Universal covering maps
and radial variation

by

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Preprint-Nr.: 36

1998
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June 18\textsuperscript{th}, 1998

\textsuperscript{1}Supported by NSF grant DMS-9423746.

\textsuperscript{2}Supported by the Austrian Academy of Sciences, APART-Programm.
1 Introduction and Statement of Results

We let $E \subseteq \mathbb{C}$ be a closed set with two or more points. By the uniformization theorem there exists a Fuchsian group of Moebius transformations such that $\mathbb{C} \setminus E$ is conformally equivalent to the quotient manifold $\mathbb{D}/G$. The universal covering map $P : \mathbb{D} \to \mathbb{C} \setminus E$ is then given by $P = \tau \circ \pi$, where $\pi$ is the natural quotient map onto $\mathbb{D}/G$ and $\tau$ is the conformal bijection between $\mathbb{C} \setminus E$ and $\mathbb{D}/G$. In this paper we will show that there exists $e^{i\beta} \in \mathbb{T}$ such that

$$\int_0^1 |P^*(re^{i\beta})|dr < \infty.$$  

Considering $u = \log |P'|$, one obtains (1.1) from variational estimates.

**Theorem 1** There exists $e^{i\beta} \in \mathbb{T}$ and $M > 0$ such that for $r < 1$,

$$u(re^{i\beta}) < -\frac{1}{M} \int_0^r |\nabla u(\rho e^{i\beta})|d\rho + M.$$

The class of universal covering maps contains two extremal cases. The case where $\mathbb{C} \setminus E$ is simply connected and the case where $E$ consists of two points. We considered the simply connected case in [J-M] where we proved that Anderson’s conjecture is true. The second case is easier; well known estimates for the Poincaré metric on the triply punctured sphere give (1.1) when $P$ is the universal covering of $\mathbb{C} \setminus \{0, 1\}$.

In the course of the proof of Theorem 1 we measure the thickness of $E$ at all scales, and we are guided by the following philosophy. If, at some scale, the boundary $E$ appears to be thick then, locally, the universal covering map behaves like a Riemann map. On the other hand, if $E$ appears to be thin, then, locally, the Poincaré metric of $\mathbb{C} \setminus E$ behaves like the corresponding Poincaré metric of $\mathbb{C} \setminus \{0, 1\}$. With the right estimates for the transition from the thick case to the thin case, this philosophy leads to a rigorous proof. Our proof also shows the existence of a very large set of angles $\beta$ for which Theorem 1 holds.

The following propositions present the main technical results of this paper. Each proposition gives estimates on the radial variation of $u = \log |P'|$. The hypothesis of Proposition 1 covers the case when to an observer at $w = P(\zeta)$ the boundary $E$ looks like a connected set. The hypothesis of Proposition 2 covers the case when the boundary $E$ looks like an isolated point. To express these alternatives analytically, we use the function

$$M(\zeta) = \sup_{z \in \Gamma(\zeta)} |\nabla u(z)|(1 - |z|)$$
where
\[ T(\zeta) = \{ w \in \mathbb{D} : |w - \zeta| \leq 1 - |\zeta|, (1 - |\zeta|)/2 \leq 1 - |w| \leq 1 - |\zeta| \}. \]

The first alternative corresponds to the case where \( u = \log |P'| \) satisfies a Bloch condition near \( \zeta \). The second alternative causes the failure of Bloch estimates near \( \zeta \). Correspondingly the proof of Proposition 1 uses the condition

\[ M(\zeta) \leq \text{some constant}, \]

whereas Proposition 2 requires that

\[ M(\zeta) \geq \text{a very large constant}. \]

Further combinatorial considerations provide the tools for an iterative solution of Theorem 1 based on repeated applications of Propositions 1 and 2.

In both Proposition 1 and 2 the following family of curves plays an important role. We let \( L \geq 1 \) be a positive integer, and we let \( z_1, z_2 \in \mathbb{D}, |z_1| < |z_2| \). Then \( \Gamma(z_1, z_2, L) \) is the collection of all radial line segments

\[ \gamma = \{ s \in \mathbb{D} : |z_1| < |s| \leq |z_2| \} \cap (0, t), \]

where \( t \in \mathbb{D} \) satisfies \( |t| = |z_2|, |t - z_2| \leq 2^L(1 - |z_2|) \) and where \((0, t)\) denotes the ray connecting \( 0 \in \mathbb{D} \) to \( t \in \mathbb{D} \).

We let \( M, L \) be positive integers and we fix a point \( \zeta \in \mathbb{D} \). Under the hypothesis that \( M(\zeta) < C \), the universal covering map \( P \) behaves locally like a Riemann map. Hence in the proof of Proposition 1 we work with stopping time arguments and J. Bourgain’s estimate for the radial variation of positive harmonic functions.

**Proposition 1** There exist \( C_1 \geq 1 \) so that the following holds. If \( L \geq C_1 \) and if \( M(\zeta) \leq M_1/2^{C_1L} \) then there exists \( q \in \bar{\mathbb{D}} \) such that

a) \[ \int_{\gamma} |\nabla u(w)||dw| \leq C_1 M_1 L, \text{ for } \gamma \in \Gamma(\zeta, q, L). \]

b) \[ \text{If } |q| < 1 \text{ then } u(q) - u(\zeta) \leq -M_1/LC_1. \]

c) \[ 1 - |q| \leq (1 - |\zeta|)/2 \text{ and } |q - \zeta| \leq (1 - |\zeta|)2^L. \]
The constant $C_0 \geq 1$ appearing in the formulation of Proposition 1 is specified in Section 2. Let us assume temporarily that the point $q$ obtained by Proposition 1 also satisfies the condition that $M(q) \leq M_1/2^{C_0 L}$. Then we could apply Proposition 1 again with $\zeta$ replaced by $q$. Doing this would start an iteration leading to the desired variational estimates for $u$ — until a point is reached for which $M(q) > M_1/2^{C_0 L}$. Proposition 2 explains what we do when $M(q) > M_1/2^{C_0 L}$: First using group invariance of $P$ we replace $q$ by a (specially chosen) point $w$ such that

$$|u(w) - u(q)| \leq C_2 + d(\mathbb{D}, w, q).$$

Then using geometric estimates for the hyperbolic metric in $\mathbb{C} \setminus E$ we prove variational estimates for $u$ along the radius that connects $w$ to the boundary of $\mathbb{D}$. Note that as stated Proposition 2 does not give any information about how close $w$ is to $q$. Only later, when we exploit that the machinery underlying the proof of Proposition 1 is composed of stopping time arguments, are we able to show that $M(q) > M_1/2^{C_0 L}$ implies a bound like

$$d(\mathbb{D}, w, q) \leq C_4 L.$$

(This is done in Lemma 3 of Section 3.)

**Proposition 2** There exists $C_2 \geq 1$ so that for $q \in \mathbb{D}$ and $M(q) \geq M_0$ there exist $w, v \in \mathbb{D}$ so that the following holds.

