Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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Preprint-Nr.: 58 1998

solv-int/9811016

The supersymmetric Camassa-Holm equation and geodesic flow on the superconformal group

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Abstract

We study a family of fermionic extensions of the Camassa-Holm equation. Within this family we identify three interesting classes: (a) equations, which are inherently hamiltonian, describing geodesic flow with respect to an H^1 metric on the group of superconformal transformations in two dimensions, (b) equations which are hamiltonian with respect to a different hamiltonian structure and (c) supersymmetric flow equations. Classes (a) and (b) have no intersection, but the intersection of classes (a) and (c) gives a candidate for a new supersymmetric integrable system. We demonstrate the Painlevé property for some simple but nontrivial reductions of this system.

1 Introduction

Recently there has been substantial interest in the Camassa-Holm (CH) equation [1, 2]:

$$u_t - \nu u_{xxt} = \kappa u_x - 3uu_x + \nu(uu_{xxx} + 2u_x u_{xx}) \ . \tag{1.1}$$

This equation has been proposed as a model for shallow water waves. It is believed to be integrable (having bihamiltonian structure), but due to the nonlinear dispersion term, uu_{xxx}, it exhibits more general wave phenomena than other integrable water wave equations such as KdV. In particular it admits a class of nonanalytic weak solutions known as *peakons*, as well as finite time blow-up of solutions.

Geometrically, the relationship of CH to KdV is rather deeper: Both are regularisations of the Euler equation for a one dimensional compressible fluid (Monge or inviscid Burgers equation),

$$u_t = -3uu_x \ . \tag{1.2}$$

This latter equation describes geodesic motion on the group of diffeomorphisms of the circle $\mathrm{Diff}(S^1)$ [3] with respect to a metric induced by an L^2 norm, $\int u^2 dx$, on the associated algebra. If the group is centrally extended to the Bott-Virasoro group, the KdV equation arises [4, 5, 6, 7]. On the other hand, if the metric is changed to one induced by an H^1 norm, $\int (u^2 + \nu u_x^2) dx$, the CH equation arises [8, 9, 10]. Both these 'deformations' have a regularising effect on solutions of (1.2), which exhibit discontinuous shocks.

Thus KdV and CH arise in a unified geometric setting; both are integrable systems which describe geodesic flows. This raises an important question: What features of the underlying geometry give rise to integrability? In general, geodesic flows of this type are *not* integrable: the Euler equation for fluid flow in more than one spatial dimension is an example [3]. Indeed, for the latter, Arnold has suggested a relationship between negative sectional curvatures and non-predictability of the flow. Is integrability also geometrically determined?

One further example of an integrable bihamiltonian system arising from a geodesic flow has been discussed in the pioneering paper of Ovsienko and Khesin [4]. Using the superconformal group with an L^2 type metric, they obtained the so-called kuperKdV system of Kupershmidt [11]. This is a fermionic extension of KdV: it describes evolution of functions valued in (the odd or even parts of) a grassmann algebra. In fact, as we will see below, taking a general L^2 type metric on the superconformal group gives rise to a one parameter family of fermionic extensions of KdV, which includes not only kuperKdV, but also the superKdV system of Mathieu and Manin-Radul [12, 13]. The latter is integrable: it has only a single hamiltonian structure, but unlike kuperKdV it is supersymmetric, a property which is widely believed to contribute to integrability. It remains a mystery as to why, of the one parameter family of geodesic flow equations associated with L^2 type metrics on the superconformal group, only two specific choices of the parameter give rise to integrable systems.

The main purpose of this paper is to investigate geodesic flow equations obtained from H^1 type norms on the superconformal group; more generally we consider the following family of fermionic extensions of CH:

$$\begin{aligned} u_t - \nu u_{xxt} &= \kappa_1 u_x + \kappa_2 u_{xxx} + \beta_1 u u_x + \beta_2 u_x u_{xx} + \beta_3 u u_{xxx} + \gamma_1 \xi\xi_{xx} + \gamma_2 \xi_x \xi_{xxx} + \gamma_3 \xi\xi_{xxxx} \\ \xi_t - \mu \xi_{xxt} &= \sigma_1 \xi_x + \sigma_2 \xi_{xxx} + \epsilon_1 u_x \xi + \epsilon_2 u \xi_x + \rho_1 u \xi_{xxx} + \rho_2 u_x \xi_{xx} + \rho_3 u_{xx} \xi_x + \rho_4 u_{xxx} \xi \ . \end{aligned} \quad (1.3)$$

Here $u(x,t)$ and $\xi(x,t)$ are fields valued, respectively, in the even and odd parts of a grassmann algebra, and $\{\nu,\mu,\kappa_1,\kappa_2,\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,\gamma_3,\sigma_1,\sigma_2,\epsilon_1,\epsilon_2,\rho_1,\rho_2,\rho_3,\rho_4\}$ are parameters. By rescaling u and ξ it is possible to set $\beta_1 = -3$ and $\gamma_1 = 2$ (assuming that they are nonzero), and we shall do this throughout. In addition it is possible to eliminate up to two further parameters by rescaling the coordinates x, t.

We derive three interesting classes of systems of the form (1.3). In section 2, we consider geodesic flow on the superconformal group with an H^1 type metric; the resulting systems have a natural hamiltonian structure, or more precisely, since the fields are grassmann algebra valued, a graded hamiltonian structure. In section 3 we identify a class of systems having a different hamiltonian structure. Unfortunately the latter has no intersection with the class of section 2, so there does not seem to be a bihamiltonian fermionic extension of CH. In section 4 we consider systems of the form (1.3) that are invariant under supersymmetry transformations between u and ξ. This class has nontrivial intersections with both the classes of sections 2 and 3. In particular, there is a unique supersymmetric geodesic flow system, which is a promising candidate for being integrable. In section 5 we show that two reductions of this system have the Painlevé property.

A trivial integrable CH system of the form (1.3), which is not incorporated in the classes of sections 2,3, and 4, and which we shall not discuss further, is the odd linearisation of the bosonic CH system (1.1)

$$\begin{aligned} u_t - \nu u_{xxt} &= \kappa u_x - 3 u u_x + \nu (u u_{xxx} + 2 u_x u_{xx}) \ , \\ \xi_t - \nu \xi_{xxt} &= \kappa \xi_x - 3(\xi u)_x + \nu(\xi u_{xxx} + u \xi_{xxx} + 2(\xi_x u_x)_x) \ . \end{aligned} \quad (1.4)$$

Replacing u by $u + \frac{\kappa}{3}$ and considering the limit $\nu \to 0$, $\kappa \to \infty$, with $\nu\kappa = 3$, yields the system

$$\begin{aligned} u_t &= -3 u u_x + u_{xxx} \ , \\ \xi_t &= -3(\xi u)_x + \xi_{xxx} \ . \end{aligned} \quad (1.5)$$

This trivial fermionic extension of KdV has appeared often in the literature (see e.g. [12]).

