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Singular solutions of the capillary problem

by

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Abstract

The problem of determining the surface interface of fluid partly filling a semi-infinite capillary tube closed at one end is considered, in the absence of gravity. It is known that solutions of the formal equations for a solution as a graph covering the base do not in general exist. In this paper it is shown that whenever smooth solutions fail to exist there will nevertheless exist a solution in a singular sense suggested by physical intuition. In some configurations of particular interest, the procedure leads to unique determination of the singular solution. However, uniqueness cannot in general be expected, as is shown by example. Further examples show a) that singular solutions may appear also when smooth solutions exist, and b) they may fail to occur in that case, depending on the particular geometry.

AMS Classifications: Primary 76B45; 53A10; 53C80. **Secondary** 49Q10; 35J60, 35B65

1. Introductory remarks.

We consider in this paper the classical problem of determining the configuration of liquid that covers the base and partially fills a semi-infinite cylindrical tube, composed of homogeneous solid material and of general section Ω. In a gravity field directed downwards along the generators of the cylinder toward the base, one is led to the equation

$$\text{div } Tu = \kappa u + \lambda, \ Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \tag{1.1}$$

for the height $u(x, y)$ of the liquid surface $\mathcal{S}$ over Ω, with κ a positive constant and λ a Lagrange parameter arising from an eventual volume constraint. If gravity vanishes then $\kappa = 0$. For background discussion, see, e.g., [F 3] Chapter 1.

On the boundary Σ of Ω, one finds the condition

$$\nu \cdot Tu = \cos\gamma \tag{1.2}$$

with ν unit exterior normal on Σ, and γ the (physically determined) angle between the surface interface $\mathcal{S} : u(x, y)$ and the vertical cylinder walls Z along the contact manifold. We may assume $0 \leq \gamma < \pi/2$; if $\gamma = \pi/2$ then the problem admits the (uniquely determined) trivial solution $u \equiv const$, while the case $\pi/2 < \gamma \leq \pi$ reduces to the indicated one by the transformation $u \to -u$.

The first general existence theorem for this problem was given by Emmer [E 1]; Emmer's conditions were later relaxed in some ways by Finn and Gerhardt [FG 1]. In both works, however, the condition $\kappa > 0$ is essential for the discussion. This is not an accident of the methods, as it was shown by Concus and Finn [CF 1] that if $\kappa = 0$ (gravity absent) then existence of classical solutions cannot in general be expected; solutions can fail to exist even for convex analytic domains. Physically, what may occur is the existence of a critical γ_{cr} in the range $0 < \gamma_{cr} < \pi/2$, such that smooth bounded solutions exist when $\gamma_{cr} < \gamma < \pi/2$, while if $0 < \gamma < \gamma_{cr}$ the fluid will climb up the walls in regions of relatively large boundary curvature, until either a portion of the base becomes uncovered or the top of the container is reached. (This change in behavior can under some circumstances be discontinuous in the sense that the bound is uniform in $\gamma_{cr} \leq \gamma < \pi/2$, see [CF 1], [CF 2]; it should be noted also that the climbing does not always occur at points of maximum boundary curvature.) Conceptually there is a connection with the *soluzioni generalizzate* introduced by M. Miranda [M 1] with regard to minimal surfaces. These are formal solutions in a very weak sense, and may be identically infinite on certain subsets of the domain having positive area.

Conditions sufficient for existence of smooth solutions when $\kappa = 0$ were given in [F 1, F 2]. These conditions include some configurations for which $\gamma = \gamma_{cr}$. In ensuing literature [T 1, La 1, CF 2, FF 1, FL 1, FM 1] other particular cases were examined for which $\gamma = \gamma_{cr}$ and for which it could be shown that no smooth solution exists. In these cases, it was established that a solution surface $\mathcal{S}_{cr} : u_{cr}(x, y)$ nevertheless exists as a smooth function

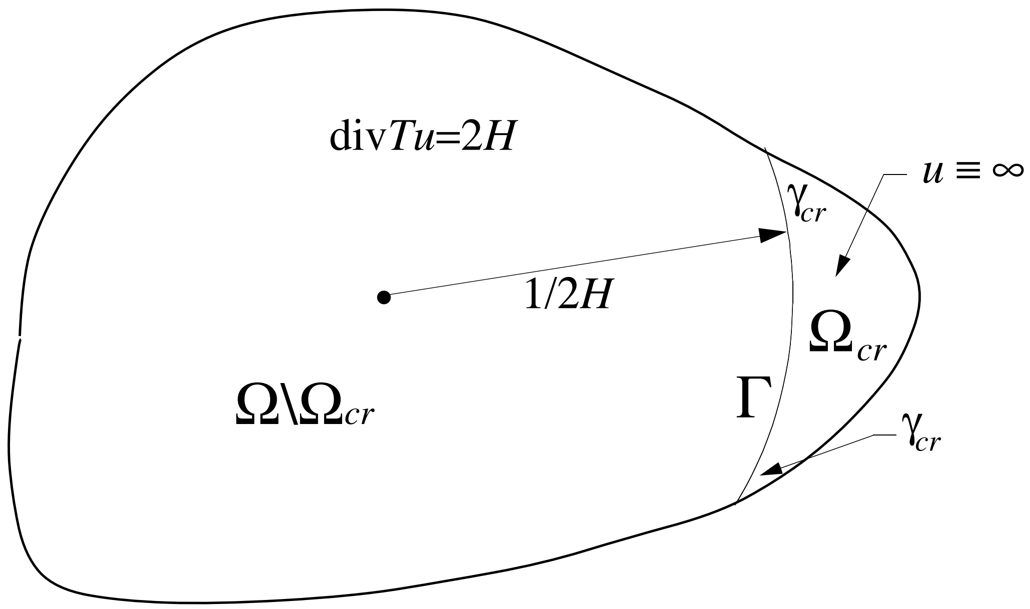

Figure 1: Regular and singular portions of a domain, $\gamma = \gamma_{cr}$

over the complement in Ω of a subdomain $\Omega_{cr} \subset \Omega$, bounded within Ω by non-intersecting circular arcs Γ of radius $R_{cr} = |\Omega|/(\,|\Sigma|\cos\gamma_{cr})$ that meet Σ in the angle γ_{cr} (see Figure 1); the surface $\mathcal{S}_{cr}$ achieves the prescribed data on $\Sigma\backslash\partial\Omega_{cr}$, and is upwardly tangent to vertical cylinder walls over Γ, which meet Σ in the same angle γ_{cr}. Such solutions are necessarily infinite on Γ, see [F 4, Theorem 2]; they are obtained as limits of solutions regular in all of Ω, and hence are *soluzioni generalizzate*, which are identically infinite throughout Ω_{cr}.

The indicated value for R_{cr} is determined by observing first that the left side of (1.1) is twice the mean curvature H of a solution surface over Ω; writing $\lambda = 2H$ and setting $\kappa = 0$ in (1.1) the resulting equation

$$\mathrm{div}Tu = 2H \tag{1.3}$$

admits as particular *soluzione generalizzata* a vertical cylinder of radius $R_{cr} = 1/2\,H$, whenever a curve Γ satisfying the requisite geometrical conditions can be found. In this respect H is the limiting value of mean curvatures such that smooth solutions exist over Ω, and thus can be determined by a (symbolic) integration over Ω and use of (1.2). The solution $u_{cr}(x, y)$ in $\Omega\backslash\Omega_{cr}$ is determined alternatively as the unique (up to an additive constant) regular solution of (1.3) in that domain, for which $\nu \cdot Tu = \cos\gamma_{cr}$ on $\Sigma\backslash\partial\Omega_{cr}$, and $\nu \cdot Tu = 1$ (i.e., $\gamma = 0$) on Γ.

It must be expected that if $\gamma_{cr} > 0$ then also for the case $0 < \gamma < \gamma_{cr}$ solutions should exist in a generalized sense. To our knowledge, the only authors to address this question in the literature were deLazzer, Langbein, Dreher and Rath [LLDR 1], who offered an empirical procedure for determining the mean curvature of fluid interfaces in closed cylindrical containers with polygonal cross-section and large height in zero gravity, in particular cases for which it is known that regular solutions fail to exist. The paper [LLDR 1] assumes without proof the existence of a singular solution of the type sought; that existence is in our view not evident. The intuition of these authors in the case of regular polygons that they considered was however correct; singular solutions having the form they surmise do in fact exist for such configurations, as we shall prove in Sec. 2.1.

In a general situation, the procedure of [LLDR 1] is not clearly defined, as it depends on a judicious guess as to where the singular set will occur. One of the contributions in the present work is to offer a procedure that leads to a singular solution of the type envisaged, in every case for which a smooth solution fails to exist. As it turns out, singular solutions can also occur in particular cases for which smooth solutions do exist, see Example 4.1. Singular solutions do not, however, exist in every case, as we show in Example 4.2.

We commence our study in Sec. 2 by considering the regular polygon domains introduced in [LLDR 1] and also another class of special domains of particular interest, for which we demonstrate the unique existence (up to an additive constant) of solutions of the form postulated in that reference. In these cases, both the subdomains of regularity and the corresponding solutions in those subdomains are uniquely determined by the conditions of the problem. In a general configuration the subdomain of regularity may, however, not be uniquely determined, as we show by example in Sec. 2.3; thus in general multiple solutions must be reckoned with. The precise conditions for uniqueness in that sense remain an open question.

In Sec. 3 we study domains of general form, and provide the asserted procedure leading to existence of singular solutions whenever solutions smooth throughout Ω fail to exist; these singular solutions are unique up to constants in a piecewise smooth subdomain of regularity (which as just noted may not be uniquely determined), and identically infinite on the complementary set in Ω. The discussion here applies to general piecewise smooth planar domains Ω bounded by a finite union $\Sigma = \bigcup_1^N \Sigma_j$ of smooth arcs which meet at angles strictly between 0 and π, and with prescribed constant contact angle data γ on the arcs. No data are prescribed at the intersection points. We determine conditions under which there will be a subdomain $\widehat{\Omega}$ strictly contained in Ω, bounded in part by subarcs $\widehat{\Sigma}$ of Σ and in part within Ω by subarcs Γ of semicircles of radius $R = 1/2\,H$ as in Figure 1, such that a solution of (1.3) will exist in $\widehat{\Omega}$, achieving the prescribed data γ on $\widehat{\Sigma}$, and boundary angle $\gamma = 0$ on Γ. Such solutions necessarily become positive infinite on Γ.

Our underlying weapon for attacking the general existence problem is the necessary and sufficient condition Theorem 7.10 of [F 3], for existence of smooth solutions. In the case of constant H considered here this theorem takes the form: *A smooth solution $u(x, y)$ exists in $\widehat{\Omega}$ if and only if the functional*

$$\Phi[\Omega^*] \equiv \int_\Omega D\chi_{\Omega^*} - \int_\Sigma \chi_{\Omega^*} \cos\gamma\, ds + 2H|\Omega^*| \tag{1.4}$$

is positive for every Caccioppoli set $\Omega^* \subset \widehat{\Omega}$, with $\Omega^* \neq \oslash,\ \Omega$. Here χ is characteristic function, and

$$2H = \frac{1}{|\widehat{\Omega}|} \int_{\partial\widehat{\Omega}} \cos\gamma\, ds. \tag{1.5}$$

In order to determine whether a given set $\widehat{\Omega}$ has this property, we show that *for the configurations considered, there exists a minimizing set for Φ in $\widehat{\Omega}$, and that this set is bounded within $\widehat{\Omega}$ by a finite number N of subarcs of semicircles of radius $1/2H$, each of which either meets Σ at a smooth point with angle γ measured within $\widehat{\Omega}$, or meets Σ at a vertex*; see Sec. 3. *The curvature vector of each subarc is directed exterior to $\widehat{\Omega}$.* It can happen that $N = 0$, in which case a smooth solution does exist.

