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ON A CONJECTURE OF WOLANSKY

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Abstract. In ths paper, we consider the following problem (P_M) :

$$\Delta u + \lambda |x|^\beta e^u = 0, \quad x \in B_R,$$

$$-\int_{\partial B_R} \frac{\partial u}{\partial \nu} = M,$$

$$u = 0 \quad on\ \partial B_R,$$

where λ is an unknown constant, $\beta > 0$, $B_R = \{x \in R^2 | |x| < R\}$, M is a prescribed constant and ν is the outer normal to the disk. Problem (P_M) arises in the evolution of self-interactioning clusters and also in prescribing Gaussian curvature problem. It is known that for $M < 8\pi$, problem (P_M) has a global minimizer solution (which is radially symmetric). We prove that for $M > 8\pi$, there exists a $\beta_c \geq 1$ such that for $\beta > \beta_c$ and $M \in (8\pi, 4(2+\beta)\pi) \backslash \{8m\pi, m = 2, ...\}$, problem (P_M) admits a non-radially symmetric solution. This partially answers a conjecture of Wolansky. Our main idea is a combination of Struwe's technique and blow-up analysis for a problem with degenerate potential.

1. Introduction

In this paper, we consider a problem proposed by Wolansky. In [W2], he conjectured

Conjecture 1.1. *Let $\mathcal{C}_\alpha$ be the circular cone obtained from the disk B_R by identifying the rays $\{0 \leq r \leq R, \theta = 0\}$ and $\{0 \leq r \leq R, \theta = \alpha\}$. Let $M > 8\pi$. Then, if $\alpha > \alpha_c$ for some $\alpha_c > 2\pi$, there is a metric $d\sigma^2$ on $\mathcal{C}_\alpha$, conformally equivalent to the Euclidean metric where*

(a) *$d\sigma^2$ is not radially symmetric,*
(b) *$d\sigma^2$ admits a constant, positive Gaussian curvature on $\mathcal{C}_\alpha$ and total curvature $= M/2$,*
(c) *$d\sigma^2$ is point-wise isometric to the Euclidean metric on $\partial \mathcal{C}_\alpha$.*

An equivalent formulation is as follows. Let us consider the following problem (P_M):

$$-\Delta u = \lambda e^{V+u}, \qquad \text{in } B_R, \tag{1.1}$$

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$$-\int_{\partial B_R} \frac{\partial u}{\partial \nu} = M, \tag{1.2}$$

$$u = 0, \qquad \text{on } \partial B_R, \tag{1.3}$$

where λ is an unknown constant, M is a prescribed constant and ν is the outer normal to the disk. When $V = \beta \log |x|$, (1.1) and (1.2) mean that $e^u |x|^\beta dx^2$ admits a constant, positive Gaussian curvature and total curvature $M/2$, while (1.3) means that $e^u|x|^\beta dx^2$ is point-wise isometric to the metric $|x|^\beta dx^2$ on ∂B_R. When $M = 8\pi$ and $e^{V(x)} \geq a > 0$, problem P_M is critical with respect to the Moser-Trudinger inequality. We refer reader to [KW], [CD], [CLMP], [CY], [CkL], [DJLW1], [W2] and references therein. When $M > 8\pi$, problem (P_M) is supercritical. If, in addition, $e^{V(x)} \geq a > 0$, we refer to [ST], [DJLW2] and [WW]. In this paper, we consider the case when $M > 8\pi$ and $V(x) = |x|^\beta$ with $\beta > 0$.

Conjecture 1.1 is equivalent to

Conjecture 1.2. *Given $M > 8\pi$, consider the Dirichlet problem (1.1)-(1.3) on the disk B_R with $V = \beta \log r$. Then there exists $\beta_c > 0$ such that (1.1)-(1.3) admits a non-radially symmetric solution, provided that $\beta > \beta_c$.*

A solution of (1.1)-(1.3) can be characterized in [W2] as a critical point of a free energy functional. For any $M > 0$, let Γ_M be a subset of the Orlicz space $\mathcal{L}\log\mathcal{L}$ defined by

$$\Gamma_M = \{\rho \in \mathcal{L}\log\mathcal{L} | \rho \geq 0 \text{ a.e. and } \int_{B_R} \rho = M\}.$$