2 Geodesic flows on the superconformal group

An inner-product $\langle .,. \rangle$ on a Lie algebra $\mathfrak{g}$ determines a right (or a left) invariant metric on the corresponding Lie group G. The equation of geodesic motion on G with respect to this metric is determined as follows [3]. Define a bilinear operator $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\Big\langle [V,W] \, , \, U \Big\rangle = \Big\langle W \, , \, B(U,V) \Big\rangle \quad , \qquad \forall \ W \in \mathfrak{g} \ . \quad (2.1)$$

The geodesic flow equation is then simply

$$U_t = B(U,U) \ . \tag{2.2}$$

In our case, $\mathfrak{g}$ is the NSR superconformal algebra, consisting of triples $(u(x), \varphi(x), a)$, where u is a bosonic field, φ is a fermionic field and a is a constant. The Lie bracket is given by

$$\begin{aligned} &\Big[(u,\varphi,a)\,,\,(v,\psi,b)\Big] \\ &= \left(uv_x - u_x v + \tfrac{1}{2}\varphi\psi \,,\, u\psi_x - \tfrac{1}{2}u_x\psi - \varphi_x v + \tfrac{1}{2}\varphi v_x \,,\, \int dx (c_1 u_x v_{xx} + c_2 u v_x + c_1 \varphi_x \psi_x + \tfrac{c_2}{4}\varphi\psi) \right) \,, \end{aligned} \tag{2.3}$$

where c_1, c_2 are constants. On this algebra, an H^1 inner product is given by

$$\begin{aligned} \Big\langle (u,\varphi,a)\,,\,(v,\psi,b) \Big\rangle &= \int dx \left(uv + \nu u_x v_x + \alpha\varphi\partial_x^{-1}\psi + \alpha\mu\varphi_x\psi \right) \ + ab \\ &= \int dx \ \ (u \ \Delta_0 \ v + \varphi \ \Delta_1 \ \psi) \ + \ ab \quad , \end{aligned} \tag{2.4}$$

where

$$\Delta_0 \ = \ 1 - \nu\partial_x^2 \quad , \quad \Delta_1 \ = \ \alpha\left(\partial_x^{-1} - \mu\partial_x\right) \ , \tag{2.5}$$

and μ, ν, α are further constants, all assumed nonzero. Writing $U{=}(u,\varphi,a)$, $V{=}(v,\psi,b)$, we find $B(U,V) = (B_0, B_1, 0)$, where

$$\begin{aligned} \Delta_0 B_0(U,V) &= -\left(2v_x\Delta_0 u + v\Delta_0 u_x + \tfrac{3}{2}\psi_x\Delta_1\varphi + \tfrac{1}{2}\psi\Delta_1\varphi_x\right) + a(c_1 v_{xxx} - c_2 v_x) \ , \\ \Delta_1 B_1(U,V) &= -\left(\tfrac{3}{2}v_x\Delta_1\varphi + v\Delta_1\varphi_x + \tfrac{1}{2}\psi\Delta_0 u\right) + a(c_1\psi_{xx} - \tfrac{c_2}{4}\psi) \ . \end{aligned} \tag{2.6}$$

The geodesic equations are therefore conveniently written in the form

$$\begin{aligned} \Delta_0 \ u_t &= \Delta_0 \ B_0(U,U) \\ \Delta_0 \ \varphi_t &= \Delta_1 \ B_1(U,U) \\ a_t &= 0 \ . \end{aligned} \tag{2.7}$$

Writing $\varphi = \lambda\xi_x$, where λ is a constant satisfying $\lambda^2 = \frac{4}{3\alpha}$, this yields the system

$$\begin{aligned} u_t - \nu u_{xxt} &= \kappa_1 u_x + \kappa_2 u_{xxx} - 3uu_x + \nu(uu_{xxx} + 2u_x u_{xx}) + 2\xi\xi_{xx} + \tfrac{2\mu}{3}\xi_x\xi_{xxx} \ , \\ \xi_t - \mu\xi_{xxt} &= \tfrac{\kappa_1}{4\alpha}\xi_x + \tfrac{\kappa_2}{\alpha}\xi_{xxx} - \tfrac{3}{2}u_x\xi - (1 + \tfrac{1}{2\alpha})u\xi_x + \mu u\xi_{xxx} + \tfrac{3\mu}{2}u_x\xi_{xx} + \tfrac{\nu}{2\alpha}u_{xx}\xi_x \ . \end{aligned} \tag{2.8}$$

Here κ_1, κ_2 are independent parameters determined by a, c_1, c_2 . This is evidently a 5 parameter class of systems of type (1.3).

Setting ξ to zero in (2.8) yields the CH result of [8, 9, 10]. If instead we choose μ, ν to vanish, the H^1 norm becomes an L^2 norm; then choosing κ_1 to be zero and rescaling κ_2 to 1 we obtain the following 1 parameter fermionic extension of KdV:

$$\begin{aligned} u_t &= u_{xxx} - 3uu_x + 2\xi\xi_{xx} \ , \\ \xi_t &= \tfrac{1}{\alpha}\xi_{xxx} - \tfrac{3}{2}u_x\xi - (1 + \tfrac{1}{2\alpha})u\xi_x \ . \end{aligned} \tag{2.9}$$

Modulo rescalings, the superKdV of Mathieu and Manin-Radul is obtained by taking $\alpha = 1$. The kuperKdV system arises by taking $\alpha = \frac{1}{4}$, the choice made in [4]. Other values of the parameters give systems which are not believed to be integrable (see however [14]).