To fix the ideas, we examine in the next following sections some cases of particular interest for which the constructions can be effected explicitly.

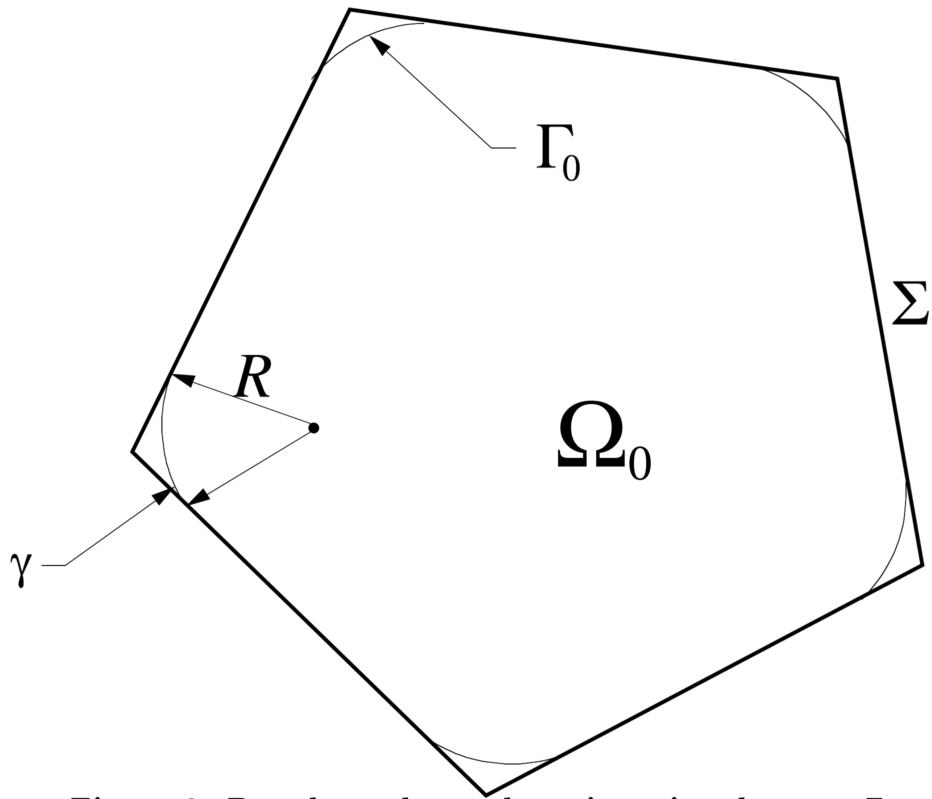

Figure 2: Regular polygonal section; singular arcs Γ_0; subdomain Ω_0

2.1. Regular polygons.

We study here the construction effected in [LLDR 1], with a view to establishing the existence of appropriate singular solutions of (1.3, 1.2) in the constructed subdomain of a regular N-gon Ω. Specifically, we assume constant prescribed data γ in the range $0 < \gamma < \pi/N$ prescribed on $\Sigma = \partial\Omega$. For such data it is known [CF 1] that no regular solution can exist in Ω. We cut off the corners with circular arcs $\Gamma_0 = \{\Gamma_j\}$, $j = 1, \ldots N$, of radius R and meeting the boundary segments in angle γ, as indicated in Figure 2, leaving a domain Ω_0 bounded in part by the arcs Γ_0 and in part by subsegments Σ_0 of Σ. Here $R = 1/2H$ is determined by the necessary condition for existence of a solution of (3) in Ω_0, achieving data γ on Σ_0 and data $\gamma = 0$ on Γ_0, that

$$\int\limits_{\Omega_0} \mathrm{div} Tu\, dx = 2H|\Omega_0| = \oint\limits_{\Sigma_0 \cup \Gamma_0} \nu \cdot Tu\, ds = |\Sigma_0|\cos\gamma + |\Gamma_0|. \tag{2.1}$$

Following [LLDR 1], we obtain a quadratic equation for R, one root of which is spurious; the other root yields the (necessary) condition

$$R = \frac{1}{2H} = \frac{a\sqrt{\tan\frac{\pi}{N}}}{\sqrt{\tan\frac{\pi}{N}}\cos\gamma + \sqrt{\sin\gamma\cos\gamma + \frac{\pi}{N} - \gamma}} \tag{2.2}$$

where a is distance from the polygon's center to its sides. From Theorem 6.13 of [F 3] adapted to the present configuration, we see that a solution will indeed exist if for every non-trivial subset Ω^* of Ω_0, bounded within Ω_0 by subarcs Γ of semicircles of the same radius R and which meet two interior points of Σ_0 in angle γ, or meet any interior point of Γ_0 in angle zero, or which terminate in one or more juncture points of Γ_0 with Σ_0, there will hold $\Phi[\Omega^*] > 0$. In fact, the initial two categories of intersection can be excluded directly by geometric considerations, which we proceed to verify.

Any arc Γ of radius R that meets one of the arcs Γ_0 in angle zero would have to coincide with that arc. Hence we may restrict ourselves to the case of an arc that meets Σ_0 at two interior points in angle γ. Denoting the boundary segments in Σ_0 by $\{e_j\}$, we consider such an arc initiating at an interior point of e_1, which it meets in angle γ, and terminating interior to a segment e_j. We choose Cartesian coordinates with the center of the polygon at the origin and e_1 orthogonal to the positive x-axis. Since Γ_1 and Γ have the same radius and the same contact angle with e_1, we can obtain Γ by rigid horizontal motion of Γ_1 in the positive direction. Thus, the center of Γ_1 will be displaced positively from its original center. But since Γ_1 intersects both e_1 and e_2 in angle γ, its center lies on the angle bisector at the vertex v_2; this line passes through the origin and lies in the second and fourth quadrants. Now Γ meets both e_1 and e_j in angle γ, and thus its center lies either on a line bisecting one of the vertices (if i is even) or on a line bisecting one of the sides (if i is odd). In either case its center lies on a line through the origin. If $i \leq N/2 + 1$ this line lies either in the second or fourth quadrant or is collinear with the x-axis. In the fourth quadrant, the line containing the center of Γ lies to the negative x-direction of the line containing the center of Γ_1. However, we know that the former center is obtained by an e_1 translation of the center of Γ_1 in the positive x-direction. Thus, the center of Γ lies either in the fourth quadrant or on the x-axis. It follows that the center of Γ is at least distance a from e_1. But its distance from e_1 is exactly $R\cos\gamma$. We conclude $R\cos\gamma \geq a$, hence

$$\frac{\sqrt{\tan\frac{\pi}{N}}\cos\gamma}{\sqrt{\tan\frac{\pi}{N}}\cos\gamma + \sqrt{\sin\gamma\cos\gamma + \frac{\pi}{N} - \gamma}} \geq 1, \tag{2.3}$$

contradicting $0 \leq \gamma \leq \pi/N$. Thus no such arc can exist, for $i \leq N/2+1$.

If $i > N/2+1$ and N is even, all cases are excluded since Γ would subtend at least π radians, which is impossible as Γ must be a proper subarc of a semicircle [F 3, Theorem 6.11]. If N is odd, all cases are excluded for the same reason, except for the single case $i = (N+3)/3$. But in this event, we observe that the minimum distance between e_1 and $e_{(N+3)/2}$ is at least

$$d = a\left(1 + \sec\frac{\pi}{N} - \frac{\sin^2\frac{\pi}{N}}{\cos\frac{\pi}{N}}\right). \tag{2.4}$$

Thus, we must have $2R \geq d$, from which

$$\frac{2\sqrt{\tan\frac{\pi}{N}}}{\cos\gamma\sqrt{\tan\frac{\pi}{N}} + \sqrt{\sin\gamma\cos\gamma + \frac{\pi}{N} - \gamma}} \geq \sqrt{\tan\frac{\pi}{N}}\left(1 + \sec\frac{\pi}{N} - \frac{\sin^2\left(\frac{\pi}{N}\right)}{\cos\frac{\pi}{N}}\right). \tag{2.5}$$

Using the bound $0 < \gamma < \pi/N$ and noting $N \geq 4$, we calculate that the left side of (2.5) is bounded above by 1.5, while the right side is bounded below by 1.7. Thus we are led also in this case to a contradiction, and we conclude that no such arc can exist.

We consider finally the possibility of an arc of the specified radius, that terminates at a juncture point p of one of the singular arcs Γ_0 with the boundary Σ of the polygon. We assert that this cannot occur in a minimizing configuration. For, following the reasoning on p. 153 of [F 3] and denoting by γ', γ'' the angles on the two sides of Γ at the intersection point, we must have either $\gamma' \geq \gamma$, $\gamma'' \geq \pi$, or else $\gamma' \geq 0$, $\gamma'' \geq \pi - \gamma$, depending on the orientation of Γ, see Figure 3. The first case is geometrically not possible, while in the second case, Γ would have to coincide with Γ_0.

We conclude that there can be no minimizing arcs in Ω_0, which means that the minimum for the functional Φ is achieved by the entire domain Ω_0.

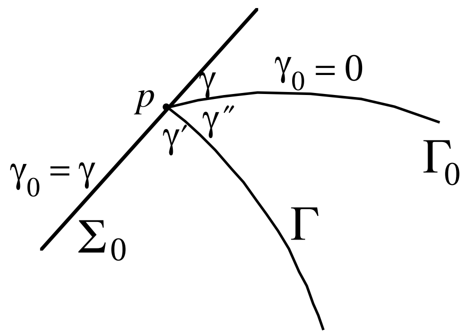

Figure 3: Extremal arc Γ through juncture point p

But $\Phi[\Omega_0] = 0$, and thus every non-trivial Caccioppoli subset of Ω_0 yields a positive Φ. Existence now follows from the general theorem at the end of Section 1.

2.2. Ice cream cones; existence.

As a second example, we consider "ice cream cone" domains Ω determined by two line segments each of length a which form an angle 2ϑ, and capped by a circular arc tangent to both segments at their end points, see Figure 4a. We may assume the arc has unit radius. This configuration has a particular interest, as whenever $\vartheta+\gamma \geq \pi/2$ a regular solution to (2,3) is given explicitly as a spherical cap, with center on the vertical through the center of the circular arc. We consider again the case $\gamma < \pi/2 - \vartheta$, and we seek to cut off the corner with a circular arc Γ of radius $R = 1/2H$ which meets the linear boundary segments in angle γ (Figure 4a), in such a way that a singular solution will exist on the side of Γ opposite to the corner, and which tends to positive infinity on Γ. As above, we note that the possibility of any solution on the side including the corner is excluded by Theorem 6.2 of [F 3].

