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by

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1. Introduction

The paper deals with homogenization of a boundary condition for the heat equation which is rapidly oscilating between Dirichlet and Neumann type. Since the area fraction of the set where Dirichlet conditions are prescribed is order 1, in the first term of the expansion the Neumann condition is forgotten. In order to see the asymptotic effect of the Neumann boundary condition on the solution one has to go to the second term of the expansion. In two dimensions and in a special geometric setting this problem was already considered in [6].

Here we try to present a "most general" framework where deterministic homogenization methods can still be applied to calculate the second term in the asymptotic expansion.

The condition we arrive at is roughly speaking asymptotic almost periodicity and as a special example quasi-periodicity of the characteristic function of the set on the boundary where Dirichlet conditions hold. Almost periodicity has to be understood in the metric given by L^2 capacity (see Assumption (P)). The homogenization method we employ is a blow up technique combined with a suitably strong uniqueness result for solutions - in this case of a boundary problem for the Laplace equation in half space. The technique is similar to the one used for convergence of saturation in the matrix in [3].

For other approaches for homogenization of boundary conditions see e.g. [8] and references therein.

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Our study initially, in Section 2, parallels the corresponding investigation in [6] concerning basic a priori estimates. As a simple consequence one obtains the first term in the ε–expansion. The small parameter ε quantitatively describes the oscilation between Dirichlet and Neuman type boundary conditions. Section 3 investigates harmonic functions in the half–space subject to the mixed boundary conditions. Here the main result says that if a solution with homogeneous boundary conditions is sublinear, then it is the trivial solution. Using the blow up argument, Sections 4,5 provide a detailed analysis of a weak limit of the rescaled difference of the solution and its first term in the expansion divided by ε. The second term in the ε–expansion is determined in Section 6 provided the mean value over the hyperplane $x_n = 0$ of the limit function studied in Section 4 exists. The key understanding of circumstances, under which the mean value of a harmonic function with given mixed boundary condition on the hyperplane exists, is capacity, introduced and utilized in Section 7. Finally, Section 8 presents some applications.

We finish this introductory section by introducing some notation. For a typical point in $\mathbb{R}^n$ we write $x = (x_1, \dots, x_n)$ or $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1})$ and we set $\mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n \mid x_n > 0\}$, $\Omega_T \equiv \Omega \times (0,T)$, $\mathcal{N}_T \equiv \mathcal{N} \times (0,T)$, etc. An open ball in $\mathbb{R}^n$ with center x and radius r will be denoted by

$$B(x,r) \equiv \{y \in \mathbb{R}^n \mid |x-y| < r\} \quad \text{and} \quad B(r) \equiv B(0,r)\,.$$

Similarly in $\mathbb{R}^{n-1}$, i.e.

$$B(x',r) \equiv \{y' \in \mathbb{R}^{n-1} \mid |x'-y'| < r\} \quad \text{and} \quad B'(r) \equiv B(0',r)\,.$$

and we set $B_+(r) \equiv B(r) \cap \mathbb{R}^n_+$. An open cube in $\mathbb{R}^n$ with a center x and the side $2r$ we denote $C(x,r) \equiv \{y \in \mathbb{R}^n \mid |x_i - y_i| < r\,,\ i = 1,2,\dots,n\}$ and $C(r) \equiv C(0,r)$. Analogously, $C(x',r)$ and $C'(r)$ in $\mathbb{R}^{n-1}$. $\bar{E}$ denotes the closure of E. Next, set $A \triangle B \equiv A \cup B \setminus A \cap B$ for $A, B \subset \mathbb{R}^{n-1}$. We denote the average of f over the set $Q \subset \mathbb{R}^n$ and over its boundary by

$$\fint_Q f\,dx = \frac{1}{\mathcal{L}^n(Q)} \int_Q f(x)\,dx\,,$$
$$\fint_{\partial Q} f\,dH^{n-1}(x) = \frac{1}{\mathcal{H}^{n-1}(\partial Q)} \int_{\partial Q} f(x)\,dH^{n-1}(x)\,,$$

where $\mathcal{L}^n$ is n dimensional Lebesgue measure in $\mathbb{R}^n$ and $\mathcal{H}^{n-1}$ is $n-1$ dimensional Hausdorf measure in $\mathbb{R}^n$. $\alpha(k) = \pi^{k/2}/\Gamma(1+k/2)$ denotes volume of the unit ball in $\mathbb{R}^k$. By ε we shall denote small parameter finally tending to zero and by $\rightharpoonup$ weak convergence. The function spaces we use are rather familiar and we omit the definitions. Finally, by $J\Phi$ we shall denote Jacobian of Φ.

2. The first term in ϵ–expansion

In this section we study the problem

$$\begin{aligned} &\partial_t u_\varepsilon - \Delta u_\varepsilon = f_\varepsilon(x,t) && \text{in} \quad \Omega_T, \\ &\partial_\nu u_\varepsilon + \sigma_\varepsilon u_\varepsilon = \vartheta_\varepsilon(x,t) && \text{on} \quad \mathcal{N}_T^\varepsilon, \\ &u_\varepsilon = 0 && \text{on} \quad \mathcal{D}_T^\varepsilon, \\ &u_\varepsilon = u_0^\varepsilon && \text{on} \quad \Omega \times \{t=0\} \end{aligned} \tag{2.1}$$

subject to the following assumptions:

(A) *$\Omega \subset \mathbb{R}^n$ is open, bounded and connected set with Lipschitz boundary $\partial\Omega$.*

(B) *For any $0 < \varepsilon < 1$ let*

$$\partial\Omega \equiv \mathcal{D}^\varepsilon \cup \mathcal{N}^\varepsilon ,$$

where $\mathcal{D}^\varepsilon$ is a relatively open portion of $\partial\Omega$ fulfilling the following structure condition: There exist two numbers

$$0 < \delta < \rho$$

such that for any $x^0 \in \partial\Omega$ there is $x \in \mathcal{D}^\varepsilon$ satisfying

$$B(x, \varepsilon\delta) \cap \partial\Omega \subset B(x^0, \varepsilon\rho) \cap \mathcal{D}^\varepsilon . \tag{2.2}$$

(C) *$f_\varepsilon, f, f^1 \in L^2(\Omega_T)$ and such that*

$$F_\varepsilon \equiv \frac{f_\varepsilon - f}{\varepsilon} \rightharpoonup f^1 \qquad \text{in} \quad L^2(\Omega_T) \quad \text{as} \quad \varepsilon \to 0 .$$

(D) *$\sigma_\varepsilon, \partial_t\sigma_\varepsilon \in L^\infty(\partial\Omega_T)$ and there exists a positive constant C independent of ε such that $\|\sigma_\varepsilon\|_{L^\infty(\partial\Omega_T)} \le C$.*

(E) *$\vartheta_\varepsilon, \vartheta, \partial_t\vartheta_\varepsilon \in L^2(\partial\Omega_T)$ and such that*

$$\vartheta_\varepsilon \rightharpoonup \vartheta \qquad \text{in} \quad L^2(\partial\Omega_T) \quad \text{as} \quad \varepsilon \to 0 .$$

(F) *u_0^ε, $u_0 \in W^{1,2}(\Omega)$, $u_0 = 0$ on $\partial\Omega$, $u_0^\varepsilon = 0$ on $\mathcal{D}^\varepsilon$, $u_0^1 \in L^2(\Omega)$ and such that*

$$\mathcal{P}_\varepsilon \equiv \frac{u_0^\varepsilon - u_0}{\varepsilon} \rightharpoonup u_0^1 \qquad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \varepsilon \to 0 .$$

Note that some of the above assumptions can be relaxed, but for simplicity we do not give the most general hypotheses.

