Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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Preprint-Nr.: 45 1998

SELF DUALITY EQUATIONS FOR GINZBURG-LANDAU AND SEIBERG-WITTEN TYPE FUNCTIONALS WITH 6^{th} ORDER POTENTIALS

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Version 19.10.1998

Abstract. The abelian Chern-Simons-Higgs model of Hong-Kim-Pac and Jackiw-Weinberg leads to a Ginzburg-Landau type functional with a 6^{th} order potential on a compact Riemann surface. We derive the existence of two solutions with different asymptotic behavior as the coupling parameter tends to 0, for any number of prescribed vortices. We also introduce a Seiberg-Witten type functional with a 6^{th} order potential and again show the existence of two asymptotically different solutions on a compact Kähler surface. The analysis is based on maximum principle arguments and applies to a general class of scalar equations.

0. Introduction

Let Σ be a compact Riemann surface with a line bundle L. For a unitary connection $D_A = d + A$ on L with curvature F_A, and a section ϕ of L, we have the Ginzburg-Landau functional

$$GL(A, \phi) = \int_\Sigma (|D_A\phi|^2 + |F_A|^2 + \frac{1}{4}(1 - |\phi|^2)^2) * 1.$$

This functional can be rewritten as

$$GL(A, \phi) = \int_\Sigma (|(D_1 + iD_2)\phi|^2 + (F_A - \frac{1}{2}(1 - |\phi|^2))^2) * 1 + 2\pi \deg L,$$

see *e.g.* [J; §9.1][1] for details. This reformulation shows that absolute minimizers satisfy the self duality equations

$$(D_1 + iD_2)\phi = 0$$
$$F = \frac{1}{2}(1 - |\phi|^2).$$

Key words and phrases. Chern-Simons-Higgs model, Ginzburg-Landau functional, Seiberg-Witten functional, self duality equations, exponential nonlinearity.

[1]Here, however, in agreement with the physics literature $A = -iA_\alpha dx^\alpha$, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, $F_A = -\frac{i}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta$, $F = F_{12}$, $D_\alpha = \partial_\alpha - iA_\alpha$. We assume w.l.o.g. that the degree of L, $\deg L$, is nonnegative.

The first equation says that ϕ is a holomorphic section of the line bundle L.

The self duality mechanism still works if we introduce a coupling parameter ε as follows

$$\begin{aligned} GL_\varepsilon(A,\phi) &= \int_\Sigma (|D_A\phi|^2 + \varepsilon^2|F_A|^2 + \frac{1}{4\varepsilon^2}(1-|\phi|^2)^2) * 1 \\ &= \int_\Sigma (|(D_1+iD_2)\phi|^2 + (\varepsilon F - \frac{1}{2\varepsilon}(1-|\phi|^2))^2) * 1 \\ &\quad + 2\pi \deg L. \end{aligned}$$

The self duality equations then are

$$\begin{aligned} (D_1+iD_2)\phi &= 0 \\ \varepsilon^2 F_{12} &= \frac{1}{2}(1-|\phi|^2). \end{aligned}$$

$N := \deg L$ is the degree of L and determines the number of zeroes $p_1, \cdots, p_N$ (counted with multiplicity) of ϕ. With

$$u := \log |\phi|^2,$$

the equations are reduced to the single scalar equation

$$\Delta u = \frac{1}{2\varepsilon^2}(e^u - 1) + 4\pi \sum_{i=1}^N \delta_{p_i},$$

where δ_{p_i} is the Dirac functional based at p_i.

It follows from the analysis of Taubes [T1] that there exists $\varepsilon_c > 0$ such that for $0 < \varepsilon < \varepsilon_c$, this equation has a unique solution u_ε for any prescribed set of vortices $p_1, \cdots, p_N$. Hong-Jost-Struwe [HJS] carried out the asymptotic analysis of u_ε for $\varepsilon \to 0$. In the limit, $|\phi_\varepsilon|$ tends to 1 away from the vortices, and the curvature F_{A_ε} becomes a sum of delta distributions centered at the vortices. Thus, the line bundle is degenerated into a flat bundle with a covariantly constant section with N singular points where the curvature concentrates.

As described in [J; §9.1], the self duality mechanism works in still more generality, namely, we may replace the parameter ε by an arbitrary real function $\gamma(\phi)$ of ϕ and consider

$$\begin{aligned} GL_\gamma(A,\phi) &= \int_\Sigma (|D_A\phi|^2 + \gamma(\phi)^2|F_A|^2 + \frac{1}{4\gamma(\phi)^2}(1-|\phi|^2)^2) * 1 \\ &= \int_\Sigma (|(D_1+iD_2)\phi|^2 + (\gamma(\phi)F - \frac{1}{2\gamma(\phi)}(1-|\phi|^2))^2) * 1 \\ &\quad + 2\pi \deg L. \end{aligned}$$

For the choice

$$\gamma(\phi) = \frac{\varepsilon}{|\phi|},$$

we obtain the Chern-Simons-Higgs functional introduced by Hong-Kim-Pac [HKP] and Jackiw-Weinberg [JW] for the time independent vortex solutions of an abelian Chern-Simons-Higgs model on $\mathbb{R}^{2,1}$, namely

$$\begin{aligned} CS(A,\phi) &= \int_\Sigma (|D_A\phi|^2 + \frac{\varepsilon^2}{|\phi|^2}|F_A|^2 + \frac{1}{4\varepsilon^2}|\phi|^2(1-|\phi|^2)^2) * 1 \\ &= \int_\Sigma (|(D_1+iD_2)\phi|^2 + (\frac{\varepsilon}{|\phi|}F - \frac{1}{2\varepsilon}|\phi|(1-|\phi|^2))^2) * 1 \\ &\quad + 2\pi \deg L. \end{aligned}$$

Absolute minimizers satisfy the following self duality equations

$$(D_1 + iD_2)\phi = 0 \tag{0.1}$$

$$\varepsilon^2 F = \frac{1}{2}|\phi|^2(1-|\phi|^2). \tag{0.2}$$

The first authors to consider this problem on a compact Riemann surface, namely a torus, were Caffarelli-Yang [CY]. They introduced a sub/supersolution method to construct a solution $(A^1_\varepsilon, \phi^1_\varepsilon)$ for every positive ε below some critical threshold ε_c above which no solution exists. For $\varepsilon \to 0$, this solution has the same asymptotic behavior as one of the Ginzburg-Landau model described above. Tarantello [Ta] then showed the existence of a second solution $(A^2_\varepsilon, \phi^2_\varepsilon)$ for $0 < \varepsilon < \varepsilon_c$ (as follows from [DJLW1], there may exist more than two solutions). For the case of one vortex, $N = 1$, she was able to analyze the asymptotic behavior of a second solution; ϕ^2_ε converges to 0 uniformly for $\varepsilon \to 0$, and after rescaling, one obtains a solution of an interesting mean field equation whose geometric significance remains to be explored. The method was restricted to $N = 1$ because it was of a variational nature and depended on the Moser-Trudinger inequality. The case $N = 2$ represents a borderline case for this inequality and was treated in [DJLW1, DJLW2] and [NT]. In the present paper, we construct a second solution for which we are able to perform the asymptotic analysis for an arbitrary number N of vortices, thereby completing this line of investigation.

As in the quoted previous papers, by putting

$$v := \log|\phi|^2,$$

we reduce the above system to the single scalar equation

$$\Delta v = \frac{4}{\varepsilon^2}e^v(e^v - 1) + 4\pi\sum_{j=1}^N \delta_{p_j},$$

or with u_0 being the corresponding Green function, *i.e.* the solution of

$$\Delta u_0 = -\frac{4\pi N}{|\Sigma|} + 4\pi\sum_{j=1}^N \delta_{p_j},$$

$$\int_\Sigma u_0 = 0,$$

$u = v - u_0$ satisfies

$$\Delta u = \frac{4}{\varepsilon^2} K e^u (K e^u - 1) + A, \tag{0.3}$$

with $K = e^{u_0}$, $A = \frac{4\pi N}{|\Sigma|}$. This is the equation we shall study in some generality, namely on an arbitrary compact Riemannian manifold.

Our result for the Chern-Simons-Higgs problem then is

Theorem 0.1. *For $N > 0$, $p_1, \cdots, p_N \in \Sigma$ and $0 < \varepsilon < \varepsilon_c$, there are solutions $(A^1_\varepsilon, \phi^1_\varepsilon)$ and $(A^2_\varepsilon, \phi^2_\varepsilon)$ of (0.1) - (0.2) such that for $\varepsilon \to 0$*

(1) $|\phi^1_\varepsilon| \to 1$ *on every $\Omega \subset\subset \Sigma \setminus \{p_1, \cdots, p_N\}$;*
(2) $|\phi^2_\varepsilon| \to 0$ *almost everywhere.*

(1) of course is the result of Caffarelli-Yang [CY]. The first solution corresponds to a topological, the second one to a non- topological solution of the field equations.

As already indicated, our method works in any dimension. Therefore, we now introduce a functional on a 4-manifold, namely a generalization of the Seiberg-Witten functional with 6^{th} order potential obtained by the same type of self duality mechanism as above to which our method also applies, at least if the manifold is Kähler.

First, we recall some facts from the Seiberg-Witten theory (for more details, see [J], [JPW] and [S]). Let (X, g) be a compact, oriented four-dimensional manifold with a Riemannian metric g, and $P_{SO(4)} \to X$ its oriented orthonormal frame bundle. Let $spin^c(4)$ be the $U(1)$ extension of $SO(4)$, namely,

$$1 \to U(1) \to spin^c(4) \to SO(4) \to 1.$$

A $spin^c$-structure on the Riemannian Manifold (X, g) is a lift of the structure group $SO(4)$ to $spin^c(4)$, i.e. there is a principle $spin^c(4)$-bundle $P_{spin^c(4)} \to X$ such that there is a bundle map

$$\begin{array}{ccc} P_{spin^c(4)} & \longrightarrow & P_{SO(4)} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \end{array}$$

It is well-known that any compact, oriented four-manifold admits a $spin^c$-structure.