3 Hamiltonian flows

Like KdV, CH has bihamiltonian structure, and this accounts for its integrability. We might hope that for some choices of parameters the system (2.8) should also have a bihamiltonian structure. One hamiltonian structure follows automatically from the geometric origins of the system [3]. Explicitly, introducing new variables, $m = u - \nu u_{xx}$ and $\eta = \xi - \mu\xi_{xx}$, (2.8) takes the form

$$\begin{pmatrix} m_t \\ \eta_t \end{pmatrix} = \mathcal{P}_2 \begin{pmatrix} \frac{\delta\mathcal{H}_2}{\delta m} \\ \frac{\delta\mathcal{H}_2}{\delta\eta} \end{pmatrix} \tag{3.1}$$

where

$$\mathcal{P}_2 = \begin{pmatrix} \kappa_2\partial_x^3 + \kappa_1\partial_x - \partial_x m - m\partial_x & \frac{1}{2}\partial_x\eta + \eta\partial_x \\ -\partial_x\eta - \frac{1}{2}\eta\partial_x & \frac{3}{4\alpha}(\frac{\kappa_1}{4} + \kappa_2\partial_x^2) - \frac{3m}{8} \end{pmatrix} \tag{3.2}$$

and the hamiltonian functional is given succinctly by the H^1 inner product on the algebra,

$$\mathcal{H}_2 \;=\; \tfrac{1}{2}\left\langle U\,,\,U\right\rangle \;=\; \tfrac{1}{2}\int dx\,\left(u^2 + \nu u_x^2 + \tfrac{4}{3}(\xi_x\xi + \mu\xi_{xx}\xi_x)\right) \quad . \tag{3.3}$$

This generalises the so-called *second Hamiltonian structure* of KdV and its fermionic extensions [11, 12]. Checking (3.1) is straightforward: the Euler-Lagrange derivatives $\frac{\delta\mathcal{H}_2}{\delta m}, \frac{\delta\mathcal{H}_2}{\delta\eta}$ are defined by

$$\delta\mathcal{H}_2 = \int\, dx\, \left(\frac{\delta\mathcal{H}_2}{\delta m}\,\delta m + \frac{\delta\mathcal{H}_2}{\delta\eta}\,\delta\eta\right)\ , \tag{3.4}$$

from which it follows immediately that $\frac{\delta\mathcal{H}_2}{\delta m} = u$ and $\frac{\delta\mathcal{H}_2}{\delta\eta} = \frac{4}{3}\xi_x$.

To investigate the possibility of systems amongst (2.8) having another hamiltonian form, we look at systems of the form

$$\begin{pmatrix} m_t \\ \eta_t \end{pmatrix} = \mathcal{P}_1 \begin{pmatrix} \frac{\delta\mathcal{H}_1}{\delta m} \\ \frac{\delta\mathcal{H}_1}{\delta\eta} \end{pmatrix}\ , \tag{3.5}$$

where

$$\mathcal{P}_1 = \begin{pmatrix} \partial_x(1-\nu\partial_x^2) & 0 \\ 0 & -\frac{\epsilon_1}{2}(1-\mu\partial_x^2) \end{pmatrix}\ . \tag{3.6}$$

Here ϵ_1 is a constant and $\mathcal{H}_1$ is a functional generalising the KdV *first Hamiltonian*,

$$\begin{aligned}\mathcal{H}_1 \;&=\; \int dx\,\Big(-\tfrac{1}{2}u^3 - \tfrac{\beta_3}{2}uu_x^2 - \tfrac{\kappa_2}{2}u_x^2 + \tfrac{\kappa_1}{2}u^2 + \tfrac{\sigma_1}{\epsilon_1}\xi\xi_x + \tfrac{\sigma_2}{\epsilon_1}\xi\xi_{xxx} \\ &\qquad +2u\xi\xi_x + (\gamma_2-\gamma_3)u\xi_x\xi_{xx} + \gamma_3 u\xi\xi_{xxx}\Big)\ .\end{aligned} \tag{3.7}$$

This is the most general functional of this type, up to rescalings of u and ξ. Since $\delta m{=}(1-\nu\partial_x^2)\delta u$, we have $(1-\nu\partial_x^2)\frac{\delta\mathcal{H}_1}{\delta m} = \frac{\delta\mathcal{H}_1}{\delta u}$, and similarly $(1-\mu\partial_x^2)\frac{\delta\mathcal{H}_1}{\delta\eta} = \frac{\delta\mathcal{H}_1}{\delta\xi}$. Thus equations (3.5) take the simple form

$$\begin{aligned} u_t - \nu u_{xxt} &= \partial_x\left(\tfrac{\delta\mathcal{H}_1}{\delta u}\right) \\ &= \kappa_1 u_x + \kappa_2 u_{xxx} - 3uu_x + \beta_3(2u_x u_{xx} + uu_{xxx}) + 2\xi\xi_{xx} + \gamma_2\xi_x\xi_{xxx} + \gamma_3\xi\xi_{xxxx} \\ \xi_t - \mu\xi_{xxt} &= \epsilon_1\left(\tfrac{\delta\mathcal{H}_1}{\delta\xi}\right) \\ &= \sigma_1\xi_x + \sigma_2\xi_{xxx} + \epsilon_1(u_x\xi + 2u\xi_x) + \epsilon_1(2\gamma_3-\gamma_2)u\xi_{xxx} + \tfrac{3}{2}\epsilon_1(2\gamma_3-\gamma_2)u_x\xi_{xx} \\ &\quad + \tfrac{1}{2}\epsilon_1(4\gamma_3-\gamma_2)u_{xx}\xi_x + \tfrac{1}{2}\epsilon_1\gamma_3 u_{xxx}\xi\ . \end{aligned} \tag{3.8}$$

This is a 10 parameter class of systems of the form (1.3). Comparing with (2.8), we see that the only bihamiltonian systems occur when $\{\mu{=}\nu{=}\beta_3{=}\gamma_2{=}\gamma_3{=}0\,,\,\epsilon_1{=}-\frac{3}{2}\,,\,\sigma_1{=}\kappa_1\,,\,\sigma_2{=}4\kappa_2\}$, which is equivalent to (2.8) with $\{\mu{=}\nu{=}0\,,\,\alpha{=}\frac{1}{4}\}$, i.e. the kuperKdV system. Thus, no new bihamiltonian systems arise.