Now we state the rate of convergence of solutions u_ε of (2.1) to the solution u of the following problem:

$$\begin{aligned} &\partial_t u = \Delta u + f(x,t) && \text{in} \quad \Omega_T, \\ &u = 0 && \text{on} \quad \partial\Omega_T \\ &u = u_0 && \text{on} \quad \Omega \times \{t=0\}, \end{aligned} \tag{2.3}$$

Theorem 2.1. *Assume that* (A)–(F) *are satisfied. Then there exists a positive constant C, independent of ε, such that*

$$\max_{0\le t\le T} \int_\Omega |u_\varepsilon - u|^2(x,t)\,dx + \int_{\Omega_T} |\nabla(u_\varepsilon - u)|^2(x,t)\,dx\,dt \le C\varepsilon \ ,$$

$$\int_{\partial\Omega_T} |u_\varepsilon - u|^2(x,t)\,dH^{n-1}(x)\,dt + \int_{\Omega_T} |u_\varepsilon - u|^2(x,t)\,dx\,dt \le C\varepsilon^2 \ ,$$

$$\max_{0\le t\le T} \int_\Omega |u_\varepsilon - u|^2(x,t)\phi(x)\,dx + \int_{\Omega_T} |\nabla(u_\varepsilon - u)|^2(x,t)\phi(x)\,dx\,dt \le C\varepsilon^2$$

and

$$\operatorname{ess\,sup}_{0\le t\le T} \int_\Omega |\nabla(u_\varepsilon - u)|^2(x,t)\phi^3(x)\,dx + \int_{\Omega_T} |\partial_t(u_\varepsilon - u)|^2(x,t)\phi^3(x)\,dx\,dt \le C\varepsilon^2 \ ,$$

where ϕ is the principal eigenfunction of the problem

$$\begin{aligned} &\Delta\phi + \lambda\phi = 0 && \textit{in} \quad \Omega, \\ &\phi = 0 && \textit{on} \quad \partial\Omega\,, \end{aligned}$$

with the corresponding principal eigenvalue $\lambda = \lambda_1 > 0$.

Proof. This is almost exactly like the proof of the corresponding assertion of [6]. Therefore we prove here only the following

Proposition 2.2. *Assume* (A), (B) *are satisfied. Then there exists a positive constant C such that*

$$\|u\|_{L^2(\partial\Omega)} \le C\sqrt{\varepsilon}\,\|u\|_{W^{1/2,2}(\partial\Omega)} \tag{2.4}$$

for any $u \in W^{1,2}(\Omega)$, $u = 0$ on $\mathcal{D}^\varepsilon$.

Proof of Proposition 2.2. 1. Since $\partial\Omega$ is compact, it can be covered by a finite number of open sets $U^1, \dots, U^m$ such that $\partial\Omega \cap U^\ell$ is the graph of a Lipschitz function and $\Omega \cap U^\ell$ is on one side of $\partial\Omega \cap U^\ell$. To be more precise, for each $\ell \in \{1, \dots, m\}$ there exist an Euclidean coordinate system $e_1^\ell, \dots, e_n^\ell$ in $\mathbb{R}^n$, positive numbers r^ℓ, h^ℓ and a Lipschitz continuous function $\gamma^\ell : \mathbb{R}^{n-1} \to \mathbb{R}$, $\gamma^\ell(0) = 0$, such that if we denote $x = \sum_{i=1}^n x_i^\ell e_i^\ell$ for $x \in \mathbb{R}^n$,

$$U^\ell \equiv \left\{ x \in \mathbb{R}^n \mid |x_i^\ell| < r^\ell \text{ and } |x_n^\ell - \gamma^\ell(x_1^\ell, \dots, x_{n-1}^\ell)| < h^\ell \right\},$$

$$\partial\Omega \cap U^\ell \equiv \left\{ x \in \bar{\Omega} \mid x_n^\ell = \gamma^\ell(x_1^\ell, \dots, x_{n-1}^\ell) \right\}$$

and if $0 < x_n^\ell - \gamma^\ell(x_1^\ell, \dots, x_{n-1}^\ell) < h^\ell$, then $x \in \Omega$. Note next that

$$\int_{\partial\Omega} |u(x)|^2 \, dH^{n-1}(x) \le \sum_{\ell=1}^m \int_{\partial\Omega \cap U^\ell} |u(x)|^2 \, dH^{n-1}(x) \, .$$

2. Fix now $\ell \in \{1, \dots, m\}$ and denote

$$v(y') \equiv u\left(\sum_{i=1}^{n-1} y_i e_i^\ell + \gamma^\ell(y') e_n^\ell \right) \, , \; y' = (y_1, \dots, y_{n-1}) \, .$$

Then

$$\int_{\partial\Omega \cap U^\ell} |u(x)|^2 \, dH^{n-1}(x) = \int_{C'(r)} |v(y')|^2 a(y') \, dy' \, ,$$

where $C'(r) \equiv \{ y' \in \mathbb{R}^{n-1} \mid |y_i| < r \, , \; i = 1, \dots, n-1 \}$ and $a(y') = (1 + |D\gamma(y')|^2)^{1/2}$. For simplicity of notation we shall not indicate the dependence of the data on ℓ.

