Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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N=2 strings

by

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Preprint-Nr.: 4 1998
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Abstract: We discuss the precise relation of the open N=2 string to a self-dual Yang-Mills (SDYM) system in 2+2 dimensions. In particular, we review the description of the string target space action in terms of SDYM in a “picture hyperspace” parametrised by the standard vectorial $\mathbb{R}^{2,2}$ coordinate together with a commuting spinor of $SO(2,2)$. The component form contains an infinite tower of prepotentials coupled to the one representing the SDYM degree of freedom. The truncation to five fields yields a novel one-loop exact lagrangean field theory.

1. Introduction The relation to self-dual Yang-Mills (SDYM) of the critical open N=2 string has recently been elaborated by us [1] in view of the particular picture degeneracy and global $SO(2,2)$ properties of the physical spectrum of string states. There has been much discussion in the literature of this relationship since Ooguri and Vafa [2] first mooted the idea that the self-duality equations,

$$F_{\mu\nu} - \frac{i}{2} \epsilon_{\mu\nu\rho\lambda} F_{\rho\lambda} = 0,$$

(1)

with the field strengths taking values in the Chan-Paton Lie algebra, describe what at that stage appeared to be the single dynamical degree of freedom of the open N=2 string. (We shall not give all relevant references here, referring to [1] for further references). The comparison has been based on determinations of tree-level amplitudes for the two theories, so light-cone gauge action principles for SDYM have played a central role in the discussion. In two-spinor notation, using the splitting of the $\mathbb{R}^{2,2}$ “Lorentz algebra”,

$$so(2, 2) \cong sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})' \iff x^\mu \gamma^\alpha_{\mu} = x^\alpha = \begin{pmatrix} x^0 + x^3 & x^1 + x^2 \\ x_1 - x_2 & x^0 - x^3 \end{pmatrix},$$

(2)

the three (real) SDYM equations take the form

$$F_{\alpha\beta} \equiv \frac{i}{2} \left( \partial_{[\alpha} \gamma_{\beta]} A_{\gamma} + [A_{\alpha}, A_{\beta}] \right) = 0$$

(3)

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In components, with the spinor indices $\alpha, \beta$ taking values $+$ and $-$, we have
\begin{align*}
 F^{\alpha+} & \equiv \partial_{\alpha+} A^{+\gamma}_{\gamma} + \frac{1}{2} [A^{+\gamma}_{\gamma}, A^{+\gamma}_{\gamma}] = 0 \\
 F^{\alpha-} & \equiv \frac{1}{2} \left( \partial_{\alpha-} A^{-\gamma}_{\gamma} + \partial_{\gamma}^{-\gamma} A^{+\gamma}_{\gamma} + [A^{+\gamma}_{\gamma}, A^{-\gamma}_{\gamma}] \right) = 0 \\
 F^{-\alpha-} & \equiv \partial_{\gamma}^{-\gamma} A^{-\gamma}_{\gamma} + \frac{1}{2} [A^{-\gamma}_{\gamma}, A^{-\gamma}_{\gamma}] = 0 \ .
\end{align*}

Clearly, the $(++)$ equation affords the generalised light-cone gauge $A^{+\gamma}_{\gamma} = 0$ in which $F^{\alpha-}$ becomes homogeneous. Two strategies now suggest themselves. First, resolving the (inhomogeneous) $(---)$ equation in the Yang fashion,
\begin{equation}
 A^{-\alpha}_{\alpha} = e^{-\phi} \partial^{-\alpha}_{\alpha} e^{+\phi} \ , \tag{5}
\end{equation}
the $(+-)$ equation describes the $\phi$-dynamics in the form of the (non-polynomial) Yang equation
\begin{equation}
 \partial^{+\dot{\alpha}} (e^{-\phi} \partial^{-\alpha}_{\alpha} e^{+\phi}) = 0 \ . \tag{6}
\end{equation}
Second, the (homogeneous) $(+-)$ equation is instead fulfilled in terms of a Leznov prepotential, writing
\begin{equation}
 A^{-\alpha}_{\alpha} = \partial^{\pm}_{\alpha} \varphi^{--} \ , \tag{7}
\end{equation}
which then must satisfy $F^{--} = 0$, tantamount to the (quadratic) Leznov equation,
\begin{equation}
 \Box \varphi^{--} - \frac{1}{2} [\partial^{+\dot{\alpha}} \varphi^{--}, \partial^{\pm}_{\alpha} \varphi^{--}] = 0 \ . \tag{8}
\end{equation}