1. $M(w)/C_2 \leq M(q) \leq M(w)C_2$.
2. $|u(w) - u(q)| \leq C_2 + d(\mathbb{D}, w, q)$.
3. If $|v| < 1$, then $M_0/2 \leq M(v) \leq 2M_0$.
4. $(1 - |v|)(1 - |w|) \leq 2^{(-M(q) + M_0)/C_2} C_2$ and $w$ lies on the ray $(0, v)$.
5. If $\gamma \in \Gamma(w, v, L)$, and if $w_1, w_2$ are points on $\gamma$, with $(1 - |w_1|)/(1 - |w_2|) \geq 4$, then

$$u(w_2) - u(w_1) \leq -M(w_1)/C_2 + C_2 L.$$

Now we describe in more detail the relative positions of $q, w$ and $v$. In the case when $|v| = 1$, the point $w$ is the top of the horocycle that is tangent to $\mathbb{T}$ at $v$ and contains $q$. If $v < 1$, there exist $\zeta_1, \zeta_2 \in \mathbb{T}$ so that $w$ is the top of the horocycle $S(q)$ containing $\zeta_1, \zeta_2$ and $q$. The point $v$ is then the top of another horocycle $S_0$ underneath $S(q)$ that contains $\zeta_1, \zeta_2$ and satisfies
$M_0/2 \leq M(v) \leq 2M_0$. We remark also that $w$ will be hyperbolically very close to $k(q)$, for a suitably chosen $k \in G$. And we will see that therefore the right hand side of part (b) does not depend on $M(q)$. This is useful since we apply Proposition 2 when $M(q)$ is a very large constant.

Repeatedly applied, Propositions 1 and 2 give the following result.

**Proposition 3** There exists $C_3 \geq 1$ so that for $L > C_3$ and $M = 4C_1^2C_2L^2$, a sequence of points $s_k \in \mathbb{D}$ can be found satisfying the following conditions.

a) \[ u(s_k) - u(s_{k-1}) \leq -\frac{1}{M} \int_\gamma |\nabla u(w)||dw| \text{ for } \gamma \in \Gamma(s_{k-1}, s_k, L), \]

b) \[ 1 - |s_k| \leq (1 - |s_{k-1}|)/4 \text{ and } |s_k - s_{k-1}| \leq 2L(1 - |s_{k-1}|). \]

## 2 Bloch estimates and Stopping time Lipschitz domains

In this section we will recapitulate and extend our arguments from [J–M]. In the first paragraphs of this section we discuss the tools necessary to define and analyze stopping time Lipschitz domains. Then we give the proof of Proposition 4 which implies Proposition 1.

We begin by describing a deep result of J. Bourgain [B]. It plays an important role in the proof of Proposition 1. We fix a positive harmonic function $g$ in $\mathbb{D}$, and an interval $I \subset \mathbb{T}$ such that $m(\mathbb{T} \setminus I) \leq L^{-2}$. For $\dot{e}^\alpha \in I$ we let $\Sigma(\dot{e}^\alpha, L)$ be the collection of curves in $\mathbb{D}$ which remain in a Stolz cone with vertex $\dot{e}^\alpha$ and opening angle $\pi - L$, and have an $L$–Lipschitz parametrization. More precisely the curves in $\Sigma(\dot{e}^\alpha, L)$ admit the following representation,

$$
\gamma(r) = re^{ie^\alpha e^{\theta(r)}}, \quad 0 \leq r \leq 1,
$$

where $|\theta(r)| < L(1 - r)$ and $|\theta'(r)| < L$. Then the following holds.

**Theorem (J. Bourgain)** There exists $\dot{e}^\alpha \in I$ such that

a) \[ g(\dot{e}^\alpha) \leq g(0) \left(1 + \frac{1}{coL^2}\right) - co \int_0^1 |\nabla g(\dot{e}^\alpha r)|dr, \]

where $co > 0$ is universal, and such that

b) \[ \int_\gamma |\nabla g(w)||dw| \leq CLg(0), \]
whenever $\gamma \in \sum(e^{i\alpha}, L)$. The constant $C \geq 1$ is universal.

Next we recall the result that a Bloch function is bounded on a dense set of radii. We fix a $C_0$-Lipschitz domain $W \subseteq \mathbb{D}$. For $w \in W$ we let $s = \text{dist}(w, \partial W)$ and we choose $w_0 \in \partial W$ such that $|w - w_0| = s$. Let $r$ be contained in the intersection $\{y \in \mathbb{D} : |w_0 - y| = s/10\} \cap \partial W$, and let $I = \{y \in \mathbb{D} : |r - y| \leq s/100\} \cap \partial W$. Let $h$ be harmonic in $W$, and $M = \sup \{|
abla h(z)| \text{dist}(z, \partial W) : z \in W\}$. In the construction below we use the following theorem [P, Proposition 4.6].

**Theorem (Ch. Pommerenke)** There exists a geodesic $\gamma$ in $W$, connecting $w$ to a point in $I$ such that for $z \in \gamma$, $|h(w) - h(z)| \leq MA_0$, where $A_0 > 0$ is universal.

The constant $A_0$ appearing in the above theorem is a fixed multiple of $1/\omega(I, W, w)$, where $\omega(I, W, w)$ denotes the harmonic measure of $I$ in $W$ evaluated at $w$. The upper bound for $A_0$ comes from Beurling’s minorisation of harmonic measure.

Finally we discuss an estimate which controls the growth rate of

$$M(\zeta) = \sup_{z \in \Gamma(\zeta)} |\nabla u(z)|(1 - |z|).$$

We let $g$ be a Moebius transform without fixed points in $\mathbb{D}$. Then on $\mathbb{D}$ the function $\log d_{\mathbb{D}}(z, g(z))$ is Lipschitz with respect to the hyperbolic metric $d_{\mathbb{D}}$. Taking into account that $u = \log |P|$ where $P$ is actually a universal covering map, we obtain our next Lemma from the above remark and (3.1), (3.2) below.

**Lemma 1** There exists a universal $K > 0$ such that the function $\log M(z)$ is $K$-Lipschitz with respect to $d_{\mathbb{D}}$.

We have completed the discussion of the preliminaries and will now describe the construction of stopping time Lipschitz domains. For the rest of this paper we fix $u = \log |P|$. We also fix constants $M_1, L \in \mathbb{N}$ such that $M_1 > L > A_0$. We let $C_0 = K^2$, $\zeta \in \mathbb{D}$, and we assume that

$$M(\zeta) \leq M_1/2^{L C_0}.$$

Around $\zeta$, we wish to construct a large Lipschitz domain on which $u - u(\zeta)$ is bounded below and satisfies a Bloch estimate. This is done in two steps each of which uses stopping time procedures on dyadic intervals.

We define the box around $\zeta$ as follows,

$$D(\zeta) = \{w \in \mathbb{D} : |\zeta/|\zeta| - w/|w|| \leq 2^L (1 - |\zeta|), \text{ and } 1 - |w| \leq 2^L (1 - |\zeta|)\}.$$
Note that the four sides and the four angles of the box $D(\zeta)$ are of the same size. For a dyadic interval $I \subset \mathbb{T}$ we let $T(I) = \{w \in \mathbb{D} : w/|w| \in I, |I|/2 \leq 1 - |w| \leq |I|\}$ and $M(I) = \sup \{M(\zeta) : \zeta \in T(I)\}$. Defining the first stopping time we let $\mathcal{E} = \{I_j\}$ be the collection of maximal dyadic intervals $\subseteq \mathbb{T}$ that satisfy $T(I) \cap D(\zeta) \neq \emptyset$ and

$$M(I) \geq M_i/LA_0.$$ We let $E(I)$ be the Euclidean convex hull of $\{w \in \mathbb{D} : 1 - |w| = |I|, w/|w| \in I\}$ and $16I \subseteq \mathbb{T}$, where $16I$ is the interval with the same midpoint as $I$ and $|16I| = 16|I|$. Our first Lipschitz domain is given as

$$\mathcal{L}(\zeta) = D(\zeta) \setminus \bigcup_{I \in \mathcal{E}} E(I).$$

On $\mathcal{L}(\zeta)$, the function $u = \log |P'|$ satisfies a Bloch estimate. Indeed for $z \in \mathcal{L}(\zeta)$ we have by construction $|\nabla u(z)|(1 - |z|) \leq M_i/LA_0$, and therefore $|\nabla u(z)|\text{dist}(z, \partial \mathcal{L}(\zeta)) \leq M_i/LA_0$.