3. Take $C'(r)$ and divide it into k_ε^{n-1} subcubes of side length $1/k_\varepsilon$, where

$$k_\varepsilon = \left[\frac{r}{\varepsilon\rho} \right] \, .$$

Call these subcubes $Q_1, \dots, Q_{k_\varepsilon^{n-1}}$. Then

$$\int_{\partial\Omega \cap U^\ell} |u(x)|^2 \, dH^{n-1}(x) = \sum_{j=1}^{k_\varepsilon^{n-1}} \int_{Q_j} |v(y')|^2 a(y') \, dy' \, .$$

Observe now

$$\int_{Q_j} |v(y')|^2 a(y') \, dy' = \int_{Q_j} \frac{|v(y') - v(z')|^2}{|y' - z'|^n} \, |y' - z'|^n \; a(y') \, dy' \tag{2.5}$$

for any $z' \in D_j^\varepsilon$, where

$$D_j^\varepsilon \equiv \left\{ y' \in Q_j \mid x = \sum_{i=1}^{n-1} y_i e_i^\ell + \gamma^\ell(y') e_n^\ell \in \mathcal{D}^\varepsilon \right\} .$$

We claim that there exists a positive constant K, independent of j and ε, such that

$$\mathcal{L}^{n-1}(D_j^\varepsilon) \geq K(\delta\varepsilon)^{n-1} . \tag{2.6}$$

Indeed, (B) yields the the existence $x^j \in \mathcal{D}^\varepsilon$ such that

$$B(x^j, \varepsilon\delta) \cap \partial\Omega \subset \mathcal{D}_j^\varepsilon \equiv \left\{ x \in \mathcal{D}^\varepsilon \mid \quad x = \sum_{i=1}^{n-1} y_i e_i^\ell + \gamma^\ell(y') e_n^\ell \,,\ y' \in D_j^\varepsilon \right\} .$$

Then, however,

$$\begin{aligned} &\mathcal{L}^{n-1}\left(B'\left(\varepsilon\delta / \sqrt{1 + \mathrm{Lip}^2(\gamma)} \right) \right) \\ &\leq \mathcal{H}^{n-1}(\mathcal{D}_j^\varepsilon) = \int_{D_j^\varepsilon} \sqrt{1 + |D\gamma(y')|^2}\, dy' \ \leq\ C_1\, \mathcal{L}^{n-1}(D_j^\varepsilon) \end{aligned}$$

and the estimate (2.6) follows. Note that by $\mathrm{Lip}\,(\gamma)$ we mean Lipschitz constant of γ.

4. Integrate now (2.5) over D_j^ε through z' and one easily arrives at

$$\int_{Q_j} |v(y')|^2 a(y')\, dy' \leq \varepsilon\, \frac{C_2\, \rho^{n-1}}{\delta^{n-1}} \int_{D_j^\varepsilon} \int_{Q_j} \frac{|v(y') - v(z')|^2}{|y' - z'|^n}\, a(y') a(z')\, dy'\, dz' .$$

Summing on j gives

$$\begin{aligned} &\int_{\partial\Omega \cap U^\ell} |u(x)|^2\, dH^{n-1}(x) \\ &\leq C_3\, \varepsilon \int_{\partial\Omega \cap U^\ell} \int_{\partial\Omega \cap U^\ell} \frac{|u(y) - u(z)|^2}{|y - z|^n}\, dH^{n-1}(y)\, dH^{n-1}(z) , \end{aligned}$$

and finally, adding up through $k = 1, \ldots, m$ we arrive at (2.4). ■

3. Harmonic functions on the half-space

This section studies harmonic functions in the half space subject to the mixed boundary condition. We will here mostly deal with the boundary-value problem

$$\begin{aligned} &\Delta\omega = 0 && \text{in} \quad \mathbb{R}^n_+ , \\ &\omega = 0 && \text{on} \quad \mathbb{D}_0 \equiv \mathbb{D} \times \{x_n = 0\} , \\ &\partial_\nu \omega = g && \text{on} \quad \mathbb{N}_0 \equiv \mathbb{N} \times \{x_n = 0\} , \end{aligned} \tag{3.1}$$

where $g \in \mathbb{R}$ is given, $\partial_\nu \omega = -\partial\omega/\partial x_n$, $\mathbb{N} = \mathbb{R}^{n-1} \setminus \mathbb{D}$ and an open set $\mathbb{D} \subset \mathbb{R}^{n-1}$ is asymptotically dense in $\mathbb{R}^{n-1}$ in the sense of the following definition.

Definition 3.1. *A set $\mathbb{D} \subset \mathbb{R}^{n-1}$ is said to be asymptotically dense in $\mathbb{R}^{n-1}$ if there exist positive numbers δ, ρ such that any ball with radius ρ in $\mathbb{R}^{n-1}$ contains at least one ball with radius δ entirely lying in $\mathbb{D}$, i.e. for any $y' \in \mathbb{R}^{n-1}$ there exists at least one point $x' \in B(y', \rho)$ such that*

$$B(x', \delta) \subset B(y', \rho) \cap \mathbb{D}. \tag{3.2}$$

Definition 3.2. *We say $\omega \in W^{1,2}_{loc}(\mathbb{R}^n_+)$ is a weak solution of Problem (3.1) if $\omega = 0$ on $\mathbb{D}_0$ and the integral identity*

$$\int_{\mathbb{R}^n_+} \nabla\omega\nabla\psi \, dx \;=\; \int_{\mathbb{R}^{n-1}} g\,\psi\;(x', 0)\, dx' \tag{3.3}$$

holds for any $\psi \in W^{1,2}_{loc}(\mathbb{R}^n_+)$, $\psi = 0$ on $\mathbb{D}_0$, with a compact support in $\bar{\mathbb{R}}^n_+$.

Note

$$\omega = -g\, x_n$$

is a solution of Problem (3.1). We will see in a moment, that there is also a bounded solution of our problem. Thus, the difference of those solutions satisfies

$$\begin{aligned} \Delta\omega &= 0 && \text{in} \quad \mathbb{R}^n_+ \,, \\ \omega &= 0 && \text{on} \quad \mathbb{D}_0 \,, \\ \partial_\nu\omega &= 0 && \text{on} \quad \mathbb{N}_0 \,, \end{aligned} \tag{3.4}$$

and does not equal zero. The main result of this section says, however, that if a solution of (3.4) is sublinear, then it is the trivial solution.

Theorem 3.3. *Assume ω is a solution of Problem (3.4) in the sense of Definition 3.2. Suppose further ω satisfies*

$$\liminf_{R\to\infty} \fint_{B_+(R)} |\omega|\, dx \,/\, R \;=\; 0\,. \tag{3.5}$$

Then

$$\omega \;\equiv\; 0\,.$$

Proof. 1. First of all, if we define

$$u(x) \equiv \begin{cases} \omega(x) & \text{if } x_n \geq 0 \\ \omega(x', -x_n) & \text{if } x_n < 0 \end{cases},$$

then $|u(x)|$ is a weak subharmonic function, i.e. $|u| \in W^{1,2}_{loc}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n_+} \nabla|u|\, \nabla\psi\, dx \;\leq\; 0$$

for any $\psi \in W^{1,2}_{loc}(\mathbb{R}^n)$, $\psi \geq 0$, with a compact support. Therefore,

$$|u(x)| \leq \fint_{B(x,R)} |u(y)|\, dy$$

and thus,

$$\sup_{B_+(R)} |\omega| \leq 2^n \fint_{B_+(2R)} |\omega|\, dx\,. \tag{3.6}$$

Now let $\{R_i\}_{i=1}^{\infty}$, $R_i \to \infty$ as $i \to \infty$ be such that

$$\lim_{i\to\infty} \fint_{B_+(2R_i)} |\omega|\, dx\,/\,R_i \;=\; 0\,. \tag{3.7}$$

Our intention next is to deal with the problem in the half ball

$$\begin{aligned} \Delta\omega &= 0 && \text{in} \quad B_+(R)\,, \\ \omega &= 0 && \text{on} \quad \mathbb{D}_0 \cap B(R)\,, \\ \partial_\nu\omega &= 0 && \text{on} \quad \mathbb{N}_0 \cap B(R)\,, \end{aligned}$$

assuming that

$$\sup_{B_+(R)} |\omega| = M\,. \tag{3.8}$$

2. *Claim #1: There exist constants* $0 < \gamma < 1$, $0 < d < \infty$ *such that*

$$\sup_{B'\left(R-4k-\sum_{j=1}^k d_j\right)\times(0,1)} |\omega| \;\leq\; \gamma^k M \tag{3.9}$$

for

$$d_j \equiv \frac{d}{\gamma^j}$$

$j \in \{1,\ldots,k\}$, *whenever* $4k + \sum_{j=1}^k d_j < R$. *The constants* γ, d *are independent of* R, M.