We note that the systems (3.8) can be obtained from a Lagrangian. Introducing a potential f defined by $u{=}f_x$, they are Euler-Lagrange equations for the functional

$$\begin{aligned}\mathcal{L} \;=\; \int dx\, \Big(& \tfrac{1}{2}(f_x-\nu f_{xxx})f_t+\tfrac{1}{\epsilon_1}(\xi-\mu\xi_{xx})\xi_t+\tfrac{1}{2}f_x^3+\tfrac{\beta_3}{2}f_xf_{xx}^2+\tfrac{\kappa_2}{2}f_{xx}^2-\tfrac{\kappa_1}{2}f_x^2 \\ & -\tfrac{\sigma_1}{\epsilon_1}\xi\xi_x-\tfrac{\sigma_2}{\epsilon_1}\xi\xi_{xxx}-2f_x\xi\xi_x+(\gamma_3-\gamma_2)f_x\xi_x\xi_{xx}-\gamma_3f_x\xi\xi_{xxx}\Big)\;. \end{aligned} \tag{3.9}$$

4 Supersymmetric flows

Define a fermionic superfield $\Phi(x,\vartheta)=s\xi+\vartheta u$ and superderivative $D=\frac{\partial}{\partial\vartheta}+\vartheta\partial_x$, where s is a nonzero parameter and ϑ is an odd coordinate. The most general superfield equation having component content of the form (1.3) is the 8 parameter system,

$$\begin{aligned}\left(1-\nu D^4\right)\Phi_t \;=\; & \kappa_1D^2\Phi+\kappa_2D^6\Phi-\tfrac{2}{s^2}\Phi D^3\Phi+\left(\tfrac{2}{s^2}-3\right)D\Phi D^2\Phi+\left(\tfrac{\gamma_3}{s^2}{+}\beta_3\right)D\Phi D^6\Phi \\ & -\tfrac{\gamma_3}{s^2}\Phi D^7\Phi+\left(\beta_3{+}\tfrac{\gamma_3-\gamma_2}{s^2}\right)D^2\Phi D^5\Phi+\left(\beta_2{-}\beta_3{+}\tfrac{\gamma_2-\gamma_3}{s^2}\right)D^3\Phi D^4\Phi\;,\end{aligned} \tag{4.1}$$

where $\{\nu,s,\kappa_1,\kappa_2,\beta_2,\beta_3,\gamma_2,\gamma_3\}$ are parameters. The component equations are,

$$\begin{aligned} u_t-\nu u_{xxt} \;&=\; \kappa_1u_x+\kappa_2u_{xxx}-3uu_x+\beta_2u_xu_{xx}+\beta_3uu_{xxx}+2\xi\xi_{xx}+\gamma_2\xi_x\xi_{xxx}+\gamma_3\xi\xi_{xxxx}\;,\\ \xi_t-\nu\xi_{xxt} \;&=\; \kappa_1\xi_x+\kappa_2\xi_{xxx}-\tfrac{2}{s^2}u_x\xi+\left(\tfrac{2}{s^2}-3\right)u\xi_x+\left(\tfrac{\gamma_3}{s^2}+\beta_3\right)u\xi_{xxx}\\ &\quad+\left(\beta_2-\beta_3+\tfrac{\gamma_2-\gamma_3}{s^2}\right)u_x\xi_{xx}+\left(\tfrac{\gamma_3-\gamma_2}{s^2}+\beta_3\right)u_{xx}\xi_x-\tfrac{\gamma_3}{s^2}u_{xxx}\xi\;.\end{aligned} \tag{4.2}$$

These systems are by construction invariant under the supersymmetry transformations

$$\delta u \;=\; \tau\xi_x\;,\quad \delta\xi \;=\; \frac{\tau u}{s^2}\;, \tag{4.3}$$

where τ is an odd parameter. The superKdV limit, namely $\{\nu,\beta_2,\beta_3,\gamma_2,\gamma_3,\kappa_1\}$ all zero, yields, modulo rescalings, the one-parameter family of systems studied by Mathieu [12].

By comparing (4.2) and (3.8) it is straightforward to extract systems which are both supersymmetric and have hamiltonian form (3.5),(3.6). Taking $s^2{=}2$ in (4.2), $\{\nu{=}\mu\,,\,\sigma_1{=}\kappa_1\,,\,\sigma_2{=}\kappa_2\,,\,\epsilon{=}-1\}$ in (3.8), and $\{\beta_2{=}2\beta_3\,,\,\beta_3=\gamma_2{-}\frac{5}{2}\gamma_3\}$ in both, we obtain the systems,

$$\begin{aligned} u_t-\nu u_{xxt} \;&=\; \kappa_1u_x+\kappa_2u_{xxx}-3uu_x+(\gamma_2-\tfrac{5}{2}\gamma_3)(2u_xu_{xx}+uu_{xxx})\\ &\quad+2\xi\xi_{xx}+\gamma_2\xi_x\xi_{xxx}+\gamma_3\xi\xi_{xxxx}\;,\\ \xi_t-\nu\xi_{xxt} \;&=\; \kappa_1\xi_x+\kappa_2\xi_{xxx}-u_x\xi-2u\xi_x+(\gamma_2-2\gamma_3)\,u\xi_{xxx}\\ &\quad+\tfrac{3}{2}(\gamma_2-2\gamma_3)\,u_x\xi_{xx}+\tfrac{1}{2}(\gamma_2-4\gamma_3)\,u_{xx}\xi_x-\tfrac{1}{2}\gamma_3u_{xxx}\xi\;.\end{aligned} \tag{4.4}$$

These may be expressed in superfield form (4.1) with the above choice of parameters. The manifestly supersymmetric hamiltonian form is given by

$$M_t = \widehat{\mathcal{P}}_1 \, \frac{\delta \widehat{\mathcal{H}}_1}{\delta M} \quad , \quad M = \Phi - \nu D^4 \Phi \ , \tag{4.5}$$

with

$$\widehat{\mathcal{P}}_1 \;=\; D(1-\nu D^4) \ , \tag{4.6}$$

$$\begin{aligned} \widehat{\mathcal{H}}_1 \;=\; \int dx d\vartheta \ \Big(& \tfrac{\kappa_1}{2}\Phi D\Phi - \tfrac{\kappa_2}{2} D^2\Phi D^3\Phi - \tfrac{1}{2}\Phi(D\Phi)^2 \\ & + \tfrac{1}{4}\gamma_3 \Phi (D^3\Phi)^2 + \tfrac{1}{4}(\gamma_2 - 2\gamma_3)(D\Phi)^2 D^4\Phi \Big) \ . \end{aligned} \tag{4.7}$$

Since the KdV reduction of (4.4) (with $\kappa_1 = \gamma_2 = \gamma_3 = 0$) is not believed to be integrable, we have not explored this class of systems further.