Next we will remove the points $w \in \mathcal{L}(\zeta)$ for which $u(w) - u(\zeta) < -M_i/2$. This will be achieved by the following stopping time procedure. Let $\mathcal{V} = \{J\}$ be the collection of maximal dyadic intervals $J$ for which $T(J) \cap \mathcal{L}(\zeta) \neq \emptyset$ and there exists $v \in T(J)$ for which

$$u(v) - u(\zeta) < -M_i/2.$$ Using Pommerenke’s theorem we will extract the information encoded in the stopping time collection $\mathcal{V}$. This requires some preparation. For $J \in \mathcal{V}$ we denote by $w$ the point in $T(J)$ which satisfies $u(w) - u(\zeta) < -M_i/2$, and which is of smallest possible modulus. Let $s = \text{dist}(w, \partial \mathcal{L}(\zeta))$ and choose $w_0 \in \partial \mathcal{L}(\zeta)$ such that $|w - w_0| = s$. Also let $I_i = \{v \in \mathbb{D} : |v - w_i| \leq s/100\} \cap \partial \mathcal{L}(\zeta)$, where $w_1, w_2$ are the points in the intersection $\{y : |w_0 - y| = s/10\} \cap \partial \mathcal{L}(\zeta)$. By Pommerenke’s theorem there exists $y_i \in I_i$ such that for each $z$ on the $\mathcal{L}(\zeta)$–geodesic connecting $w$ to $y_i$ we have the upper bound

$$|u(w) - u(z)| \leq M_i/L.$$ We call this geodesic $\gamma_i$. For $i \in \{1, 2\}$ we let $R_i$ be the straight line segment $(w, y_i)$. Note that our construction gives the straight line segments $R_1, R_2$ in $\mathcal{L}(\zeta)$. Moreover for any point $v \in R_i$, there exists $z \in \gamma_i$ such that $z$ and $v$ can be connected by a curve in $\mathcal{L}(\zeta)$ and the $d_\mathbb{D}$–length of this curve is $\leq K_1$. The constant $K_1 > 0$ is universal. In particular $K_1$ does not depend on our choice of $L$. This gives the following estimate for the deviation of $u$ along $R_i$,

$$|u(w) - u(v)| \leq M_i/L + M_iK_1/LA_0, \text{ for } v \in R_i.$$ 

(2.1)
We let $R_3 \subset \partial \mathcal{L}(\zeta)$ be the shorter arc in $\partial \mathcal{L}(\zeta)$ that connects $y_1$ and $y_2$. Finally we define $V(J) \subseteq \mathcal{L}(\zeta)$ to be the domain in $\mathcal{L}(\zeta)$ that is bounded by $R_1, R_2, R_3$, and we put,

$$W(\zeta) = \mathcal{L}(\zeta) \setminus \bigcup_{J \in \mathcal{V}} V(J).$$

From now on we will only consider $L \geq 4 + 4K_1/A_0$. The following list describes the basic properties of the domain $W(\zeta)$, and contains additional important information about the stopping time intervals in $\mathcal{E}$ and $\mathcal{V}$.

Remarks.

1. If $z \in W(\zeta)$, then $M(z) < M_1/LA_0$ and $u(z) - u(\zeta) > -M_1/2$.

2. The boundary of $W(\zeta)$ can be canonically decomposed into four very simple pieces: Two vertical line segments in $\partial D(\zeta)$, a horizontal line segment in $\partial D(\zeta)$, and a piece that is contained in the graph of a Lipschitz function defined on $T$. To see this we only need to recall and compare the definitions of $E(I)$ and $V(J)$.

3. It follows from the stopping rule defining $\mathcal{E}$ that $M(I)/M(\zeta) \geq 2^{L_{co}}/LA_0$, whenever $I \in \mathcal{E}$. Comparing this estimate with Lemma 1 we find that the intervals $I \in \mathcal{E}$ satisfy $|I| \leq (1 - |\zeta|)/8$.

4. For $I \in \mathcal{E}$ and $q \in E(I)$, we have $q/|q| \in 16I$.

5. The stopping rule for $\mathcal{V}$ together with Lemma 1 implies that any $J \in \mathcal{V}$ satisfies $|J| \leq (1 - |\zeta|)/8$.

6. Let $I \in \mathcal{V}$ and assume that $q \in \partial V(I)$ and $|q| < 1$. Then by our choice of $L > 4 + 4K_1/A_0$ and by (2.1) we obtain that $u(q) - u(\zeta) < -M_1/4$. We point out that this upper bound for the difference $u-u(\zeta)$ on $\partial V(I) \cap \mathcal{D}$ is comparable to the lower bound of that difference in the entire domain $W(\zeta)$. Indeed by Remark 1, for $z \in W(\zeta)$ we have $u(z) - u(\zeta) > -M_1/4$.

$W(\zeta)$ is the domain we will be working with, in this section. The following subset of $\partial W(\zeta)$ is important for the construction below. It contains the points that play a role in the Future.

$$F(\zeta) = \{ w \in \partial W(\zeta) : |\zeta/|\zeta| - w| < 2^L(1 - |\zeta|) \text{ and } 1 - |w| \leq (1 - |\zeta|)/2 \}$$
It follows from Remark 3) and 5) that $F(\zeta)$ is connected. Moreover, by Beurling, we have the following minorization of harmonic measure

$$\omega(F(\zeta), W(\zeta), \zeta) \geq 1 - L^{-2}.$$ 

The main result of this section is the following Proposition.

**Proposition 4** There exists $q \in F(\zeta)$ such that,

1) \[ \int_\gamma |\nabla u(w)||dw| \leq CM_1L, \text{ for } \gamma \in \Gamma(\zeta, q, L). \]

2) \[ \text{If } |q| < 1, \text{ then } u(q) - u(\zeta) < M_1 \left( \frac{C}{c_0L^2} - \frac{c_0}{CL_A} \right), \]

where $C \geq 1$ is universal, and where $c_0, A_0 > 0$ are the constants appearing, in Bourgain’s theorem resp. Pommerenke’s theorem.