With notation as in the figure, we are to determine H by the relation

$$2H|\Omega_0| = |\Gamma_0| + |\Omega_0| \cos\gamma \tag{2.6}$$

in which

$$2|\Omega_0| = (\pi + 2\vartheta)a^2 \tan^2\vartheta + (\pi - 2\vartheta - 2\gamma)R^2 \frac{\cos\gamma\cos(\vartheta+\gamma)}{\sin\vartheta} \tag{2.7}$$

$$|\Sigma_0| = 2a + (\pi + 2\vartheta)a\tan\vartheta - 2R\frac{\cos(\vartheta+\gamma)}{\sin\vartheta} \tag{2.8}$$

$$|\Gamma_0| = R(\pi - 2\vartheta - 2\gamma). \tag{2.9}$$

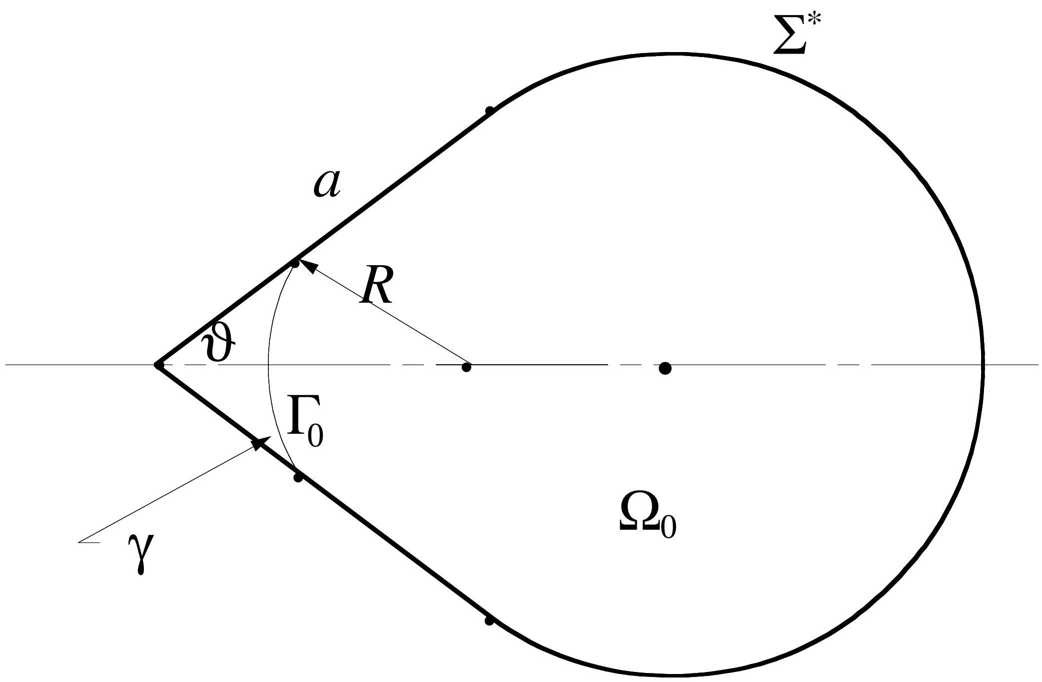

Figure 4a: Ice cream cone domain; singular arc Γ_0

We are led again to a quadratic equation for H, which we write in the form

$$xH^2 + yH + z = 0 \tag{2.10}$$

where

$$x = a^2(\pi + 2\vartheta)\tan^2\vartheta + 2a^2\tan\vartheta \tag{2.11}$$

$$y = -2a\cos\gamma - a(\pi + 2\vartheta)\tan\vartheta\cos\gamma \tag{2.12}$$

$$4z = -(\pi - 2\vartheta - 2\gamma) + 2\frac{\cos\gamma\cos(\vartheta+\gamma)}{\sin\vartheta}. \tag{2.13}$$

For the discriminant Δ of (2.10), we find

$$\frac{1}{a^2}\Delta = (\pi + 2\vartheta)^2 \tan^2\vartheta \cos^2\gamma + 2(\sin 2\gamma + (\pi - 2\vartheta - 2\gamma)\,) \tag{2.14}$$

$$\cdot(\,(\pi + 2\vartheta)\tan^2\vartheta + \tan\theta)$$

from which we conclude $\Delta > 0$, so that two real solutions appear; they are given by

$$H^{\pm} = \frac{2\cos\gamma + (\pi + 2\vartheta)\tan\vartheta\cos\gamma \pm \sqrt{\Delta}}{2a(\pi + 2\vartheta)\tan^2\vartheta + 4a\tan\vartheta}. \tag{2.15}$$

We must examine these solutions with regard to consistency with the geometric assumptions under which they were derived. Specifically, we must ensure that the arc Γ_0 actually meets the boundary segments between the vertex and the circular cap. Denoting the distance from the vertex to the intersections with Γ_0 by δ, we must thus show $\delta < a$. We find

$$\delta = R\frac{\cos(\vartheta+\gamma)}{\sin\vartheta} \tag{2.16}$$

and we thus require that

$$H \geq \frac{\cos(\vartheta+\gamma)}{2a\sin\vartheta}. \tag{2.17}$$

But (2.15) yields the bound

$$H^{+} > \frac{\cos\gamma}{2a\tan\vartheta}. \tag{2.18}$$

Since both ϑ and γ lie between 0 and $\pi/2$, it is clear that

$$\frac{\cos\gamma}{2a\tan\vartheta} \geq \frac{\cos(\vartheta+\gamma)}{2a\sin\vartheta} \tag{2.19}$$

and we see that H^{+} provides an effective solution to the problem.

To study H^{-}, we consider the function

$$F(\vartheta,\gamma) \equiv H^{-} - \frac{\cos(\vartheta+\gamma)}{2a\sin\vartheta} \tag{2.20}$$

over the region of admissible data, which is bounded between the coordinate axes and the line $\vartheta + \gamma = \pi/2$. On that line, $F(\vartheta, \gamma) = 0$, while below the line there holds $F(\vartheta, \gamma) < 0$. Thus, the solution H^- is extraneous, and we conclude that there is a unique arc Γ_0 meeting the boundary segments and satisfying the required conditions. For this arc, we have

$$\Psi[\Omega_0] \equiv |\Gamma_0| + |\Sigma^*| \cos\gamma - 2H|\Omega_0| = 0 \tag{2.21}$$

with $2H = 1/R$, see Figure 4a.

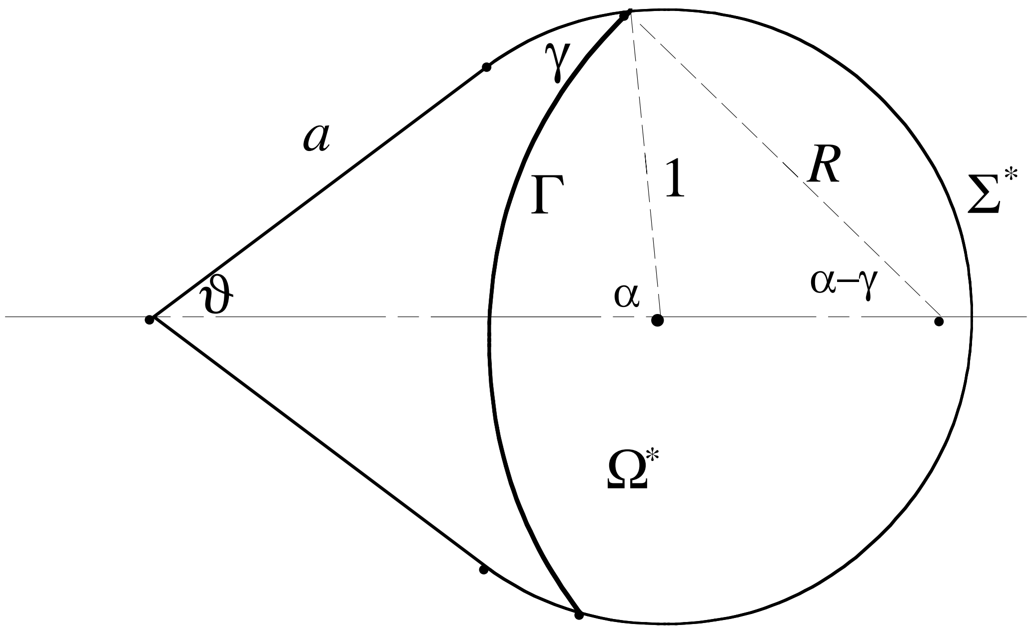

Figure 4b: Ice cream cone; extremal domain Ω^*.

We consider now the possibility of arcs Γ that intersect the circular portion of Σ. A number of cases can arise, as indicated in Figures 4b,c,d.

In Figure 4b, a solution on the side of Γ including the corner is excluded by Theorem 6.2 of [F 3]. A necessary condition for a solution in the complementary domain Ω^* is that $\Psi[\Omega^*] = 0$, see (2.21). We translate Γ rigidly to the right, keeping R fixed and allowing γ to change. We find, denoting initial values with subscript zero and setting

$$\Psi(\cdot;\alpha) = |\Gamma(\alpha)| + |\Sigma^*(\alpha)|\cos\gamma_0 - \frac{1}{R}|\Omega^*(\alpha)|, \tag{2.22}$$

$$R\sin(\alpha-\gamma) = \sin\alpha, \qquad \frac{d\gamma}{d\alpha} = 1 - \frac{\cos\alpha}{R\cos(\alpha-\gamma)} \tag{2.23}$$

$$|\Gamma| = 2(\alpha-\gamma)R, \qquad \frac{d|\Gamma|}{d\alpha} = 2\left(\frac{\cos\alpha}{\cos(\alpha-\gamma)}\right) \tag{2.24}$$

$$|\Sigma^*| = 2(\pi-\alpha), \qquad \frac{d|\Sigma^*|}{d\alpha} = -2. \tag{2.25}$$

Denoting by x the intersection point of Γ with the horizontal, relative to the center of Σ^*, we find

$$x = -\cos\alpha - (1-\cos(\alpha-\gamma)\,)R \tag{2.26}$$

$$\frac{d|\Omega^*|}{dx} = -2R\sin(\alpha-\gamma), \qquad \frac{d|\Omega^*|}{d\alpha} = -2R\tan(\alpha-\gamma)\sin\gamma \tag{2.27}$$

and thus

$$\frac{\cos(\alpha-\gamma)}{2}\,\frac{d\Psi}{d\alpha} = \cos\alpha - \cos\gamma_0\cos(\alpha-\gamma) + \sin\gamma\sin(\alpha-\gamma). \tag{2.28}$$

Since the arc Γ is a proper subarc of a semicircle, we have $0 < (\alpha-\gamma) < \pi/2$, and thus by (2.23) $\gamma > \gamma_0$ throughout the range $\alpha_0 \le \alpha \le \pi$. Therefore,

$$\frac{d\Psi}{d\alpha} < \frac{2}{\cos(\alpha-\gamma)}(\cos\alpha - \cos\alpha) = 0. \tag{2.29}$$

Since $\Psi(\pi) = 0$, we find that the initial value of Ψ cannot vanish, and thus no singular solution can exist in the configuration considered.

Next we study a subdomain Ω^* as in Figure 4c. Allowing Γ to move to the right, we obtain as above that Ψ decreases. But at the extreme position, $\Psi = 0$. Thus, initially $\Psi > 0$, which means there is no singular solution in Ω^*.