Define a functional $F : \Gamma_M \to \mathbb{R}$ by

$$F(\rho) = \int_{B_R} \rho \log \rho + \int_{B_R} V\rho + \frac{1}{2}\int_{B_R} U_\rho \rho \tag{1.4}$$

with $U_\rho \in H_0^1(B_R)$ satisfying

$$\Delta U_\rho = \rho. \tag{1.5}$$

A solution of (1.1)-(1.3) is a critical point of F in Γ_M. For problem of free energy functionals, we refer to [BB] and [CSW]. In [W2] Wolansky observed that one can construct a mountain pass value as follows. If $8\pi < M < 4\pi(2 + \beta)$, all radial solutions of

$$\Delta u + |x|^\beta e^u = 0 \qquad \text{in } B_R \tag{1.6}$$

with (1.2)-(1.3) can be explicitly determined (see section 2 below). In fact, in this case, (1.6) admits a unique radial solution u_M (cf [W1]). Furthermore, for large $\beta > 0$, he proved that $\rho_M = -\Delta u_M$ is a local minimizer of F in Γ_M. On the other hand, when $M > 8\pi$, F is unbounded in Γ_M. Hence one can construct a mountain pass value, see the definition in [W2] or section 2 below. However, the functional F is lack of compactness when $M > 8\pi$. It is difficult to analyze the Palais-Smale sequence ([AE] or [P]) of F or the heat equation of (1.1)-(1.3).

One can also use another formulation of the problem, see [Ws]. For convenience, here we use a third formulation which is motivated in studying the Moser-Trudinger inequality ([M]) in [DJLW1], [CSW] and [Wg]. Let us consider the following functional

$$J_M(u) = \frac{1}{2}\int_{B_R} |\nabla u|^2 - M \log \int_{B_R} |x|^\beta e^u \tag{1.7}$$

in the space $H_0^1(B_R)$. It is also easy to check that the Euler-Lagrange equation of J_M is (1.1)-(1.3). J_M and F are almost equivalent, see section 2 below or [CSW]. So we can also define a mountain pass value for J_M. Again, there is the problem of analyzing the Palais-Smale sequence of J_M.

In this paper, we first use a trick of Struwe [St1] to show that for certain dense subset Λ of $(8\pi, (2+\beta)4\pi)$, the mountain pass value discussed above is achieved for J_M with $M \in \Lambda$. Then we consider convergence of solutions of (1.1)-(1.3). By generalizing the results in [LS] (see also [NS], [Su], [Wj], [WW] and [Y]), we show that

Theorem 1.3. *There exists a $\beta_c > 0$ such that for any $\beta > \beta_c$ and $M \in (8\pi, (2+\beta)4\pi)\backslash\{8m\pi, m = 2, ...\}$, the conclusion of Conjecture 1.2 holds.*

This means that expect the set $\cup_{m\in\mathbb{N}}\{8\pi m\}$ Conjectures 1.1 and 1.2 are true. On the other hand, it is very difficult to handle the problem when $M \in \{8\pi m | m \in \mathbb{N}\} \cap (8\pi, 4\pi(2+\beta))$. This method was also used in [ST], [DJLW2] and [WW].

Equation (1.1) with $V = \beta \log |x|$ can also be interpreted as an equation on a surface with conical singularities. For such a problem, we refer reader to [CL2] and references therein.

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2. Existence for a dense subset

First we list an obvious Lemma.

Lemma 2.1. *A critical solution of J_M in $H_0^1(B_R)$ satisfies (1.1)-(1.3).*

Proof. Clearly, a critical solution of J_M in $H_0^1(B_R)$ satisfies

$$-\Delta u = M \frac{e^{V+u}}{\int_{B_R} e^{V+u}} \tag{2.1}$$

and (1.3). Integrating (2.1), we get (1.2). (2.1) is (1.1) with

$$\lambda = M(\int_{B_R} e^{V+u})^{-1}.$$

□

Lemma 2.1 means that F and J_M are equivalent as the first derivative.