**Proof.** Let $f : \mathbb{D} \to W(\zeta)$ be the Riemann map normalized such that $f(0) = \zeta$. Recall that $F(\zeta)$ is connected and that $u(F(\zeta), W(\zeta), \zeta) \geq 1 - L^{-2}$. Hence $A = f^{-1}(F(\zeta))$ is an interval such that $m(T \setminus A) \leq L^{-2}$. By Remark 1 the pullback

$$g(w) = u(f(w)) - u(f(0)) + M_1$$

is a positive harmonic function in $\mathbb{D}$. Applying Bourgain’s theorem gives $e^{i\alpha} \in A$ such that

$$g(e^{i\alpha}) \leq g(0) \left( 1 + \frac{1}{c_0L^2} \right) - c_0 \int_0^1 |\nabla g(re^{i\alpha})|dr.$$ 

As $g(0) = M_1$, this is the same as

(2.3) \[ u(f(e^{i\alpha})) - u(f(0)) \leq \frac{M_1}{c_0L^2} - c_0 \int_{\gamma_0} |\nabla u(w)||dw|, \]

where $\gamma_0 = f((0, e^{i\alpha}))$. The second part of Bourgain’s theorem gives

$$\int_\gamma |\nabla g(w)||dw| \leq CM_1L, \quad \text{for } \gamma \in \sum(e^{i\alpha}, L).$$

With a change of variables we rewrite this line as follows,

(2.4) \[ \int_{f(\gamma)} |\nabla u(w)||dw| \leq CM_1L, \text{ for } \gamma \in \sum(e^{i\alpha}, L). \]
The admissible curves in (2.4) are \( f(\gamma) \) with \( \gamma \in \Sigma(e^{i\alpha}, L) \). Below we will use estimates on harmonic measure to show that the straight line segments in \( \Gamma(\zeta, f(e^{i\alpha}), L) \) are also admissible curves. In fact we will show that (2.4) implies,

\[
(2.5) \quad \int_{\sigma} |\nabla u(w)||dw| \leq CM_1 L, \quad \text{for } \sigma \in \Gamma(\zeta, q, L).
\]

Now we let \( q = f(e^{i\alpha}) \). Note that we chose the interval \( A \) such that \( f(e^{i\alpha}) \) is contained in \( F(\zeta) \). By construction the set \( F(\zeta) \) splits canonically into three subsets carrying different pieces of information: The subset that intersects \( \mathcal{T} \). The subset where \( u - u(\zeta) < -M_1/4 \). And the set of points \( z \) for which we know that somewhere in the Stolz cone centered at \( z \) the Bloch constant was larger than \( M_1/A_0 L \). Accordingly we continue by distinguishing between the following three cases:

a) \( |q| = 1 \).

b) \( |q| < 1 \) and there exists \( I \in \mathcal{V} \) such that \( q \in \partial V(I) \).

c) \( |q| < 1 \) and \( q \in \partial \mathcal{L}(\zeta) \).

Note that these cases cover all possibilities for \( q \in F(\zeta) \). Treating different cases by different means, we will now verify that \( q = f(e^{i\alpha}) \) satisfies the conclusion of Proposition 4.

ad a) If \( q = f(e^{i\alpha}) \) satisfies \( |q| = 1 \) then we only have to show that

\[
\int_{\sigma} |\nabla u(w)||dw| \leq CM_1 L, \quad \text{for } \sigma \in \Gamma(\zeta, q, L).
\]

This however is just the estimate in (2.5).

ad b) By Remark 6 we have that \( u(q) - u(\zeta) < -M_1/4 \). When we combine this estimate with the variational estimate in (2.5) we obtain the assertions of Proposition 4. Note that in case b) the resulting decay of \( u \) is much better than claimed or needed.

ad c) By Remark 4 there exists an interval \( I \in \mathcal{E} \), such that \( q/|q| \in 16I \). Hence \( T(I) \) is contained in a Stolz cone with vertex \( q \). As \( I \in \mathcal{E} \) we have

\[
M(I) \geq M_1/L A_0.
\]

In \( W(\zeta) \), the geodesic \( \gamma_0 = f((0, e^{i\alpha})) \) passes through a fixed enlargement of \( T(I) \). Moreover \( \gamma_0 = f((0, e^{i\alpha})) \) is a \( C^2 \) curve with uniform constants in \( T(I) \). Hence by a simple normal families argument,

\[
\int_{\gamma_0} |\nabla u(w)||dw| \geq \frac{M_1}{C L A_0}.
\]
where $C > 0$ is universal, and in particular independent of $L$. We insert the last estimate into (2.3) and obtain

$$u(q) - u(\zeta) < M_1 \left( \frac{C}{\alpha_0 L^2} - \frac{\alpha_0}{CLA_0} \right).$$

We have dealt with all possible cases, and Proposition 4 is proven, provided that (2.4) implies (2.5). To show this implication we use the following lemma which is folklore.

We let $I, J$ be adjacent intervals in $\mathbb{T}$ which have $e^{i\alpha}$ as endpoint and $m(I) = m(J) = m(\mathbb{T})/2$. Their images under the Riemann map $f$ are $A = f(I)$ respectively $B = f(J)$. Let $\gamma \subset W(\zeta)$. Using lower bounds for the harmonic measures of $A$ and $B$ we obtain useful information about the location of $f^{-1}(\gamma)$.

**Lemma (Folklore)** *If for any $z \in \gamma$, $w(A, W(\zeta), z) \geq 1/L$ and $w(B, W(\zeta), z) \geq 1/L$, then $f^{-1}(\gamma)$ is contained in a Stolz cone of vertex $e^{i\alpha}$ and of opening angle $\pi - 1/CL$.***

We can now show that (2.4) implies (2.5). We choose $\gamma \in \Gamma(\zeta, q, L)$, i.e., $\gamma$ is of the form

$$\{s : |q| < |s| < |\zeta| \cap (0, t),$$

where $t$ satisfies $|t| = |q|$, $|t - q| \leq 2^L(1 - |q|)$. By elementary geometry and Beurling’s minorization of harmonic measure we find $t_1 \in \gamma$, whose hyperbolic distance to $t$ is $\leq LC$, and so that for each $z \in \gamma_1 = \gamma \cap (t_1, 0)$ we have the estimates $w(A, W(\zeta), z) \geq \eta/L$ and $w(B, W(\zeta), z) \geq \eta/L$, with an universal $\eta > 0$. The above folk lemma and the Koebe distortion theorem imply that $f^{-1}(\gamma_1)$ is a curve in $\Sigma(e^{i\alpha}, CL)$. Hence by (2.4)

$$\int_{\gamma_1} |\nabla u(w)||dw| \leq CM_1 L.$$

Finally for $\gamma_2 = \gamma \cap (t_1, t)$ we estimate

$$\int_{\gamma_2} |\nabla u(w)||dw| \leq M(t_1)d_\Delta(t, t_1) \leq CM_1.$$

**Remark.** We will use Proposition 4 to deduce Proposition 1. Therefore it is important that the constant appearing in condition 2) of Proposition 4,

$$(2.6) \quad \left( \frac{C}{\alpha_0 L^2} - \frac{\alpha_0}{CLA_0} \right),$$

is negative and independent of $M_1$. But for $L$ large enough the expression in (2.6) is just a small perturbation of $-\alpha_0/CLA_0$. Here our argument really needs the additional freedom gained
by introducing the parameter $L$. It now follows that Proposition 4 implies Proposition 1 when we choose $L > 2C_0^2A_0/c_0^2$ and $C_1 = 2CA_0/c_0$. Note that such a choice is compatible with our previous lower bound on $L$.