Before proving *Claim #1*, let us introduce two auxiliary assertions.

3. *Claim #2: Assume*

$$|\omega| \le K$$

in $B'(L) \times (0,4)$, $4 < L \le R$. *Then*

$$|\omega| \le \beta K$$

in $B'(L-4) \times (0,1)$ *for some* $0 < \beta < 1$, *independent of* R, K *and* L.

Note that for simplicity of notation we have put $\rho = 1$ in Definition 3.1 which applies here.

Proof of Claim #2: It suffices to prove the following statement. *Let* $z = (z', 0)$ *be such that* $B(z,4) \subset B(R)$ *and*

$$\sup_{B_+(z,4)} |\omega| \le K\,.$$

Then

$$\sup_{B_+(z,1)} |\omega| \le \beta K \tag{3.10}$$

for some $0 < \beta < 1$.

As $\mathbb{D}$ is asymptotically dense, there is $y = (y', 0) \in \mathbb{D}_0$ such that $B(y', \delta) \subset \mathbb{D}$ and $B(y,3) \subset B(R)$. Moreover

$$\begin{aligned}
\Delta\omega &= 0 && \text{in} \quad B_+(y,3)\,,\\
\omega &= 0 && \text{on} \quad \mathbb{D}_0 \cap B(y,3)\,,\\
\partial_\nu \omega &= 0 && \text{on} \quad \mathbb{N}_0 \cap B(y,3)
\end{aligned}$$

and

$$\sup_{B_+(y,3)} |\omega| \le K\,.$$

If we compare ω with the solutions v_+, v_- of the problem

$$\begin{aligned}
\Delta v &= 0 && \text{in} \quad B_+(y,3)\,,\\
v &= 0 && \text{on} \quad B(y',\delta) \times \{x_n = 0\}\,,\\
\partial_\nu v &= 0 && \text{on} \quad (B(y',3) \setminus B(y',\delta)) \times \{x_n = 0\}\,,\\
v &= \pm K && \text{on} \quad \partial B_+(y,3) \cap \mathbb{R}^n_+\,,
\end{aligned}$$

we obtain

$$-\beta K \le v_- \le \omega \le v_+ \le \beta K$$

on $B_+(y,2)$ for some $0 < \beta < 1$, i.e. (3.10) holds and the assertion of *Claim #2* follows easily.

4. *Claim #3: Let ω, R, M be as above, $|\omega| \leq M$ on $B_+(R)$ and let $0 < \varepsilon < 1$, $0 < c < \infty$ and $0 < \alpha < 1$ be given. Moreover, choose a constant $\ell > 0$ such that*

$$\frac{(1-\alpha)4c}{\ell} \leq \varepsilon\,.$$

Then

$$|\omega| \leq (\alpha + \varepsilon)M \tag{3.11}$$

on $B'(L-\ell) \times (0,4)$, whenever

$$|\omega| \leq \alpha M$$

on $B'(L) \times \{x_n = 0\}$.

Proof of Claim #3: Let $z = (z', 0)$ and $\varrho > 0$. Consider first the problem

$$\begin{aligned} \Delta u_\varrho &= 0 && \text{in} \quad B_+(z, 2\varrho)\,, \\ u_\varrho &= 0 && \text{on} \quad B(z', 2\varrho) \times \{x_n = 0\}\,, \\ u_\varrho &= 1 && \text{on} \quad \partial B_+(z, 2\varrho) \cap \mathbb{R}^n_+\,. \end{aligned} \tag{3.12}$$

Then there exist positive constants c_1, c_2 (independent of ϱ) such that

$$\frac{c_1}{\varrho} \leq \frac{\partial u_\varrho}{\partial x_n} \leq \frac{c_2}{\varrho} \tag{3.13}$$

for any $x \in B_+(z, \varrho)$. This follows easily, if we first consider $\varrho = 1$ and then put $u_\varrho(x) \equiv u_1(x/\varrho)$.

Assume now that $|\omega| \leq M$ on $B_+(R)$, $|\omega| \leq \alpha M$ on $B'(L) \times \{x_n = 0\}$ and ℓ is choosen in such a way that

$$\frac{(1-\alpha)c_2 4}{\ell} \leq \varepsilon$$

for given ε. Let z in (3.12) be such that

$$|z| \leq L - 2\ell\,.$$

Then it is not difficult to see that

$$|\omega| \leq \alpha M + (1-\alpha)M u_\ell$$

in $B_+(z, 2\ell)$. Hence,

$$|\omega| \leq \alpha M + (1-\alpha)M 4c_2/\ell$$

on $B^+(z,\ell) \cap \{x \mid x_n \leq 4\}$, i.e. $|\omega| \leq (\alpha + \varepsilon)M$. As this holds for any $z = (z', 0)$, $|z| \leq R - 2\ell$, (3.11) holds on $B'(R-\ell) \times (0,4)$. *Claim #3* is proved.

5. *Proof of Claim #1:* According to *Claim #2*,

$$\sup_{B'(R-4)\times(0,1)} |\omega| \le \beta M$$

for given $0 < \beta < 1$. Put $c = c_2$ (see (3.13)) in *Claim #3* and we have

$$\sup_{B'(R-4-\ell)\times(0,4)} |\omega| \le (\beta + \varepsilon_1) M \,,$$

where $\varepsilon_1 \ell = 4c$. Choose $\varepsilon_0 > 0$ such small that

$$\gamma \equiv \beta(1+\varepsilon_0) < 1$$

and set $\varepsilon_1 = \beta\varepsilon_0$. Then

$$\sup_{B'(R-4-d_1)\times(0,4)} |\omega| \le \gamma^1 M \,, \tag{3.14}$$

for

$$d_1 = \frac{4c}{\beta\varepsilon_0} = \frac{4c(1+\varepsilon_0)}{\varepsilon_0}\frac{1}{\gamma^1} \,.$$

Now we apply *Claim #2* again, and we arrive at

$$\sup_{B'(R-8-d_1)\times(0,1)} |\omega| \le \beta\gamma M \,,$$

that due to *Claim #3* gives

$$|\omega| \le (\beta\gamma + \varepsilon_2) M$$

on $B'(R-8-d_1-d_2)\times(0,4)$. Taking $\varepsilon_2 = \gamma\beta\varepsilon_0$,

$$|\omega| \le \beta\gamma(1+\varepsilon_0) M = \gamma^2 M$$

and

$$d_2 = \frac{4c}{\beta\varepsilon_0\gamma} = \frac{4c(1+\varepsilon_0)}{\varepsilon_0}\frac{1}{\gamma^2} \,.$$

Hence, by iteration we get the required estimate (3.9) for

$$d_j \equiv \frac{d}{\gamma^j} \,, \quad d = \frac{4c(1+\varepsilon_0)}{\varepsilon_0} \,.$$

This proves *Claim #1*.