Lemma 2.2. *For any $\rho \in \Gamma_M$ there exists $u \in H_0^1(B_R)$ with $F(\rho) \geq J_M(u) + M \log M$, while for any $u \in H_0^1(B_R)$ there exists some $\rho \in \Gamma_M$ with*

$$J_M(u) - F(\rho) = \frac{1}{2} \int_{B_R} |\nabla(u - w)|^2, \tag{2.2}$$

where $w \in H_0^1(B_R)$ satisfies

$$-\Delta w = \rho.$$

Proof. The same argument in [CSW] gives the proof. For convenience of the reader, we give the proof of the second statement. For any $u \in H_0^1(B_R)$, let $\rho \in \Gamma_M$ defined by

$$\rho = M \frac{e^{V+u}}{\int_{B_R} e^{V+u}}.$$

A direct computation shows

$$J_M(u) - F(\rho) = \frac{1}{2} \int_{B_R} |\nabla(u - w)|^2,$$

where w was defined above. $\square$

Let u_M be the unique radially symmetric solution of (1.1)-(1.3) (see [W1] and [W2]). In fact, u_M has following form

$$u_M(x) = \log \frac{2(2+\beta)^2\sigma}{1+\sigma|x|^{2+\beta}} - \log \frac{2(2+\beta)^2\sigma}{1+\sigma|R|^{2+\beta}}$$

with

$$\sigma = \frac{M}{4\pi(2+\beta) - M}.$$

If $M \in (0, 4\pi(2+\beta))$, then $\sigma \in (2/\beta, +\infty)$. Let $\rho_M = -\Delta u_M$. From [W2], for large $\beta > 0$, ρ_M is a local minimizer of F in Γ_M. By Lemma 2.2, u_M is also a local minimizer of J_M in $H_0^1(B_R)$. In fact, we have

Lemma 2.3. *For any $M \in (8\pi, 4\pi(2+\beta))$, there exists a $\delta_M > 0$ such that*

$$\begin{aligned} D^2 J_M(u_M)(\xi, \xi) &= \int_{B_R} |\nabla \xi|^2 - M \frac{\int_{B_R} |x|^\beta e^{u_M} \xi^2}{\int_{B_R} |x|^\beta e^{u_M}} + M \frac{(\int_{B_R} |x|^\beta e^{u_M} \xi)^2}{(\int_{B_R} |x|^\beta e^{u_M})^2} \\ &\geq \delta_M \int_{B_R} |\nabla \xi|^2 \end{aligned}, \tag{2.3}$$

for any $\xi \in H_0^1(B_R)$ with $\xi \not\equiv 0$.

Proof. First, from Lemma 2.2 and the result of [W2], one can show that

$$D^2 J_M(u_M)(\xi, \xi) > 0 \qquad \text{for any } \xi \in H_0^1(B_R) \text{ with } \xi \not\equiv 0. \tag{2.4}$$

Then, we claim that there exists a $\delta_M > 0$ such that (2.3) holds. If the claim were false, then there exists a sequence $\xi_i \in H_0^1(B_R)$ satisfying

(a) $\int_{B_R} |\nabla \xi_i|^2 = 1$ for any $i = 1, 2, ...,$
(b) $D^2 J_M(u_M)(\xi_i, \xi_i) \to 0$ as $i \to \infty$.

From (a), we know that there exists $\xi_0 \in H_0^1(B_R)$ such that

(i) ξ_i converges to ξ_0 weakly in $H_0^1(B_R)$,

(ii) ξ_i converges to ξ_0 strongly in $L^p(B_R)$ for any $p > 1$ and almost everywhere.

From (i), (ii) and (b), we can show that $D^2 J_M(u_M)(\xi_0, \xi_0) = 0$. It follows from (2.4) that $\xi_0 = 0$, which is impossible. $\square$

On the other hand, since the best constant for the Moser-Trudinger inequality is 8π, J_M is unbound below when $M > 8\pi$. In fact, let $x_0 = (\frac{R}{2}, 0)$. We now define

$$v_0(x) = \chi(|x - x_0|) \log \frac{32\epsilon^2}{(\epsilon^2 + |x - x_0|^2)^2}, x \in B_R,$$

where $\chi(t) = 1$ for $|t| \leq R/4$ and $\chi(t) = 0$ for $|t| > R/2$. It is easy to see that $v_0 \in H_0^1(B_R)$ and moreover by simple computations (see Lemma 2.2 of [WW]),

$$J_M(v_0) = 2(8\pi - M) \log \frac{1}{\epsilon} + O(1)$$

where $|O(1)| \leq C$ as $\epsilon \to 0$.