### 3 When Bloch estimates fail

In this section we prove Proposition 2. We recall that there exits a Fuchsian group $G$ without elliptic elements so that $\mathbb{C} \setminus E$ is conformally equivalent to $\mathbb{D}/G$. The universal covering map is $P = \tau \circ \pi$ where $\pi$ is the natural projection, and $\tau$ is the conformal bijection between $\mathbb{D}/G$ and $\mathbb{C} \setminus E$. The density of the hyperbolic metric on $\Omega = \mathbb{C} \setminus E$ is given by

\[
\lambda_\Omega(P(z))|P'(z)| = \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}.
\]

By the result of A.F Beardon and Ch. Pommerenke [B-P], the density $\lambda_\Omega$ admits the following geometric estimate,

\[
\lambda_\Omega(v_0) \sim \frac{1}{\text{dist}(v_0, E)(\beta(v_0) + 1)}, \quad v_0 \in \mathbb{C} \setminus E,
\]

where

\[
\beta(v_0) = \inf \left\{ \log \left| \frac{v_0 - a}{a - b} \right| : |v_0 - a| = \text{dist}(v_0, E) \text{ and } a, b \in E \right\}.
\]

If for a given $v_0 \in \mathbb{C} \setminus E$ the infimum in the definition of $\beta(v_0)$ is attained in $a, b \in E$, then one of the following cases holds. (We let $K(a, r)$ denote the open disk with radius $r > 0$ and center $a$.)

**P1:** There exists $B, \eta \in \mathbb{R}_0^+$ such that $\mathbb{C} \setminus E \supset K(a, B) \setminus \bar{K}(a, \eta)$, $\eta < B^{-1}$, $b \notin K(a, B)$ and $\beta(v_0) \sim \log |\text{dist}(v_0, E)/B|$.

**P2:** There exists $\eta > 0$ such that $\Omega \supset K(a, \eta^{-1}) \setminus K(a, \eta)$, $a, b \in K(a, \eta)$ and $\beta(v_0) \sim |\log(\text{dist}(v_0, E)/\eta)|$.

We define these cases as giving rise to pictures; for example we will say that we see picture P1 at $v_0$ if P1 holds.

The following geometric lemma will be very useful when we study the decay of $\log |P'|$ along radial line segments. We consider the following annuli centered at $a \in E$,

\[
A_k = \{ v \in \mathbb{C} : \text{dist}(v_0, E)/2^{k+1} \leq |a - v| \leq \text{dist}(v_0, E)/2^k \}, \text{ for } k \in \mathbb{N}_0.
\]
We will only use these $A_k$ when $\beta(v_0)$ is large and in this case the annuli $A_k$ are disjoint from $E$ when $k \leq C_1\beta(v_0)$. We also remark that these annuli allow us to trace the changes of the hyperbolic metric in $\Omega = \mathbb{C} \setminus E$, as we approach the boundary of $\Omega$. In fact, by (3.2), the density of the hyperbolic metric remains essentially constant on each of the $A_k$, and the corresponding value can be computed from $k$ and $\beta(v_0)$. The formulas are given in the proof below.

Lemma 2 Let $s = \text{dist}(v_0, E)$ and let $\gamma : [0, 1] \to K(a, s) \cap \mathbb{C} \setminus E$ be a curve satisfying the following conditions:

1. $\gamma(0) = v_0$.

2. The linear measure of $\gamma \cap A_k$ is bounded by $C\text{diam}A_k$, $k \in \mathbb{N}$.

3. There exists $c < 1/2$ so that if $\gamma(t) \in A_k$ and $t_1 > t$ then $\gamma(t_1) \notin A_k$.

4. $4 > \int_{\gamma} \lambda_\Omega(w)|dw| > 1/4$.

Then $|\gamma(1) - a|/|v_0 - a| \leq C^2\beta(v_0)/c$ and $\beta(\gamma(1)) \leq C\beta(v_0)$, where $C \geq 1$ is universal.

Proof. First we consider the case when we see the picture P1 at $v_0$. There exists a smallest $\eta \geq 0$ so that P1 holds. We denote it by $\epsilon \geq 0$. Now we determine how $\beta(v)$ changes when $v$ moves through the annuli $A_k$. For $v \in A_k$, we have $\text{dist}(v, E) = |v - a| \sim |v_0 - a|/2^k$. Let $k_0 \in \mathbb{N}$ be the first integer for which $|v_0 - a|/2^{k_0} \leq \sqrt{B\epsilon}$. One observes that $\beta(v)$ increases as $v$ moves through the first $k_0$ annuli, and after that $\beta(v)$ decreases until it reaches $\sim 0$. In fact, for $v \in A_k$ and $k \leq k_0$ we have $1 + \beta(v) \sim 1 + \beta(v_0) + k$. For $k \geq k_0$ we have $1 + \beta(v) \sim \max\{1, \beta(v_0) + 2k_0 - k\}$. We let $l \in \mathbb{N}$ be the smallest integer for which

$$\gamma \subset \bigcup_{k=1}^{l} A_k.$$ 

The rest of the proof is used to show that $l$ is comparable to $C\beta(v_0)$. We let $\gamma_k = \gamma \cap A_k$ and we need to consider only the case when $k_0 < l$. Then using hypothesis 2) we estimate as follows.

$$\int_{\gamma} \lambda_\Omega(v)|dv| = \sum_{k=1}^{l} \int_{\gamma_k} \lambda_\Omega(v)|dv| \sim \sum_{k=0}^{k_0} \int_{\gamma_k} \frac{|dv|}{\text{dist}(v, E)(1 + \beta(v_0) + k)}$$

$$+ \sum_{k=k_0}^{l} \int_{\gamma_k} \frac{|dv|}{\text{dist}(v, E)(1 + \beta(v_0) + 2k_0 - k)}$$

$$\sim \sum_{k=0}^{k_0} \frac{1}{\beta(v_0) + k} + \sum_{k=k_0}^{l} \frac{1}{\beta(v_0) + 2k_0 - k}$$

$$\sim \left|\log \frac{(\beta(v_0) + k_0)^2}{\beta(v_0)(\beta(v_0) - l + 2k_0)}\right|. $$
Next using that \( \int_\gamma \lambda_\Omega(v) |dv| \geq 1/4 \) we obtain

\[
\beta(v_0)(\beta(v_0) - l + 2k_0)e^{1/C} \leq (\beta(v_0) + k_0)^2.
\]

A simple calculation, using \( k_0 \leq l \), gives \( l \geq \beta(v_0)/2 \). Hypothesis (3) gives the estimate

\[
\frac{|\gamma(1) - a|}{|\gamma(0) - a|} \leq 2^{-l/C}.
\]

Combining this with \( 2^{-l/C} < 2^{-\beta(v_0)/2C} \) gives the first conclusion of the lemma when we “see” P1 at \( v_0 \) and \( k_0 < l \). Finally we remark that the above line of inequalities can be reversed and we obtain also

\[
\int_\gamma \lambda_\Omega(v) |dv| \geq \log \frac{(\beta(v_0) + k_0)^2}{\beta(v_0)(\beta(v_0) - l + 2k_0)}.
\]

Hence if \( \int \lambda_\Omega(v) < 4 \) then, by a simple calculation, \( l \leq C/\beta(v_0) \). This gives the second conclusion of Lemma 2. If we see P2 at \( v_0 \) then

\[
1 + \beta(v) \sim \max\{1, \beta(v_0) - k\},
\]

for all \( k \), and \( v \in A_k \). Hence this case corresponds to \( k_0 = 0 \) in the above consideration, and the above calculation can simply be repeated, setting \( k_0 = 0 \).

\[ \blacksquare \]

**Proof of Proposition 2.** We are given \( q \in \mathbb{D} \). The first part of the proof consists of constructing the points \( w \in \mathbb{D} \), \( v \in \overline{\mathbb{D}} \). The construction is based on the following estimate which holds when \( M(q) \geq 1 \),

\[
(3.3) \quad \frac{1}{CM(q)} \leq \inf_{g \in G} d_\mathbb{D}(q, g(q)) \leq \frac{C}{M(q)}.
\]

The right hand side of (3.3) follows from Lemma 1 and Koebe’s distortion estimate by rescaling. The left hand side is obtained from univalence criteria by rescaling. See [M, Proposition 1.3] for an elementary univalence criterion that suffices here.