Finally, we consider the configuration of Figure 4d. We can exclude this case again by moving the right hand arc to the right, observing that Ψ decreases in this motion, and that the end configuration is that of Figure 4b, for which $\Psi > 0$.

Let us introduce now the functional (1.4) $\Phi[\Omega^*]$, $\Omega^* \subset\subset \Omega_0$. Using that $\Psi[\Omega_0] = 0$, one shows easily that if $\Psi[\Omega^*]$ is defined by

$$\Psi[\Omega^*] \equiv \int_\Omega D\chi_{\Omega^*} + \int_\Sigma \chi_{\Omega^*} \cos\gamma ds - 2H|\Omega^*| \tag{2.30}$$

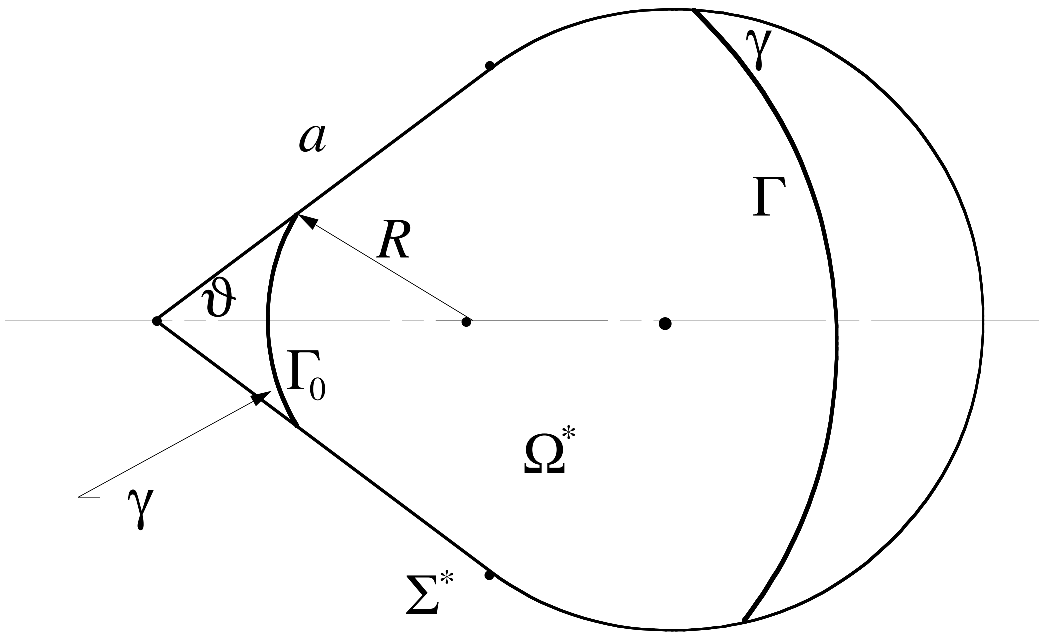

Figure 4c: Ice cream cone; extremal domain Ω^*.

with H defined by (1.5), then

$$\Psi[\Omega^*] = \Phi[\Omega_0 \backslash \Omega^*] \tag{2.31}$$

all $\Omega^* \subset \Omega_0$. It follows that the minimizing problems for the two functionals over subsets of Ω_0 are equivalent, with the minimizer for Ψ being the complement in Ω_0 of the minimizer for Φ. *The subsets Ω determined by the circular arcs Γ introduced above are exactly the extremal sets for Ψ (up to* *rigid motions of* Γ *that do not affect* Ψ*). Since we have shown that* Ψ *is positive on all these sets, it follows from the general theorem at the end of Sec. 1 that a solution with the prescribed data exists over* Ω_0*. This is the singular solution whose existence was to be proved.*

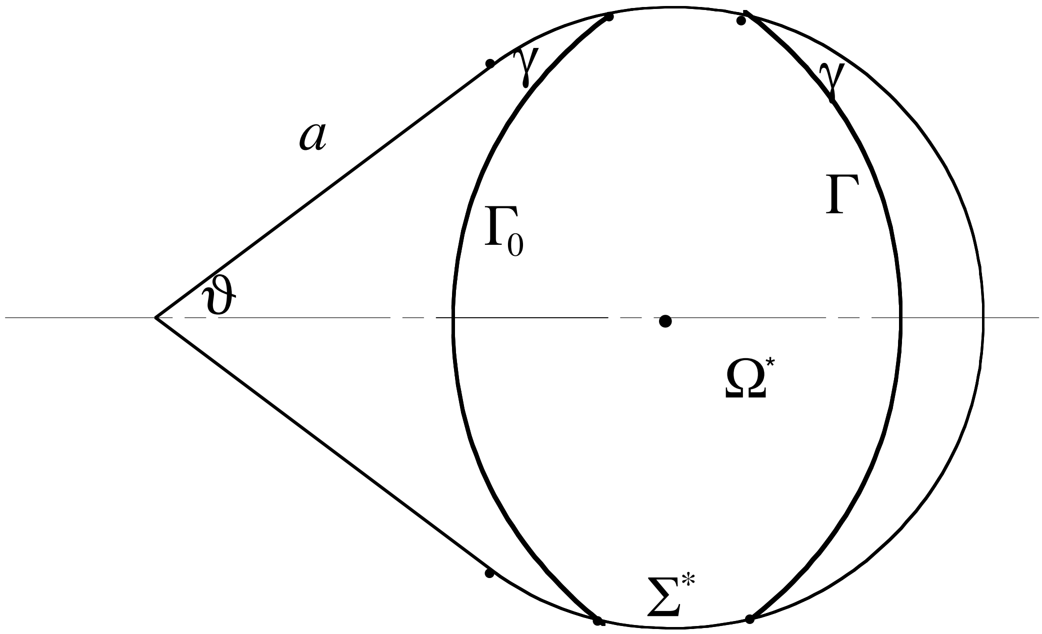

Figure 4d: Ice cream cone; extremal domain Ω^*.

2.3. Ice cream cones; uniqueness and non-uniqueness.

We observe from the above discussion that *the domain* Ω_0 *is uniquely determined. Also the singular solution is uniquely determined in* Ω_0*, as follows from Theorem 3.1 of* [FLu 1]. (We note that we cannot use the simpler Theorem 5.1 of [F 3] to this purpose, as the proof of that theorem required solutions in $W^{1,1}(\Omega_0)$, which is not known in the present case.) We now show by example that *by altering the initial geometry, we can obtain configurations for which the base domain Ω_0 is not uniquely determined.*

Our underlying observation is that the requirement $\Psi[\Omega_0] = 0$ involves only properties of the region Ω_0 itself and its boundary. Thus, having found Ω_0, we are free to alter the remaining part of the initial domain in any way, without altering that property. We consider two ice cream domains $\Omega^{(1)}$ and $\Omega^{(2)}$, with different opening angles $\phi^{(1)}$ and $\phi^{(2)}$, and we choose γ so that $\phi_1 + \gamma < \pi/2$, $\phi_2 + \gamma < \pi/2$. By the discussion of the preceding section, we may construct subdomains $\Omega_0^{(1)}, \Omega_0^{(2)}$ admitting the respective singular solutions $u^{(1)}, u^{(2)}$. Following a similarity transformation, we may assume that the vertical heights at the two intersection points with the respective arcs $\Gamma^{(1)}, \Gamma^{(2)}$ are equal for the two domains. Since $\phi^{(1)} \neq \phi^{(2)}$, we will then have $H^{(1)} \neq H^{(2)}$. We reflect one of the domains in a vertical axis, position the two resultant domains as in Figure 5 and take as initial domain Ω the domain bounded by the simple outer curve. This will be a piecewise smooth domain with two re-entrant vertices.

The two singular solutions $u^{(1)}, u^{(2)}$, originally determined for the individual domains $\Omega^{(1)}$ and $\Omega^{(2)}$, now provide two distinct singular solutions for the domain Ω.

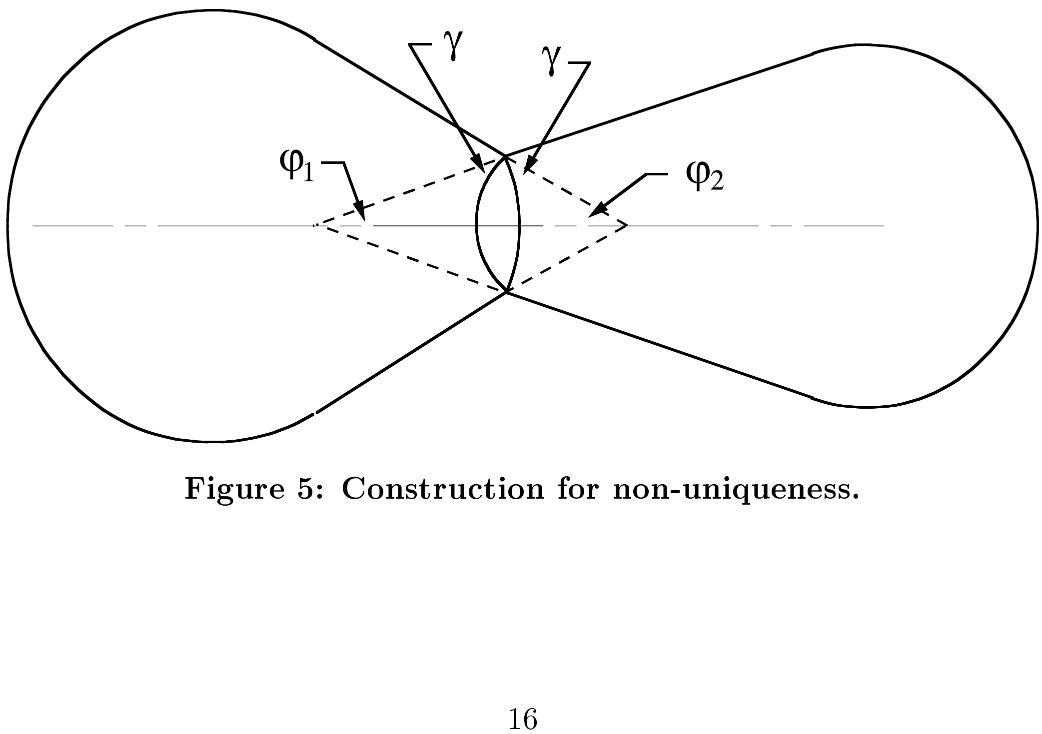

Figure 5: Construction for non-uniqueness.