Since u_M is a local minimizer of J_M and $J_M(v_0) < J_M(u_M)$ for ϵ small, it is now natural to define a mountain pass value by

$$\alpha_M = \inf_{l \in \mathcal{L}_M} \max_{t \in [0,1]} J_M(l(t)),$$

where $\mathcal{L}_M$ consists of all paths connecting u_M and v_0, i.e.

$$\mathcal{L}_M = \{l \in C^0([0,1], H_0^1(B_R)) | l(0) = u_M \text{ and } l(1) = v_0\}.$$

The definition of α_M was suggested in [W2]. If α_M is a critical value, we then obtain a non-radially symmetric solution. The main difficulty is the lack of compactness of the functional J_M when $M > 8\pi$. To overcome this, we use a technique introduced by Struwe [St1]. However, to apply the method of Struwe [St1], we have to show the monotonicity of α_M with respect to M, which is difficult since the definition of $\mathcal{L}_M$ depends on M. With the help of Lemma 2.3, we now define another mountain-pass value.

Consider any small interval I in $(8\pi, 4\pi(2+\beta))$. In such a small subinterval, we can show that for any $M \in I$, $\delta_M > \delta_0 > 0$, where δ_M is obtained in Lemma 2.3 and δ_0 can be made a fixed constant. Let us fix some $M_0 \in I$. And we can also choose v_0 in the definition of $\mathcal{L}_M$ such that $J_M(v_0) < J_M(u_M)$ for any $M \in I$. Now we change the definition of $\mathcal{L}_M$ and α_M a little bit as follows

$$\mathcal{L}_I = \{l \in C^0([0,1], H_0^1(B_R)) | l(0) = u_{M_0} \text{ and } l(1) = v_0\}$$

and

$$\alpha'_M = \inf_{l \in \mathcal{L}_I} \max_{t \in [0,1]} J_M(l(t)). \tag{2.5}$$

Note that we can choose $|I|$ so small so that

$$\begin{aligned} J_M(v) - J_M(u_{M_0}) &= J_M(v) - J_M(u_M) + J_M(u_M) - J_M(u_{M_0}) \\ &\geq \delta_0 \int_{B_R} |\nabla(v-u_M)|^2 + O(|M-M_0|) \geq \eta_0 > 0 \end{aligned}$$

for $\|v-u_{M_0}\|_{H_0^1(B_R)} = \rho_0$, where $\eta_0 > 0, \rho_0 > 0$ are small fixed numbers (which can be chosen to be independent of I). Hence α'_M defines a Mountain-Pass value and moreover

$$\alpha'_M > \eta_0 + J_M(u_{M_0}) > J_M(u_M).$$

Since the definition of $\mathcal{L}_I$ is independent of $M \in I$, we now have the required monotonicity.

Lemma 2.4. $\frac{\alpha'_M}{M}$*, as a function in* I*, is monotone.*

Proof. It is trivial by definition. See, for instance, [ST] or [WW]. □

Applying Struwe's method in [St1,2], (see [ST], [WW] for similar arguments), we obtain that

Proposition 2.5. *There exists a dense subset* $\Lambda \subset (8\pi, 4\pi(2+\beta))$ *such that, for any* $M \in \Lambda$, α'_M *is achieved by* $w_M \in H_0^1(B_R)$*, which, in particular, is a solution of* P_M *(and is different from* u_M*).*

3. Convergence

Let $M_0 \in (8\pi, 4(2+\beta)\pi)\backslash\{8m\pi, m=2,...\}$. By Proposition 2.5, there exists a sequence $M_i \in \Lambda$ and $M_i \to M_0$ as $i \to \infty$. Let u_i be the solutions of (2.1) and (1.3) corresponding to M_i constructed in Proposition 2.5. From results of [BM] and [LS] (see also [L]), if $e^V \geq a > 0$ for some constant a, then either, u_i converges in $H_0^1(B)$, or, $M_0 = 8\pi m$ for some positive integer m. However, here $e^V = |x|^\beta$, we can not apply their results directly. So we need to generalize their results.