Let $Q = P_{spin^c(4)}/spin(4)$ be a principle $U(1)$-bundle. $W = P_{spin^c(4)} \times_{spin^c(4)} \mathbb{C}^4$ and $L = Q \times_{U(1)} \mathbb{C}$ resp. is the associated spinor bundle and the line bundle resp.. W can be decomposed globally as W^+ and W^-. Locally,

$$W^\pm = S^\pm \otimes L^{1/2}.$$

Here $S^\pm$ is a spinor bundle with respect to a local $spin$-structure on X. Both $S^\pm$ and $L^{1/2}$ are locally defined.

There exists a Clifford multiplication

$$TX \times W^+ \to W^-$$

denoted by $e \cdot \phi \in W^-$ for $e \in TX$ and $\phi \in W^+$. Here TX is the tangent bundle of X. A connection on the bundle W^+ can be defined by the Levi-Civita connection and a connection on L. The "twisted" Dirac operator $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is defined by

$$D_A = \sum_{i=1}^{4} e_i \cdot \nabla_A.$$

Here, $\Gamma(W^\pm)$ is the space of sections of $W^\pm$, $\{e_i\}$ is an orthonormal basis of TX and ∇_A is a connection on W^+ induced by the Levi-Civita connection and a connection A on the line bundle L. Let $\mathcal{A}(L)$ be the space of the hermitian connections of the line bundle L.

The Seiberg-Witten functional is defined for pairs $(A, \phi) \in \mathcal{A}(L) \times \Gamma(W^+)$:

$$SW(A,\phi) = \int_X (|\nabla_A \phi|^2 + |F_A^+|^2 + \frac{R}{4}|\phi|^2 + \frac{1}{8}|\phi|^4) * 1,$$

where F_A^+ is the self-dual part of the curvature of A, and R is the scalar curvature of X. Using the Weitzenböck formula, this can be rewritten as

$$SW(A,\phi) = \int_X (|D_A \phi|^2 + |F_A^+ - \frac{1}{4}\langle e_i e_j \phi, \phi\rangle e^i \wedge e^j|^2) * 1,$$

where $\{e^i\}$ is the dual basis of an orthonormal basis $\{e_i\}$. ¿From this reformulation, one directly sees the self duality involved: Absolute minimizers satisfy the Seiberg-Witten equations

$$\begin{aligned} D_A \phi &= 0, \\ F_A^+ &= \frac{1}{4}\langle e_i e_j \phi, \phi\rangle e^i \wedge e^j. \end{aligned}$$

Now, first of all, the Seiberg-Witten functional may be perturbed by adding 2-forms σ, η in the functional:

$$\begin{aligned} SW_{\sigma,\eta}(A,\phi) &= \int_X (|D_A\phi|^2 + |(F_A^+ - \sigma) - \frac{1}{4}((\langle e_i e_j \phi, \phi\rangle e^i \wedge e^j - \eta)|^2) * 1 \\ &= \int_X (|\nabla_A \phi|^2 + |F_A^+|^2 + \frac{R}{4}|\phi|^2 \\ &\qquad + |\eta - \langle e_i e_j \phi, \phi\rangle e^i \wedge e^j|^2 + 2\langle F_A^+, \eta - \sigma\rangle) * 1. \end{aligned}$$

Secondly, the self duality mechanism still works if we insert a real-valued function $\gamma(\phi)$ of ϕ in the following manner:

$$\begin{aligned} SW_{\sigma,\eta,\gamma}(A,\phi) &= \int_X (|\nabla_A \phi|^2 + \gamma(\phi)^2 |F_A^+|^2 + \frac{R}{4}|\phi|^2 \\ &\qquad + \frac{1}{\gamma(\phi)^2}|\eta - \langle e_i e_j \phi, \phi\rangle e^i \wedge e^j|^2 + 2\langle F_A^+, \eta - \sigma\rangle) * 1 \\ &= \int_X (|D_A \phi|^2 \\ &\qquad + |\gamma(\phi)(F_A^+ - \sigma) - \frac{1}{4\gamma(\phi)}((\langle e_i e_j \phi, \phi\rangle e^i \wedge e^j - \eta)|^2) * 1. \end{aligned}$$

In analogy with the Chern-Simons-Higgs functional discussed above, we choose

$$\gamma(\phi) = \frac{\varepsilon}{|\phi|}$$

for a real parameter $\varepsilon > 0$. This choice seems to lead to the most natural and interesting theory, and so we study the following Seiberg-Witten type functional with 6^{th} order potential

$$\begin{aligned} L(A,\phi) &= \int_X (|D_A\phi|^2 + \left|\frac{\varepsilon}{|\phi|}(F_A^+ - \sigma) - \frac{|\phi|}{4\varepsilon}(\langle e_i e_j \phi, \phi\rangle e^i \wedge e^j - \eta)\right|^2) * 1 \\ &= \int_X (|\nabla_A \phi|^2 + \frac{\varepsilon^2}{|\phi|^2}|F_A^+|^2 + \frac{R}{4}|\phi|^2 \\ &\quad + \frac{|\phi|^2}{\varepsilon^2}|\eta - \langle e_i e_j \phi, \phi\rangle e^i \wedge e^j|^2 + 2\langle F_A^+, \eta - \sigma\rangle) * 1. \end{aligned}$$

Given $\eta \in \wedge^2 T^* X$, we thus consider the self duality equations that are satisfied by minimizers of $L(A,\phi)$, namely

$$\begin{aligned} D_A\phi &= 0, \\ F_A^+ - \sigma &= \frac{1}{4\varepsilon^2}|\phi|^2(\langle e_i e_j \phi, \phi\rangle e^i \wedge e^j - \eta). \end{aligned} \tag{0.4}$$

The Seiberg-Witten functional as described above exhibits a strong structural similarity with the Ginzburg-Landau functional; namely it contains a squared covariant derivative of the scalar field, a squared curvature of the vector field and a 4^{th} order potential term for the scalar field. In fact, the Ginzburg-Landau functional can be considered as a dimensional reduction of the Seiberg-Witten functional. The analogy goes further. In the Ginzburg-Landau functional with parameter ε, one sees that for $\varepsilon \to 0$, the (unique) solution $(A_\varepsilon, \phi_\varepsilon)$ concentrates at the prescribed vortices, in the sense that ϕ_ε converges to 1 uniformly away from those vortices, and the curvature F_{A_ε} tends to a Delta distribution supported at the vortices, see [HJS]. Taubes [T2, T3, T4] showed that on a symplectic manifold, with η being the symplectic 2-form, the Seiberg-Witten functional with parameter ε (*i.e.* $\gamma(\phi) = \varepsilon$ in our above notation) for $\varepsilon \to 0$ exhibits a similar limiting behavior in the sense that now a concentration along a set of pseudo-holomorphic curves occurs. Recently, Lin and Rivière [LR] were able to obtain such a concentration analysis in a general context in arbitrary dimension.

As we discussed above, the Chern-Simons-Higgs functional exhibits a richer asymptotic structure than the Ginzburg-Landau functional, in the sense that we are able to show in this paper the existence of two very different types of asymptotic solutions for $\varepsilon \to 0$, for any number of vortices. As the structural relation between our functional L and the Chern-Simons-Higgs functional is completely analogous to the one between the Seiberg-Witten functional and the Ginzburg-Landau one, we also expect an analogously rich asymptotic behavior for L. In the present paper, we perform the corresponding analysis in the case where X is a Kähler surface. In this case, our self duality equations admit a reduction to a single scalar valued equation of the same type as (0.3), to be derived in section 1. We shall prove

Theorem 0.2. *Let (X,ω) be a compact Kähler surface with a $spin^c$-structure induced by a hermitian line bundle $E \to X$, and let K be the canonical line bundle of the Kähler surface X. Let $\eta = \omega$, and $\sigma = F_{A_{can}}$, where A_{can} is the canonical connection on K^* induced by the Kähler metric. There exists ε_c with $\frac{1}{\varepsilon_c^2} > \frac{1}{\mathrm{Vol}(X)} 64\pi c_1(E)\cdot[\omega]$ such that for any $\varepsilon < \varepsilon_c$ the equation (0.4) admits two solutions $(A^1_\varepsilon, \phi^1_\varepsilon)$ and $(A^2_\varepsilon, \phi^2_\varepsilon)$, with the following asymptotic behavior:*

(1) $|\phi^1_\varepsilon| \to 1$ *almost everywhere, as* $\varepsilon \to 0$;
(2) $|\phi^2_\varepsilon| \to 0$ *almost everywhere, as* $\varepsilon \to 0$.

Technically, our approach will be based on maximum principle arguments. Variational arguments do not seem to work already in the case of the Chern-Simons-Higgs functional for more than two vortices, because the case of two vortices is the limiting case for Moser-Trudinger inequality as explained above. For the functional L, a 6^{th} order potential term can not be controlled by a squared derivative via a Sobolev type embedding theorem. In fact, in physical terms, our functional L will lead to a nonrenormalizable theory, and so no general approach applies. Our point here, however, is that although we are beyond the range of embedding theorems, there still exists a finer internal structure that allows to draw interesting consequences.

We expect, however, that a similar result also holds in the general case of a symplectic 4-manifold X; necessarily, the analysis needs to be some what different as one has to deal with vector valued equations. We speculate that the expected two types of asymptotic regimes will lead to topological applications by allowing to relate topological quantities identified by the two different asymptotic solutions.