Now we select a group element \( g \in G \) such that \( d_\mathbb{D}(q, g(q)) \leq CM(q)^{-1} \). As \( G \) does not contain elliptic elements, there are either one or two fixed points of \( g \) on \( \mathbb{T} \). Each case requires a different construction to obtain \( w, v \).

We first treat the case where \( g \) has two fixed points in \( \mathbb{T} \). Let \( \zeta_1, \zeta_2 \in \mathbb{T} \) be the fixed points of \( g \), and let \( A \) be the hyperbolic geodesic connecting \( \zeta_1 \) to \( \zeta_2 \). We let \( S(q) \) be the hypercycle in \( \overline{\mathbb{D}} \) which contains \( \zeta_1, \zeta_2 \) and \( q \). Now we let \( K \subset \mathbb{D} \) be the region which is bounded by the
axis \( A \) of \( g \) and the interval \( I \subset \mathbb{T}, m(I) \leq m(\mathbb{T})/2 \), whose endpoints are \( \zeta_1, \zeta_2 \). We consider the hypercycle

\[
S_0 = \{ s \in K : \sinh(d_\mathbb{D}(s, g(s))) = \sinh(d_\mathbb{D}(q, g(q)))M(q)/M_0 \}
\]

and the ray \( R \) that connects \( 0 \in \mathbb{D} \) to the midpoint of \( I \). Note that the hypercycle \( S_0 \) is well defined; it lies underneath the axis \( A \), and also underneath \( S(q) \). Depending on the position of \( q \) relative to \( A \) the hypercycle \( S(q) \) may be above or underneath the axis \( A \). We point out however that when we apply Proposition 2 the hypercycle \( S(q) \) will be above the axis \( A \), and the point \( q \) we use will be close to the top of \( S(q) \). (See Lemma 3 below.) Now we define

\[
(3.4) \quad w = R \cap S(q), \quad v = R \cap S_0.
\]

We turn to the case when \( g \in G \) has one fixed point \( \zeta_1 \in \mathbb{T} \). The first step is again the construction of \( w \in \mathbb{D}, v \in \mathbb{T} \). We let \( S(q) \) be the horocycle through \( q \in \mathbb{D} \) and \( \zeta_1 \in \mathbb{T} \). Without loss of generality we may assume that \( 0 \in \mathbb{D} \) is not contained in the disk bounded by \( S(q) \). Then we define

\[
(3.5) \quad w = S(q) \cap (0, \zeta_1), \quad v = \zeta_1.
\]

Again we point out that we will only apply this when \( q \) is near the top of the horocycle.

The following properties of \( w, v \) are easily verified:

\[
(3.6) \quad C^{-1} \leq M(w)/M(q) \leq C,
\]

\[
(3.7) \quad \text{if } |v| < 1, \text{ then } C^{-1} \leq M(v)/M_0 \leq C,
\]

\[
(3.8) \quad 1 - |v|^2/1 - |w|^2 \leq 2^{-M(q)+M_0},
\]

\[
(3.9) \quad |u(q) - u(w)| \leq C + |\log((1 - |w|^2)/(1 - |q|^2))|.
\]

As \( S(q), S_0 \) are levels for \( s \mapsto \sinh d_\mathbb{D}(s, g(s)) \), (3.6) and (3.7) follow from (3.3). Condition (3.8) is a consequence of elementary circle geometry. To verify (3.9) we exploit group invariance of \( P \). We choose \( m \in \mathbb{Z} \) so that for \( k = g^m \)

\[
(3.10) \quad d_\mathbb{D}(k(q), w) \leq CM^{-1}(q).
\]
This is possible by (3.3). As $P = P \circ k$ we obtain $k'(q)P'(k(q)) = P'(q)$. Consequently
\[ \log |P'(q)| - \log |P'(k(q))| = \log |k'(q)|, \]
and $1 - |w|^2/2(1 - |q|^2) \leq |k'(q)| \leq 1 - |w|^2/1 - |q|^2$. By (3.10) we have
\[ |u(k(q)) - u(w)| \leq M(w)d(w, k(q)) < C. \]
Clearly, the last two estimates give (3.9):
\[ |u(w) - u(q)| \leq C + |\log((1 - |w|^2)/(1 - |q|^2))|. \]
So far we have verified conditions a) - d) of Proposition 2. The remaining condition e) follows from our next proposition.

We let $R$ be the radial line segment connecting $w$ and $v$, that is, $R = (w, v)$. When a point moves along $R$ towards the boundary of $\mathbb{D}$ we observe the following decrease of $u = \log |P'|$:

**Proposition 5** If $z_1, z_2 \in R$ satisfy $1/32 \leq 1 - |z_2|/1 - |z_1| \leq 1/4$, then $u(z_2) - u(z_1) \leq \lambda(z_1)/C + C$, where $C > 0$ is universal.

**Proof.** By choice of $R$, the line segment $t \mapsto z_1 + t(z_2 - z_1)$ minimizes the $\lambda_\mathbb{D}$-distance between the hypercycles (respectively horocycles) $S(z_1)$ and $S(z_2)$. Therefore among all curves connecting $P(z_1)$ and $P(z_2)$ the following,
\[ \gamma : t \mapsto P(z_1 + t(z_2 - z_1)), \]
has minimal length with respect to the hyperbolic metric on $\mathbb{C} \setminus E$. And so $\gamma$ satisfies conditions 1) - 4) of Lemma 2, with $\gamma(0) = v_0 = P(z_1)$ and $\gamma(1) = P(z_2)$. To verify condition 2 of Lemma 2 we first note that for each $A_k$ and $z, z' \in A_k$,
\[ C^{-1} \lambda_\Omega(z) \leq \lambda_\Omega(z') \leq C\lambda_\Omega(z). \]
If condition 2 would fail then we could make a new curve with the same initial point and same last point as $\gamma$, and such that the hyperperbolic length of this new curve is less than the hyperbolic length of $\gamma$. The same argument proves also that condition 3 holds.

Applying Lemma 2 to our curve $\gamma$ gives the following estimates.
\[ \beta(P(z_2)) \leq C\beta(P(z_1)), \]
and
\[
|a - P(z_2)|/|a - P(z_1)| \leq C 2^{-\beta(z_1)/C}.
\]
Combining these estimates with (3.1) and (3.2) we obtain
\[
\frac{|P'(z_2)|}{|P'(z_1)|} = \frac{\lambda_\Omega(P(z_1))(1 - |z_1|^2)}{\lambda_\Omega(P(z_2))(1 - |z_2|^2)} \leq C \frac{|a - P(z_2)| (\beta(P(z_2)) + 1)}{|a - P(z_1)| (\beta(P(z_1)) + 1)} \leq C 2^{-\beta(P(z_1))/C}.
\]
We remark that by rescaling and normal families \( M(z_1) \leq C \beta(P(z_1)) \); this completes the proof of Proposition 5.

Finally we conclude the proof of Proposition 2: Conditions a) – d) of Proposition 2 follow from (3.5) – (3.8). We will now verify condition e), using Proposition 5, Lemma 1, (3.9) and (3.10).