6. Finally, we are now prepare to prove Theorem 3.3.

Choose first integers k_i such that

$$\nu R_i \;\le\; 4k_i + \sum_{j=1}^{k_i} d_j \;\le\; R_i/2\,,$$

where R_i are given by (3.7). It is not difficult to see that this is possible if ν is sufficiently small, say $2\nu \le \gamma$. Moreover, note that $k_i \le \kappa \ln R_i$ for some positive constant κ. Next, *Claim #1* yields

$$\sup_{B'(R_i/2)\times(0,1)} |\omega| \;\le\; \frac{\sup_{B_+(R_i)} |\omega|}{\nu R_i - 4k_i + \text{const.}}$$

that together with (3.6) yields

$$\sup_{B'(R_i/2)\times(0,1)} |\omega| \;\le\; C \fint_{B_+(2R_i)} |\omega|\, dx \,/\, R_i$$

for some positive constant C independent of i. Making use (3.7) one easily gets $\omega \equiv 0$ on $\mathbb{R}^n_+$. ■

Theorem 3.4 (Existence of weak solution). *There exists a bounded weak solution of (3.1). Moreover, this solution is positive in $\mathbb{R}^n_+$ if $g > 0$ and $\mathbb{N}_0 \neq \emptyset$.*

Proof. The bounded solution of Problem (3.1) arises as the limit of weak solutions to the problems on bounded domains:

$$\begin{aligned} \Delta u_k &= 0 && \text{in} \quad B_+(k)\,, \\ u_k &= 0 && \text{on} \quad \mathbb{D}_0 \cap B(k)\,, \\ u_k &= 0 && \text{on} \quad S_k\,, \\ \partial_\nu u_k &= 1 && \text{on} \quad \mathbb{N}_0 \cap B(k)\,, \end{aligned} \tag{3.15}$$

as $k \longrightarrow \infty$, where $S_k \equiv \{x \in \bar{B}_+(k) \mid \; |x| = k \text{ and } x_n > 0\}$.

Note, that without loss of generality we can consider $g = 1$, as otherwise the result follows by multiplication.

Lemma 3.5. *There exists a positive constant M, independent of k, such that*

$$0 < u_k \le M \qquad \text{on} \;\; B_+(k)\,. \tag{3.16}$$

Proof of Lemma 3.5. 1. As $\mathbb{D}$ is supposed to be asymptotically dense in $\mathbb{R}^{n-1}$, the following statement follows.

Claim #1: There exists a sequence $\{z_i'\}_{i=1}^{\infty}$, $z_i' \in \mathbb{R}^{n-1}$ *such that* $B(z_i', \delta) \subset \mathbb{D}$,

$$B(z_i', 4\rho) \cap B(z_j', 4\rho) = \emptyset \ \ (i \neq j) \quad \text{and} \quad \mathbb{R}^{n-1} \subset \bigcup_{i=1}^{\infty} C(z_i', 10\rho)\,.$$

2. Next, let v be a bounded nonnegative solution of the problem

$$\begin{aligned} \Delta v &= 0 && \text{in} \quad \mathbb{R}^n_+\,, \\ v &= f && \text{on} \quad \mathbb{R}^{n-1} \times \{x_n = 0\}\,, \end{aligned} \tag{3.17}$$

where $f \in C^{\infty}(\mathbb{R}^{n-1})$, $0 \leq f \leq 1$ and

$$f(x') = \begin{cases} 0 & \text{if } x' \in \bigcup_{i=1}^{\infty} B(z_i', \delta/2)\,, \\ 1 & \text{if } x' \in \mathbb{R}^{n-1} \setminus \bigcup_{i=1}^{\infty} B(z_i', \delta)\,. \end{cases}$$

We claim that there exists a constant $0 < \beta < 1$ such that

$$0 < v(x) \leq 1 - \beta \qquad \text{on} \quad \bigcup_{i=1}^{\infty} B(z_i, 2\rho)\,, \tag{3.18}$$

where

$$z_i = (z_i', 4\rho)\,.$$

To see this, suppose the contrary and denote by $\{z_k'\}_{k=1}^{\infty}$ the sequence of those points of $\{z_i'\}_{i=1}^{\infty}$ for which

$$\max_{\bar{B}(z_k, 2r)} v(x) \longrightarrow 1 \qquad \text{as} \quad k \to \infty\,,$$

$z_k = (z_k', 4\rho)$. Write

$$Q \equiv B'(4\rho) \times (0, 6\rho)$$

and define

$$w_k(x) \equiv v(x' + z_k', x_n)$$

for $x \in Q$. So we get the sequence of harmonic functions in Q such that

$$0 \leq w_k \leq 1 \qquad \text{on} \quad \bar{Q}\,,$$

$$w_k(x_k) \longrightarrow 1 \qquad \text{as} \quad k \to \infty$$

for $x_k \in \bar{B}(a, 2\rho)$, $a = (0', 2\rho)$ and

$$\fint_{\partial Q} w_k(x)\, dH^{n-1}(x) \ \leq \ 1 - \alpha \tag{3.19}$$

for some $0 < \alpha < 1$ and each k. Hence, there exists a subsequence $\{w_{k_i}\}_{i=1}^{\infty}$ converging uniformly on $\bar{B}\,(a, 2\rho)$, to a harmonic function w and $w(x_\infty)= 1$, where $x_{k_i} \to x_\infty$ as $i \to \infty$. Then, however, $w \equiv 1$ on Q – a contradiction to (3.19), as $w_{k_i} \to w$ weakly star in $L^\infty(\partial Q)$. Indeed, note that

$$\int_{\partial Q} w_{k_i} \partial_\nu \psi \;=\; \int_Q w_{k_i} \Delta \psi$$

for any sufficiently smooth test function ψ such that $\psi = 0$ on ∂Q. We can let $i \to \infty$ to obtain

$$\int_{\partial Q} w \partial_\nu \psi \;=\; \int_Q \Delta \psi \;=\; \int_{\partial Q} \partial_\nu \psi \,.$$

Assuming, that $\partial_\nu \psi \equiv 1$ on ∂Q we obtain

$$\fint_{\partial Q} w(x)\, dH^{n-1}(x) \;=\; 1 \;,$$

that contradicts to (3.19), as we have mentioned above.