The paper is organized as follows. In Section 1, we derive the reduction to a scalar valued equation of the equation (1.8), if X is a Kähler surface. In Section 2, we show the existence of two solutions. The first solution is obtained by the super/subsolution of Caffarelli-Yang [CY]. The second solution is constructed with the help of the mountain pass method for some associated functional. We use a heat flow to construct the required deformation. This constitutes the main technical innovation of the present paper compared to previous works on the Chern-Simons-Higgs functional. Section 3 then establishes the different asymptotic behavior of the two types of solutions.

The research for this paper was carried out at the Max-Planck Institute for Mathematics in the Sciences in Leipzig. The first, third, and fifth author thank the institute for generous hospitality and good working conditions.

1. The self duality equations on a Kähler surface

In this section we shall derive the self duality equations for our generalized Seiberg-Witten functional on Kähler surfaces.

Let (X,ω,J) be a Kähler surface with Kähler metric $g(v,w)=\omega(v,Jw)$. The tangent bundle of X carries a canonical $spin^c$ structure with

$$W_{can} = \wedge^{0,*}T^*X, \quad L_{can} = K^* = \wedge^{0,2}T^*X,$$

where K is the canonical line bundle of X. The Levi-Civita connection of the Kähler metric induces a canonical connection A_{can} on the line bundle L_{can}, and the curvature tensor considered as a 2-form $F_{A_{can}}$ of type (1,1), represents the first Chern class of the line bundle, namely,

$$[\frac{i}{2\pi}F_{A_{can}}] = c_1(L_{can}) = -c_1(K).$$

Let $E \to X$ be a hermitian line bundle over X, and consider the *spin*^{c}-structure corresponding to the line bundle

$$L_E = K^* \otimes E^2.$$

Then we have the spinor bundles

$$W_E^+ = (\wedge^{0,0} \oplus \wedge^{0,2}) \otimes E,$$
$$W_E^- = \wedge^{0,1} \otimes E,$$

where $\wedge^{p,q} = \wedge^{p,q}T^*X$.

Let $\mathcal{A}(E)$ be the space of hermitian connections on E. For $B \in \mathcal{A}(E)$, we have an induced connection

$$A = A_{can} + B^2 \in \mathcal{A}(L_E)$$

on the line bundle $L_E = K^* \otimes E^2$, with curvature 2-form given by

$$F_A = F_{A_{can}} + 2F_B.$$

The self duality equations (0.4) reduce to the following equations for the pair (B, Φ),

$$\begin{aligned}
&\bar\partial_B \phi_0 + \bar\partial_B^* \phi_2 = 0,\\
&4(F_B - \sigma)^{0,2} = \frac{1}{\varepsilon^2}\bar\phi_0\phi_2|\Phi|^2 + \eta^{0,2},\\
&4i(F_{A_{can}} + 2F_B - \sigma)_\omega = \frac{1}{\varepsilon^2}|\Phi|^2(|\phi_2|^2 - |\phi_0|^2 + \eta_\omega),
\end{aligned} \tag{1.1}$$

where $\Phi = (\phi_0, \phi_2) \in (\wedge^{0,0} \otimes E) \times (\wedge^{0,2} \otimes E)$, and the perturbations σ, $\eta \in i\wedge^{2,+}$ are self-dual 2-forms with respect to the Kähler metric g. η_ω is the component of η in the direction of ω.

Remark 1.1. Without the $|\Phi|^2$ term the equation (1.1) is exactly the Seiberg-Witten equation on a Kähler manifold. As in that case (see [S]), we have

Proposition 1.2. *Let X be a connected Kähler surface, and σ, $\eta \in \wedge^{1,1} \cap \wedge^{2,+}$. Then for any solution (B, Φ) of the equation (1.1) either $\phi_0 = 0$ or $\phi_2 = 0$.*

Proof. The proof is same as in the Seiberg-Witten case, since we need only the first two equations of (1.1) to get the conclusion.

Applying the operator $\bar\partial_B$ to the first equation of (1.1), and using $\bar\partial_B\bar\partial_B = F_B^{0,2}$ and the second equation of (1.1), we have

$$\bar\partial_B\bar\partial_B^*\phi_2 = -\bar\partial_B\bar\partial_B\phi_0 = -F_B^{0,2}\phi_0 = -\frac{1}{4\varepsilon^2}|\phi_0|^2|\Phi|^2\phi_2.$$

Now take the L^2-product with ϕ_2 to get

$$\int_X |\bar{\partial}_B^* \phi_2|^2 + \frac{1}{4\varepsilon^2}|\phi_0|^2|\phi_2|^2|\Phi|^2 = 0.$$

Then this yields

$$\bar{\partial}_B^* \phi_2 = 0, \quad \bar{\partial}_B \phi_0 = 0, \quad \text{and } |\phi_0|^2|\phi_2|^2 = 0.$$

By the unique continuation theorem for the Dirac operator, we obtain the conclusion. □

As in the Seiberg-Witten case, which one of the two sections ϕ_0, ϕ_2 vanishes is determined by the topology of the line bundle L_E, if $\sigma = 0$, and $\eta = 0$.

Proposition 1.3. *Let X be a connected Kähler surface and let (B, Φ) be a solution of the equation* (1.1) *with $\sigma = 0$, $\eta = 0$. Then*

$$\begin{aligned} (2c_1(E) - c_1(K))[\omega] < 0 &\iff \phi_0 \neq 0, \quad \phi_2 = 0; \\ (2c_1(E) - c_1(K))[\omega] > 0 &\iff \phi_0 = 0, \quad \phi_2 \neq 0. \end{aligned}$$

Proof. Integrating the third equation of (1.1) over X,

$$\begin{aligned} &\frac{1}{\varepsilon^2}\int_X (|\phi_2|^4 - |\phi_0|^4)\frac{\omega \wedge \omega}{2} \\ =&\int_X 2i(F_{A_{can}} + 2F_B) \wedge \omega \\ =&\int_X 4\pi(c_1(K^*) + 2c_1(E)) \wedge \omega \\ =&4\pi(2c_1(E) - c_1(K))[\omega], \end{aligned}$$

where $\frac{\omega\wedge\omega}{2}$ is the volume form of the Kähler metric g. The conclusion follows directly from the above equation. □

Remark 1.4. The situation for $\eta \neq 0$ is different from the one of Proposition 1.3. Let $\eta = k\omega$, if $k >> 1$ or $\varepsilon^2 << 1$, and $(2c_1(E) - c_1(K) - \sigma)[\omega] > 0$. Then we shall get another type of solution of the equations (1.1), see Theorem 2.1.

If we assume that $\phi_2 = 0$, $\eta = k\omega \in \wedge^{1,1}$, and $\sigma \in \wedge^{1,1}$, then the equations (1.1) reduce to the following equations,

$$\begin{aligned} &\bar{\partial}_B \phi_0 = 0 \\ &F_B^{0,2} = 0 \\ &4i(F_{A_{can}} + 2F_B - \sigma)_\omega = \frac{1}{\varepsilon^2}|\phi_0|^2(-|\phi_0|^2 + k). \end{aligned} \tag{1.2}$$

¿From complex geometry, we know that the equations

$$\begin{aligned} \bar{\partial}_B \phi_0 &= 0 \\ F_B^{0,2} &= \bar{\partial}_B\bar{\partial}_B = 0 \end{aligned} \tag{1.3}$$

always admit a solution (B, ϕ_0).

The equations (1.2) and (1.3) are invariant under a unitary gauge transformation, and the equations (1.3) are also invariant under a real gauge transformation, hence (1.3) is invariant under the complexified gauge group $C^\infty(X, \mathbb{C}^*)$. This fact can be seen from the following computation.

Let $u : X \to \mathbb{C}^*$; u acts on the pair (B, ϕ_0) by

$$\begin{aligned} u^* B &= B + u^{-1}\bar{\partial} u - \bar{u}^{-1}\partial \bar{u} \\ u^*\phi_0 &= u^{-1}\phi_0 \end{aligned}$$

Then we have

$$\begin{aligned} \bar{\partial}_{u^*B}(u^*\phi_0) &= u^{-1}\bar{\partial}_B u(u^{-1}\phi_0) \\ &= u^{-1}\bar{\partial}_B(\phi_0) \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} F_{u^*B} &= d(u^*B) \\ &= d(B + u^{-1}\bar{\partial}u - \bar{u}^{-1}\partial\bar{u}) \\ &= F_B + (\partial + \bar{\partial})(u^{-1}\bar{\partial}u - \bar{u}^{-1}\partial\bar{u}). \end{aligned}$$

Let u be a real gauge, i.e. $u = e^{-\theta}$, $\theta : X \to \mathbb{R}$. We have

$$u^{-1}\bar{\partial}u - \bar{u}^{-1}\partial\bar{u} = -\bar{\partial}\theta + \partial\theta,$$

and

$$\begin{aligned} F_{u^*B} &= F_B + (\partial + \bar{\partial})(-\bar{\partial}\theta + \partial\theta) \\ &= F_B + 2\bar{\partial}\partial\theta \\ &= F_B - 2\partial\bar{\partial}\theta. \end{aligned} \tag{1.5}$$

¿From (1.4) and (1.5), $(u^*B, u^*\phi_0)$ is a solution of (1.3), if (B, ϕ_0) is a solution of (1.3). Then $(u^*B, u^*\phi_0)$ satisfies (1.2) if and only if

$$4i(F_{A_{can}} + 2F_{u^*B} - \sigma)_\omega = \frac{1}{\varepsilon^2}|u^*\phi_0|^2(-|u^*\phi_0|^2 + k). \tag{1.6}$$

¿From (1.5),

$$\begin{aligned} 4i(F_{A_{can}} + 2F_{u^*B} - \sigma)_\omega &= 4i(F_{A_{can}} + 2F_B - \sigma + 2(F_{u^*B} - F_B))_\omega \\ &= 4i(F_{A_{can}} + 2F_B - \sigma)_\omega - 16i(\partial\bar{\partial}\theta)_\omega, \end{aligned}$$

on the other side

$$4i(\partial\bar{\partial}\theta)_\omega = -d^*d\theta = \Delta\theta,$$

where Δ is the negative Laplace operator, i.e. $\Delta = -d^*d$.