In a similar fashion, we may look for choices of parameter sets for which the geodesic flow equations of section 2 are also supersymmetric. Comparing (2.8) with (4.2), we see that the choice $\{\mu{=}\nu,\ \alpha{=}1\,,\ \kappa_1{=}0\}$ in the former and $\{s^2{=}\frac{4}{3}\,,\ \beta_2{=}2\nu\,,\ \beta_3{=}\nu\,,\ \gamma_2{=}\frac{2\nu}{3}\,,\ \gamma_3{=}\kappa_1{=}0\}$ in the latter, yields the two-parameter system of supersymmetric geodesic flow equations:

$$\begin{aligned} u_t - \nu u_{xxt} &= \kappa_2 u_{xxx} - 3uu_x + 2\xi\xi_{xx} + \nu(uu_{xxx} + 2u_x u_{xx}) + \tfrac{2\nu}{3}\xi_x\xi_{xxx} \ , \\ \xi_t - \nu\xi_{xxt} &= \kappa_2 \xi_{xxx} - \tfrac{3}{2}(u\xi)_x + \nu(u\xi_{xxx} + \tfrac{3}{2}u_x\xi_{xx} + \tfrac{1}{2}u_{xx}\xi_x) \ . \end{aligned} \tag{4.8}$$

We shall call this system, with $\kappa_2{=}0$ and $\nu{\neq}0$, the *supersymmetric Camassa-Holm equation* (superCH). In section 5, we present some evidence for its integrability. The system (4.8) reduces to superKdV, upon setting ν to zero, and to CH, upon setting ξ to zero and translating u.

Not surprisingly, the systems (4.8) arise as geodesic flow equations precisely when the metric (2.4) on the NSR superconformal algebra is supersymmetric. Then, the calculations of section 2 can be performed using superfields. Specifically, writing $\mathcal{U} = u + \vartheta\phi$ and $\mathcal{V} = v + \vartheta\psi$, the bracket (2.3) takes the form

$$\Big[(\mathcal{U},a)\,,\,(\mathcal{V},b)\Big] = \Big(\mathcal{U}D^2\mathcal{V} - \mathcal{V}D^2\mathcal{U} + \tfrac{1}{2}D\mathcal{U}D\mathcal{V}\ ,\ c_1\int dx d\vartheta\, D^2\mathcal{U}D^3\mathcal{V}\Big) \tag{4.9}$$

and the inner product (2.4) may be written

$$\Big\langle(\mathcal{U},a)\,,\,(\mathcal{V},b)\Big\rangle = \int dx d\vartheta \ \Big(\mathcal{U}D^{-1}\mathcal{V} + \nu D^2\mathcal{U}D\mathcal{V}\Big) \ + ab \ . \tag{4.10}$$

The superspace bilinear operator $\widehat{B}$ is given by $\widehat{B}\Big((\mathcal{U},a),(\mathcal{V},b)\Big) = (\widehat{B}_0, 0)$, where $\widehat{B}_0$ satisfies

$$(1-\nu D^4)D^{-1}\widehat{B}_0 \;=\; c_1 a D^5\mathcal{V} - \tfrac{3}{2}D^2\mathcal{V}(1{-}\nu D^4)D^{-1}\mathcal{U} - \tfrac{1}{2}D\mathcal{V}(1{-}\nu D^4)\mathcal{U} - \mathcal{V}(1{-}\nu D^4)D\mathcal{U} \ . \tag{4.11}$$

Writing $c_1 a{=}\kappa_2$ and $\mathcal{U} = D\Phi$, the geodesic flow equations $(\mathcal{U}_t, a_t) = \widehat{B}\Big((\mathcal{U},a),(\mathcal{U},a)\Big)$ yield

$$(1-\nu D^4)\Phi_t \;=\; \kappa_2 D^6\Phi - \tfrac{3}{2}(\Phi D^3\Phi + D\Phi D^2\Phi) + \nu\Big(D\Phi D^6\Phi + \tfrac{1}{2}D^2\Phi D^5\Phi + \tfrac{3}{2}D^3\Phi D^4\Phi\Big) \ . \tag{4.12}$$

We thus recover the subsystem of (4.1) having component content (4.8). Equation (4.12) has superfield hamiltonian formulation,

$$M_t = \widehat{\mathcal{P}}_2 \, \frac{\delta \widehat{\mathcal{H}}_2}{\delta M} \quad , \quad M = \Phi - \nu D^4 \Phi \ , \tag{4.13}$$

with

$$\begin{aligned} \widehat{\mathcal{P}}_2 &= \kappa_2 D^5 - \tfrac{1}{2} DMD - D^2 M - MD^2 \ , && (4.14) \\ \widehat{\mathcal{H}}_2 &= \tfrac{1}{2} \Big\langle (D\Phi, 0) \, , \, (D\Phi, 0) \Big\rangle \ = \ \tfrac{1}{2} \int dx d\vartheta \ \Phi DM \ . && (4.15) \end{aligned}$$

5 Painlevé integrability of superCH systems

In this section we investigate, in more detail, the supersymmetric geodesic flow system (4.8) with $\nu{=}1$ and $\kappa_2 = 0$,

$$\begin{aligned} m_t &= -2mu_x - um_x + 2\eta\xi + \tfrac{2}{3}\eta_x \xi_x \ , \qquad & m &= u - u_{xx} \ , \\ \eta_t &= -\tfrac{3}{2}\eta u_x - \tfrac{1}{2} m \xi_x - u\eta_x \ , & \eta &= \xi - \xi_{xx} \ . \end{aligned} \tag{5.1}$$

We shall consider the two simplest possible choices for the grassmann algebra in which the fields are valued, viz. algebras with one or two odd generators. Taking the algebra to be finite dimensional is a very convenient tool for preliminary investigations of systems with grassmann algebra-valued fields. Manton [15] recently studied some simple supersymmetric classical mechanical systems in this way and he introduced the term 'deconstruction' to denote a component expansion in a grassmann algebra basis. In [16] we investigate fermionic extensions of KdV in a similar fashion.