The light-cone gauge explicitly breaks the global $SO(2,2)$ covariance of eq. (3) to $GL(1,\mathbb{R}) \otimes SL(2,\mathbb{R})$. In a Cartan-Weyl basis for $sl(2,\mathbb{R})$ consisting of a diagonal hyperbolic generator $L_{+-}$ and two parabolic generators $L_{\pm\pm}$, the unbroken $gl(1,\mathbb{R})$ generator is $L_{+-}$ in the Yang but $L_{++}$ in the Leznov case.

Non-covariant action principles for (6) or (8) yield themselves using merely the prepotentials,
\begin{align*}
 S_{\text{Yang}} & = \mu^2 \int d^4 x \operatorname{Tr} \left\{ -\frac{1}{2} \Box \phi + \frac{1}{3} \phi \partial^{(+\delta} \phi \partial^{-\alpha)} \phi + \mathcal{O}(\phi^4) \right\} \tag{9} \\
 S_{\text{Leznov}} & = \mu^2 \int d^4 x \operatorname{Tr} \left\{ -\frac{1}{2} \varphi^{--} \Box \varphi^{--} + \frac{1}{3} \varphi^{--} [\partial^{+\dot{\alpha}} \varphi^{--}, \partial^{\pm}_{\alpha} \varphi^{--}] \right\} \tag{10}
\end{align*}
with some mass scale $\mu$. Alternatively, Lagrange multipliers facilitate the construction of dimensionless actions, for example,
\begin{equation}
 S_{\text{CS}} = \int d^4 x \operatorname{Tr} \left\{ -\varphi^{++} \Box \varphi^{--} + \varphi^{++} [\partial^{+\dot{\alpha}} \varphi^{--}, \partial^{\pm}_{\alpha} \varphi^{--}] \right\} \tag{11}
\end{equation}
which was shown to be even one-loop exact by Chalmers and Siegel [3].

The tree-level amplitudes following from these actions are extremely simple. Since we are dealing with massless fields in $2+2$ dimensions, the on-shell momenta factorise,
\begin{equation}
 k^{\alpha\beta} k_{\alpha\beta} = 0 \quad \iff \quad k^{\alpha\dot{\beta}} k^{\dot{\alpha}\beta} = k^{\alpha} k^{\dot{\alpha}} \ . \tag{12}
\end{equation}
The on-shell three-point functions $A_3(k_1, k_2, k_3)$ can be read off as

$$A_3^{\text{Yang}} = f^{abc} \kappa_1^{\alpha} \kappa_2^{\beta} \kappa_3^{\gamma} , \quad A_3^{\text{Leznov}} = f^{abc} \kappa_1^{\alpha} \kappa_2^{\beta} \kappa_3^{\gamma}$$

where $\sum_i k_i = 0$ and $f^{abc}$ are the structure constants of the gauge group. Surprisingly, the four-point Feynman diagrams sum to zero, in virtue of a quartic contact interaction in the Yang case. It is believed that all higher tree amplitudes vanish on-shell. The version of Chalmers and Siegel leads to the same tree-level amplitudes as the Leznov action, although in the former case one of the legs needs to be the multiplier field. Interestingly, this two-field theory does not allow diagrams beyond one loop. As we shall describe below, both (10) and (11) are related to the target space effective action for the open $N=2$ string.