3. General domains.

We study sections Ω of cylinders bounded by piecewise smooth curves Σ, that are assumed to be uniformly in C^1 except for a finite number N_P of protruding corners of opening $2\alpha_i$, $0 < 2\alpha_i < \pi$, and a finite number N_R of re-entrant corners, $\pi < 2\alpha_i \leq 2\pi$. We suppose that for some boundary angle γ in $0 < \gamma < \pi/2$ there exists a Caccioppoli set $\widehat{\Omega} \subset \Omega$, $\widehat{\Omega} \neq \Omega, \oslash$, such that the functional

$$\Phi(\Omega^*; H) \equiv |\Gamma| - |\Sigma^*| \cos\gamma + 2H|\Omega^*| \tag{3.1}$$

will be non-positive when

$$2H = \frac{|\Sigma|}{|\Omega|} \cos\gamma \tag{3.2}$$

and $\Omega^* = \widehat{\Omega}$ up to a null set. Here, $|\Gamma| = \int_\Omega |D\chi_{\Omega^*}|$ (χ = characteristic function) is the perimeter in Ω of Ω^*, and Σ^* is the trace on Σ of Ω^*, see, e.g., [G 1, MM 1, EG 1] for background details on the associated "BV theory", and [F 3] Chapters 6 and 7 for background material on the functional. We note that this requirement is equivalent to the nonexistence of a smooth solution of (1.3) in Ω that satisfies the boundary condition (1.2) in a reasonable sense, see [F 3], Theorem 7.10. We intend to prove:

Theorem 3.1: *Under the conditions just indicated, there exist H_0, $0 < H_0 \leq H$ and a subdomain $\Omega_0 \subset \Omega$, $\Omega_0 \neq \oslash, \Omega$, bounded in Ω by a finite number of subarcs Γ_0 of semicircles of common radius $1/(2H_0)$, all intersecting Σ at smooth points and in angle γ (measured exterior to Ω_0) or else at re-entrant vertices with angles $\geq \gamma$, and a solution $u(x,y)$ of*

$$\operatorname{div} Tu = 2H_0 \tag{3.3}$$

in Ω_0, such that $\nu \cdot Tu = \cos\gamma$ on $\Sigma^* = \partial\Omega_0 \cap \Sigma$, and $\nu \cdot Tu \to 1$ as Γ_0 is approached from within Ω_0. Further, $u \to +\infty$ for any approach to Γ_0 from within Ω_0. The solution $u(x,y)$ is determined within Ω_0 up to an additive constant.

In this result, the curvature vectors of the arcs Γ_0 are directed into Ω_0, and the angles γ are measured exterior to Ω_0, as is the case in Figure 1. In the expression $\nu \cdot Tu$ near Γ_0, ν is chosen as the vector directed along the radial line from the center of Γ_0. We note that no differentiable function can yield the value 1 for $\nu \cdot Tu$; the result $\nu \cdot Tu \to 1$ on Γ_0 means that the solution surface $\mathcal{S}$ is tangent vertically upward to the circular cylinders over Γ_0, that is, $\mathcal{S}$ meets those cylinders asymptotically in angle zero. The result $u \to +\infty$ on Γ_0 means that $\mathcal{S}$ is asymptotic at positive infinity to the vertical cylinders over Γ_0, as Γ_0 is approached within Ω_0.

We prove the theorem in several steps. We prove first:

Lemma 3.1: *Suppose that Ω contains no vertices (so that $\Sigma \in C^1$), and suppose that for some H_0, there exists a subset $\widehat{\Omega}_0 \subset \Omega$, $\widehat{\Omega}_0 \neq \oslash, \Omega$, that minimizes the "adjoint" functional*

$$\Psi(\Omega^*; H_0) \equiv |\Gamma| + |\Sigma^*| \cos\gamma - 2H_0|\Omega^*| \tag{3.4}$$

and for which $\Psi(\widehat{\Omega}_0; H_0) = 0$. Then

(i) the boundary $\widehat{\Gamma}_0$ of $\widehat{\Omega}_0$ in Ω consist of a finite number of disjoint subarcs of semicircles, each of radius $1/2H_0$, with curvature vectors directed into $\widehat{\Omega}_0$ and meeting Σ in angles γ measured exterior to $\widehat{\Omega}_0$, and

(ii) there exists $\Omega_0 \subset \widehat{\Omega}_0$, bounded in $\widehat{\Omega}_0$ by arcs Γ_0 of the same radius, for which the theorem holds.

Proof:

(i) Since $\widehat{\Omega}_0$ is minimizing, we conclude from a theorem of Massari [Ma 1] that it is bounded by Ω by analytic arcs, and we obtain from Lemmas 6.4, 6.9 and Theorem 6.11 of [F 3] (adapted to the functional Ψ) that these arcs are disjoint strict subarcs of semicircles $\widehat{\Gamma}_0$, each of radius $1/(2H_0)$ and meeting Σ in angles γ. The indicated orientation relations must hold, as otherwise one shows easily that the configuration could not minimize. Since $\gamma > 0$, there exists $\delta > 0$ such that each arc of $\widehat{\Gamma}_0$ subtends an arc of length at least δ on Σ, and thus the number of such arcs must be finite.

(ii) We choose $\widehat{\Omega}_0$ as an initial candidate for the existence domain of the sought regular solution, and we denote by $\widehat{\Phi}, \widehat{\Psi}$ the functionals Φ, Ψ restricted to subsets of $\widehat{\Omega}_0$, which we take as base domain, imposing boundary data $\gamma = 0$ on each of the arcs. The formal definition for Ψ changes to

$$\widehat{\Psi}(\Omega^*; H_0) \equiv |\Gamma| + |\Sigma^*| \cos\gamma + |\Gamma^*| - 2H_0|\Omega^*| \tag{3.5}$$

where Γ^* is the trace on $\widehat{\Gamma}_0$ of Ω^*, and we see that the values achieved by $\widehat{\Psi}$ on subsets of $\widehat{\Omega}_0$ are identical to those achieved by Ψ. That is not the case for the original functional Φ, which when restricted to $\widehat{\Omega}_0$ becomes

$$\widehat{\Phi}(\Omega^*; H_0) \equiv |\Gamma| - |\Sigma^*| \cos\gamma - |\Gamma^*| + 2H_0|\Omega^*|. \tag{3.6}$$

From the relation $\Psi(\widehat{\Omega}_0; H_0) = 0 = \widehat{\Psi}(\widehat{\Omega}_0; H_0)$ we obtain at once

$$\widehat{\Phi}(\Omega^*; H_0) = \widehat{\Psi}(\widehat{\Omega}_0 \backslash \Omega^*; H_0) \tag{3.7}$$

and thus if for every Caccioppoli subset $\Omega^* \subset \widehat{\Omega}_0$ with $\Omega^* \neq \widehat{\Omega}_0, \oslash$ there holds $\Psi(\Omega^*; H_0) > 0$, we obtain also $\Phi(\Omega^*; H_0 > 0$ for every such subset; as a consequence, we may apply Theorem 7.10 of [F 3] to obtain the existence of a solution $u(x, y)$ in $\widehat{\Omega}_0$ of $\mathrm{div} Tu = 2H_0$, with $\nu \cdot Tu = \cos\gamma$ on $\Sigma^* = \partial\Omega \cap \Sigma$, $\nu \cdot Tu \to 1$ on the boundary arcs Γ_0, as asserted in the theorem.

It may however occur that a subset Ω^* exists for which $\Psi(\Omega^*; H_0) = 0$. In this event, Theorem 7.10 of [F 3] yields the nonexistence of such a solution in $\widehat{\Omega}_0$. Should that happen, we proceed by constructing a "locally smallest" domain on which the functional vanishes.

We begin by covering $\widehat{\Omega}_0$ with a net of squares $T_\epsilon(i,j)$ of side length ϵ, $0 < \epsilon < 1$, and restrict attention to the index subset $(i,j) \in \mathcal{I}_\epsilon^0$ for which the squares $T_\epsilon(i,j)$ lie interior to $\widehat{\Omega}_0$. Under our assumptions, there would exist a positive

$$\epsilon_1 = \text{lub}\, \{\epsilon : \exists \widehat{\Omega} \subset \widehat{\Omega}_0,\, \widehat{\Omega} \neq \oslash,\, \widehat{\Psi}(\widehat{\Omega}; H_0) = 0$$
$$\text{and } \widehat{\Omega} \cap \mathcal{T}_\epsilon^0(i,j) = \oslash \text{ for some } (i,j) \in \mathcal{I}_\epsilon^0\}.$$

Corresponding to this ϵ_1, there would be a maximal number N_1 of squares in $T_\epsilon(i,j)$ that can be excluded in this way by some such $\widehat{\Omega}$, and we denote by $\widehat{\Omega}_1$ a particular non-null subset of $\widehat{\Omega}_0$, for which $\widehat{\Psi}(\widehat{\Omega}_1; H_0) = 0$ and for which N_1 squares of $T_{\epsilon_1}(i,j)$ lie in the closure of $\widehat{\Omega}_0$ and exterior to $\widehat{\Omega}_1$. Since $\widehat{\Omega}_1$ is minimizing for $\widehat{\Psi}(\Omega^*; H_0)$ over $\widehat{\Omega}_0$, it is bounded in $\widehat{\Omega}_0$ by subarcs $\widehat{\Gamma}_1$ of semicircles of radius $1/2H_0$. No two of these subarcs can terminate in the same boundary point, as then the configuration could easily be modified to reduce the value of $\widehat{\Psi}$, contradicting the minimizing property. Nor can any of them terminate in an interior point of $\widehat{\Gamma}_0$, as it would have to meet that arc in angle zero and hence would coincide with that arc, since it has the same radius.

We consider next the functional $\widehat{\Psi}_1$, restricted to sets in $\widehat{\Omega}_1$. If there is no $\widehat{\Omega} \subset \widehat{\Omega}_1$ such that $\widehat{\Omega} \neq \oslash$, $\widehat{\Omega}_1$ and $\widehat{\Psi}_1(\widehat{\Omega}; H_0) = 0$, then Theorem 7.10 of
[F 3] yields that the set $\widehat{\Omega}_1$ satisfies the requirements of the theorem, as in the reasoning above. Otherwise we proceed to construct a decreasing sequence $\epsilon_j \to 0$ and corresponding sequence $\widehat{\Omega}_j \subset \widehat{\Omega}_{j-1}$, $\widehat{\Omega}_j \neq \oslash$, Ω_{j-1}. The $\widehat{\Omega}_j$ are bounded from zero in measure, since it follows from $\widehat{\Psi}_j(\widehat{\Omega}_j; H_0) = 0$ and the isoperimetric inequality that

$$C\sqrt{|\widehat{\Omega}_j|} + |\widehat{\Sigma}_j| \cos\gamma - 2H|\widehat{\Omega}_j| \leq 0$$

for some fixed $C > 0$. Further, each $\widehat{\Omega}_j$ is bounded in Ω by a finite number N_j of subarcs $\widehat{\Gamma}_j$ of semicircles of radius $1/2H_0$. There holds $N_j < N(\Omega; \gamma; H_0)$ independent of j, as each arc subtends on Σ a length bounded from zero, depending only on Ω, on γ, and on H_0. Thus, a subsequence of the $\widehat{\Omega}_j$ converge to a limit set $\Omega_0 \neq \oslash$, and the corresponding functional $\widehat{\Psi}_0(\Omega_0; H_0) = 0$. In view of the continuous transition, Ω_0 is again minimizing, and is bounded in Ω by subarcs of semicircles of radius $1/2H_0$ that meet Σ in angle γ. If there were to exist $\widehat{\Omega} \subset \Omega_0$, with $\widehat{\Omega} \neq \oslash$, Ω_0 and for which $\widehat{\Psi}_0(\widehat{\Omega}; H_0) = 0$, that would contradict the maximality of the number N_j of excluded squares, for some j. Thus, $\widehat{\Psi}_0(\widehat{\Omega}; H_0) > 0$ for every such subset $\widehat{\Omega}$, from which we conclude from $\widehat{\Psi}_0(\Omega_0; H_0) = 0$ that $\widehat{\Phi}_0(\widehat{\Omega}; H_0) > 0$ for every such subset, and finally from Theorem 7.10 of [F 3] that Ω_0, H_0 have the properties required by the theorem. □

It is not clear whether there exist configurations for which the indicated infinity of steps in this procedure is actually required, or whether the procedure will always terminate after a finite number of steps. We remark, however, that it cannot be expected that the initially given domain Ω_0 will always provide a configuration that works. An example is provided by the "canonical proboscis" domains studied in [FF 1, FL 1, FM 1], see Figure 6. The proboscis shape has the property that a continuum of "extremal arcs" Γ

meeting Σ in the prescribed angle γ appear as translates of a single circular arc. Any of the subdomains consisting of the portion of the entire region to the left of any of the arcs Γ would serve equally well as the initial Ω_0. In this particular example, the procedure will terminate after a finite number of steps, yielding as Ω_0 the region to the left of the extremal arc joining the two intersections of the circular portion of the boundary with the proboscis.