5.1 First deconstruction of superCH

We first consider the superCH system (5.1) with fields taking values in the simplest grassmann algebra with basis $\{1, \tau\}$, where τ is a single fermionic generator. In this case the fermionic fields may be expressed as $\xi = \tau\xi_1$, $\eta = \tau\eta_1$, where ξ_1 and η_1 are standard (i.e. commuting, c-number) functions, as are u and m in this simple case. Since $\tau^2 = 0$, the fermionic bilinear terms do not contribute and we are left with the system

$$\begin{aligned} m_t &= -2mu_x - um_x \ , \qquad & m &= u - u_{xx} \\ \eta_{1t} &= -\tfrac{3}{2}\eta_1 u_x - \tfrac{1}{2} m \xi_{1x} - u\eta_{1x} \ , & \eta_1 &= \xi_1 - \xi_{1xx} \ . \end{aligned} \tag{5.2}$$

Further analysis is simplified by changing coordinates as described in [17]. Writing $m{=}p^2$, the first equation of (5.2) takes the form $p_t = (-pu)_x$, which suggests new coordinates y_0, y_1 defined via

$$dy_0 = p\,dx - pu\,dt \ , \quad dy_1 = dt \ , \tag{5.3}$$

or dually, via

$$\tfrac{\partial}{\partial x} = p \tfrac{\partial}{\partial y_0} \ , \quad \tfrac{\partial}{\partial t} = \tfrac{\partial}{\partial y_1} - pu \tfrac{\partial}{\partial y_0} \ . \tag{5.4}$$

Implementing this coordinate change and eliminating the functions u and ξ_1 , the remaining equations for p and $q \equiv \eta_1$ are:

$$p^2\dot{p}'' - p(\dot{p}p'' + \dot{p}'p') + \dot{p}p'^2 - 2p^3p' - \dot{p} = 0 \ , \tag{5.5}$$

$$\begin{aligned} \dot{q}'' - \frac{3p'}{p}\dot{q}' - \frac{3\dot{p}}{2p}q'' + \left(\frac{4p'^2}{p^2} - \frac{2p''}{p} - \frac{1}{p^2}\right)\dot{q} + \left(\frac{15p'\dot{p}}{2p^2} - \frac{3\dot{p}'}{p} - \frac{p}{2}\right)q' & \\ +3\left(\frac{\dot{p}p'' + 2p'\dot{p}'}{p^2} - \frac{4\dot{p}p'^2}{p^3} - p'\right)q \quad = \quad 0 \ . & \end{aligned} \tag{5.6}$$

Here the dot and prime denote differentiations with respect to y_1 and y_0 respectively. We note: (a) thanks to supersymmetry (4.3), if p is a solution of (5.5), then $q{=}p^2$ is a solution of (5.6); and (b) under the substitution $q = p^{3/2}r$, (5.6) takes the substantially simpler form

$$\dot{r}'' + \left(\frac{p'^2}{4p^2} - \frac{p''}{2p} - \frac{1}{p^2}\right)\dot{r} - \frac{p}{2}r' - \frac{3p'}{4}r = 0 \ . \tag{5.7}$$

The system (5.5),(5.6) passes the WTC Painlevé test.

Proof: Equation (5.5) is a rescaled version of the Associated Camassa-Holm equation of [17]. Consideration of solutions with $p(y_0, y_1) \sim p_0(y_0, y_1)\phi(y_0, y_1)^n$ near $\phi(y_0, y_1) = 0$, for some $n \neq 0$, yields $n = -2$ or $n = 1$ as the possible leading orders of Laurent series solutions. We need to perform the WTC Painlevé test [18] for both these types of series. The first type, namely, Laurent series solutions exhibiting double poles on the singular manifold $\phi(y_0, y_1) = 0$, have already been considered in [19]. These take the form

$$p = \frac{2\phi'\dot{\phi}}{\phi^2} - \frac{\dot{\phi}'}{\phi} + p_2 + p_3\phi + p_4\phi^2 + \dots \ , \tag{5.8}$$

where ϕ, p_2, p_4 are arbitrary functions of y_0, y_1, and

$$\begin{aligned} p_3 \quad = \quad \frac{-1}{2\phi'^2\dot{\phi}^2} \quad & \left(\phi'^2\dot{\phi}\dot{p}_2 + \phi'\dot{\phi}^2p_2' - \left(\phi'^2\ddot{\phi} - 2\phi'\dot{\phi}\dot{\phi}' + \phi''\dot{\phi}^2\right)p_2\right. \\ & \left. - \left(\phi'\dot{\phi}\ddot{\phi}'' - \phi'\ddot{\phi}\dot{\phi}'' - \dot{\phi}\phi''\ddot{\phi}' + \ddot{\phi}\phi''\dot{\phi}'\right)\right) \ . \end{aligned} \tag{5.9}$$

We have, at present, no explanation of the remarkable symmetry of these expressions under interchange of the independent variables. The second type of solutions have a simple zero on the singular manifold $\phi(y_0, y_1) = 0$. They take the form

$$p = \pm\frac{\phi}{\phi'} + p_2\phi^2 + p_3\phi^3 + \dots \ , \tag{5.10}$$

where ϕ, p_2, p_3 are arbitrary functions. The verification of the consistency of both these types of expansions is straightforward. This completes the WTC test for equation (5.5).

It remains to look at the equation (5.6). Although linear in q, it is *not* automatically Painlevé. The movable poles and zeros in p give rise to movable poles in the coefficient functions of the linear equation for q, and we need to examine the resulting singularities of q. If p has a pole on ϕ=0, then near ϕ=0 we have $p \sim 2\dot{\phi}\phi'/\phi^2$, and equation (5.6) takes the form

$$\dot{q}'' + \left(\frac{6\phi'}{\phi}+\ldots\right)\dot{q}' + \left(\frac{3\dot{\phi}}{\phi}+\ldots\right)q'' + \left(\frac{4\phi'^2}{\phi^2}+\ldots\right)\dot{q} + \left(\frac{11\phi'\dot{\phi}}{\phi^2}+\ldots\right)q' + \left(O\left(\frac{1}{\phi^2}\right)\right)q = 0 \ .$$

Thus the equation has a solution with $q \sim \phi^n$ if $n(n-1)(n-2)+9n(n-1)+15n = 0$, giving $n=-4,-2,0$. It follows that in the case when p is given by the series (5.8), no inconsistencies will arise near the double poles of p if (5.6) has a series solution of the form

$$q = \frac{q_0}{\phi^4} + \frac{q_1}{\phi^3} + \frac{q_2}{\phi^2} + \frac{q_3}{\phi} + q_4 + \ldots \tag{5.11}$$

with q_0, q_2, q_4 arbitrary. The consistency of such a solution can easilly be verified using a symbolic manipulator. Using MAPLE we find that

$$q_1 = \frac{2\phi'' q_0 - \phi' q_0'}{\phi'^2} \ . \tag{5.12}$$

The explicit expression for q_3 is too lengthy to be given here.