2. $N=2$ Open Strings The spectrum of world-sheet fields in the NSR formulation of $N=2$ strings consists of the 2d $N=2$ supergravity multiplet, whose conformal gauge fixing produces the standard set of $N=2$ superconformal ghosts, plus $N=2$ matter fields $(X^{\alpha\beta}, \Psi^{\alpha\beta})$. The computation of open string amplitudes requires the evaluation of correlation functions for appropriate choices of physical external states on Riemann surfaces with handles, boundaries, punctures, and a harmonic $U(1)$ gauge field background with instantons. The result is to be integrated over the moduli of the Riemann surface and the $U(1)$ gauge field, and finally one is to sum over the topologies labelled by the Euler and $U(1)$ instanton numbers. The relative cohomology of the BRST operator determines the string external states, which are annihilated by the commuting $N=2$ Virasoro and the anticommuting $N=2$ antighost zero modes. The resulting spectrum has the following quantum numbers:

- total ghost number $u \in \mathbb{Z}$
- target space momentum $k^{\alpha\beta}$
- total picture $\pi \in \mathbb{Z}$
- picture twist $\Delta \in \mathbb{R}$
- $gl(1,\mathbb{R}) \oplus sl(2,\mathbb{R})$ quantum numbers $m, (j', m')$

These quantum numbers are however redundant, since they have interrelationships:

$$k \cdot k = 0 \quad (i.e. \ k^{\alpha\beta} = \kappa^{\alpha} \kappa^{\beta} ), \quad u - \pi = 1, \quad j' = m' = 0, \quad m \text{ runs in integral steps from } -j \text{ to } +j, \quad \text{where } j := \frac{\pi}{2} + 1.$$ 

The physical spectrum consists of just one $SL(2,\mathbb{R})'$ singlet for each value of $\pi$, $\Delta$, and $(\kappa, \kappa)$. There is still a certain redundancy, since the pictures $(\pi \geq \pi_0, \Delta)$ can be reached from $(\pi_0, 0)$ by applying spectral flow $S$ and picture raising $P^\alpha$, which commute with the BRST operator and effect the mappings

$$(\pi, \Delta) \xrightarrow{S(\rho)} (\pi, \Delta + 2\rho) , \quad (\pi, \Delta) \xrightarrow{P^\alpha} (\pi + 1, \Delta)$$

with $\rho \in \mathbb{R}$. Because the string path integral integrates over the twists of the $U(1)$ gauge bundle, it averages over the spectral flow orbits. The $S$-equivalent states therefore ought to be identified and we may choose the $\Delta=0$ representative. Picture lowering can also be constructed, except on zero-momentum states. In essence, all
physical states (with $k \neq 0$) can be generated starting from the canonical picture $\pi = -2$ (i.e. $j=0$), and the result is symmetric under the “Poincaré duality” $\pi \rightarrow -4 - 2\pi$ (i.e. $j \rightarrow -j$):

$$
\begin{array}{c|cccccccc}
\pi & \cdots & -5 & -4 & -3 & -2 & -1 & 0 & +1 & \cdots \\
\hline
j & \cdots & \frac{\alpha \beta \gamma}{\alpha} & -1 & \frac{\alpha \beta}{\alpha} & 0 & \frac{\alpha \beta}{\alpha} & +1 & \frac{\alpha \beta \gamma}{\alpha} & \cdots .
\end{array}
$$

(15)

The states form $SL(2, \mathbb{R})$ tensors of rank $2|j|$ (spin $|j|$), because the picture-raising operator $P^a$ carries a spinor index. There is no contradiction with the above statement of unit multiplicity, since all states in a given $SL(2, \mathbb{R})$ multiplet are related to each other, albeit in a non-local fashion. The open spinor indices are just carried by normalisation factors multilinear in $\kappa^a$, with $P^a$ increasing the spin by $\frac{1}{2}$:

$$
P^{a_1}P^{a_2} \ldots P^{a_{2j}} |(0); k\rangle = |a_1a_2 \ldots a_{2j}; k\rangle \propto \kappa^{a_1}\kappa^{a_2} \ldots \kappa^{a_{2j}} |(j); k\rangle .
$$