Using the above computations, we rewrite the equation (1.6) in the following way

$$4\Delta\theta = \frac{1}{\varepsilon^2}e^{2\theta}|\phi_0|^2(e^{2\theta}|\phi_0|^2 - k) + 4i(F_{A_{can}} + 2F_B - \sigma)_\omega. \tag{1.7}$$

To simplify the equation (1.7), let us set

$$\begin{aligned} v :=& 2\theta \\ u_0 :=& \ln |\phi_0|^2, \qquad \text{or } e^{u_0} = |\phi_0|^2, \end{aligned}$$

u_0 is the Green function for the divisor D defined by the zero set of ϕ_0, namely (see [GH])

$$\Delta u_0 = -4i(F_B)_\omega + 4\pi\delta_D.$$

Set $\lambda := \frac{1}{2\varepsilon^2}$. The equation (1.7) assumes the following form:

$$\Delta v = \lambda e^{v+u_0}(e^{v+u_0} - k) + 2i(F_{A_{can}} + 2F_B - \sigma)_\omega. \tag{1.8}$$

For simplicity, we put

$$\sigma \cdot [\omega] := \int_X \frac{i\sigma}{2\pi} \wedge \omega.$$

Proposition 1.5. *A necessary condition for the existence of a solution for* (1.8) *is*

$$\lambda k^2 > \frac{16\pi}{\mathrm{Vol}(X)}(2c_1(E) - c_1(K) - \sigma) \cdot [\omega].$$

Proof. Rewrite the left side of (1.8)

$$\begin{aligned} \Delta v &= \lambda e^{v+u_0}(e^{v+u_0} - k) + 2i(F_{A_{can}} - \sigma + 2F_B)_\omega \\ &= \lambda(e^{v+u_0} - \frac{k}{2})^2 - \frac{\lambda}{4}k^2 + 2i(F_{A_{can}} - \sigma + 2F_B)_\omega. \end{aligned}$$

Integrating the equation over X, we obtain

$$0 = \int_X \lambda(e^{v+u_0} - \frac{k}{2})^2 - \frac{\lambda}{4}k^2 \,\mathrm{Vol}(X) + 4\pi(2c_1(E) - c_1(K) - \sigma)[\omega].$$

Hence we have

$$\lambda k^2 > \frac{16\pi}{\mathrm{Vol}(X)}(2c_1(E) - c_1(K) - \sigma)[\omega].$$

□

2. Existence of solutions

In this section, we consider the equation

$$\Delta u = \lambda e^{u+u_0+v_0}(e^{u+u_0+v_0} - 1) + A \tag{2.1$_\lambda$}$$

for a constant A and a smooth function v_0 with

$$\int_X v_0 = 0$$

and the Green function u_0 corresponding to some subvariety D of real codimension 2. For example, equation (0.3) is of this type. Also, equation (1.8) is of this form as we now wish to explain: Let $\lambda = \frac{\kappa^2}{2}$, and put $k = 1$ for simplicity,

$$A := \frac{4\pi}{\mathrm{Vol}(X)}(2c_1(E)[\omega] - c_1(K)[\omega] - \sigma[\omega]),$$

and let v_1, v_2 be the solutions of

$$\begin{aligned}\Delta v_1 &= 4i(F_B)_\omega - A_1,\\ \Delta v_2 &= 2i(F_{A_{can}} - \sigma)_\omega - A_2,\end{aligned}$$

with $A_1 = \frac{8\pi}{\mathrm{Vol}(X)}c_1(E)[\omega]$ and $A_2 = \frac{4\pi}{\mathrm{Vol}(X)}(-c_1(K) - \sigma)[\omega]$, normalized by the condition

$$\int_X v_j = 0, \quad \text{for } j = 1, 2,$$

and put

$$v_0 = v_1 + v_2.$$

Returning to the general case, the Green function u_0 satisfies an equation of the type

$$\Delta u_0 = -\Lambda + 4\pi\delta_D,$$

where Λ is smooth with $A_1 := \int_X \Lambda = 4\pi \,\mathrm{Vol}(D)$, and we let v_1 be the solution of $\Delta v_1 = \Lambda - A_1$ with $\int_X v_1 = 0$, $A_2 := A - A_1$, $v_2 := v_0 - v_1$. Since u_0 is the Green function for the subvariety D, the method of Caffarelli-Yang [CY] yields the existence of a first solution of (2.1):

Theorem 2.1. *For λ sufficiently large, the equation $(2.1)_\lambda$ admits a maximal solution $\underline{u}_\lambda$ with $\underline{u}_\lambda + u_0 + v_0 < \bar{v}$, where $\bar{v}$ is a smooth function defined below.*

Proof. Let $\bar{v}$ be a smooth function satisfying

$$\Delta(-v_2 + \bar{v}) \leq A_2 + \lambda e^{\bar{v}}(e^{\bar{v}} - 1). \tag{2.2}$$

Such $\bar{v}$ exists, and in fact we can choose $\bar{v} \geq 0$.

Choose a constant $K \geq 2\lambda e^{2\bar{v}}$.

We want to use induction to construct a sequence w_k that converges to a solution of the equation (2.1).

Put $w_0 = -(u_0 + v_1) + (-v_2 + \bar{v})$. It is clear that $w_0(x) \to +\infty$, as $x \to x_0 \in D$. We have

$$\begin{aligned}(\Delta - K)w_0 &= \Delta w_0 - Kw_0\\ &= -\Delta(u_0 + v_1) + \Delta(-v_2 + \bar{v}) - Kw_0\\ &\leq -8\pi\delta_D + A_1 + A_2 + \lambda e^{\bar{v}}(e^{\bar{v}} - 1) - Kw_0.\end{aligned}$$

Now set

$$(\Delta - K)w_k = \lambda e^{u_0+v_0+w_{k-1}}(e^{u_0+v_0+w_{k-1}} - 1) + A_1 + A_2.$$

Then we have

$$\begin{aligned}(\Delta - K)(w_1 - w_0) =&(\Delta - K)w_1 - (\Delta - K)w_0\\ \geq&\lambda e^{u_0+v_0+w_0}(e^{u_0+v_0+w_0} - 1) + A_1 + A_2\\ &- (-8\pi\delta_D + A_1 + A_2 + \lambda e^{\bar{v}}(e^{\bar{v}} - 1))\\ &= 0,\end{aligned}$$

for any $x \in X \setminus D$.

Let $B_\varepsilon(D) = \{x \in X \mid \operatorname{dist}(x, D) \leq \varepsilon\}$ be the ε-neighborhood of D, and $X_\varepsilon = X \setminus B_\varepsilon(D)$. Since

$$w_0(x) \to +\infty, \text{ as } x \to x_0 \in D,$$

we have

$$w_1 - w_0 < 0, \text{ on } \partial X_\varepsilon.$$

The maximum principle implies that $w_1 - w_0 < 0$ on X_ε. This implies the first step of the induction:

$$w_1 - w_0 < 0 \text{ on } X.$$

Next, by induction assumption $w_k - w_{k-1} < 0$, and we want to prove $w_{k+1} - w_k < 0$.

We compute

$$\begin{aligned}&(\Delta - K)(w_{k+1} - w_k)\\ =&\lambda e^{w_k+u_0+v_0}(e^{w_k+u_0+v_0} - 1)\\ &- \lambda e^{w_{k-1}+u_0+v_0}(e^{w_{k-1}+u_0+v_0} - 1) - K(w_k - w_{k-1})\\ =&\lambda e^{2(u_0+v_0)}(e^{2w_k} - e^{2w_{k-1}}) - \lambda e^{u_0+v_0}(e^{w_k} - e^{w_{k-1}}) - K(w_k - w_{k-1})\\ \geq&\lambda e^{(2u_0+v_0)}(e^{2w_k} - e^{2w_{k-1}}) - K(w_k - w_{k-1}) \text{ since } w_k - w_{k-1} < 0\\ =&2\lambda e^{2u_0+2v_0+2w}(w_k - w_{k-1}) - K(w_k - w_{k-1})\\ &\qquad \text{for a } w, \text{ with } w_k \leq w \leq w_{k-1} < \cdots < w_0\\ \geq&2\lambda e^{2u_0+2v_0+2w_0}(w_k - w_{k-1}) - K(w_k - w_{k-1})\\ =&2\lambda e^{2\bar{v}}(w_k - w_{k-1}) - K(w_k - w_{k-1})\\ =&(2\lambda e^{2\bar{v}} - K)(w_k - w_{k-1}) \geq 0,\end{aligned}$$

and again by the maximum principle, we get

$$w_{k+1} - w_k < 0.$$

We inductively get a monotonically decreasing sequence

$$w_{k+1} < w_k < \cdots < w_1 < w_0.$$

Let w_- be a subsolution of the equation

$$\Delta w_- \geq \lambda e^{u_0+v_0+w_-}(e^{u_0+v_0+w_-} - 1) + A_1 + A_2.$$

Such a subsolution exists for sufficient large λ, see Lemma 2.2 below.

Now we want to show that the subsolution w_- is a lower bound for the sequence w_k. We proceed by induction as above.