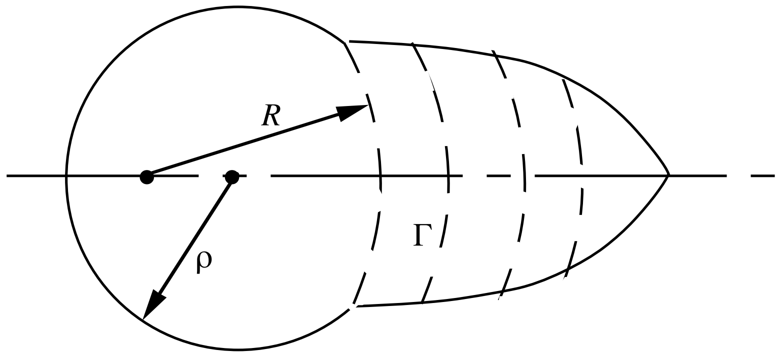

Figure 6: Proboscis domain

Lemma 3.2: *Suppose again that no vertices appear, and suppose there exists* $\widehat{\Omega} \subset \Omega$, $\widehat{\Omega} \neq \oslash, \Omega$, *such that* $\Phi(\widehat{\Omega}; H) \leq 0$, *with* H *defined by (6). Then there exists* $\widehat{\Omega}_0 \subset \Omega$, *with the properties described in Lemma 3.1.*

Proof: If $\Phi(\widehat{\Omega}; H) = 0$ and $\widehat{\Omega}$ minimizes, we simply set $\widehat{\Omega}_0 = \Omega \backslash \widehat{\Omega}$, $H_0 = H$. Suppose there exists $\widehat{\Omega} \subset \Omega$ such that $\Phi(\widehat{\Omega}; H) < 0$. This functional is clearly bounded below, and it is lower semicontinuous in BV norm [F 3, Lemma 6.3]; thus, there is a minimizing set, which we again denote by $\widehat{\Omega}$. The set $\widehat{\Omega}$ cannot be the null set or all of Ω, as Φ vanishes on those sets. $\widehat{\Omega}$ is bounded in Ω by a finite number of subarcs $\widehat{\Gamma}$ of semicircles, of radius $1/2H$, that meet Σ in angles γ. The complementary set $\widehat{\Omega}_1 = \Omega \backslash \widehat{\Omega}$ is minimizing for the adjoint functional $\Psi(\Omega^*; H)$ in Ω, and $\Psi(\widehat{\Omega}_1; H) = \Phi(\widehat{\Omega}; H) < 0$.

Since $\widehat{\Omega}_1 \neq \oslash$, there exists H_1 with $0 < H_1 < H$, such that $\Psi(\widehat{\Omega}_1; H_1) = 0$. The set $\widehat{\Omega}_1$ is not a minimizer for the functional $\Psi(\Omega^*; H_1)$, as the boundary arcs of a minimizer would have curvature $2H_1 \neq 2H$. As with the functional Φ above, a minimizer Ω_1 does, however, exist and is bounded in Ω by a finite number of subarcs of semicircles, each of radius $R_1 = 1/2H_1$ and meeting Σ in angle γ; we label this set Γ_1.

The set Ω_1 cannot be the null set, as $\Psi(\oslash; H_1) = 0$ whereas $\Psi(\Omega_1; H_1) < 0$. Suppose $\Omega_1 = \Omega$; then $|\Sigma| \cos\gamma - 2H_1|\Omega| < 0$, and since $H_1 < H$ there follows $|\Sigma| \cos\gamma - 2H|\Omega| < 0$, contradicting (3.2). Thus, Ω_1 is a strict non-trivial subset of Ω, and $|\Gamma_1| + |\Sigma_1| \cos\gamma - 2H_1|\Omega_1| < 0$, Σ_1 being the trace on Σ of Ω_1.

We continue the procedure, choosing $H_2 < H_1$ so that $\Psi(\Omega_1; H_2) = 0$ and then minimizing this functional, etc. We obtain in this way a decreasing sequence of positive H_n, tending to a limit H_0. At each step, we have $|\Gamma_n| + |\Sigma_n| \cos\gamma - 2H_n|\Omega_n| < 0$. Thus by isoperimetric inequality, there is a positive constant C such that $C - 2H_n\sqrt{\Omega_n} < 0$, and from this we conclude both that $H_0 > 0$ and that the Ω_n are bounded below in measure by a positive constant. Since Ω_n is a minimizing set for $\Psi(\Omega_n; H_{n+1})$, it is bounded in Ω by "extremal" subarcs of semicircles Γ_n of common radius $1/2H_{n+1}$ ([F 3, Theorems 6.10, 6.11]). Since $\Sigma \in C^{(1)}$ and each of the Γ_n meets Σ at two points in angle $\gamma, 0 < \gamma < \pi/2.$, each arc of Γ_n subtends on Σ a length bounded from zero, depending only on γ and on the geometry. We conclude that the total number of boundary arcs Γ_n remains less than a fixed bound throughout the procedure. We may thus choose a subsequence $j(n)$ so that the arcs $\Gamma_{j(n)}$ converge strictly throughout Ω, to the boundary in Ω of a limit set $\widehat{\Omega}_0$; from the continuity of Ψ with respect to such convergence, we have $\Psi(\widehat{\Omega}_0; H_0) = \lim_{n\to\infty} \Psi(\Omega_{j(n)}; H_{j(n)+1}) = 0$. Further, since the $H_{j(n)+1}$ and the minimizing sets $\Omega_{j(n)+1}$ corresponding to those values are both converging, there holds $\lim_{n\to\infty} \Psi(\Omega_{j(n)}; H_{j(n)}) = 0$.

Clearly the limit set $\widehat{\Omega}_0$ is bounded in Ω by a finite number of subarcs of semicircles $\widehat{\Gamma}_0$ of radius $1/2H_0$, each of which meets Σ in angle γ. We assert now that for all $\Omega^* \subset \widehat{\Omega}_0$ there holds $\Psi(\Omega^*; H_0) \geq 0$. For if there were an $\Omega^* \subset \widehat{\Omega}_0$ with $\Psi(\Omega^*; H_0) = -\omega^2 < 0$, there would hold for all sufficiently large n $\Psi(\Omega^* \cap \Omega_{j(n)}; H_{j(n)}) < -\omega^2/2$. But $\Omega_{j(n)}$ minimizes for $H_{j(n)}$, hence $\Psi(\Omega_{j(n)}; H_{j(n)}) < -\omega^2/2$, contradicting the limiting behavior just established. □

We next consider configurations in which a finite number N_R of re-entrant corners are present. This does not affect the lower semicontinuity of the functionals Φ and Ψ; the only change that occurs is that the boundary arcs Γ_n in Ω of the minimizing sets need not all be disjoint, but that (at most) two such arcs could meet at a corner point. An inspection of the proofs of Lemmas 3.1 and 3.2 shows that no change is needed.

We consider finally the general case in which additionally a finite number N_P of protruding vertices appear. The major new difficulty that presents itself is that the existence of minimizing configurations for $\Phi(\Omega^*; H_j)$ and $\Psi(\Omega^*; H_j)$ may no longer follow from established literature, as the conditions for Lemma 6.3 in [F 1] will not be fulfilled when the opening angle 2α at a corner is such that $\alpha + \gamma < \pi/2$. We continue to know, however, that any minimizing set $\widehat{\Omega}$ is bounded in Ω by "extremal" curves Γ_j that are subarcs of semicircles of radius $1/2H_j$, which if they terminate at smooth points of Σ must meet Σ in angle γ. In fact, the following lemma shows that this is the only case we need consider.

Lemma 3.3: *No extremal arc of a minimizing configuration, either for Φ or for Ψ, can terminate at a protruding vertex.*

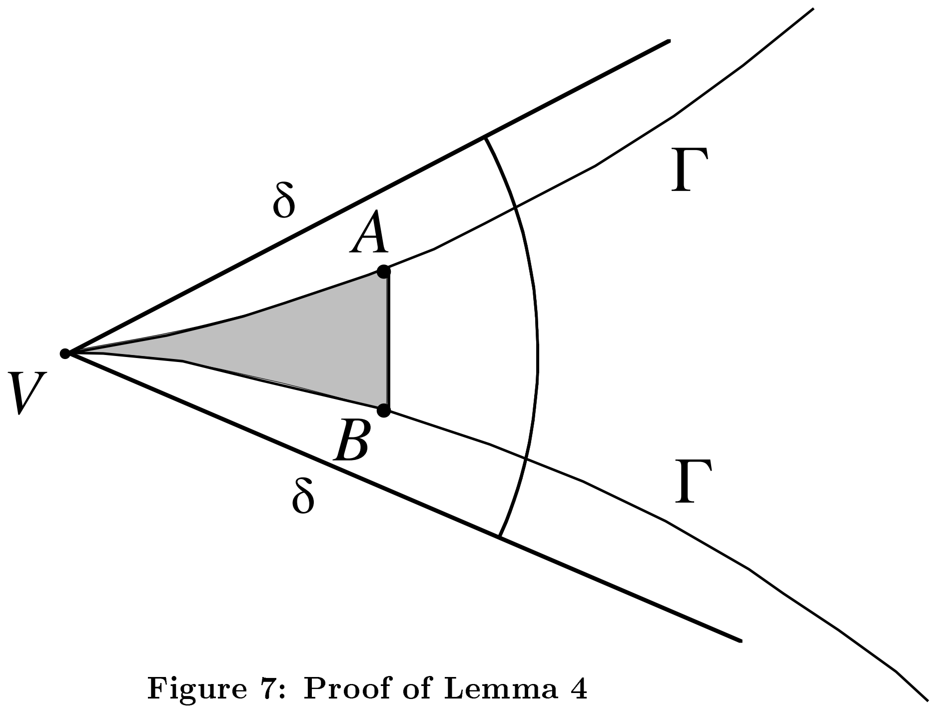

Figure 7: Proof of Lemma 4

Proof: For definiteness we consider a functional $\Phi(\Omega^*; H_j)$. We suppose that at least one of the boundary arcs Γ of a minimizing set $\widehat{\Omega}$ enters a vertex V. This arc must extend from V, either to another vertex or to a point where it meets Σ in angle γ. Since all such arcs have the same curvature, there is a disk B_δ of radius δ about V, such that all extremal arcs that start at

V cannot again meet Σ interior to B_δ. We choose δ small enough that all such arcs will appear essentially linear, in a sense that will be clear from the discussion.