Let \( \Lambda \in \Gamma(w, v, L) \) and choose \( w_1, w_2 \in \Lambda \) such that \( (1 - |w_1|)/(1 - |w_2|) > 4 \). As above we denote \( R = (w, v) \). Let us first treat the case when \( |v| = 1 \). In that case \( \Gamma(w, v, L) \) contains only one element namely \( R \), and applying Proposition 5 to \( \Lambda = R = (w, v) \) gives condition e) of Proposition 2.

Next we consider the case when \( |v| < 1 \). This condition implies that our group element \( g \) has two fixed points \( \zeta_1, \zeta_2 \in \mathbb{T} \). For \( i \in \{1, 2\} \) we let \( z_i \in R \) be the top of the hypercycle containing \( w_i \) and the fixed points \( \zeta_1, \zeta_2 \in \mathbb{T} \). As in (3.6) we have \( M(w_i)/C \leq M(z_i) \leq M(w_i)C \). Combining (3.9) and (3.10) we obtain \( |u(z_2) - u(w_1)| \leq CL \). Applying Proposition 5 to \( z_1, z_2 \) gives \( u(z_2) - u(z_1) \leq -M(z_1)/C + C \). Summing up we obtain that
\[
u(w_2) - u(w_1) \leq -M(w_1)/C_2 + C_2L.
\]

We will now link the Lipschitz domains of Section 2 to elements of the above construction. Recall that we have isolated the following connected subset on the boundary of our Lipschitz domain \( W(\zeta) \),
\[
F(\zeta) = \{ w \in \partial W(\zeta) : |\zeta/|\zeta| - w| < 2^L(1 - |\zeta|) \text{ and } 1 - |w| \leq (1 - |\zeta|)/2 \}.
\]
We recall also that for \( q \in \mathbb{D} \) we started the proof of Proposition 2 by selecting a group element \( g \in G \) satisfying \( d_{\mathbb{D}}(q, g(q)) \leq CM(q)^{-1} \). Then we defined \( S(q) \) to be the horocycle containing \( q \) and the fixed points \( \zeta_1, \zeta_2 \) of \( g \), when \( g \) was hyperbolic. In the case of a parabolic \( g \), \( S(q) \) was the horocycle through \( q \) that was tangent to \( T \) at the (sole) fixed point of \( g \). In our next lemma we will utilize again that \( W(\zeta) \) is the result of stopping time arguments, and we find that for \( q \in F(\zeta) \) the top of \( S(q) \) is close to \( q \), whenever \( M(q) \) is a large constant.

**Lemma 3** Let \( q \in F(\zeta) \), and assume that \( M(q) \geq M_1/2^{G_0 L} \). Let \( w \in \mathbb{D} \) be the top of \( S(q) \). Then in \( \mathbb{D} \) the hyperbolic distance between \( q \) and \( w \) is bounded by \( C_4 L \).

**Proof.** We assume to the contrary that the lemma is false. Under this assumption we will construct a long sequence of points \( w_i \in W(\zeta) \) so that \( M(w_0) \geq M_1/C_2 2^{L C_0} \) and \( M(w_i) \geq 2^i M(w_0) \). On the other hand the points \( w_i \in W(\zeta) \) satisfy the stopping time condition \( M(w_i) \leq M_1/L A_0 \). This gives a contradiction when the sequence of points is long enough.

Now we assume that \( d_{\mathbb{D}}(q, w) > CL \) for arbitrary large \( C \). We let \( R_0 \) be the straight line segment \( R \cap W(\zeta) \) where \( R \) is the straight line connecting \( w \) to \( v \). We recall that \( 0, w \) and \( v \) are points on the same radial ray. As \( d_{\mathbb{D}}(q, w) > CL \), there exists \( \tau > 0 \) depending only on the Lipschitz constants of \( W(\zeta) \), such that the hyperbolic diameter of \( R_0 \) is \( \geq \tau CL \). Therefore we find points \( w_0 = w, w_1, \ldots, w_{i_0} \) on \( R_0 \) with \( 1 - |w_{i+1}|^2 / 1 - |w_i|^2 < \eta \) and \( i_0 \geq \eta \tau CL \). It follows from [Be, Section 7.35] and an elementary calculation that the displacement function decreases at a geometric rate on \( R_0 \). Hence

\[
d_{\mathbb{D}}(w_{i+1}, g(w_{i+1})) \leq \eta d_{\mathbb{D}}(w_i, g(w_i)), \quad i \leq i_0.
\]

If moreover \( \eta > 0 \) is small enough, it follows from (3.3) that

\[
M(w_i) \geq 2^i M(w_0), \quad i < i_0.
\] (3.11)

Finally, it follows from our hypothesis on \( M(q) \) and condition (a) of Proposition 2, that

\[
M(w_0) \geq M_1/C_2 2^{C_0 L}.
\] (3.12)

On the other hand, in Section 2 the stopping time Lipschitz domain was constructed such that for \( w_i \in W(\zeta) \), we have \( M(w_i) \leq M_1/L A_0 \). This contradicts (3.11) and (3.12) for \( i_0 \) large enough, and the assumption was that we can make \( i_0 \) as large as we please.
4 Selecting good rays

In this section we first prove Proposition 3 and then Theorem 1. The inductive construction of the points \( \{s_k\} \) in Proposition 3 is based on repeated application of Proposition 1 and 2. These propositions can interact when the constants \( M_0, M_1, L \) are specified as follows. We recall that we have imposed the lower bound \( L > 4 + 4K_1/A_0 \) in Section 2 during the construction of the domains \( W(\zeta) \), and that later, in the remark following the proof of Proposition 4, we have chosen \( L \) such that also \( L > 2C^2A_0/c_0^2 \). Now we let \( M_0 > 1 \) be such that

\[
-M_0/C_2 + C_2 \leq -M_0/2C_2 \leq -1,
\]

where \( C_2 \geq 1 \) is the constant appearing in Proposition 2. Finally we take \( M_1 \) large enough so that \( M_1/2C_0L \geq 2M_0 \) and

\[
-M_1/LC_1 + 4C_4L \leq -M_1/2LC_1.
\]

We will verify Proposition 3 with \( C_3 = \max\{4+4K_1/A_0, 2C^2A_0/c_0^2\} \) and \( M = 4C^2C_2L^2 \). The proof begins with the inductive construction of the sequence \( \{s_k\} \). Assuming, as we may that for \( u = \log|P'|, u(0) = 0, \) and \( |\nabla u(0)| = 1 \) we take \( s_0 = 0 \). We assume that \( s_0, \ldots, s_n \) have been constructed such that the conclusion of Proposition 3 holds, and such that \( M(s_n) \leq M_1/2C_0L \). Now we determine \( s_{n+1} \) as follows.

We start by constructing the stopping time Lipschitz domain \( W(s_n) \) and apply Proposition 1, to obtain \( q \in F(s_n) \) such that

\[
u(q) - u(s_n) \leq -M_1/C_1 L,
\]

when \( |q| < 1 \), and

\[
\int_{\gamma} |\nabla u(z)||dz| \leq M_1 LC_1,
\]

for \( \gamma \in \Gamma(s_n, q, L) \). Now we consider three cases:

1. If \( |q| = 1 \) then we put \( s_{n+1} = q \) and we stop the construction.

2. If \( |q| < 1 \) and if \( M(q) \leq M_1/2C_0L \) then we put \( s_{n+1} = q \). By (4.3) and (4.4) the induction step is completed. We may continue with the construction of the next point.
3. If $|q| < 1$ and $M(q) > M_1/2^{C_0!}$ then we apply Proposition 2 to $q \in \mathbb{D}$ and obtain $w \in \mathbb{D}$, $v \in \overline{\mathbb{D}}$ for which the conclusion of Proposition 2 hold. We define $s_{n+1} = v$. In the next paragraph we will verify that $s_{n+1}$ satisfies the conclusion of Proposition 3.