Suppose now that p has a zero on ϕ=0. Near this, $p \sim \pm\phi/\phi'$ and equation (5.6) has the structure

$$\dot{q}'' - \left(\frac{3\phi'}{\phi}+\ldots\right)\dot{q}' - \left(\frac{3\dot{\phi}}{2\phi}+\ldots\right)q'' + \left(\frac{3\phi'^2}{\phi^2}+\ldots\right)\dot{q} + \left(\frac{15\phi'\dot{\phi}}{2\phi^2}+\ldots\right)q' - \left(\frac{12\phi'^2\dot{\phi}}{\phi^3}+\ldots\right)q = 0 \ .$$

Thus (5.6) has a solution with $q \sim \phi^n$ if $n(n-1)(n-2) - \frac{9}{2}n(n-1) + \frac{21}{2}n - 12 = 0$, giving $n = \frac{3}{2}, 2, 4$. The appearance of a half-integer here is not considered a violation of the Painlevé test (see e.g. [20]). The half integer value of n gives rise to a series solution of (5.6), near a zero of p, of the form

$$q = q_0\phi^{\frac{3}{2}} + q_1\phi^{\frac{5}{2}} + q_2\phi^{\frac{7}{2}} + \ldots \tag{5.13}$$

with q_0 arbitrary, and $q_1, q_2, \ldots$ determined by q_0 (and the arbitrary functions arising in the series (5.10) for p). The two integer values of n tell us that we need to check the consistency of solutions of (5.6) taking the form

$$q = Q_0\phi^2 + Q_1\phi^3 + Q_2\phi^4 + \ldots \tag{5.14}$$

with two arbitrary functions Q_0 and Q_2. This is indeed consistent; using MAPLE we obtain

$$Q_1 \ = \ \pm 2\phi' Q_0 p_2 - \frac{1}{3\phi'^2\dot{\phi}}\left(2\phi'^2\dot{Q}_0 + 2\phi''\dot{\phi}Q_0 + \phi'\dot{\phi}Q_0' + 4\phi'\dot{\phi}'Q_0\right) \ , \tag{5.15}$$

with the choice of $\pm$ depending on the choice in (5.10). The general solution of (5.6) near a zero of p, with three arbitrary functions, is a linear combination of the series (5.13) and (5.14). Thus the system (5.5),(5.6) passes the WTC test. □

The WTC test is strong evidence for the complete integrability of the system (5.5),(5.6). This in turn demonstrates that superCH indeed has some integrable content.

5.2 Second deconstruction of superCH

We now consider the system (5.1) with fields taking values in a grassmann algebra with two anticommuting fermionic generators, τ_1, τ_2. Expanding in the basis $\{1, \tau_1, \tau_2, \tau_1\tau_2\}$,

$$\begin{aligned} u &= u_0 + \tau_1\tau_2 u_1 , \quad & \xi &= \tau_1\xi_1 + \tau_2\xi_2 , \\ m &= m_0 + \tau_1\tau_2 m_1 , \quad & \eta &= \tau_1\eta_1 + \tau_2\eta_2 , \end{aligned} \tag{5.16}$$

where the functions $u_0, u_1, m_0, m_1, \xi_1, \xi_2, \eta_1, \eta_2$ are all standard, we obtain the system:

$$m_{0t} = -2m_0u_{0x} - u_0m_{0x} , \qquad m_0 = u_0 - u_{0xx} , \tag{5.17}$$

$$\eta_{it} = -\tfrac{3}{2}u_{0x}\eta_i - \tfrac{1}{2}m_0\xi_{ix} - u_0\eta_{ix} , \qquad \eta_i = \xi_i - \xi_{ixx} , \qquad i = 1,2 , \tag{5.18}$$

$$\begin{aligned} m_{1t} &= -2m_1u_{0x} - 2m_0u_{1x} - u_0m_{1x} - u_0m_{1x} \\ &\quad +2(\eta_1\xi_2 - \eta_2\xi_1) + \tfrac{2}{3}(\eta_{1x}\xi_{2x} - \eta_{2x}\xi_{1x}) , \qquad m_1 = u_1 - u_{1xx} . \end{aligned} \tag{5.19}$$

Supersymmetry (4.3) tells us that given a solution u_0, m_0 of (5.17), we can solve the remaining equations by taking $\xi_i = \alpha_i u_0$, $\eta_i = \alpha_i m_0$ (i=1, 2), $u_1 = \beta u_{0x}$ and $m_1 = \beta m_{0x}$, where $\alpha_1, \alpha_2, \beta$ are arbitrary constants.

We handle the system (5.17)-(5.19) following the procedure of the previous section. Writing m_0=p^2 and changing coordinates to y_0, y_1, the system can be written:

$$u_0' = \left(\frac{1}{p}\right)^{\cdot} , \qquad u_0 = p^2 - p\left(\frac{\dot p}{p}\right)' , \tag{5.20}$$

$$\xi_i' = \frac{3\eta_i\dot p}{p^4} - \frac{2\dot\eta_i}{p^3} , \qquad \xi_i = \eta_i + p\left(\frac{3\eta_i\dot p}{p^3} - \frac{2\dot\eta_i}{p^2}\right)' , \qquad i = 1,2 , \tag{5.21}$$

$$\begin{aligned} \left(\frac{m_1}{p^2}\right)^{\cdot} &= -(2u_1p)' + \left(\frac{8(\dot\eta_1\eta_2 - \dot\eta_2\eta_1)}{3p^3}\right)' + \left(\frac{4(\eta_1'\eta_2 - \eta_2'\eta_1)}{3p^3}\right)^{\cdot} , \\ & m_1 = u_1 - p(pu_1')' . \end{aligned} \tag{5.22}$$