Theorem 3.1. *Let* u_i *be a sequence of solutions of equations (2.1) and (1.3) with* $M = M_i$ *and* $M_i \to M_0$ *as* $i \to \infty$*. If* $M_0 \in (8\pi, 4\pi(2+\beta))$*, then there are two possibilities:*

(1) u_i *converges to* u_0 *in* H^1,
(2) $\max u_i \to +\infty$ *as* $i \to \infty$*. In this case,*

$$M_0 = 8\pi m$$

for some integer m*.*

Proof. From the result of [LS], we only have to consider the case that the blow-up point is the origin. That is, there exists a sequence $\{x_i\}$ such that $u_i(x_i) \to +\infty$ and $x_i \to 0$ as $i \to \infty$. And for simplicity, assume there is no other blow-up point. Hence, we may assume that $u_i(x_i) = \max u_i$. Consider the following rescaling argument (see also [NT]). Set

$$\delta_i = \exp(-\frac{1}{2+\beta}\{u_i(x_i) + \log M_i - \log \int_{B_R} |x|^\beta e^{u_i}\}).$$

Clearly, $\delta_i \to 0$. There are two possibilities:

(i) $\lim_{i\to\infty} \frac{|x_i|}{\delta_i} = \infty$;

(ii) $\lim_{i\to\infty} \frac{|x_i|}{\delta_i} = c$ for some constant $c \geq 0$.

If case (ii) occurs, we may assume that

$$\frac{x_i}{\delta_i} \to x_0 \in \mathbb{R}^2.$$

Define $v_i(x) = u_i(\delta_i x + x_i) - u_i(x_i)$ in $\delta_i^{-1}(B_R - x_i) := \{y \in \mathbb{R}^2 | \delta_i y + x_i \in B_R\}$. Clearly, v_i satisfies

$$-\Delta v_i = |x + \frac{x_i}{\delta_i}|^\beta e^{v_i} \qquad \text{in } \delta_i^{-1}(B_R - x_i) \tag{3.1}$$

and

$$\int_{\delta_i^{-1}(B_R - x_i)} |x + \frac{x_i}{\delta_i}|^\beta e^{v_i} = M_i. \tag{3.2}$$

Since $\delta_i \to \infty$, $\delta_i^{-1}(B_R - x_i) \to \mathbb{R}^2$. The result of [BM] can be applied to obtain that $v_i \to v_0$ in $C^k_{loc}(\mathbb{R}^2)$, where v_0 satisfies that

$$-\Delta v_0 = |x + x_0|^\beta e^{v_0} \qquad \text{in } \mathbb{R}^2, \tag{3.3}$$

$$\int_{\mathbb{R}^2} |x + x_0|^\beta e^{v_0} \leq M \tag{3.4}$$

and

$$v_0 \leq 0 \qquad \text{for any } x \in \mathbb{R}^2. \tag{3.5}$$

We now have

Lemma 3.2. *If v is a solution of (3.3)-(3.5), then*

$$\int_{\mathbb{R}^2} |x + x_0|^\beta e^v = (2 + \beta) 4\pi.$$

Proof. The idea of the proof is similar to one in [CL3]. So we only give the sketch.

We may assume that $x_0 = 0$. Set

$$w(x) = \int_{\mathbb{R}^2} \frac{1}{2\pi} (\log|x - y| - \log(|y| + 1)) |y|^\beta e^v(y) dy \tag{3.6}$$

and

$$\gamma = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^\beta e^v dx.$$

Clearly,

$$-\Delta w(x) = |x|^\beta e^v \quad \text{in } \mathbb{R}^2.$$

A potential analysis as in [CL1] and [CL3] gives

$$-\gamma \log|x| - c \le w(x) \le -\gamma \log|x| + c \quad \text{for } x \in \mathbb{R}^2, \tag{3.7}$$

for some constant $c > 0$. Let $\tilde{w} = v - w$. From (3.6), (3.5) and (3.7), we know that $\tilde{w}$ is a harmonic function with $\tilde{w}(x) \le c \log|x| + c$ for any $x \in \mathbb{R}^2$. Hence $\tilde{w} = c_1$ for some constant c_1. It follows that

$$v = \int_{\mathbb{R}^2} \frac{1}{2\pi} (\log|x-y| - \log(|y|+1))|y|^\beta e^v(y) dy$$

