| a^k_1 \right|_{\text{BMO}(P_r(z_1))} + \left| d^k a^k_1 \right|_{L^{2:1}(P_r(z_1))} \leq C \left( \frac{R}{h} \right) \left | a^k_1 \right|_{\text{BMO}(P_r(z_1))} + \left | a^k_1 \right|_{\text{BMO}(P_r(z_1))},
\]

The analogous estimate
\[
\left| a^k_2 \right|_{\text{BMO}(P_r(z_1))} \leq C \left| d^k a^k_2 \right|_{L^{2:1}(P_r(z_1))} \leq C \left | \theta^{-h} \right|_{L^{2:1}(P_r(z_1))}
\]
is obtained exactly as in the continuous case from [2], Theorem 16.1., and Poincaré’s inequality.

Observe that the local energy inequality Lemma 3.2 implies that
\[
\limsup_{h \to 0} \left | \theta^{-h} \right|_{L^{2:1}(P_r(z_1))} \leq C R^{-1} \mu(P_R(z_0)).
\] (28)

Indeed, for any \(r < R\), any \(z_1 = (t_1,x_1)\) such that \(P_r(z_1) \subset P_R(z_0)\), if \(3r < R\) by Lemma 3.2 we have
\[
(4r)^{-1} \left | \theta^{-h} \right|_{L^2(P_r(z_1))} \leq \sup_{t_1-r < t < t_1+r} \int_{Q_r(x_1)} e_k(u^h(t)) \leq \int_{Q_{2r}(x_1)} e_k(u^h(t_1 - r)) + o(1) \\
\leq \int_{Q_{3r}(x_1)} e_k(u^h(t_1 - r)) + o(1) \leq R^{-1} \int_{t_1-r}^{t_1+r} \int_{Q_{2r}(x_1)} e_k(u^h(t)) dt + o(1) \\
\leq R^{-1} \int_{P_{3r}(z_0)} e_k(u^h(t)) + o(1) \leq R^{-1} \mu(P_{3r}(z_0)) + o(1)
\]
where \(o(1) \to 0\) as \(h \to 0\).

If \(R/3 \leq r \leq R\), clearly
\[
(3r)^{-1} \left | \theta^{-h} \right|_{L^2(P_r(z_1))} \leq R^{-1} \left | \theta^{-h} \right|_{L^2(P_r(z_1))} \leq R^{-1} \mu(P_{3r}(z_0)) + o(1),
\]
where \(o(1) \to 0\) as \(h \to 0\).

Regarding \(\omega_{ij}^k\), we now introduce the bilinearly interpolated frame to split
\[
\omega_{ij}^k = \langle \Omega_{ij}^1 \rangle^k + \langle \Omega_{ij}^2 \rangle^k - \langle \Omega_{ij}^3 \rangle^k.
\] (29)

**Lemma 4.4.** For any \(h > 0\) there holds \(d^h \langle \Omega_{ij}^1 \rangle^k, \langle \Omega_{ij}^2 \rangle^k, \langle \Omega_{ij}^3 \rangle^k \rangle \in \mathcal{H}^1 (T^3)\) and
\[
d^h \langle \Omega_{ij}^1 \rangle^k \to d(e_i, e_j) = d\omega_{ij}
\]
in \(\mathcal{H}^1 (T^3)\) as \(h \to 0\) suitably.
Proof. In view of the identity $d^b \circ d^b = 0$, we have

$$d^b(d^b e^h_i, \tau^b_j) = \partial^b_\alpha (\partial^b_\beta e^h_i, \tau^b_j) dx^\alpha \wedge dx^\beta$$

$$= \langle \partial^b_\beta e^h_i (\cdot + h_x), \partial^b_\beta e^h_i \rangle dx^\alpha \wedge dx^\beta$$

$$= d^b(e^h_i, h_x)$$

for any $q \in \mathbb{R}^n$. Exactly as in [4], Theorem 2.1, we may therefore show that

$$d^b(d^b e^h_i, \tau^b_j) \in \mathcal{H}^1(T^3)$$

with

$$\|d^b(d^b e^h_i, \tau^b_j)\|_{\mathcal{H}^1} \leq CE_h(e^h) \leq C,$$

where we also used Proposition 2.2 iii). Since the space $VMO(T^3)$, the dual of $\mathcal{H}^1(T^3)$ is separable, we conclude that $(d^b(d^b e^h_i, \tau^b_j))_{h>0}$ is relatively weakly-$s$ sequentially compact. But, as $h \to 0$ suitably,

$$d^b(d^b e^h_i, \tau^b_j) \to \omega_{ij}$$

in the sense of distributions.