First we check that

$$\begin{aligned}\Delta(w_- - w_0) &= \Delta(w_- + u_0 + v_1 + v_2 - \bar{v})\\ &\geq \lambda e^{u_0+v_1+v_2+w_-}(e^{u_0+v_1+v_2+w_-} - 1) - \lambda e^{\bar{v}}(e^{\bar{v}} - 1)\\ &= \lambda e^{w_- - w_0 + \bar{v}}(e^{w_- - w_0 + \bar{v}} - 1) - \lambda e^{\bar{v}}(e^{\bar{v}} - 1)\\ &= \lambda e^{2\bar{v}}(e^{2(w_- - w_0)} - 1) - \lambda e^{\bar{v}}(e^{w_- - w_0} - 1).\end{aligned}$$

¿From the maximum principle, we have

$$w_- - w_0 < 0.$$

By induction, we suppose that $w_- - w_k < 0$. We want to prove that $w_- - w_{k+1} < 0$.

$$\begin{aligned}&(\Delta - K)(w_- - w_{k+1})\\ \geq&\lambda e^{u_0+v_0+w_-}(e^{u_0+v_0+w_-} - 1) + A_1 + A_2 - Kw_-\\ &- \lambda e^{u_0+v_0+w_k}(e^{u_0+v_0+w_k} - 1) - A_1 - A_2 + Kw_k\\ =&\lambda e^{2(u_0+v_0)}(e^{2w_-} - e^{2w_k}) - e^{u_0+v_0}(e^{w_-} - e^{w_k}) - K(w_- - w_k)\\ \geq&\lambda e^{2(u_0+v_0)}(e^{2w_-} - e^{2w_k}) - K(w_- - w_k)\\ =&2\lambda e^{2(u_0+v_0)+2w}(w_- - w_k) - K(w_- - w_k)\\ &\qquad\qquad \text{for a } w \text{ with } w_- \leq w \leq w_k < \cdots < w_0\\ \geq&2\lambda e^{2(u_0+v_0)+2w_0}(w_- - w_k) - K(w_- - w_k)\\ =&(2\lambda e^{2\bar{v}} - K)(w_- - w_k) \geq 0,\end{aligned}$$

for any $x \in X \setminus D$, where the third and last inequalities are from the inductive assumption.

¿From the maximum principle, we obtain the conclusion

$$w_- \leq w_{k+1}.$$

Combining the two inductions, we get a monotonically decreasing sequence that is bounded from both sides by smooth functions, namely

$$w_- < w_{k+1} < w_k < \cdots < w_1 < w_0.$$

Then by the standard bootstrap argument, w_k converges to a solution $\underline{u}_\lambda$ of the equation (2.1) in C^k, for any $k \geq 0$. From the argument of Caffarelli-Yang and Tarantello [Ta; p3776], this solution is the maximal one. □

We now proceed to derive the lemma utilized above.

Lemma 2.2. *For λ sufficiently large, there exists a subsolution w_- of*

$$\Delta w_- \geq \lambda e^{u_0+v_0+w_-}(e^{u_0+v_0+w_-} - 1) + A.$$

Proof. Recall that $B_\varepsilon(D)$ is the ε-neighborhood of D. Let f_ε be a smooth function with $0 \leq f_\varepsilon \leq 1$, $f_\varepsilon = 1$ on $B_\varepsilon(D)$, and $f_\varepsilon = 0$ on $X \setminus B_{2\varepsilon}(D)$.

Let $c > 0$ be a constant, and define a new function

$$g_{\varepsilon,c} = (A+c)f_\varepsilon - \frac{1}{\mathrm{Vol}(X)}\int_X (A+c)f_\varepsilon.$$

The function $g_{\varepsilon,c}$ has the following properties:

(1) $\int_X g_{\varepsilon,c} = 0$;

(2) $g_{\varepsilon,c} \geq A$, on $B_\varepsilon(D)$, for ε sufficiently small and c sufficiently large.

The first one results from the definition of $g_{\varepsilon,c}$, and the second one can be seen from the following computation.

$$\begin{aligned} g_{\varepsilon,c} &\geq A + c - (A+c)\frac{\mathrm{Vol}(B_{2\varepsilon}(D))}{\mathrm{Vol}(X)} \\ &= A + c(1 - \frac{\mathrm{Vol}(B_{2\varepsilon}(D))}{\mathrm{Vol}(X)}) - A\frac{\mathrm{Vol}(B_{2\varepsilon}(D))}{\mathrm{Vol}(X)} \\ &\geq A, \end{aligned}$$

if ε is sufficiently small, and c is sufficiently large.

A solution w of the equation

$$\Delta w = g_{\varepsilon,c},$$

is unique up to additive a constant, and we may therefore choose a solution w_- with $e^{u_0+v_0+w_-} < 1$ on X.

On $B_\varepsilon(D)$,

$$\begin{aligned} \Delta w_- = g_{\varepsilon,c} &\geq A \\ &\geq \underbrace{\lambda e^{u_0+v_0+w_-}(e^{u_0+v_0+w_-} - 1)}_{\leq 0} + A. \end{aligned}$$

On $X \setminus B_\varepsilon(D)$, let

$$\begin{aligned} \mu_0 &= \inf\{e^{u_0+v_0+w_-} \mid x \in X \setminus B_\varepsilon(D)\}, \\ \mu_1 &= \sup\{e^{u_0+v_0+w_-} \mid x \in X \setminus B_\varepsilon(D)\}. \end{aligned}$$

Obviously

$$0 < \mu_0 < \mu_1 < 1.$$

Let $c_0 = -\mu_1(\mu_0 - 1)$, then

$$e^{u_0+v_0+w_-}(e^{u_0+v_0+w_-} - 1) \leq \mu_1(\mu_0 - 1) = -c_0 < 0.$$

Choosing $\lambda > 0$ sufficiently large, we have

$$g_{\varepsilon,c} \geq \lambda e^{u_0+v_0+w_-}(e^{u_0+v_0+w_-} - 1) + A.$$

Hence, we get a subsolution w_- for λ sufficiently large. $\square$

Corollary 2.3. *If $2i(F_{A_{can}} - \sigma)_\omega \geq 0$, then there exists a critical value $\lambda_c \geq 4A$ such that for every $\lambda > \lambda_c$ the equation $(1.8)_\lambda$ admits a maximal solution $\underline{u}_\lambda$ with $\underline{u}_\lambda + u_0 + v_0 < 0$, while for $\lambda < \lambda_c$ the equation $(1.8)_\lambda$ admits no solution.*

Proof. If $2i(F_{A_{can}} - \sigma)_\omega \geq 0$, then we can choose $\bar{v} = 0$, since

$$\Delta v_2 = 2i(F_{A_{can}} - \sigma)_\omega - A_2 \geq -A_2.$$

This is the inequality (2.2) for $\bar{v} = 0$.

Let

$$\lambda_c := \inf\{\lambda \geq 4A \mid \text{ the equation } (2.1)_\lambda \text{ is solvable } \}.$$

If u_λ is a solution of $(2.1)_\lambda$, then u_λ is a subsolution of $(2.1)_{\lambda_1}$ for any $\lambda_1 > \lambda$, since

$$\begin{aligned}\Delta u_\lambda &= \lambda e^{u_\lambda + u_0 + v_0}(e^{u_\lambda + u_0 + v_0} - 1) + A \\ &= \lambda_1 e^{u_\lambda + u_0 + v_0}(e^{u_\lambda + u_0 + v_0} - 1) + A \\ &+ \underbrace{(\lambda - \lambda_1) e^{u_\lambda + u_0 + v_0}(e^{u_\lambda + u_0 + v_0} - 1)}_{\geq 0} \\ &\geq \lambda_1 e^{u_\lambda + u_0 + v_0}(e^{u_\lambda + u_0 + v_0} - 1) + A.\end{aligned}$$

¿From the proof of Theorem 2.1, the existence of the maximal solution of $(2.1)_\lambda$ depends on the existence of a subsolution of $(2.1)_\lambda$. By the definition of λ_c, the equation $(2.1)_\lambda$ admits a maximal solution for any $\lambda > \lambda_c$, and admits no solution for any $\lambda < \lambda_c$. □

Tarantello [Ta] proved that in the two-dimensional case, (2.1) has a second solution, and this solution (or else a third one) is known to have a different asymptotic behavior at least in the cases of one and two vortices, see [Ta], [DJLW1], [DJLW2], [NT]. The method, however, does not extend to higher dimensions, because we then do not have a Palais-Smale condition anymore. In this section, we develop a heat equation method that yields a second solution of (2.1) in any dimension.

We recall the equation as

$$(2.3)_\lambda \qquad \Delta u = \lambda e^{u + u_0 + v_0}(e^{u + u_0 + v_0} - 1) + A.$$

Let $\underline{u}_\lambda$ be the solution obtained in Theorem 2.1 by using the super/subsolution method, or the solution obtained in Corollary 2.3 for any $\lambda > \lambda_c$. We choose a fixed subsolution ψ_0 of equation $(2.3)_\lambda$ for λ sufficiently large, or $\lambda > \lambda_c$ in the case of Corollary 2.3.

We define a partial ordering in $L^{1,2}(X) \cap C^0(X)$ by $f_1 > f_2$ (resp. $f_1 \geq f_2$) if $f_1(x) > f_2(x)$ (resp. $f_1(x) \geq f_2(x)$) for all $x \in X$. If $f_1 > f_2$, we define

$$[f_2, f_1] := \{g \in L^{1,2}(X) \cap C^0(X) \mid f_2 \leq g \leq f_1\},$$

and

$$[f_2, f_1) := \{g \in L^{1,2}(X) \cap C^0(X) \mid f_2 \leq g < f_1\}.$$

Here, possibly $f_1 = +\infty$ or $f_2 = -\infty$.