(16)

The NSR formulation of $N=2$ strings introduces a complex structure in the target space, which explicitly breaks $SO(2, 2) \rightarrow GL(1, \mathbb{R}) \otimes SL(2, \mathbb{R})$. Individual pieces of an $n$-point amplitude are only $SL(2, \mathbb{R})$ invariant, and contributions from the $M$-instanton $U(1)$ background carry a $gl(1, \mathbb{R})$ weight equal to $M$. Surprisingly, the path integral measure constrains the instanton sum to $|M| \leq J \equiv n-2$ at tree level. Moreover, the weight factors built from the string coupling $e$ and the instanton angle $\theta$ conspire to restore $SO(2, 2)$ invariance of the instanton sum if $\sqrt{e} \cos \frac{\theta}{2}, \sin \frac{\theta}{2}$ is assumed to transform as an $SL(2, \mathbb{R})$ spinor! This spinor simply parametrises the choices of complex structure, and it may be Lorentz-rotated to $(1, 0)$. Henceforth we shall remain in such a frame where $e=1$ and $\theta=0$. It has the virtue that only the highest $SL(2, \mathbb{R})$ weights $m_i = j_i$, $i = 1, \ldots, n$, occur and only the maximal instanton number sector, $M = J$, contributes.

The tree-level open string on-shell amplitudes may then be found to be

$$
A^{\text{string}}_3 = f^{abc} \kappa^+_1 \kappa^+_2 \kappa^+_3 \kappa^-_4 = A^{\text{Leznoy}}_3
$$

(17)

$$
A^{\text{string}}_4 \propto \kappa^+_1 \kappa^+_2 \kappa^+_3 \kappa^+_4 (\kappa^+_1 \kappa^+_2 \kappa^+_3 \kappa^-_4 t + \kappa^+_2 \kappa^+_3 \kappa^-_4 \kappa^-_1 s) = 0
$$

(18)

$$
A^{\text{string}}_{n>4} = 0 .
$$

(19)

They are independent of the external $SL(2, \mathbb{R})$ spins $j_i$, as long as $\sum_{i=1}^n j_i = J \equiv n-2$. Clearly, the Leznoy version (13) of SDYM is reproduced. However, a covariant description needs to take the entire tower of states in (15) into account.

3. **Target Space Actions**

Physical string states correspond to target space (background) fields, whose on-shell dynamics is determined by the string scattering amplitudes. In particular, the string three-point functions directly yield cubic terms in the effective target space action. In the present case, the correspondence reads

$$
|++ \ldots +\rangle \leftrightarrow \varphi^{-\ldots-} (j \geq 0) , \quad |-- \ldots -\rangle^t \leftrightarrow \varphi^{++\ldots+} (j < 0)
$$

(20)
and we denote the fields by $\varphi_j$. Then, the target space effective action for the infinite tower $\{\varphi_j\}$ is

$$S_\infty = \int d^4x \, \text{Tr} \left\{ -\frac{1}{2} \sum_{j \in \mathbb{Z}/2} \varphi(-j) \Box \varphi(+j) + \frac{i}{3} \sum_{j_1 + j_2 + j_3 = 1} \varphi(j_1) \left[ \partial^{+\dot{\alpha}} \varphi(j_2) \right. \left, \partial^{+\dot{\alpha}} \varphi(j_3) \right] \right\}$$

$$= \int d^4x \, \text{Tr} \left\{ -\frac{1}{2} \Phi^{--} \Box \Phi^{--} + \frac{1}{6} \Phi^{--} \left[ \partial^{+\dot{\alpha}} \Phi^{--}, \partial^{+\dot{\alpha}} \Phi^{--} \right] \right\}_{\eta^4} \quad (21)$$