(3.7) and (3.4) imply that

$$\gamma > 2 + \beta. \tag{3.8}$$

Furthermore, we can also estimate

$$r\frac{\partial v}{\partial r} = \gamma + o(1)$$

and

$$\frac{\partial v}{\partial \theta} = o(1)$$

as $|x| \to \infty$. We now apply the following Pohozaev identity for (3.3)

$$R(\frac{1}{2}\int_{\partial B_R} |\nabla v|^2 - \int\int_{\partial B_R} |\frac{\partial v}{\partial r}|^2) = R\int_{\partial B_R} |x|^\beta e^v - (2+\beta)\int_{B_R} |x|^\beta e^v.$$

Taking limit on the both sides of the previous identity and using the estimates obtained above, we have

$$\gamma = 2(2+\beta).$$

This proves the Lemma. $\square$

Remark. (1) Lemma 3.2 is also true without (3.5).

(2) When $-2 < \beta \le 0$, all solutions of (3.3)-(3.4) are classified, see [CL1] and [CL3].

(3) When $\beta > 0$, solutions of (3.3)-(3.4) may not be radially symmetric. Actually, in [CK2,p230] examples of non-symmetric solutions were given when $\beta \ge 2$ is an integer. These solutions satisfy all conditions in Lemma 2.3, and the conclusion certainly.

Completion of the proof of Theorem 3.1. From Lemma 3.2 and the argument above, we get a contradiction since $M < 4(2+\beta)\pi$. Hence we have, if the blow-up happens,

$$\lim_{i\to\infty} \frac{|x_i|}{\delta_i} = +\infty, \tag{3.9}$$

which means that x_i approaches the origin much slower than bubbling off.

Now we want to show, in this case, that $M_0 = 8\pi m$ for some integer m. Let

$$\gamma_i = |x_i|^{-\beta/2} \exp(-\frac{1}{2}(u_i(x_i) + \log M_i - \log \int_{B_R} e^{u_i})).$$

It is easy to see, from (3.9), that

$$\frac{|x_i|}{\gamma_i} \to +\infty \qquad \text{as } i \to \infty. \tag{3.10}$$

Define $w_i(x) = u_i(\gamma_i x + x_i) - u_i(x_i)$ in $\gamma_i^{-1}(B_R - x_i)$. One readily shows that w_i satisfies in $\gamma_i^{-1}(B_R - x_i)$

$$-\Delta w_i = |\frac{\gamma_i x}{|x_i|} + \frac{x_i}{|x_i|}|^\beta e^{w_i} \tag{3.11}$$

and

$$\int_{\gamma_i^{-1}(B_R - x_i)} |\frac{\gamma_i x}{|x_i|} + \frac{x_i}{|x_i|}|^\beta e^{w_i} = M_i. \tag{3.12}$$

Brezis-Merle's result implies that w_i converges to w_0 in $C^k_{loc}(\mathbb{R}^2)$, where w_0 is a entire solution of

$$-\Delta w_0 = e^{w_0} \text{ with } \int_{\mathbb{R}^2} e^{w_0} \le M$$

and

$$w_0(0) = 0 \text{ and } w_0(x) \le 0.$$

It follows from the classification result of Chen-Li that $w_0 = -2\log(1 + |x|^2/8)$.

We can also interpret the above as follows. Let $v_i(x) = u_i(x) + \beta \log |x|$. Clearly $v_i(x)$ satisfies

$$\Delta v_i = M_i \frac{e^{v_i}}{\int_{B_R} e^{v_i}} \qquad \text{in } B_R \backslash \{0\},$$

$$\gamma_i = \exp\{\frac{1}{2}(v_i(x_i) + \log M_i - \log \int_{B_R} e^{v_i})\},$$

$$\frac{|x_i|}{\gamma_i} \to +\infty, \qquad \text{as } i \to \infty.$$

Set $\tilde{v}_i(x) = v_i(\gamma_i x + x_i) - v_i(x_i)$. It is clear that

$$\tilde{v}_i(x) = w_i(x) + \beta \log |\frac{\gamma_i}{|x_i|} x + \frac{x_i}{|x_i|}|.$$

Hence $\tilde{v}_i$ converges to w_0 in any compact subset of $\mathbb{R}^2$. The argument above holds for local maximum of u_i.