Set

$$S_\lambda = \left\{u \text{ is a solution of } (2.1)_\lambda \;\middle|\; u \in (\psi_0, \underline{u}_\lambda]\right\}.$$

Clearly, $S_\lambda \neq \emptyset$, since $\underline{u}_\lambda \in S_\lambda$.

Lemma 2.4. *There exists* $\underline{u}_\lambda^1 \in S_\lambda$ *such that* $S_\lambda \cap (\psi_0, \underline{u}_\lambda^1] = \{\underline{u}_\lambda^1\}$, *i.e. there is no solution* u *of* $(2.3)_\lambda$ *with* $u \in (\psi_0, \underline{u}_\lambda^1)$.

Remark 2.5. We believe that $S_\lambda = \{\underline{u}_\lambda\}$, at least for large λ.

Proof of Lemma 2.4. For any $u \in S_\lambda$, define $\mu(u) = \min_{x\in X}(u(x) - \psi_0(x))$ and $\mu_0 = \inf_{u\in S_\lambda} \mu(u)$. By Lemma 2.9 below, it is easy to show that S_λ is compact, see also [Ta]. It follows that there is a $\underline{u}_\lambda^1 \in S_\lambda$ such that $\mu_0 = \mu(\underline{u}_\lambda^1)$. Assume that $\mu_0 = \underline{u}_\lambda^1(x_0) - \psi_0(x_0)$. We claim that $S_\lambda \cap (\psi_0, \underline{u}_\lambda^1] = \{\underline{u}_\lambda^1\}$. Assume by contradiction that there is another solution $v \in (\psi_0, \underline{u}_\lambda^1]$. We have, by the definition of μ_0,

$$v(x) \le \underline{u}_\lambda^1(x) \text{ and } v(x_0) = \underline{u}_\lambda^1(x_0).$$

The maximum principle implies that $v = \underline{u}_\lambda^1$, a contradiction. $\square$

By Lemma 2.4, we may assume $S_\lambda = \{\underline{u}_\lambda\}$. Now we consider the following functional

$$J_\lambda(u) = \int_X \frac{1}{2}|\nabla u|^2 + \frac{1}{2}\lambda(e^{u+u_0+v_0} - 1)^2 + Au \tag{2.9}$$

in

$$X_\lambda = (-\infty, \underline{u}_\lambda] \cap C^1(X).$$

We want to show that $\underline{u}_\lambda$ is a strict local minimizer of J_λ in X_λ. We first show

Lemma 2.6.

$$J_\lambda(\underline{u}_\lambda) = \inf_{g\in(\psi_0, \underline{u}_\lambda]} J_\lambda(g).$$

Proof. Minimizing J_λ in $(\psi_0, \underline{u}_\lambda]$, we can obtain a solution v by a standard method (see Appendix in [Ta]) such that $v \in (\psi_0, \underline{u}_\lambda]$. ¿From the discussion above, $v = \underline{u}_\lambda$. Hence,

$$J_\lambda(\underline{u}_\lambda) = \inf_{g\in(\psi_0, \underline{u}_\lambda]} J_\lambda(g).$$

$\square$

Remark 2.7. ¿From Lemma 2.4, $\underline{u}_\lambda$ is a local minimizer of J_λ in X_λ with respect to the C^1-norm, *i.e.*, there exists a δ_0 such that if $u \in X_\lambda$ with $\|\underline{u}_\lambda - u\|_{C^1} < \delta_0$, then $J_\lambda(\underline{u}_\lambda) \le J_\lambda(u)$. Actually, we shall show in the sequel that $\underline{u}_\lambda$ is a strict local minimizer of J_λ.

To achieve this, we first discuss the heat equation with respect to $(2.3)_\lambda$,

$$(2.5)_\lambda, \qquad \begin{cases} u_t = \Delta u - \lambda e^{u+u_0+v_0}(e^{u+u_0+v_0} - 1) - A \\ u(\cdot, 0) = g_0 \end{cases}$$

which will be also used to construct deformations below.

Lemma 2.8. *For any $g_0 \in X_\lambda$, there exists a $T \in (0, \infty]$ such that $(2.5)_\lambda$ admits a solution $u(\cdot, t)$ in $[0, T)$, and either $\lim_{t\to T} J_\lambda(u(t)) = -\infty$, or $J_\lambda(u(t)) \geq c > -\infty$ for any $t \in [0, T)$, in this case $T = +\infty$ and $u(\cdot, \infty) = \lim_{t\to+\infty} u(\cdot, t)$ is a solution of the equation $(2.3)_\lambda$. Moreover, solutions of the equation $(2.5)_\lambda$ continuously depend on initial functions.*

To prove Lemma 2.8, we need the standard apriori estimates for parabolic equations. Here we first prove an auxiliary lemma.

Lemma 2.9. *For any $u \in X_\lambda$, let $f = \Delta u - \lambda e^{u+u_0+v_0}(e^{u+u_0+v_0} - 1) - A$. If $\|f\|_{L^2} < c_1$, then $\|\nabla u\|_{L^2} \leq c_3$; If, in addition, $|J_\lambda(u)| < c_2$, then $\|u\|_{L^{1,2}} \leq c_4$, for some constants c_3 and c_4 depending only on the geometry of the manifold X, the constants c_1, c_2, λ, A and $\|\bar{v}\|_{L^\infty}$.*

Proof. For simplicity, we set

$$h = \lambda e^{u+u_0+v_0}(e^{u+u_0+v_0} - 1) + A.$$

First we know

$$u + u_0 + v_0 \leq \underline{u}_\lambda + u_0 + v_0 < \bar{v}, \text{ for any } u \in X_\lambda,$$

hence we have

$$\|h\|_{L^\infty} = \|\lambda e^{u+u_0+v_0}(e^{u+u_0+v_0} - 1) + A\|_{L^\infty} \leq c,$$

where c depends on λ, A and $\|\bar{v}\|_{L^\infty}$.

Taking the L^2-product of u with the equation $f = \Delta u - h$ yields

$$\int_X fu = -\int_X |\nabla u|^2 + hu.$$

Integrating the equation $f = \Delta u - h$, we have

$$\int_X f + h = 0.$$

Let

$$\bar{u} = \frac{1}{\mathrm{Vol}(X)} \int_X u$$

be the mean value of u. Combining the two equations, we have

$$\begin{aligned}
\int_X |\nabla u|^2 &= \int_X (f + h)u \\
&= \int_X (f + h)(u - \bar{u}) \\
&\leq \varepsilon \int_X |u - \bar{u}|^2 + \frac{1}{\varepsilon} \int_X |f + h|^2 \\
&\leq \frac{\varepsilon}{\lambda_1} \int_X |\nabla u|^2 + \frac{1}{\varepsilon} \int_X |f + h|^2,
\end{aligned}$$

where ε is some positive constant, and λ_1 is the first positive eigenvalue of the Laplace operator Δ. Choosing an ε with $\frac{\varepsilon}{\lambda_1} \leq \frac{1}{2}$, we obtain

$$\int_X |\nabla u|^2 \leq c(\varepsilon, \lambda_1) \int_X |f+h|^2 \leq c.$$

On the other side,

$$J_\lambda(u) = \int_X \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{2}\int_X (e^{u+u_0+v_0}-1)^2 + A\int_X u,$$

and we rewrite the above equation

$$\bar{u} = \frac{1}{A\,\mathrm{Vol}(X)}\left(J_\lambda(u) - \frac{1}{2}\int_X |\nabla u|^2 - \frac{\lambda}{2}\int_X (e^{u+u_0+v_0}-1)^2\right),$$

to get

$$|\bar{u}| < c,$$

and our conclusion

$$\|u\|_{L^{1,2}} \leq c(\int_X |\nabla u|^2 + \bar{u}^2) \leq c,$$

where c depends λ, A, $\|\bar{v}\|_{L^\infty}$, c_1 and c_2. □

Proof of Lemma 2.8. As in the proof of Lemma 2.9, we set

$$h = \lambda e^{u+u_0+v_0}(e^{u+u_0+v_0}-1) + A.$$

Recall the equation (2.5)

$$\begin{cases} u_t = \Delta u - h \\ u(\cdot, 0) = g_0 \qquad \text{for } g_0 \in X_\lambda. \end{cases} \tag{2.6}$$

First we know that if $g_0 \in X_\lambda$, then $u_t \in X_\lambda$, for any $t \in [0,T)$, where T is the maximal existence time of the solution, since $\underline{u}_\lambda$ is a solution of the equation $\Delta u - h = 0$. If $\lim_{t\to T} J_\lambda(u(\cdot,t)) = -\infty$, we have shown the first statement of Lemma. So we assume that $\lim_{t\to T} J_\lambda(u(\cdot,t)) = c_0 > -\infty$. Then by Lemma 2.9, we have

$$\sup_{0\leq t<T} \|h(\cdot,t)\|_{L^\infty} \leq c.$$

By the general theory of parabolic equations, one may have the following estimates for a solution u of (2.6) (see *e.g.* [L]):

$$\|u(\cdot,t)\|_{C^{1,\alpha}} \leq c(\sup_{0\leq t<T}\|h(\cdot,t)\|_{L^\infty} + \sup_{0\leq t<T}\|u(\cdot,t)\|_{L^2}), \tag{2.7}$$

and

$$\|u(\cdot,t)\|_{C^{2,\alpha}} + \|\frac{\partial u}{\partial t}(\cdot,t)\|_{C^\alpha} \leq c(\sup_{0\leq t<T}\|h(\cdot,t)\|_{c^\alpha} + \sup_{0\leq t<T}\|u(\cdot,t)\|_{L^2}), \tag{2.8}$$

where c is a constant independent of time t. By the bootstrap argument, from (2.7) we have $u \in C^{1,\alpha}$, then from the definition of h, we have $h \in C^{\alpha}$. Again from (2.8) we have $u \in C^{2,\alpha}$. Then from the standard argument for the heat equation and the estimate (2.8), we can get $T = +\infty$, namely, the equation (2.5) admits a unique global solution $u(\cdot, t)$.

