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**Desingularization of a hyperelliptic curve associated with a doubly periodic Dirac potential**

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# DESINGULARIZATION OF A HYPERELLIPTIC CURVE ASSOCIATED WITH A DOUBLY PERIODIC DIRAC POTENTIAL

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ABSTRACT

Let $(q(x), p(x))$ be a doubly periodic Dirac potential; denote by $L$ the lattice of its periods. The associated spectral curve is a (possibly singular) hyperelliptic covering of the elliptic curve $\mathbb{C}/L$. We show that by varying $L$ we can obtain a smooth spectral curve of the same genus.

## 1. INTRODUCTION

In the last twenty-five years, the theory of Riemann surfaces, and in particular the theory of the Abel map from a Riemann surface to the associated Jacobian and its inversion in terms of theta functions, has had enormous and surprising applications on the theory of periodic solutions of completely integrable equations like the Korteweg-de-Vries equation and the Nonlinear Schrödinger equation (see [BBEIM] for a comprehensive reference). We have here a very interesting example of a direct application of "pure" mathematics to a set of problems in very "applied" mathematics.[1]

Recent work by Gesztesy and Weikard has reminded us that applications go both ways. In [GW1] a problem of hyperelliptic curves is solved via Floquet theoretic and spectral theoretic methods for an associated Schrödinger operator. It is this very work and its sequel which has inspired the idea of seeking further answers to particular problems in hyperelliptic curves by using methods that have risen in the analytic theory of completely integrable equations.

In this short note, we show that by using a gauge transformation introduced in a recent preprint of Gesztesy and Weikard [GW2], we can remove the singularities of a spectral hyperelliptic curve associated to a doubly periodic Dirac potential by varying its period lattice (and keeping genus constant).

## 2. THE DIRAC OPERATOR AND THE AKNS HIERARCHY

We begin by defining the Dirac operator and reviewing some basic facts about the AKNS hierarchy as introduced by Al'ber and described by Gesztesy and Weikard in [GW2].

Suppose that $q, p$ are meromorphic on $\mathbb{C}$ and consider the Dirac matrix-valued differential expression

[1] With direct technological applications, indeed; the semi-classical limit of the focusing NLS equation, which can be analyzed using the algebro-geometric theory of the associated Lax operator, describes optical shocks and wave breaking in the nonlinear proppagation of laser pulses in optical fibres.

$$L = J\frac{d}{dx} + Q(x),$$
$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{1.1}$$
$$Q(x) = \begin{pmatrix} 0 & -iq(x) \\ ip(x) & 0 \end{pmatrix}.$$

We define functions $f_l, g_l, h_l$ by the following recurrence relations:

$$f_{-1} = 0,\ g_0 = 1,\ h_{-1} = 0,$$
$$f_{l+1} = \frac{i}{2}f_{l,x} - iqg_{l+1},$$
$$g_{l+1,x} = pf_l + qh_l, \tag{1.2}$$
$$h_{l+1} = -\frac{i}{2}h_{l,x} + ipg_{l+1},$$
$$l = -1, 0, 1, ....$$

The functions $f_l, g_l, h_l$ are polynomials in the variables $p, q, p_x, q_x, ...$ and some integration constants $c_1, c_2, ....$ Assigning weight $k+1$ tp $p^{(k)}$ and $q^{(k)}$ one finds that $f_l, g_{l+1}, h_l$ are homogeneous of weight $l+1$.

Let

$$P_{n+1} = -\Sigma_{l=0}^{n+1}(g_{n-l+1}J + iA_{n-l})L^l,$$
$$where\ A_l = \begin{pmatrix} 0 & -f_l \\ h_l & 0 \end{pmatrix}.$$

The AKNS hierarchy is the collection of the evolution equations

$$\frac{d}{dt}L(t) - [P_{n+1}(t), L(t)] = 0,\ n = 0, 1, 2, ..... \tag{1.3}$$

and the stationary AKNS hierarchy is defined by

$$[P_{n+1}, L] = 0,\ n = 0, 1, .....$$

We introduce the polynomials with respect to $E \in \mathbb{C}$,

$$F_n(E, x) = \Sigma_{l=0}^{n} f_{n-l}(x)E^l,$$
$$G_{n+1}(E, x) = \Sigma_{l=0}^{n+1} g_{n+1-l}(x)E^l.$$

One can see that (1.3) implies

$$P_{n+1}^2 = -R_{2n+2}(L) = \Pi_{m=0}^{2n+1}(E - E_m),$$

for some $E_m, m = 0, ..., 2n+1 \in \mathbb{C}$. That is, whenever $P_{n+1}$ and $L$ commute, they define a (possibly singular) hyperelliptic curve $K_n$ of genus $N$. In such a case $Q$ (or $(q,p)$) is called an algebro-geometric AKNS potential.

## 3. A GAUGE TRANSFORMATION AND THE VARIATION OF THE PERIOD LATTICE

If a hyperelliptic curve associated to a Dirac operator is singular, that means that the spectrum of $L$ consists of a disjoint union of arcs, at least some of which have common endpoints. Assume, for simplicity, that the only such common point is $\tilde{E}$. The following Theorem enables us to desingularize the curve associated with $(q,p)$, though not keeping the genus constant.

THEOREM 2.1.[GW2] Suppose $(q,p)$ is a meromorphic algebro-goemetric AKNS potential associated with the hyperelliptic curve

$$K_n = [(E,V)|V^2 = R_{2n+2}(E) = \Pi_{m=0}^{2n+1}(E - E_m)],$$

which has only one singular point at $(\tilde{E}, 0)$, i.e. $R_{2n+2}$ has a zero of order $r \geq 2$ at $\tilde{E}$. Let

$$\tilde{p} = \frac{G_{n+1}(\tilde{E}, x)}{F_n(\tilde{E}, x)}, \tag{2.1}$$
$$\tilde{q} = -q_x(x) - 2i\tilde{E}q(x) + q(x)^2\tilde{p}(x).$$

Then $(\tilde{q}, \tilde{p})$ is a meromorphic algebro-geometric AKNS potential associated with the hyperelliptic curve

$$\tilde{K}_{\tilde{n}} = [(E,V)|V^2 = \tilde{R}_{2n-2s+4(E)} = (E - \tilde{E})^{2-2s} R_{2n+2}(E)],$$

for some $2 \leq s \leq r/2 + 1$. In particular, the two curves have the same structure away from $\tilde{E}$.

PROOF: [GW2], p.20.

REMARK 1. $\tilde{K}$ is less singular than $K$; by iterating, if necessary, we end up with a curve that is non-singular, also denoted by $\tilde{K}$.

REMARK 2. Note that if $(q, p)$ is doubly periodic, $(\tilde{q}, \tilde{p})$ is also doubly periodic, with the same period lattice $L$.

We can now vary $L$ to $L'$ in such a way that the corresponding $\tilde{K}'$ remains non-singular and has the same genus as $\tilde{K}$, while $\tilde{E}$ is no more a branch point. By applying the inverse of the gauge transformation defined in the statement of the Theorem above (and choosing $L'$ appropriately) we end up with $p', q', K'$ such that $p', q'$ are doubly meromorphic and $K'$ is non-singular and has the same genus as $K$. We end up with

THEOREM 2.2. Let $(q, p)$ be a doubly periodic Dirac potential of period lattice $L$ and with associated spectral curve $K$. By varying $L$ infinitesimally we can desingularize $K$ into a non-singular curve $K'$ of the same genus.

PROOF: The gauge transformation is invertible ([GW2] eq.(3.28)).

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