Now we can follow the argument given in [LS] to show the Theorem. Applying the argument of Lemma 4 in [LS] to v_i and the above argument, we have:

There exists m $(1 \le m \le M/8\pi)$ sequences of points $\{x_i^{(j)}\}_{j=0}^{m-1}$ in B_R and m sequences of positive numbers $\{k_i^{(j)}\}_{j=0}^{m-1}$ with $\lim x_i^{(j)} = 0$ and $\lim k_i^{(j)} = \infty$ $(0 \le j \le m-1)$ such that for any $0 \le j \le m-1$

$$u_i(x_i^{(j)}) + \beta \log |x_i^{(j)}| = \max_{|x - x_i^{(j)}| \le k_i^{(j)} \gamma_i^{(j)}} (u_i(x) + \beta \log |x|) \to \infty, \tag{3.13}$$

where $\gamma_i^{(j)} = |x_i^{(j)}|^{-\beta/2} e^{-\frac{1}{2}(u_i(x_i^{(j)}) + \log m_i - \log \int_{B_R} e^{u_i})}$ with

$$\lim_{i \to \infty} \frac{|x_i^{(j)}|}{\gamma_i^{(j)}} \to \infty, \tag{3.14}$$

$$B_{2k_i^{(j)}\gamma_i^{(j)}}(x_i^{(j)}) \cap B_{2k_i^{(j')}\gamma_i^{(j')}}(x_i^{(j')}) = \emptyset, \text{ for any } j \ne j', \tag{3.15}$$

$$\frac{\partial}{\partial t} u_i(ty + x_i^{(j)})|_{t=1} < 0, \text{ for any } \gamma_i^{(j)} \le |y| \le 2k_i^{(j)} \gamma_i^{(j)}, \tag{3.16}$$

$$\lim_{i \to \infty} \int_{B_{2k_i^{(j)}\gamma_i^{(j)}}(x_i^{(j)})} |x|^\beta e^{u_i} = \lim_{i \to \infty} \int_{B_{k_i^{(j)}\gamma_i^{(j)}}(x_i^{(j)})} |x|^\beta e^{u_i} = 8\pi, \tag{3.17}$$

$$\max_{x \in \bar{B}_R} \{u_i(x) + \beta \log |x| + 2 \log \min_{0 \le j \le m-1} |x - x_i^{(j)}|\} \le C, \text{ for any } i. \tag{3.18}$$

From (3.13)-(3.18), the argument in [LS] applies to our case to get the conclusion. □

Proof of Theorem 1.3. From Theorem 3.1 and Proposition 2.5, we have a second solution for any $M \in (8\pi, 4\pi(2+\beta))/\{8m\pi, m = 2, \cdots\}$. It is non-radially symmetric. This gives the proof of Theorem 1.3. □

References

[A] Aubin, T., *Nonlinear analysis on manifolds*, Springer-Verlag, 1982.

[AE] Aubin, T. and Ekeland, I., *Applied Nonlinear Analysis*, Wiley, New York, 1984.

[BB] Berestycki, H. and Brezis, H., *On a free boundary problem arising in plasma physics,*, Nonlinear Anal. **4** (1980), 415–436.

[BM] Brezis, H. and Merle, F., *Uniform estimates and blow up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Diff. Equat. **16** (1991), 1223–1253.

[CLMP] Caglioti, E., P. L. Lions, C. Marchioro and M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Part II*, Commun. Math. Phys. **174** (1995), 229-260.

[CY] Chang, A. S. Y. and Yang, P., *Conformal deformation of metrics on S^2*, J. Diff. Geom. **23** (1988), 259–296.

[CkL] Chang, K. C. and Liu, J. Q., *On Nirenberg's problem*, International J. Math. **4** (1993), 35–57.

[CK] Chanillo, S. and Kiessling, M. K-H, *Conformally invariant systems of nonlinear pde of Liouville type*, Geom. Func. Anal. **5** (1995), 924–947.

[CK2] Chanillo, S. and Kiessling, M. K-H, *Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry*, Comm. Math. Phys. **160** (1994), 217-238.