Now we will prove the second conclusion of Lemma 2.8. First we note that it is enough to prove

$$\|u(\cdot, t)\|_{L^2} \leq c, \text{ for all } t, \tag{2.9}$$

where c is independent of t. Since from (2.8) and (2.9) we have uniform estimates of $\|u(\cdot,t)\|_{C^{2,\alpha}}$ and $\|\frac{\partial u}{\partial t}(\cdot,t)\|_{C^{\alpha}}$, then the limit $\lim_{t\to\infty} u(\cdot,t)$ exists, and letting $u(\cdot,\infty) = \lim_{t\to\infty} u(\cdot,t)$, $u(\cdot,\infty)$ is a solution of the equation (2.1).

We claim that

$$\|\frac{\partial u}{\partial t}(\cdot,t)\|_{L^2} < c,$$

where c is independent of t.

First we assume that $|J_\lambda(u(\cdot,t))| \leq c$, otherwise $u(\cdot,t)$ will go to $-\infty$. Second, by a simple computation the functional is decreasing along the flow, i.e.,

$$\frac{d}{dt} J_\lambda(u(\cdot,t)) = -\int_X \left|\frac{\partial u}{\partial t}\right|^2 dx.$$

Integrating over the time t,

$$J_\lambda(u(\cdot,T)) - J_\lambda(u(\cdot,0)) = -\int_0^T \int_X \left|\frac{\partial u}{\partial t}\right|^2 dxdt.$$

By the assumption of the finiteness of the energy, we have

$$\int_0^\infty \int_X \left|\frac{\partial u}{\partial t}\right|^2 dxdt < c.$$

In order to get inequality (2.7), we first differentiate $\|u(\cdot,t)\|_{L^2}^2$ with respect to t,

$$\begin{aligned}
\frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2 &= 2\left\langle \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\right\rangle_{L^2} \\
&= 2\left\langle \Delta\frac{\partial u}{\partial t} - \frac{\partial h}{\partial t}, \frac{\partial u}{\partial t}\right\rangle_{L^2} \\
&= -2\int_X \left|\nabla \frac{\partial u}{\partial t}\right|^2 - 2\int_X \frac{\partial h}{\partial t}\frac{\partial u}{\partial t}
\end{aligned} \tag{2.10}$$

where the second equality comes from the differentiation of the equation (2.6). On the other hand, by the definition of h

$$\frac{\partial h}{\partial t} = \lambda e^{u+u_0+v_0}(2e^{u+u_0+v_0} - 1)\frac{\partial u}{\partial t}.$$

Noting $u(\cdot,t)\in X_\lambda$, *i.e.*, $u(\cdot,t)+u_0+v_0<\bar v$, we have the pointwise estimate

$$\left|\frac{\partial h}{\partial t}(x,t)\right|\le c\left|\frac{\partial u}{\partial t}(x,t)\right|,$$

consequently,

$$\left|\int_X \frac{\partial h}{\partial t}\frac{\partial u}{\partial t}\right|\le c\int_X\left|\frac{\partial u}{\partial t}\right|^2.$$

Integrating the equation (2.10), we have the following estimate

$$\begin{aligned}\|\frac{\partial u}{\partial t}(\cdot,T)\|^2_{L^2}-\|\frac{\partial u}{\partial t}(\cdot,0)\|^2_{L^2}&\le c\int_0^T\int_X|\frac{\partial u}{\partial t}(x,t)|^2dxdt\\&\le c\int_0^\infty\int_X|\frac{\partial u}{\partial t}(x,t)|^2dxdt,\end{aligned}$$

for any $T>0$. This proves the claim.

¿From Lemma 2.9, we have

$$\|u(\cdot,t)\|_{L^{1,2}}<c,\ \text{for all } t,$$

since we have uniform bounds for $|J_\lambda(u(\cdot,t))|$ and $\|\frac{\partial u}{\partial t}(\cdot,t)\|_{L^2}$ for all t.

Hence, we prove Lemma 2.8. □

Lemma 2.10. *There exists a $\delta>0$, such that*

$$\inf_{\|u-\underline{u}_\lambda\|_{C^1}=\delta}J_\lambda(u)>J_\lambda(\underline{u}_\lambda).$$

Proof. First choose a constant $\delta_0>0$ such that for any $u\in X_\lambda$ with $\|u-\underline{u}_\lambda\|_{C^1}\le\delta_0$, $u\in(\phi_0,\underline{u}_\lambda]$. Then we claim that for any $u\in X_\lambda$ with $\frac{1}{2}\delta_0\le\|u-\underline{u}_\lambda\|_{C^1}\le\delta_0$,

$$\sigma_0<\|\Delta u-\lambda e^{u+u_0+v_0}(e^{u+u_0+v_0}-1)-A\|_{L^2}<c.$$

Otherwise, we may obtain a solution in $(\psi_0,\underline{u}_\lambda)$.

Assume

$$\inf_{\{u\ |\ \|u-u_\lambda\|_{C^1}=\frac{3}{4}\delta_0\}}J_\lambda(u)=J_\lambda(\underline{u}_\lambda).$$

Let

$$U=\{u(\cdot,t)\ \Big|\ \|u-\underline{u}_\lambda\|_{C^1}\le\frac{3}{4}\delta_0 \text{ and } t=\frac{\delta_0}{4c}\},$$

and

$$\Gamma=\partial U=\{u(\cdot,t)\ \Big|\ \|u-\underline{u}_\lambda\|_{C^1}=\frac{3}{4}\delta_0 \text{ and } t=\frac{\delta_0}{4c}\}.$$

It is easy to check

$$\inf_{u\in\Gamma}J_\lambda(u)\le\inf_{\|u-u_\lambda\|_{C^1}=\frac{3}{4}\delta_0}J_\lambda(u)-\sigma_0\frac{\delta_0}{4c}<J_\lambda(\underline{u}_\lambda),$$

which contradicts Remark 2.7. □

Theorem 2.11. *For any $\lambda > \lambda_c$, there exists another solution $\bar{u}_\lambda$ of the equation* (2.1) *with the property that $\bar{u}_\lambda \in (-\infty, \underline{u}_\lambda)$, but $\bar{u}_\lambda \notin (\psi_0, \underline{u}_\lambda)$.*

Proof. We want to prove the theorem by using the mountain pass argument.

It is clear that the heat flow $(2.5)_\lambda$ preserves X_λ. It is trivial to see that $J_\lambda(u+c) \to -\infty$ as $c \to -\infty$. Take a $\rho > 0$, such that

$$J_\lambda(\underline{u}_\lambda - \rho) < J_\lambda(\underline{u}_\lambda).$$

Let $u(x,t;c)$ be the solution of the heat equation

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = \Delta u - \lambda e^{u+u_0+v_0}(e^{u+u_0+v_0} - 1) - A \\ u(x,0;c) = \underline{u}_\lambda(x) - c, \qquad \text{for } c \in [0,\rho] \end{array} \right. \tag{2.11}$$

We note that

$$\frac{d}{dt} J_\lambda(u(\cdot,t;c)) \le 0,$$

$J_\lambda(u(\cdot,t;c))$ is monotonically decreasing in t. In particular,

$$J_\lambda(u(\cdot,t;\rho)) \le J_\lambda(u(\cdot,0;\rho)) = J_\lambda(u-\rho) < J_\lambda(\underline{u}_\lambda) \text{ for all } t,$$

and

$$J_\lambda(u(\cdot,t;0)) = J_\lambda(\underline{u}_\lambda) \text{ for all } t,$$

since $u(x,t;0) = \underline{u}_\lambda(x)$ for any t. We consider the curve $u(\cdot,t;s)$, where $s \in [0,\rho]$ is variable, and t is the deformation parameter. By Lemma 2.10, there is a positive constant ε such that for any t there exists a $c_t \in [0,\rho]$ with

$$J_\lambda(u(\cdot,t;c_t)) \ge J_\lambda(\underline{u}_\lambda) + \varepsilon. \tag{2.12}$$

For a sequence $t_n \to +\infty$, we thus obtain a sequence $c_{t_n} \in [0,\rho]$. Since $[0,\rho]$ is compact, we assume that c_{t_n} converges to $c_0 \in (0,\rho)$. Then we have

$$J_\lambda(u(\cdot,+\infty;c_0)) = \lim_{n\to+\infty} J_\lambda(u(\cdot,t_n;c_{t_n})) \ge J_\lambda(\underline{u}_\lambda) + \varepsilon.$$

We claim that

$$J_\lambda(u(\cdot,t;c_0)) \ge J_\lambda(\underline{u}_\lambda) + \varepsilon,$$

for all t.

If the claim is not true, there is a t_0 such that

$$J_\lambda(u(\cdot,t_0;c_0)) < J_\lambda(\underline{u}_\lambda) + \varepsilon.$$

$J_\lambda(u(\cdot,t;c_0))$ is monotonically decreasing in t, and thus for any $t \ge t_0$

$$J_\lambda(u(\cdot,t;c_0)) \le J_\lambda(u(\cdot,t_0;c_0)) < J_\lambda(\underline{u}_\lambda) + \varepsilon.$$

On the other side, $u(\cdot,t;c)$ is continuous in t and c, and thus for n large enough,

$$J_\lambda(u(\cdot,t_0;c_{t_n})) < J_\lambda(\underline{u}_\lambda) + \varepsilon,$$

and for $t_n > t_0$, we have

$$J_\lambda(u(\cdot,t_n;c_{t_n})) < J_\lambda(u(\cdot,t_0;c_{t_n})) < J_\lambda(\underline{u}_\lambda) + \varepsilon,$$

This contradicts the inequality (2.12). Thus, we prove the claim.