where we have introduced a “picture hyperfield”,

$$\Phi^{--}(x, \eta^-) = \sum_j (\eta^-)^{2j} \varphi_{1-j}(x) , \quad (22)$$

depending on an extra commuting coordinate $\eta^-$, and we project the Lagrangean onto the part quartic in $\eta$. It is remarkable that the action (21) has the Leznov form in terms of the hyperfield. It not only reproduces all (tree-level) string three-point functions (17) but also yields vanishing four- and probably higher-point functions for the same reason that the Leznov action (10) does. Picture raising induces a dual action on the component fields,

$$Q^+: \quad \varphi_j \rightarrow (3-2j) \varphi_{j-\frac{1}{2}} , \quad (23)$$

which is nothing but the $\eta^-$ derivative on the hyperfield!

Three successive truncations to a finite number of fields are possible. First, keeping only $\{\varphi_-, \varphi_{-\frac{1}{2}}, \varphi_0, \varphi_{\frac{1}{2}}, \varphi_+\}$, a consistent five-field model ensues, viz.,

$$S_5 = \int d^4x \, \text{Tr} \left\{ \frac{1}{2} \partial^{+\dot{\alpha}} \varphi \partial^{\dot{\alpha}} \varphi + \partial^{+\dot{\alpha}} \varphi^+ \partial^{\dot{\alpha}} \varphi^- + \partial^{+\dot{\alpha}} \varphi^- \partial^{\dot{\alpha}} \varphi^- + \frac{1}{2} \varphi \left[ \partial^{+\dot{\alpha}} \varphi^-, \partial^{+\dot{\alpha}} \varphi^- \right] + \frac{1}{2} \varphi^{--} \left[ \partial^{\dot{\alpha}} \varphi^+, \partial^{\dot{\alpha}} \varphi^- \right] + \frac{1}{2} \varphi^{--} \left[ \partial^{\dot{\alpha}} \varphi^-, \partial^{\dot{\alpha}} \varphi^- \right] \right\} . \quad (24)$$

Second, eliminating also the fermions leaves us with three fields. Third, we may in addition kill $\varphi_0$ as well, resulting in the two-field model of Chalmers and Siegel [3]! All truncations share the one-loop exactness mentioned before.

4. Self-Duality in Hyperspace  The infinite tower of higher-spin fields which arise from the picture degeneracy parametrise simply SDYM in a hyperspace with coordinates $\{x^\alpha, \eta^\alpha, \tilde{\eta}^\dot{\alpha}\}$, with $\eta$ and $\tilde{\eta}$ commuting spinors. This commutative variant of superspace exhibits a $\mathbb{Z}_2$-graded Lie-algebra variant of the super-Poincare algebra (i.e. with all anti-commutators replaced by commutators). So the covariant target space symmetry is effectively the extension of the $\mathbb{R}^{2,2}$ Poincaré algebra by two Grassmann-even spinorial generators squaring to a translation, i.e., $[Q_\alpha, Q_\beta] = P^\alpha_\beta$ (see [1] for details). Hyperspace self-duality allows compact expression in a chiral subspace independent of the $\tilde{\eta}$ coordinates. In terms of chiral subspace gauge-covariant
derivatives $\mathcal{D}_\alpha = \partial_\alpha + A_\alpha(x, \eta)$ and $\mathcal{D}_{\alpha \dot{\alpha}} = \partial_{\alpha \dot{\alpha}} + A_{\alpha \dot{\alpha}}(x, \eta)$, the self-duality conditions take the simple form