The assumption in the third case is that $M(q) > M_1/2^{C_0!}$. By Lemma 3 this implies that $d_\mathbb{D}(w, q) \leq C_4!$. We fix $\gamma \in \Gamma(s_n, s_{n+1}, L)$, and we let $\sigma = \gamma \cap \{s : |s_n| < |s| < |q|\}$ and $\rho = \gamma \cap \{s : |w| < |s| < |s_{n+1}|\}$. Note that $\gamma = \sigma \cup \rho$. We estimate the difference $u(s_{n+1}) - u(s_n)$ by breaking it into three pieces: Recalling that $s_{n+1} = v$ and Proposition 2 (e) give

$$u(s_{n+1}) - u(w) \leq - \frac{1}{M} \int_\rho |\nabla u(z)| |dz|.$$  

Lemma 3 together with Proposition 2 (b) gives $|u(w) - u(q)| \leq C_2 + C_4! \leq 2C_4!$, and (4.1) - (4.4) imply

$$u(q) - u(s_n) + 2C_4! \leq - \frac{1}{M} \int_\sigma |\nabla u(z)| |dz|.$$  

Summing up we have,

$$u(s_{n+1}) - u(s_n) \leq u(s_{n+1}) - u(w) + u(w) - u(q) + u(q) - u(s_n) \leq - \frac{1}{M} \left( \int_\sigma |\nabla u(z)| |dz| + \int_\rho |\nabla u(z)| |dz| \right) \leq - \frac{1}{M} \int_\gamma |\nabla u(z)| |dz|.$$  

Finally we have to distinguish between the cases $|v| = |s_{n+1}| = 1$ and $|v| = |s_{n+1}| < 1$. If $|s_{n+1}| = 1$ then we stop the construction, and Proposition 3 is true in that case. If $|s_{n+1}| < 1$ then by Proposition 2 (c) we have $M(s_{n+1}) \leq M_0 2 \leq M_1/2^{C_0!}$, and we may continue to construct the next point. This completes the proof of Proposition 3.

We turn to the proof of Theorem 1. Let $\{s_k\}$ be the sequence of points given by Proposition 3. This sequence converges to a point in $T$; we denote its limit by $e^{i\beta}$. Now we let $R = (0, e^{i\beta})$ be the ray connecting 0 to $e^{i\beta}$. We will show that uniformly on $R$ the radial variation of $u$ is of the smallest possible order. More precisely we will verify that for any $\xi \in R$,

$$u(\xi) \leq - \frac{1}{M} \int_{(0, \xi)} |\nabla u(z)| |dz| + MM_1,$$  

where $M_1$ has been chosen in (4.2) and $M$ is the constant appearing in Proposition 3. We decompose $R = (0, e^{i\beta})$ as

$$R = \bigcup \gamma_k,$$

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where \( \gamma_k = R \cap \{ s \in \mathbb{D} : |s_k| \leq |s| \leq |s_{k+1}| \} \). Note that by condition b) of Proposition 3 the straight line segment \( \gamma_k \) belongs to \( \Gamma(s_k, s_{k+1}, L) \). Next we choose an arbitrary point \( \xi \in R \). Let \( k_0 \in \mathbb{N} \) be such that \( \xi \in \gamma_{k_0} \). We will treat two cases depending on how \( s_{k_0+1} \) was obtained during the proof of Proposition 3. In the first case \( s_{k_0+1} \) was obtained by an application of Proposition 1. As \( \xi \in \gamma_{k_0} \) it follows from condition a) of Proposition 1 that,

\[
|u(\xi) - u(s_{k_0})| \leq \int_{\gamma_{k_0}} |\nabla u(z)||dz| \leq C_1LM_1.
\]

Summing a telescoping series we obtain from Proposition 3,

\[
u(s_{k_0}) - u(0) \leq -\sum_{l=0}^{k_0-1} \frac{1}{M} \int_{\gamma_l} |\nabla u(z)||dz|.
\]

We let \( \rho = R \cap \{ s : |s| < |s_{k_0}| \} \). Now we estimate the difference \( u(\xi) - u(0) \) by adding the last two inequalities.

\[
u(\xi) - u(0) = u(\xi) - u(s_k) + u(s_k) - u(0)
\leq \int_{\gamma_{k_0}} |\nabla u(z)||dz| - \frac{1}{M} \int_{\rho} |\nabla u(z)||dz|
\leq C_1LM_1 - \frac{1}{M} \int_{\rho} |\nabla u(z)||dz|.
\]

In the second case \( s_{k_0+1} \) was obtained by an application of Proposition 2. This means the following: Applying Proposition 1 to \( s_{k_0} \) gives \( q \in F(s_{k_0}) \) with \( M(q) \geq M_1/2C_0L \); applying Proposition 2 to \( q \) gives \( w \in \mathbb{D}, v \in \mathbb{D} \) and \( s_{k_0+1} = v \), \( M(s_{k_0+1}) \leq 2M_0 \).

We distinguish between the cases \( (1 - |w|)/(1 - |\xi|) < 4 \) and \( (1 - |w|)/(1 - |\xi|) \geq 4 \). In the first case we estimate \( u(\xi) - u(w) \leq 4M(q) \leq M_1 \). Combining condition b) of Proposition 2 with Lemma 3 and condition b) of Proposition 1 gives

\[
u(w) - u(s_{k_0}) \leq -M_1/C_1 + 4LC_4.
\]

Now we let \( \rho = R \cap \{ s : |s| < |s_{k_0}| \} \), and using Proposition 3 we estimate as follows.

\[
u(\xi) - u(0) = u(\xi) - u(w) + u(w) - u(s_{k_0}) + u(s_{k_0}) - u(0)
\leq -\frac{1}{M} \int_{\rho} |\nabla u(z)||dz| - M_1/2C_1 + M_1
\leq -\frac{1}{M} \int_{(0, \xi)} |\nabla u(z)||dz| + M_1.
\]

Finally we consider the case where \( (1 - |w|)/(1 - |\xi|) \geq 4 \). By Proposition 2 (e),

\[
u(\xi) - u(w) \leq -\frac{1}{M} \int_\sigma |\nabla u(z)||dz|,
\]
where $\sigma = R \cap \{s : |w| < |s| < |\xi|\}$. We let $\rho = R \cap \{s : |s| < |s_k|\}$, then $(0, \xi) = \sigma \cup \rho$. Hence using Proposition 2 (b), Lemma 3 and Proposition 3 we obtain the following estimate

$$u(\xi) - u(0) = u(\xi) - u(w) + u(w) - u(s_k) + u(s_k) - u(0)$$

$$\leq -\frac{1}{M} \int_\sigma |\nabla u(z)||dz| + 2C_4L - \frac{1}{M} \int_\rho |\nabla u(z)||dz|$$

$$\leq -\frac{1}{M} \int_{(0,\xi)} |\nabla u(z)||dz| + 2C_4L.$$

This completes the proof of Theorem 1.
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