By density of $C^\infty(T^3)$ in $VMO(T^3)$, therefore we also have weak-$s$ convergence

$$d^b(d^b e^h_i, \tau^b_j) \rightharpoonup \omega_{ij}$$

in $\mathcal{H}^1(T^3)$, as claimed. \hfill \Box

From Lemma 4.3 and [8], Theorem 3.7, we hence conclude that, as $h \to 0$,

$$d^{-h}(d^{-h} e^h_i, \tau^b_j) \wedge a_j^h \to \omega_{ij} \wedge a_j + v_1 \text{ in } D',$$

(30)

where $v_1$ is a Radon measure with

$$\text{supp}(v_1) \subseteq \{z; \lim_{r \to 0} \limsup_{k \to 0} [a^h](t_{MO}(z_0)) > 0\}.$$ But by Lemma 4.3, for $r \geq h$ we have

$$[a^h](t_{MO}(z_0)) \leq C \left( \frac{r}{R} \|a^h\|_{BMO(z_0)} + \|\theta^{-h}\|_{L^\infty(t_{MO}(z_0))} \right).$$

Fixing $R > 0$, from (28) we conclude that

$$\lim_{r \to 0} \limsup_{h \to 0} [a^h]_{BMO(t_{MO}(z_0))} \leq C \limsup_{h \to 0} \|\theta^{-h}\|_{L^\infty(t_{MO}(z_0))} \leq C(R^{-1} \mu(B_{RM}(z_0))).$$

Since $R > 0$ is arbitrary, therefore $\text{supp}(v_1) \subseteq \Sigma$, as defined in Proposition 4.2.

The contribution to (27) from the second term in (29), after shifting in directions $\alpha$ and $\beta$, is

$$\partial^b_\alpha (\partial^b_\beta e^h_i, m^b_j e^h_j - \tau^b_j) \tau^b_j \partial^b_\beta a_j^h \alpha^h_{ij} dx^\alpha \wedge dx^\beta \wedge dx^\gamma = \left\{ \partial^b_\alpha \left( \partial^b_\beta (m^b_j e^h_j - \tau^b_j) \tau^b_j \partial^b_\beta a_j^h \right) \right\} dx^\alpha \wedge dx^\beta \wedge dx^\gamma = T^h + I^h.$$

Since, as $h \to 0$ suitably, $\tau^b_j a_j^h \to a^h_{ij}$ while $m^b_j e^h_j, \tau^b_j \to e_j$ in $L^p(T^3)$ for any $p < \infty$, and since $(\partial^b_\beta e^h_i)$ is bounded in $L^2(T^3)$, the first term $I^h \to 0$ in $D'(T^3)$. Observing that for any $t \in S^1, x_h \in T_h, \xi \in T, \xi \in Q^h_{0}(0)$, we have

$$(m^b_j e^h_j - \tau^b_j)(t, x) = \frac{1}{2} (m^b_j e^h_j - e_j^h)(t, x_h) - \sum_{\alpha = 1, 2} \xi^\alpha \partial^b_\alpha (e_j^h t, x_h) - \xi^1 \xi^2 \partial^b_1 \partial^b_2 e^h_j (t, x_h),$$
where we shift back, from (29) and (30) we thus obtain that

$$d^{-h} \omega_{ij}^h - a_j^h - d \omega_j \wedge a_j = \tau_{1i}^h + \tau_{2i}^h + \nu_1 + o(1),$$

where $o(1) \to 0$ in $\mathcal{D}'(T^3)$ and where

$$|\tau_{2i}^h| \leq C \sum_{a} |a^h - a^h(\cdot - h\mu_a)| e_h(e^h).$$

**Lemma 4.5.** $\tau_{1i}^h + \tau_{2i}^h \to \nu_2$ in $\mathcal{M}(T^3)$, where $\nu_2$ is a Radon measure with $\text{supp}(\nu_2) \subset \sum$, as defined in Proposition 4.2.

**Proof.** For any $\varphi \in C^0(T^3)$ we can estimate

$$|\int_{T^3} \tau_{1i}^h \varphi \, dz| + |\int_{T^3} \tau_{2i}^h \varphi \, dz| = |\int_{S^1 \times T_h} \tau_{1i}^h \varphi \, d\mu| + |\int_{S^1 \times T_h} \tau_{2i}^h \varphi \, d\mu|$$

$$\leq C \int_{S^1 \times T_h} \sum_{a} (|a^h(\cdot + h\mu_a) - u^h| + |a_j^h(\cdot + h\mu_a) - a_j^h|) (e_h(u^h) + e_h(e^h))|\varphi^h|.$$

Now by Proposition 2.2 i) and Lemma 3.2, for any $z = (t, x_h) \in S^1 \times T_h$, any $0 < h \leq 3h < r < \frac{1}{2}$ we have

$$|u^h(\cdot + h\mu_a) - u^h(t, x_h)|^2 \leq E_h(u^h(t); Q^h_{2h}(x_h)) \leq C_{r^{-1}} \int_{P_{2h}(z)} e_h(u^h) + o(1)$$

where $o(1) \to 0$ as $h \to 0$.