3. Furthermore, let $i \in \{1, 2, \ldots\}$ be arbitrary and denote by v_i the bounded solution of the problem

$$\begin{aligned} \Delta v_i &= 0 && \text{in} \quad \mathbb{R}^n_+ \setminus B(z_i, 2\rho)\,, \\ v_i &= 1 - \beta && \text{on} \quad \partial B(z_i, 2\rho)\,, \\ v_i &= 1 && \text{on} \quad \mathbb{R}^{n-1} \times \{x_n = 0\}\,. \end{aligned}$$

It follows from Hopf's lemma, that there exists a positive number ϑ such that

$$\partial_\nu v_i(x', 0) \geq \vartheta > 0 \tag{3.20}$$

for any $x' \in B(z_i', 10\rho)$. Owing to estimate (3.20) we prove

Claim #2: $\partial_\nu v \;\geq\; \vartheta > 0$ *on* $\mathbb{R}^{n-1} \setminus \bigcup_{i=1}^{\infty} B(z_i', \delta)$.

Proof of Claim #2: Fix $i \in \{1, 2, \ldots\}$. As

$$v \leq v_i \;\text{ and }\; v = v_i = 1 \;\text{ on }\; \big(\mathbb{R}^{n-1} \setminus B(z_i', \delta)\big) \times \{x_n = 0\}\;,$$

one easily gets

$$0 < \vartheta \leq \partial_\nu v_i \leq \partial_\nu v \quad \text{on} \quad \big(\mathbb{R}^{n-1} \setminus B(z_i', \delta)\big) \times \{x_n = 0\}\;.$$

As this holds for any i, the assertion of *Claim #2* follows.

4. Finally, set

$$U = \frac{v}{\vartheta}\,.$$

We utilize (3.17) and *Claim #2* to conclude that

$$\begin{aligned}
\Delta U &= 0 && \text{in} \quad \mathbb{R}^n_+\,,\\
U &= 0 && \text{on} \quad \textstyle\bigcup_{i=1}^\infty B(z_i',\delta/2)\times\{x_n=0\}\,,\\
U &\geq 0 && \text{on} \quad \textstyle\bigcup_{i=1}^\infty \left(B(z_i',\delta)\setminus B(z_i',\delta/2)\right)\times\{x_n=0\}\,,\\
\partial_\nu U &\geq 1 && \text{on} \quad (\mathbb{R}^{n-1}\setminus \textstyle\bigcup_{i=1}^\infty B(z_i',\delta))\times\{x_n=0\}\,.
\end{aligned}$$

Hence, 0 is a subsolution and U is a supersolution to Problem (3.15) for any k, i.e.

$$0 \leq u_k \leq U \qquad \text{on} \quad B_+(k)\,.$$

The fact, that $u_k > 0$ on $B_+(k)$ follows from the fact that harmonic function $u_k \not\equiv 0$ cannot attain its minimum in interior point of $B_+(k)$. This proves Lemma 3.5.

Finally, it follows from (3.16) that

$$\int_{B_+(K)} |\nabla u_k|^2 \;\leq\; M\,\mathcal{L}^{n-1}(B'(K))$$

for any $k \geq K$. Hence, by the diagonal process we obtain a sequence $\{u_{k_j}\}_{j=1}^\infty$ such that it converges to a bounded weak solution ω of Problem (3.1) an any compact subset of $\bar{\mathbb{R}}^n_+$. ■

4. Blow-up

Assume for this and next sections that $\Omega \subset \mathbb{R}^n$, that we have already supposed to be open, bounded and connected set, is C^2, i.e. near each point $x \in \partial\Omega$, $\partial\Omega$ is the graph of a C^2 function. For our purposes, however, it will be convenient to formulate it more specific.

Definition 4.1. *We say $\partial\Omega$ is C^2 if there is $r > 0$ such that for each point $x^0 \in \partial\Omega$, there exist an orthogonal matrix $\boldsymbol{O}$ and a C^2 mapping $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, $\gamma(0) = 0$, $D\gamma(0) = 0$ such that we have*

$$\Omega \cap B(x^0,r) = \{x \in \mathbb{R}^n \mid x = x^0 + \boldsymbol{O}\xi\,,\ \gamma(\xi') < \xi_n\} \cap B(x^0,r)\,. \tag{4.1}$$

Note that $\boldsymbol{O}$, γ depend on x^0 and we write down this dependence as $\boldsymbol{O}(x^0)$ and $\gamma(x^0,\cdot)$, if it needs to be.

Notation. Fix $x^0 \in \partial\Omega$, $0 < \varepsilon \ll 1$, and set

$$\Omega_\varepsilon(x^0) \equiv \{y \in \mathbb{R}^n \mid x = x^0 + \varepsilon \boldsymbol{O} y \in \Omega\},$$

$$\partial\Omega_\varepsilon(x^0) \equiv \{y \in \mathbb{R}^n \mid x = x^0 + \varepsilon \boldsymbol{O} y \in \partial\Omega\},$$

$$\mathcal{D}_\varepsilon(x^0) \equiv \{y \in \mathbb{R}^n \mid x = x^0 + \varepsilon \boldsymbol{O} y \in \mathcal{D}^\varepsilon \subset \partial\Omega\}.$$

Moreover, let χ_ε be the characteristic function of the set $\mathcal{D}_\varepsilon(x^0)$, i.e. $\chi_\varepsilon(x^0, \cdot) : \partial\Omega_\varepsilon(x^0) \to \{0, 1\}$ such that

$$\chi_\varepsilon(x^0, y) = 1 \quad \text{iff} \quad y \in \mathcal{D}_\varepsilon(x^0). \tag{4.2}$$

Note that whatever a positive constant L is, $\Omega_\varepsilon(x^0) \cap B(L)$ approximately equals the half ball $B_+(L)$ for small enough $\varepsilon > 0$. Recall that the small parameter ε shall finally tend to zero and therefore $L \in \mathbb{R}_+$ can be taken arbitrarily large.

We henceforth assume the following structure assumptions concerning the set $\mathcal{D}_\varepsilon(x^0)$, or equivalently, its characteristic function χ_ε.