No two such arcs can bound with a subarc of ∂B_δ a portion of $\widehat{\Omega}$. For were that to happen, Φ could be decreased by replacing two of the segments of Γ to the vertex by a segment $\overline{AB}$ as indicated in Figure 7; the shaded portion shows the change in $\widehat{\Omega}$ effected by the construction. We conclude that at most two extremals can end at V, and that each bounds, together with one of the sides L of Σ emanating from V, a portion of $\widehat{\Omega}$. But also this configuration cannot occur, as the same procedure would again decrease Φ. Thus at most one extremal Γ need be considered, bounding in B_δ, together with a boundary arc L that meets Γ in angle τ at V, a portion $\mathcal{P}$ of $\widehat{\Omega}$ (Figures 8,9).

Case 1; $\tau < \gamma$: We cut off an end of $\mathcal{P}$ with a segment from Γ that meets L in angle γ (Figure 8). This is easily seen to reduce Φ if the segments are small enough.

Case 2; $\tau \geq \gamma$: We adjoin a piece to $\mathcal{P}$ by replacing a small length of Γ near V by a segment from Γ to the other arc L' of Σ that emanates from V, such that the segment meets L' in angle γ. See Figure 9. Again this procedure decreases Φ. $\square$

It follows from Lemma 3.3 that once the existence of minimizing sets for the functionals is established, the procedures of the first two lemmas can be carried over without change, and the theorem will be established in the generality stated. We proceed to demonstrate the existence. In the particular case for which we are given a set $\widehat{\Omega} \neq \emptyset, \Omega$ such that $\Phi(\widehat{\Omega}; H) = 0$ and $\widehat{\Omega}$ minimizes (see the initial statement in the proof of Lemma 3.2) the proof of Lemma 3.1 proceeds without change, and the theorem is complete. We may thus assume the existence of $\widehat{\Omega}$ such that $\Phi(\widehat{\Omega}; H) < 0$.

Lemma 3.4: *Under the hypotheses of the theorem, there exists a minimizer $\widehat{\Omega}_0$ for the functional $\Phi(\Omega^*; H)$ over Ω. $\widehat{\Omega}_0$ is bounded in Ω by a finite number of "extremal" subarcs $\widehat{\Gamma}_0$ of semicircles of radius $1/2H$, each of which meets Σ in angle γ. No extremal can terminate at a protruding vertex.*

Proof: By hypothesis, we have $\mu = \operatorname{glb}_{\Omega^* \subset \Omega} \Phi(\Omega^*; H) < 0$. We begin by smoothing Σ at any protruding vertices, by inscribed circular arcs $\mathcal{T}_n$ as in

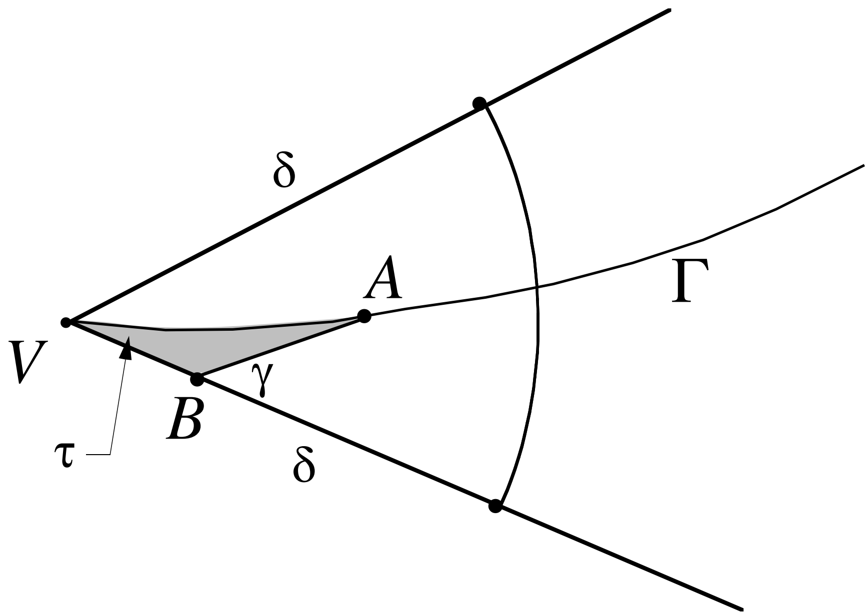

Figure 8: Reduction of Φ; Case 1.
The shaded section is removed from $\mathcal{P}$.

Figure 10; we denote by Ω_n the domain obtained when the contact points with the arcs are distance $1/n$ from the vertex V, and by μ_n the corresponding glb. Since each $\Sigma_n = \partial\Omega_n \in C^1$, there exists a minimizing set $\widehat{\Omega}_n$, bounded in Ω_n by a finite number of subarcs $\widehat{\Gamma}_n$ of semicircles.

Choose $\epsilon > 0$, and let $\widehat{\Omega}^* \subset \Omega$ be such that $\Phi(\widehat{\Omega}^*; H) - \mu < \epsilon$. Denote by Φ_n the functional Φ restricted to sets in Ω_n. We observe that given $\hat{\epsilon} > 0$ we can choose $n(\hat{\epsilon})$ so that the contribution to Φ of any sets interior to a ball of radius $1/n$ about V cannot be less than $(-\hat{\epsilon})$. We thus obtain

$$\mu_n = \Phi_n(\widehat{\Omega}_n; H) \leq \Phi_n(\widehat{\Omega}^*; H) \leq \Phi(\widehat{\Omega}^*; H) + \hat{\epsilon} \leq \mu + \epsilon + \hat{\epsilon}. \qquad (3.8)$$

Let $\widehat{\Sigma}_n$ be the trace on $\mathcal{T}_n$ of $\widehat{\Omega}_n$. We can choose $n(\hat{\epsilon})$ large enough that if $n > n(\hat{\epsilon})$ then the change in its contribution to Φ_n that arises by considering this set as part of the inner boundary set $\widehat{\Gamma}_n$ will not exceed $\hat{\epsilon}$. If that is done, then $\Phi(\widehat{\Omega}_n; H)$ becomes well defined, and we obtain

$$\mu_n = \Phi_n(\widehat{\Omega}_n; H) > \Phi(\widehat{\Omega}_n; H) - \hat{\epsilon} \geq \mu - \hat{\epsilon} \qquad (3.9)$$

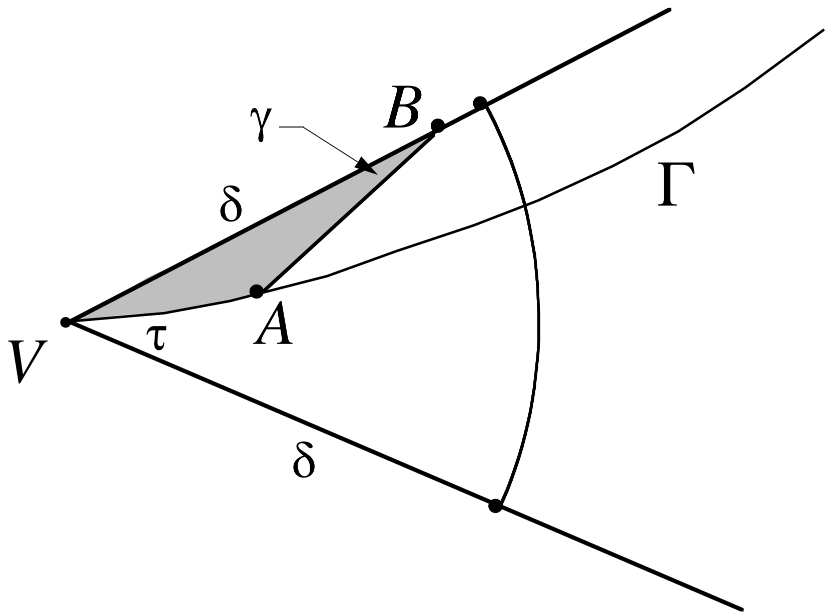

Figure 9: Reduction of Φ; Case 2.
The shaded portion is added to $\mathcal{P}$.

Since ϵ and $\hat{\epsilon}$ are arbitrary, we have proved that $\lim_{n\to\infty} \mu_n = \mu$.

Returning to our construction, we wish to let $n \to \infty$ and extract a convergent subsequence of the $\widehat{\Omega}_n$. We cannot do that directly, as the number of arcs $\widehat{\Gamma}_n$ can conceivably increase unboundedly with n. However, if we construct for fixed $\delta > 0$ a disk of radius B_δ about each vertex, and restrict attention to arcs that extend exterior to all $B_\delta(V)$ then the number stays uniformly bounded, as each such arc must subtend an arc of length bounded from zero on Σ_n. Thus, a subsequence of the $\{\widehat{\Gamma}_n\}$ can be chosen that converges throughout Ω, uniformly in $\Omega^\delta = \Omega\backslash\{B_\delta\}$ for any fixed $\delta > 0$, to a countable set of arcs $\{\widehat{\Gamma}\}$ with the same geometric properties as $\widehat{\Gamma}_n$, and bounding $\widehat{\Omega}_0$ in Ω. We must show that $\Phi(\widehat{\Omega}_0; H) \leq \mu$.

We write

$$\begin{aligned} \Phi(\widehat{\Omega}_0; H) &= \Phi_n(\widehat{\Omega}_n; H) + [\Phi_n(\widehat{\Omega}_0; H) - \Phi_n(\widehat{\Omega}_n; H)] \\ &\qquad + [\Phi(\widehat{\Omega}_0; H) - \Phi_n(\widehat{\Omega}_0; H)] \\ &= \mu_n + A_n + B_n. \end{aligned} \tag{3.10}$$

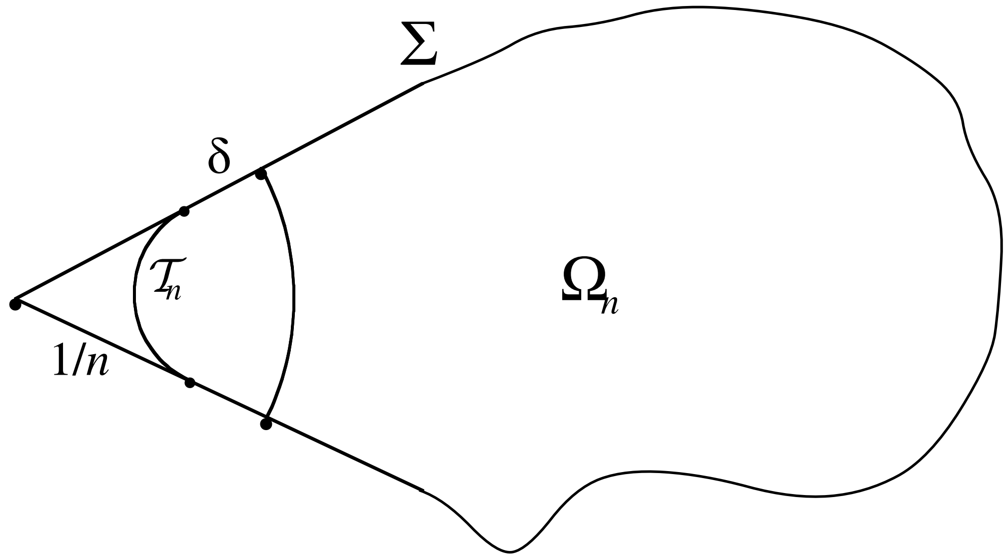

Figure 10: Smoothing at a corner.