[CD] Chen, W. X. and Ding, W. Y., *Scalar curvature on S^2*, Trans. Amer. Math. Soc. **303** (1987), 365–382.

[CL1] Chen, W. X. and Li, C., *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (1991), 615–622.

[CL2] Chen, W. X. and Li, C., *Prescribing Gaussian curvature on surfaces with conical singularities*, J. Geom. Anal. **1** (1991), 359–372.

[CL3] Chen, W. X. and Li, C., *What kinds of singular surfaces can admit constant curvature?*, Duke J. Math. **78** (1995), 437-451.

[CSW] Chipot, M., Shafrir, I. and Wolansky, G., *On the solutions of Liouville systems*, J. Diff. Equations **140** (1997), 59–105.

[CW] Chou, K. and Wan, T., *Asymptotic radial symmetry for solution of $\Delta u + e^u = 0$ in a punctured disc*, Pacific J. Math. **163** (1994), 269–275.

[DJLW1] Ding, W., Jost, J., Li, J. and Wang, G., *The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface*, Asian J. Math. **1** (1997), 230-248.

[DJLW2] Ding, W., J. Jost, J. LI and G. Wang, *Existence results for mean field equations*, Ann. Inst. Henri Poincaré, Analyse non linéaire, **16** (1999), 655-666.

[KW] Kazdan, J. and Warner, F., *Curvature functions for compact 2-manifolds*, Ann. Math. **99** (1974), 14–47.

[L] Li, YanYan, *Harnack type inequality: the method of moving planes*, Commun. Math. Phys. **200** (1999), 421-444.

[LS] Li, YanYan and Shafrir, I., *Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two*, Indiana Univ. Math. J. **43** (1994), 1255–1270.

[M] Moser, J., *A sharp form of an inequality of N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[NS] Nagasaki, K. and Suzuki, T., *Asymptotic analysis for two-dimensional elliptic problems with exponentially dominated nonlinearity*, Asymptotic Analysis **3** (1990), 173–188.

[NT] Nolasco, M. and Tarantello, G., *Double vortex condensates in the Chern-Simons-Higgs theory, preprint* (1998).

[P] Palais R. S., *Critical point theory and the minimax principle*, Global Analysis (Proc. Sympos. Pure Math.), vol. 15, 1968, pp. 185–212.

[St1] Struwe, M., *The evolution of harmonic mappings with free boundaries*, Manuscr. Math. **70** (1991), 373–384.

[St2] Struwe, M., *Multiple solutions to the Dirichlet problem for the equation of prescribed mean curvature*, Analysis, et cetera (Rabinowitz, P. H. and Zehnder, E., eds.), 1990, pp. 639-666.

[ST] Struwe, M. and Tarantello, G., *On multivortex solutions in Chern-Simons gauge theory,*, Boll. U.M.I. **8** (1998), 109–121.

[Su] Suzuki, T., *Global analysis for a two-dimensional elliptic eigenvalue problem with the exponential nonlinearity*, Ann. Inst. H. Poincaré, Anal. Non Lineaire **9** (1992), 367–398.

[Wg] Wang, G., *Moser-Trudinger inequalities and Liouville type systems,*, C. R. Acad. Sci. Paris **328** (1999), 895-900.

[WW] Wang, G. and Wei, J., *Steady State solutions of a reaction-diffusion system modeling chemotaxis, preprint* (1998).

[Ws] Wang, S., *Some nonlinear elliptic equations with subcritical growth and critical behavior*, Houston J. Math. (1990), 559-572.

[Wj] Wei,J., *Asymptotic behavior of a nonlinear fourth order eigenvalue problem*, Comm. P.D.E. **21** (1996), 1451–1467.

[W1] Wolansky, G., *On steady distributions of self-attracting clusters under friction and fluctuations*, Arch. Rat. Mech. Anal. **119** (1992), 355-391.

[W2] Wolansky, G., *On the evolution of self-interaction clusters and applications to semilinear equations with exponential nonlinearity*, J. D'Anal. Math. **59** (1992), 251–272.

[Y] Ye, D., *Une remarque sur le comportement asympotique des solutions de* $-\Delta u = \lambda f(u)$, C. R. Acad. Sci. Paris **325** (1997), 1279–1282.

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