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] = \epsilon_{\alpha \beta} F_{\dot{\alpha} \dot{\beta}}, \quad [\mathcal{D}_\alpha, \mathcal{D}_{\beta \dot{\beta}}] = \epsilon_{\alpha \beta} F_{\dot{\alpha} \dot{\beta}}, \quad [\mathcal{D}_{\alpha \dot{\alpha}}, \mathcal{D}_{\beta \dot{\beta}}] = \epsilon_{\alpha \beta} F_{\alpha \beta}.$$  \hspace{1cm} (25)

Jacobi identities yield the equations

$$\mathcal{D}_\alpha \partial F_{\alpha \beta} = 0, \quad \mathcal{D}_\alpha \partial F_{\dot{\alpha}} = 0, \quad \mathcal{D}_{\alpha \dot{\alpha}} F = \mathcal{D}_\alpha F_{\alpha \dot{\alpha}}.$$  \hspace{1cm} (26)

The first two are respectively the Yang-Mills and Dirac equations for a SDYM multiplet, and the third implies the scalar field equation $\mathcal{D}^2 F = [F_{\dot{\alpha}}, F_{\alpha \dot{\alpha}}]$. All chiral hyperfields have $\eta$ expansions, e.g.

$$A_\alpha(x, \eta) = A_\alpha(x) + \eta^\beta A_{\alpha \beta}(x) + \eta^\beta \eta^\gamma A_{\alpha \beta \gamma}(x) + \ldots.$$  \hspace{1cm} (27)

Choosing the light-cone gauge, $A^+ = 0 = A^-\alpha$, we note that all fields are defined in terms of a generalised Leznov prepotential,

$$A^- = \partial^+ \Phi^{-}, \quad A^-\beta = \partial^+ \partial^\beta \Phi^{-},$$  \hspace{1cm} (28)

$$F = \partial^+ \partial^+ \Phi^{-}, \quad F_\alpha = \partial^+ \partial^\alpha \Phi^{-}, \quad F_{\alpha \beta} = \partial^+ \partial^\alpha \partial^\beta \Phi^{-}.$$  \hspace{1cm} (29)

Since $\partial^-$ does not occur in the above, all fields are determined by the chiral ($\eta^+$-independent) part of $\Phi^{-}$. The dynamics is determined by the remaining constraints

$$[\mathcal{D}^-\alpha, \mathcal{D}^-\beta] = 0, \quad [\mathcal{D}^-, \mathcal{D}^-\beta] = 0,$$  \hspace{1cm} (30)

where the former equation is precisely the Leznov equation for $\Phi^{-}$. Choosing this to be chiral, $\Phi^{-} = \Phi^{-}(x, \eta^-)$, allows identification with (22), with action given by (21). The second equation above then merely determines the $\eta^-$ dependence of $\Phi^{-}$.

The restricted system of five fields (24) has the $SO(2, 2)$-invariant action

$$S_5^{\text{inv}} = \int d^4x \text{Tr} \left\{ \frac{1}{4} g^{\alpha \beta} F_{\alpha \beta} + \frac{1}{8} \chi^\alpha \mathcal{D}_{\alpha \dot{\alpha}} F_{\dot{\alpha}} + \frac{1}{8} \mathcal{D}^\alpha F \mathcal{D}_{\alpha \dot{\alpha}} F + \frac{1}{2} F [F_{\dot{\alpha}}, F_{\alpha \dot{\beta}}] \right\}$$  \hspace{1cm} (31)

where $g^{\alpha \beta}$ and $\chi^\alpha$ are (propagating) multiplier fields for $A_\alpha$ and $F_{\alpha \dot{\alpha}}$, respectively.

The similarity with $N=4$ supersymmetric SDYM [4] is evident, however with commuting single-multiplicity fermions replacing multiplicity 4 anticommuting ones.

To conclude, we note that theories of $N=2$ closed as well as $N=(2,1)$ heterotic strings are also intimately related to self-dual geometry and our covariant hyperspace description generalises to both these cases.

References

[1] C. Devchand and O. Lechtenfeld, 
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