Hypotheses (H). *There exists a relatively closed set $\Gamma \subset \partial\Omega$ such that $\mathcal{H}^{n-1}(\Gamma) = 0$ and for any $x^0 \in \mathcal{G} \equiv \partial\Omega \setminus \Gamma$ there exist a function*

$$\chi : \mathbb{R}^{n-1} \longrightarrow \{0, 1\},$$

a vector $\tau_\varepsilon \equiv (\tau'_\varepsilon, 0) \in \mathbb{R}^n$ and a Lipschitz one–to–one map

$$\Phi_\varepsilon : \bar{\Omega}_\varepsilon(x^0) \cap B(L) \longrightarrow \bar{\mathbb{R}}^n_+, \quad \Phi_\varepsilon(0) = 0$$

with the following properties:

(i) *The set $\mathbb{D} \equiv \{y' \in \mathbb{R}^{n-1} \mid \chi(y') = 1\}$ is asymptotically dense in $\mathbb{R}^{n-1}$.*

(ii) *Vectors τ'_ε are uniformly bounded with respect to ε, i.e. $|\tau'_\varepsilon| \le m$ for some given positive constant m independent of ε.*

(iii)

$$\Phi_\varepsilon : \partial\Omega_\varepsilon(x^0) \cap B(L) \longrightarrow \mathbb{R}^{n-1} \times \{z_n = 0\};$$

(iv)

$$\operatorname{ess\,sup}_{z \in \mathcal{R}_\varepsilon(L)} \|D\Phi_\varepsilon(\Psi_\varepsilon(z)) - \boldsymbol{I}\| \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0 \tag{4.3}$$

uniformly in $\mathcal{G}$, where

$$\Psi_\varepsilon \equiv \Phi_\varepsilon^{-1}, \quad \mathcal{R}_\varepsilon(L) \equiv \{z \in \bar{\mathbb{R}}^n_+ \mid z = \Phi_\varepsilon(y),\, y \in \bar{\Omega}_\varepsilon(x^0) \cap \bar{B}(L)\}$$

and $\|\cdot\|$ denotes the matrix norm, and finally,

(v) *for each* $\kappa \in \mathbb{Z}_+$ *there exists a relatively closed set* $\Gamma_\kappa \subset \partial\Omega$ *such that*

$$\mathcal{H}^{n-1}(\Gamma_\kappa) \le 1/\kappa\,, \quad \Gamma_{\kappa+1} \subset \Gamma_\kappa\,, \quad \Gamma = \cap_{\kappa=1}^{\infty}\Gamma_\kappa$$

and

$$\int_{B_+(R)} |\nabla H_\varepsilon|^2 \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0 \tag{4.4}$$

uniformly in

$$x^0 \in \mathcal{G}^\kappa \equiv \partial\Omega \setminus \Gamma_\kappa$$

for any fixed κ. *Here* $R < L$ *and* $H_\varepsilon \in W^{1,2}(B_+(R))$ *is such that*

$$\begin{aligned}
\Delta H_\varepsilon &= 0 && \text{in} \quad U_R \equiv B_+(R) \cap \{z \mid 0 < z_n < 1\}\,,\\
H_\varepsilon &= 0 && \text{on} \quad B_+(R) \setminus U_R\,,\\
H_\varepsilon &= 1 && \text{on} \quad \mathcal{P}_\varepsilon(R)\,,\\
\partial_\nu H_\varepsilon &= 0 && \text{on} \quad S_R\,,
\end{aligned}$$

where

$$\mathcal{P}_\varepsilon(R) \equiv \{z \in B'(R) \times \{z_n = 0\} \mid \quad |\chi_\varepsilon(\Psi_\varepsilon(z)) - \chi(z' + \tau'_\varepsilon)| = 1\}$$

and

$$S_R \equiv \partial U_R \setminus \left\{z \in \bar{B}_+(R) \mid z_n = 1\right\} \cup \mathcal{P}_\varepsilon(R)\,.$$

Notation. As the quantities χ, τ_ε, Φ_ε, Ψ_ε, $\mathcal{P}_\varepsilon$, $\mathcal{R}_\varepsilon$ depend on x^0, we shall henceforth indicate this dependence writing $\chi(x^0,\cdot)$, $\tau_\varepsilon(x^0)$, $\Phi_\varepsilon(x^0,\cdot)$, $\Psi_\varepsilon(x^0,\cdot)$, $\mathcal{P}_\varepsilon(x^0,\cdot)$ and $\mathcal{R}_\varepsilon(x^0,\cdot)$. For fix $L \in \mathbb{R}_+$, $0 < \varepsilon < 1$ we define

$$\ell_\varepsilon(L) \equiv \operatorname{ess}\sup_{x^0 \in \partial\Omega} \left(\operatorname{ess} \sup_{z \in \mathcal{R}_\varepsilon(x^0,L)} \left\| D\Phi_\varepsilon(x^0, \Psi_\varepsilon(x^0,z)) - \boldsymbol{I} \right\| \right)\,.$$

Remarks.
(1) Note that for each $L \in \mathbb{R}_+$, *(H)(iv) yields*

$$\lim_{\varepsilon \to 0} \ell_\varepsilon(L) \;=\; 0 \tag{4.5}$$

and

$$\bar{B}_+(L(1-\ell_\varepsilon)) \subset \mathcal{R}_\varepsilon(x^0,L) \subset \bar{B}_+(L(1+\ell_\varepsilon))$$

for any $x^0 \in \partial\Omega$. Indeed, according to *(iv)*

$$\Phi_\varepsilon(y) = y + \int_0^1 (D\Phi_\varepsilon(sy) - \boldsymbol{I})\, ds\, y$$

for any $y \in \bar{\Omega}_\varepsilon(x^0) \cap B(L)$ such that $sy \in \bar{\Omega}_\varepsilon(x^0)$ for $s \in [0,1]$, that together with the Inverse Function Theorem give *(1)*.
(2) The hypothesis (v) is satisfied trivially if

$$\chi_\varepsilon\left(x^0, \Psi_\varepsilon(x^0, z)\right) = \chi(x^0, z' + \tau'_\varepsilon(x^0))$$

for any $z \in \mathcal{R}_\varepsilon(x^0, L) \cap \{z \mid z_n = 0\}$.

Now we are ready to read off more detailed information concerning the weak limit of

$$W_\varepsilon \equiv \frac{u_\varepsilon - u}{\varepsilon}$$

on $\partial\Omega_T$ making use the blow-up of Ω around a point $x^0 \in \partial\Omega$ (see [5, page 198]). Recall first that W_ε is the solution of

$$\begin{aligned} \partial_t W_\varepsilon &= \Delta W_\varepsilon + F_\varepsilon(x,t) && \text{in} \quad \Omega_T\,, \\ \partial_\upsilon W_\varepsilon &= \varepsilon^{-1} g_\varepsilon(x,t) && \text{on} \quad \mathcal{N}_T^\varepsilon\,, \\ W_\varepsilon &= 0 && \text{on} \quad \mathcal{D}_T^\varepsilon\,, \\ W_\varepsilon &= \mathcal{P}_\varepsilon && \text{on} \quad \Omega \times \{t = 0\}\,, \end{aligned} \tag{4.6}$$

with

$$g_\varepsilon = \vartheta_\varepsilon - \sigma_\varepsilon u_\varepsilon - \partial_\upsilon u$$

in the following sense:

$$\begin{aligned} &\int_\Omega W_\varepsilon(\tau)\varphi_\varepsilon(\tau)\,dx - \int_0^\tau \int_\Omega \left(W_\varepsilon \partial_t \varphi_\varepsilon - \nabla W_\varepsilon \nabla \varphi_\varepsilon + F_\varepsilon \varphi_\varepsilon\right)\,dx\,dt \\ &\qquad = \int_\Omega \mathcal{P}_\varepsilon \varphi_\varepsilon(0)\,dx + \varepsilon^{-1} \int_0^\tau \int_{\partial\Omega} g_\varepsilon \varphi_\varepsilon\,dH^{n-1}(x)\,dt \end{aligned} \tag{4.7}$$

holds for any $\varphi_\varepsilon \in W^{1,2}(\Omega_T)$, $\varphi_\varepsilon = 0$ on $\mathcal{D}_T^\varepsilon$.