Applying the WTC Painlevé test to this is a mammoth task, so instead we consider the Galilean-invariant reduction and apply the Painlevé test at this level. The Galilean-invariant reduction is obtained, as usual, by restricting all functions to depend on the single variable z=y_0−vy_1 alone. Evidently the first equations of both (5.20) and (5.22) can be integrated once immediately. Then eliminating u_0 from (5.20), ξ_i from (5.21) and m_1 from (5.22), we obtain,

$$\left(\frac{p'}{p}\right)' = -\frac{p}{v} + \frac{c_1}{p} - \frac{1}{p^2} , \tag{5.23}$$

$$\eta_i''' - \frac{9p'}{2p}\eta_i'' + \left(\frac{11p}{2v} - \frac{5c_1}{p} + \frac{4}{p^2} + \frac{13p'^2}{2p^2}\right)\eta_i' - \frac{3p'}{p}\left(\frac{2p}{v} - \frac{3c_1}{p} + \frac{3}{p^2} + \frac{p'^2}{p^2}\right)\eta_i = 0, \ i\text{=}1,2, \tag{5.24}$$

$$u_1'' + \frac{p'}{p}u_1' + \left(\frac{2p}{v} - \frac{1}{p^2}\right)u_1 = d_1 + \frac{4}{p^3}(\eta_1\eta_2' - \eta_2\eta_1') , \tag{5.25}$$

where c_1, d_1 are integration constants. The equation for $p(z)$ may be integrated again after multiplying both sides by p'/p; this gives

$$p'^2 = 1 - 2c_1 p + c_2 p^2 - \tfrac{2}{v} p^3 \ , \tag{5.26}$$

where c_2 is another integration constant. This equation is well known in KdV theory. Its general solution can be written in terms of the Weierstrass $\wp$-function,

$$p(z) = -2v\wp(z) + \tfrac{1}{6} c_2 v \ , \tag{5.27}$$

where the periods of $\wp$ are determined by the coefficients c_1, c_2, v. Using (5.26), the coefficients in (5.24) can be simplified. Further, we know from supersymmetry that this equation has a solution $\eta_i = p^2$. Substituting $\eta_i = p^2 q_i$ the equation becomes a second order equation for q_i':

$$q_i''' + \frac{3p'}{2p} q_i'' + \left(-\frac{3p}{2v} - \frac{3}{2p^2} + \frac{c_2}{2} \right) q_i' \ = \ 0 \ , \qquad i = 1, 2 \ . \tag{5.28}$$

Supersymmetry (4.3) allows a reduction of the order of (5.25) as well. It implies that $u_1 = p'/p$, $\eta_i = p^2$ is a solution. So, writing $u_1 = rp'/p$, $\eta_i = p^2 q_i$ in (5.25) yields a first order equation for r':

$$r'' + \left(c_2 p - \frac{4p^2}{v} - \frac{1}{p} \right) \frac{r'}{p'} \ = \ \frac{p}{p'} \Big(d_1 + 4p(q_1 q_2' - q_2 q_1') \Big) \ . \tag{5.29}$$

Multiplying by the integrating factor p'^2/p and integrating, we obtain

$$r' = \frac{p}{p'^2} \left(d_1 p + d_2 + 4 \int (q_1 q_2' - q_2 q_1') p p' \, dz \right) \ , \tag{5.30}$$

where d_2 is a further constant of integration.

Thus the Galilean-invariant reduction of the second deconstruction of superCH takes the form of the three equations (5.26),(5.28),(5.30), to which we now apply the Painlevé test. All substitutions hitherto have been ones which do not interfere with the test. Equation (5.26) has movable double poles and movable simple zeros. Near a double pole at z_0, the series solution contains only even powers of $(z - z_0)$,

$$p(z) \ = \ -\frac{2v}{(z-z_0)^2} + \frac{c_2 v}{6} + \frac{12c_1 - c_2^2 v}{120} (z-z_0)^2 + \frac{\frac{54}{v} + c_2^3 v - 18 c_1 c_2}{3024} (z-z_0)^4 + \dots \tag{5.31}$$

and near a simple zero at z_0 ,

$$p(z) \ = \ \pm(z-z_0) - \tfrac{1}{2} c_1 (z-z_0)^2 \pm \tfrac{1}{6} c_2 (z-z_0)^3 - \tfrac{1}{24} (\tfrac{6}{v} + c_1 c_2)(z-z_0)^4 + \dots \ . \tag{5.32}$$

At both the zeros and poles of p, equation (5.28), which is just a linear third order ODE, has regular singular points. Checking Painlevé property for this reduces to doing the necessary Frobenius-Fuchs analysis at these regular singular points to check that no logarithmic singularities in the solutions q_i arise. Finally, equation (5.30) gives an explicit formula for r involving two quadratures. Here the necessary analysis involves wrtiting series expansions for the integrands near the zeros and poles of p, and checking for the absence of $1/(z - z_0)$ terms, which would give rise to logarithms on integration. We do not present all these calculations in detail; with the aid of a symbolic manipulator they are quite straightforward. We conclude that the Galilean-invariant reduction of the second deconstruction of superCH has the Painlevé property.

We note, in conclusion, that two of the equations we have encountered are interesting variants of the Lamé equation: In (5.28), the substitution $q_i' = p^{-3/4} h_i$ yields

$$h_i'' + \tfrac{3}{8}\left(\frac{p}{v} - \frac{c_2}{6} + \frac{c_1}{p} - \frac{7}{2p^2}\right) h_i \ = \ 0 \ , \tag{5.33}$$

and similarly, on writing $u_1 = p^{-1/2} k$, the homogeneous part of (5.25) takes the form,

$$k'' + \left(\frac{3p}{v} - \frac{c_2}{4} - \frac{3}{4p^2}\right) k \ = \ 0 \ . \tag{5.34}$$

By the arguments above, the latter is integrable by quadratures.

6 Outlook

In this paper we have examined fermionic extensions of the Camassa-Holm equation. In particular we have identified the superCH system (5.1), which, for low dimensional grassmann algebras displays some integrability properties. Further investigation is needed in order to determine whether the superCH system is integrable irrespective of the choice of grassmann algebra, and especially for the field theoretically interesting case with infinitely many odd generators. We have not been able to find a Lax pair for superCH. We also note that the peakon solutions of the Camassa-Holm equation do not admit supersymmetrisation (except when the grassmann algebra has just one fermionic generator); the peakon solutions are weak solutions, with a discontinuity in the first derivative, and the action of the supercharge on such functions gives objects with insufficient regularity properties to be considered as weak solutions.

Despite these open questions, our work provides a further instance of integrability arising in the setting of geodesic flow on a group manifold. It remains a pressing open problem to understand integrability from this geometric viewpoint. In this context, we should mention that the KP (and super KP) systems have yet to be presented as geodesic flow equations. If such a presentation exists, it would have a bearing on the question of whether there is a KP-type higher dimensional generalisation of Camassa-Holm (arising in a way similar to that in which KP generalises KdV).

Acknowledgements

We thank the MPI für Mathematik in den Naturwissenschaften for hospitality to J.S. during a visit to Leipzig and the Department of Mathematics and Computer Science of Bar-Ilan University for hospitality to C.D. during a visit to Israel.

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