Let $\bar{u}_\lambda = \lim_{t\to+\infty} u(\cdot,t;c_0)$. By Lemma 2.8 $\bar{u}_\lambda$ is a solution of the equation (2.1), and

$$J_\lambda(\bar{u}_\lambda) = \lim_{t\to+\infty} J_\lambda(u(\cdot,t;c_0)) \ge J_\lambda(\underline{u}_\lambda) + \varepsilon.$$

On the other side, from Lemma 2.4 $\bar{u}_\lambda \notin (\psi_0, \underline{u}_\lambda)$.

This finishes the proof of Theorem 2.11. □

3. The asymptotic behavior of the solutions

Let u_λ be the solution of the equation

$$\Delta u_\lambda = \lambda e^{u_\lambda+u_0+v_0}(e^{u_\lambda+u_0+v_0}-1)+A. \tag{3.1$_\lambda$}$$

In this section, we will study the asymptotic behavior as $\lambda\to\infty$ of the solutions of the equation $(3.1)_\lambda$ obtained in sections 1 and 2. In this section for technical reasons, we choose the perturbation $\sigma = F_{A_{can}}$. From Corollary 2.3, any solution u_λ of $(3.1)_\lambda$ satisfies the inequality $u_\lambda+u_0+v_0<0$. We will show the following theorem.

Theorem 3.1. *Let $\mu_\lambda = \frac{1}{\mathrm{Vol}(X)}\int_X u_\lambda$. If $\lambda e^{\mu_\lambda}\to\infty$ then $e^{u_\lambda+u_0+v_0}\to 1$ almost everywhere as $\lambda\to\infty$; if $\lambda e^{\mu_\lambda}\le c$ then $e^{u_\lambda+u_0+v_0}\to 0$ almost everywhere as $\lambda\to\infty$.*

We need the following Lemma.

Lemma 3.2. *Let $n=\dim X$. Then for any $1<q<\frac{n}{n-1}$, $\|\nabla u_\lambda\|_{L^q}\le c$.*

Proof. Let $q'=\frac{q}{q-1}>n$. Then

$$\|\nabla u_\lambda\|_{L^q}\le \sup\{|\int_X \nabla u_\lambda\nabla\phi| \,\Big|\, \phi\in L^{1,q'}(X), \int_X\phi=0, \|\phi\|_{L^{1,q'}(X)}=1\} \tag{3.2}$$

By the Sobolev embedding theorem we have for ϕ as in (3.2)

$$\|\phi\|_{L^\infty(X)}\le c.$$

It is clear that

$$\begin{aligned}\left|\int_X \nabla u_\lambda\nabla\phi\right| &= \left|\int_X \Delta u_\lambda\phi\right|\\ &\le \|\phi\|_{L^\infty(X)}\lambda\int_X e^{u_\lambda+u_0+v_0}(e^{u_\lambda+u_0+v_0}-1)\\ &\le c\end{aligned}$$

since $u_\lambda+u_0+v_0<0$. This proves Lemma 3.2. □

Proof of Theorem 3.1. By the Sobolev embedding theorem, we may assume that

$$u_\lambda-\mu_\lambda\to u_\infty \text{ in } L^p(X)$$

for some $p>1$.

Integrating the equation (3.1) on both sides, we get

$$\int_X(\lambda e^{u_\lambda+u_0+v_0}(e^{u_\lambda+u_0+v_0}-1))+A\,\mathrm{Vol}(X)=0,$$

hence

$$\lambda e^{\mu_\lambda}\int_X e^{u_\lambda+u_0+v_0-\mu_\lambda}(1-e^{\mu_\lambda}e^{u_\lambda+u_0+v_0-\mu_\lambda})=A\cdot\mathrm{Vol}(X).$$

If $\lambda e^{\mu_\lambda} \le c$, we have $\mu_\lambda \to -\infty$, so

$$e^{u_\lambda + u_0 + v_0} = e^{\mu_\lambda} \cdot e^{(u_\lambda - \mu_\lambda) + u_0 + v_0} \to 0 \text{ a.e. as } \lambda \to 0$$

We consider now the case that $\lambda e^{\mu_\lambda} \to \infty$. Because $u_\lambda \le -u_0 - v_0$, by the maximum principle we have $\mu_\lambda \le 1$. Hence $0 \le e^{\mu_\lambda} \le 1$. We assume that $e^{\mu_\lambda} \to \alpha$. By Fatou's Lemma, we have

$$\int_X e^{u_\infty + u_0 + v_0}(1 - \alpha e^{u_\infty + u_0 + v_0}) = 0.$$

So we have

$$e^{u_\infty + u_0 + v_0} = \frac{1}{\alpha} \quad a.e.$$

and consequently

$$u_\infty + u_0 + v_0 = \log \frac{1}{\alpha}.$$

It is clear that $\int_X u_\infty = \int_X (u_0 + v_0) = 0$. Hence $\alpha = 1$ and $u_\infty = u_0 + v_0$.

This proves the theorem. □

Theorem 3.3. *There are two solutions* $\underline{u}_\lambda$ *and* $\bar{u}_\lambda$ *of the equation* (3.1) *with the following properties:*

(1) $|e^{\underline{u}_\lambda + u_0 + v_0}| \to 1$ *a.e., as* $\lambda \to \infty$;
(2) $|e^{\bar{u}_\lambda + u_0 + v_0}| \to 0$ *a.e., as* $\lambda \to \infty$.

Proof. It is clear that the solution obtained by the super/sub-solution method satisfies

$$|e^{\underline{u}_\lambda + u_0 + v_0}| \to 1 \quad a.e., \text{ as } \lambda \to \infty.$$

It suffices to show that the second solution we obtained satisfies

$$|e^{\bar{u}_\lambda + u_0 + v_0}| \to 0 \quad a.e., \text{ as } \lambda \to \infty.$$

For simplicity, we just denote the second solution by u_λ. By Theorem 3.1, we need only show that u_λ does not converge to $-u_0 - v_0$ in $L^p(X)$ for some $p > 1$. (Note that, in the first case of Theorem 3.1, $u_\lambda \to u_\infty$ in $L^p(X)$ for some $p > 1$). We will show in the sequel that, if $u_\lambda \to -u_0 - v_0$ in $L^p(X)$ for some $p > 1$, then $u_\lambda \in [\psi_0, -u_0 - v_0]$ for large λ, where ψ_0 is the subsolution used in the proof of the existence for the second solution.

We first show that, for any $\varepsilon > 0$,

$$u_\lambda \to -u_0 - v_0 \text{ in } C^0(X \setminus B_\varepsilon(D))$$

where $B_\varepsilon(D) = \{x \in X \,|\, \operatorname{dist}(x, D) < \varepsilon\}$, and D is the zero set of ϕ_0.

In $X \setminus B_{\frac{\varepsilon}{2}}(D)$, we have

$$\Delta(u_\lambda + u_0 + v_0) \le 0,$$

since $\bar{v} = 0$ and $v_2 = 0$. By Theorem 8.17 in [GT], we get

$$(u_\lambda + u_0 + v_0)(x) \ge -c(\varepsilon)\|u_\lambda + u_0 + v_0\|_{L^p(X \setminus B_{\frac{\varepsilon}{2}}(D))},$$

for all $x \in X \setminus B_\varepsilon(D)$. Since $u_\lambda(x) \le -(u_0+v_0)(x)$ for all $x \in X$, we have

$$\begin{aligned}\|u_\lambda + u_0 + v_0\|_{C^0(X\setminus B_\varepsilon(D))} &\le c(\varepsilon)\|u_\lambda + u_0 + v_0\|_{L^p(x)} \\ &\to 0, \quad \text{as } \lambda \to \infty.\end{aligned} \tag{3.3}$$

We set $m_{\lambda,\varepsilon} = \min_{\partial B_\varepsilon(D)} u_\lambda(x)$. It is clear that $\lim_{\varepsilon\to 0}\lim_{\lambda\to\infty} m_{\lambda,\varepsilon} = \infty$. We then consider $u_\lambda(x) - m_{\lambda,\varepsilon}$ in $B_\varepsilon(D)$. Since, for $n = \dim X$,

$$\begin{aligned}\Delta(u_\lambda - m_{\lambda,\varepsilon} - \frac{A}{2n}|x|^2) &= \lambda e^{u_\lambda+u_0+v_0}(e^{u_\lambda+u_0+v_0} - 1) \\ &\le 0,\end{aligned}$$

and

$$(u_\lambda - m_{\lambda,\varepsilon} - \frac{A}{2n}|x|^2)|_{\partial B_\varepsilon(D)} \ge -\frac{A}{2n}\varepsilon^2,$$

from the maximum principle, we get

$$u_\lambda(x) - m_{\lambda,\varepsilon} - \frac{A}{2n}|x|^2 \ge -\frac{A}{2n}\varepsilon^2, \tag{3.4}$$

for all $x \in B_\varepsilon(D)$. This implies that

$$u_\lambda(x) \ge m_{\lambda,\varepsilon} - \frac{A}{2n}\varepsilon^2 > \psi_0(x), \tag{3.5}$$

for all $x \in B_\varepsilon(D)$, provided that λ is large and ε is small. (3.4) and (3.5) imply that

$$u_\lambda > \psi_0,$$

for λ sufficiently large. This is in contradiction with our construction for the second solution, namely, $u_\lambda \notin [\psi_0, \underline{u}_\lambda]$.

This finishes the proof of Theorem. □

Theorem 0.1 and Theorem 0.2 are direct consequences of Corollary 2.3, Theorem 2.11 and Theorem 3.3.

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