We know that the initial term tends to μ. In the subsequence considered, the contributions to A_n that arise from sets lying in Ω^δ tend to zero with increasing n because of the uniform convergence of the $\widehat{\Gamma}_n$ in this domain, and the corresponding contributions to B_n vanish for n large enough. We will show that interior to $B_{\hat{\delta}}$ each of the individual terms arising in A_n and in B_n is small depending only on δ.

Consider first those terms involving $\widehat{\Omega}_0$. Only a finite number of the boundary arcs $\{\widehat{\Gamma}\}$ can extend into Ω^δ. But if δ is small enough, then none of these arcs can lie entirely in $B_{\hat{\delta}}$ and fulfill the geometric conditions on radius and boundary angle (that is so even if one of the end points lies at V, without angle condition there). Thus it is clear that all contributions to the terms considered become vanishingly small, depending only on δ.

In the remaining term involving $\widehat{\Omega}_n$, many arcs of $\widehat{\Gamma}_n$ can appear with increasing n; however, these are necessarily disjoint circular arcs of common radius, joining points of $\mathcal{T}_n$, and their cumulative contribution tends to zero with δ.

Thus, by choosing first δ and then n sufficiently large depending on

δ, we can make $\Phi(\widehat{\Omega}_0; H)$ as close to μ as desired. Hence, $\widehat{\Omega}_0$ is minimizing for $\Phi(\Omega^*; H)$. Lemma 3.3 now guarantees that none of the boundary arcs $\widehat{\Gamma}$ extends to any of the vertices.

Lemma 3.5: *The conclusion of Lemma 3.4 hold for each of the boundary problems occurring in the proof of Lemma 3.2.*

Proof: The formal reasoning is identical to that of Lemma 3.4 for each of these problems.

Lemmas 3.4 and 3.5 permit us to complete the proofs of Lemmas 3.1 and 3.2 in the requisite generality, and these lemmas lead directly to the proof of the stated Theorem 3.1. □

4. Two examples.

The hypothesis that $\Phi(\Omega^*; H) \leq 0$ for some $\Omega^* \subset \Omega, \Omega^* \neq \oslash,\ \Omega$, is essential for the proof we have given of Theorem 3.1. The following examples show respectively that the hypothesis is not in general necessary, but that nevertheless it cannot in general be discarded.

Example 4.1: Chen [C1] introduced "neck domains" Ω formed by two intersecting circles of unit radius, as indicated by the solid lines in Figure 11, and he showed that $\Phi(\Omega^*; H) > 0$ for all $\Omega^* \subset \Omega, \Omega^* \neq \oslash,\ \Omega$, whenever the aperture height 2ρ is sufficiently small. Concus and Finn [CF 3] introduced "double bubble" domains as indicated in Figure 12. They showed that if ρ is small enough, then there is a value $\gamma, 0 < \gamma < \pi/2$, a circular arc Γ of radius $R > \rho$ as indicated in the figure, and a solution $u(x)$ of (1.3) in Ω^*, with $H = 1/2R$, such that $\nu \cdot Tu = \cos\gamma$ on Σ^* and $\nu \cdot Tu = 1$ on Γ; here ν is unit normal vector directed exterior to Ω^*. This solution provides a solution in the subdomain Ω^* indicated in Figure 11, which becomes vertically infinite on Γ and achieves the data γ on Σ^*. That is, *it can happen that both smooth and singular solutions occur in the identical domain.* The procedure of this

paper will locate such singular solutions, even though the hypotheses of the theorem, that $\Phi \leq 0$ for some subset, is not fulfilled. That hypothesis cannot, however, be abandoned; the following example shows that without it there may be no singular solution.

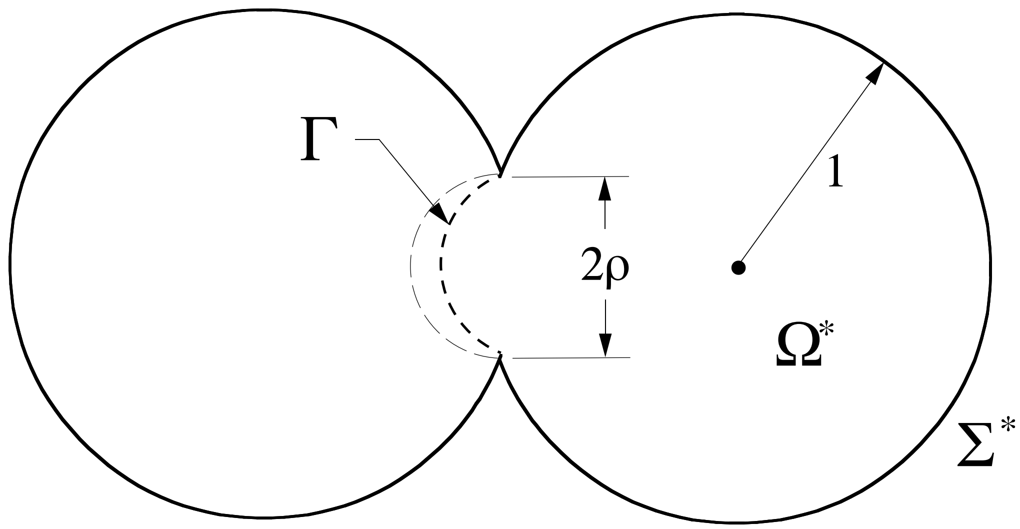

Figure 11: Neck domain bounded by solid lines; Ω^* extends to right hand dashed line.

Example 4.2: If the domain Ω is a disk, then the boundary problem (1.2), (1.3) can be solved explicitly by a spherical cap, for any data $\gamma_0, 0 \leq \gamma_0 \leq \pi/2$; hence by Theorem 6.1 of [F 3] there holds $\Phi(\Omega^*; H) > 0$, for all $\Omega^* \subset \Omega, \Omega^* \neq \oslash,\ \Omega$. Thus as in the preceding example the hypothesis fails. But in this case we assert that if $0 < \gamma_0 \leq \pi/2$ then no singular solution of the indicated type can exist in Ω. Specifically, we shall exclude the possibility of a solution of (1.3) in a subdomain Ω^* bounded by a subset $\Sigma^* \subset \Sigma$ and a set of disjoint circular arcs Γ of common radius $R = 1/2H$ in Ω, each of which meets Σ in angle γ_0, such that (1.2) holds on Σ^*, and $\nu \cdot Tu = 1$ on Γ.

We may assume that the disk Ω has unit radius. As we have seen, a necessary condition for existence of such a solution is that $\Psi(\Omega^*; H) = 0$. Since $\gamma_0 \neq 0$, there is at most a finite number N of arcs Γ.

We consider first what happens when $N = 1$; we then encounter a configuration as shown in Figure 13. We introduce the one parameter family of domains $\Omega^*(\alpha)$ and corresponding functionals $\Psi(\Omega^*(\alpha); H; \alpha) = \Psi(\cdot; \alpha)$, $\alpha_0 \leq \alpha \leq \pi$, obtained by moving Γ rigidly to the right until it degenerates to the single point (1,0). Following the discussion of the "ice cream cone" at the end of Sec. 2.2, we see that $\Psi(\cdot; \alpha_0) > 0$. Thus no solution of the projected form is possible.

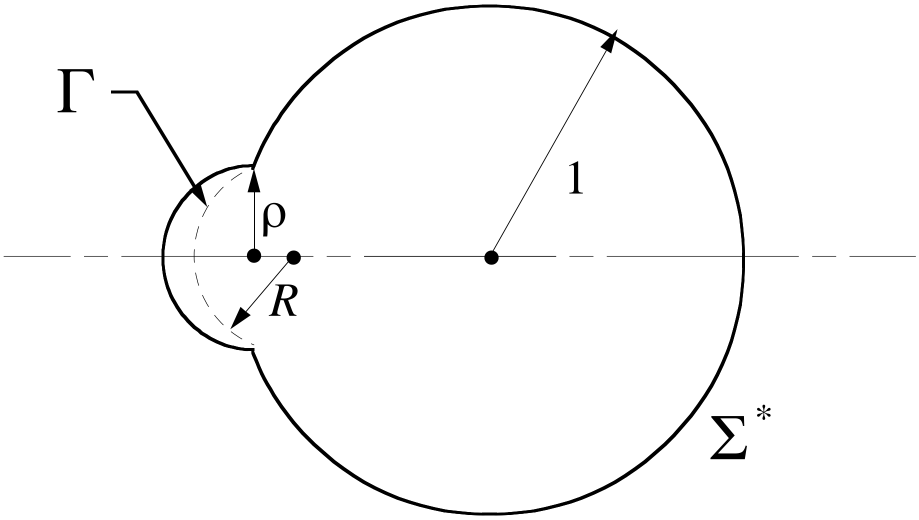

Figure 12: Double Bubble Domain.

Suppose now $N > 1$. We observe that $\Psi[\Omega^*] = \Phi[\Omega\backslash\Omega^*]$. Since $\Omega\backslash\Omega^*$ consists of a finite number of disjoint domains, each of which yields a positive Φ, the additivity properties of the Φ functional yield the result.

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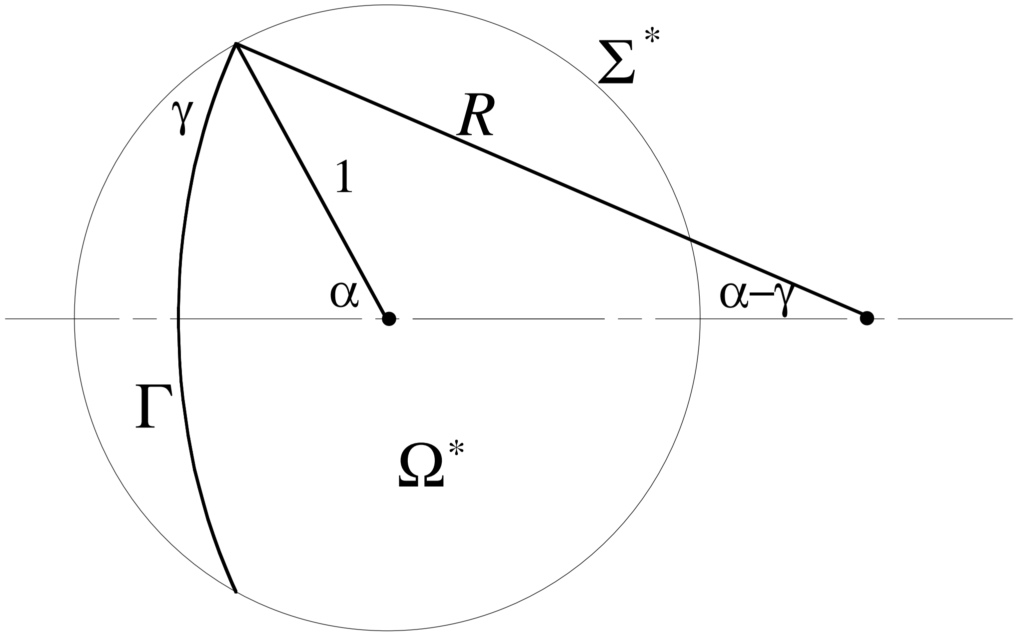

Figure 13: Configuration for Example 4.2; Case $N = 1$.

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