Similarly, for any $z = (t, x_h) \in S^1 \times T_h$, any $0 < h \leq 2h < r < R \leq \frac{1}{2}$, by Proposition 2.3 i) we can estimate

$$|a_j^h(\cdot + h\mu_a) - a_j^h(t, x_h)|^2 \leq \mu_a \int_{P_{2h}(z)} e_h(a_j^h) \leq C |d^h a_j^h|_{L^2(P_{2h}(z))}.$$

Hence by Lemma 4.3 we obtain

$$|a_j^h(\cdot + h\mu_a) - a_j^h(z)| \leq C \left( \frac{R}{R} \right) |a_j^h|_{BMO(P_{2h}(z))} + |\theta_j^h|_{L^2(P_{2h}(z))}.$$

It follows that $\tau_{1i}^h + \tau_{2i}^h \to \nu_2$ in $\mathcal{M}(T^3)$ as $h \to 0$, where $\nu_2$ is absolutely continuous with respect to $\mu$ with density

$$\frac{d\nu_2}{d\mu}(z) = \lim_{r \to 0} \frac{\nu_2(P_r(z))}{\mu(P_r(z))}$$

$$\leq C \lim_{h \to 0} \sup_{z} \left( \frac{R}{R} \right) |a_j^h|_{BMO(P_{2h}(z))} + |d^h a_j^h|_{L^2(P_{2h}(z))} \leq CR^{-1} \mu(P_{2h}(z))$$

for any $z \in T^3$.

Since $R > 0$ is arbitrary, the asserted characterization of $\text{supp}(\nu_2)$ follows. \[\square\]
This completes the proof of Theorem 4.1 if $TN$ is parallelizable. In the general case, by the results of [3] and [9] we may embed $N$ as a totally geodesic submanifold of another manifold $\tilde{N}$ with this property. As above, we now obtain weak convergence of a subsequence $u^h \to u$, where $u: \mathbb{R} \times \mathbb{R}^2 \to N \hookrightarrow \tilde{N}$ is a weak wave map into $\tilde{N}$. But then as in [11], p. 255 f., it follows that $u$ also is a weak wave map into $N$.

5. Global existence of wave maps

Theorems 3.1 and 4.1 easily give rise to the following existence result, previously established in [11] by a different method.

**Theorem 5.1.** For any $(u_0, u_1) \in H^1 \times L^2(\mathbb{R}^2; TN)$ there exists a global weak solution $u$ of the Cauchy problem (1), (2) satisfying the energy inequality

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^2} |Du(t)|^2 \, dx \leq E_0 = \frac{1}{2} \int_{\mathbb{R}^2} (|u_1|^2 + |\nabla u_0|^2) \, dx$$

for all $t$ and which continuously attains the initial data in $H^1 \times L^2$.

**Proof.** Let $u^h_0, u^h_1$ be the maps $u_0, u_1$, discretized as in Section 2.4. Note that

$$\text{dist}^2(u^h_0(x), N) \leq \int_{Q^+_{h}(x)} |u^h_0(x) - u_0(y)|^2 \, dy$$

$$\leq \int_{Q^+_{h}(x)} \int_{Q^+_{h}(x)} |u_0(y) - u_0(y')|^2 \, dy \, dy' \leq C \int_{Q^+_{h}(x)} |\nabla u_0|^2 \, dy \to 0$$

as $h \to 0$. Hence for $0 < h \leq h_0$ the range of $u^h_0$ lies in a sufficiently small tubular neighborhood of $N$ and we may project to obtain spatially discrete data $(\tilde{u}^h_0 = \pi_N \circ u^h_0, \tilde{u}^h_1 = u^h_1): M_h \to TN$ such that

$$\tilde{E}_h : = \frac{1}{2} \int_{M_h} (|\tilde{u}^h_1|^2 + |d^h \tilde{u}^h_0|^2) < \infty$$

and such that

$$(\tilde{u}^h_0, \tilde{u}^h_1) \to (u_0, u_1) \text{ in } H^1 \times L^2$$

as $h \to \infty$. In particular $\tilde{E}_h \to E_0$ as $h \to 0$.

By Theorem 3.1 now, for any $h > 0$ there exists a unique global solution $\tilde{u}^h$ of (9) with data $(\tilde{u}^h_0, \tilde{u}^h_1)_{M>0} = (\tilde{u}^h_0, \tilde{u}^h_1)$, satisfying the energy identity $E_h(\tilde{u}^h(t)) = \tilde{E}_h$ for all $t$.

By Theorem 4.1 a subsequence $(\tilde{u}^h)$ as $h \to 0$ weakly converges to a weak solution $u$ of (1), (2) with

$$E(u(t)) \leq \liminf_{h \to 0} E_h(\tilde{u}^h(t)) = E_0$$

for all $t$. In particular,

$$\limsup_{t \to 0} \frac{1}{2} \int_{\mathbb{R}^2} |Du(t)|^2 \, dx = \limsup_{t \to 0} E(u(t)) \leq E_0$$

and we conclude that $Du(t) \to Du(0)$ strongly in $L^2(\mathbb{R}^2)$ as $t \to 0$, showing that the initial data are attained continuously in $H^1 \times L^2$. \qed
References


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