Next fix $x^0 \in \mathcal{G}$ and set

$$w_\varepsilon(x^0, y, t) \equiv W_\varepsilon(x^0 + \varepsilon \boldsymbol{O}(x^0) y, t) \tag{4.8}$$

for $y \in \Omega_\varepsilon(x^0)$, $t \in (0,T)$, and similarly $\eta_\varepsilon(x^0, y, t) \equiv \varphi_\varepsilon(x^0 + \varepsilon \boldsymbol{O}(x^0) y, t)$. Moreover, take a test function φ_ε in (4.7) such that

$$\operatorname{spt} \varphi_\varepsilon \subset B(x^0, \varepsilon L)\,.$$

(4.7) then for $x = x^0 + \varepsilon \boldsymbol{O}(x^0)y$ yields

$$\begin{aligned} &\mathcal{O}_\varepsilon(x^0,s) + \int_0^s \int_{\Omega_\varepsilon(x^0)\cap B(L)} \nabla w_\varepsilon \nabla \eta_\varepsilon \, dy\, dt \\ &= \int_0^s \int_{\partial\Omega_\varepsilon(x^0)\cap B(L)} g_\varepsilon(x^0+\varepsilon \boldsymbol{O} y, t)\eta_\varepsilon(x^0,y,t)\, dH^{n-1}(y)\, dt\,, \end{aligned} \tag{4.9}$$

where ∇ means here the gradient in y variable and

$$\begin{aligned} \mathcal{O}_\varepsilon(x^0,s) \equiv \varepsilon^2 &\int_{\Omega_\varepsilon(x^0)\cap B(L)} \big(w_\varepsilon \eta_\varepsilon(s) - \mathcal{P}_\varepsilon(x^0+\varepsilon \boldsymbol{O} y)\eta_\varepsilon(0)\big)\, dy \\ - \varepsilon^2 &\int_0^s \int_{\Omega_\varepsilon(x^0)\cap B(L)} \big(w_\varepsilon \partial_t \eta_\varepsilon + F_\varepsilon(x^0+\varepsilon \boldsymbol{O} y, t)\eta_\varepsilon\big)\, dy\, dt\,. \end{aligned}$$

Later on it will be not difficult to show that

$$\int_{\partial\Omega} \mathcal{O}_\varepsilon(x^0,s)\, dH^{n-1}(x^0) \longrightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0 \tag{4.10}$$

for any $s \in (0,T)$ and therefore let us omit $\mathcal{O}_\varepsilon$ in further considerations.

Next we change variables in (4.9) utilizing **Hypotheses (H)** above and write

$$z = \Phi_\varepsilon(x^0,y)\,, \quad i.e. \quad y = \Psi_\varepsilon(x^0,z)\,. \tag{4.11}$$

Then, after a little tedious, but not difficult manipulations we arrive at

$$\begin{aligned} &\int_0^s \int_{\mathcal{G}^\kappa} \int_{\mathcal{R}_\varepsilon(L)} \sum_{l,k=1}^n a_\varepsilon^{kl} \frac{\partial V_\varepsilon}{\partial z_k} \frac{\partial \xi_\varepsilon}{\partial z_l}\, dz\, dH^{n-1}(x^0)\, dt \\ &= \int_0^s \int_{\mathcal{G}^\kappa} \int_{\partial\mathcal{R}_\varepsilon(L)} G_\varepsilon(z')\, \eta(\hat{\Psi}'_\varepsilon(z'),t) \\ &\qquad \sqrt{1+|D\zeta_\varepsilon(\Psi'_\varepsilon(z'))|^2}\, J\Psi'_\varepsilon(z')\, dz'\, dH^{n-1}(x^0)\, dt\,, \end{aligned} \tag{4.12}$$

where

$$a_\varepsilon^{kl} \equiv J\Psi_\varepsilon(z) \sum_{i=1}^n \frac{\partial \Phi_\varepsilon^k}{\partial y_i}(\Psi_\varepsilon(z)) \frac{\partial \Phi_\varepsilon^l}{\partial y_i}(\Psi_\varepsilon(z))\,,$$

$$V_\varepsilon(x^0,z,t) \equiv w_\varepsilon(x^0,\Psi_\varepsilon(x^0,z),t)\,, \quad \xi_\varepsilon(x^0,z,t) \equiv \eta_\varepsilon(x^0,\Psi_\varepsilon(x^0,z),t)\,,$$

$$\hat{\Psi}'_\varepsilon \equiv (\Psi'_\varepsilon, \zeta_\varepsilon(\Psi'_\varepsilon))\,, \quad \Psi'_\varepsilon(z') \equiv (\Psi_\varepsilon^1(z',0),\ldots,\Psi_\varepsilon^{n-1}(z',0))\,,$$

$$\zeta_\varepsilon(y') \equiv \frac{1}{\varepsilon}\gamma(\varepsilon y')\,, \quad G_\varepsilon \equiv g_\varepsilon(x^0+\varepsilon \boldsymbol{O}(x^0)\hat{\Psi}'_\varepsilon(z'),t)\,,$$

$\partial\mathcal{R}_\varepsilon(L) \equiv \Phi_\varepsilon(\partial\Omega_\varepsilon(x^0) \cap B(L))$ and κ being arbitrarily large.

Now let $\theta(x^0, z, t)$ be a sufficiently smooth function, say $\theta \in L^2(\partial\Omega_T; W^{1,2}_{loc}(\mathbb{R}^n_+))$, such that

$$\theta(x^0, z, t) = 0 \quad \text{if} \quad z \in \mathbb{D}(x^0)$$

and θ has a compact support lying in $\bar{B}_+(L-m)$, i.e.

$$\operatorname{spt}\theta \subset \bar{B}_+(L-m)\,.$$

Note that $\bar{B}_+(L-m) \subset \mathcal{R}_\varepsilon(L)$. Then

$$\varphi_\varepsilon(x,t) \equiv \theta_\varepsilon(x^0, \Psi_\varepsilon(x^0, \boldsymbol{O}^{-1}(x^0)(\frac{x-x^0}{\varepsilon})) + \tau_\varepsilon, t)\,,$$

where

$$\theta_\varepsilon(x^0, z, t) \equiv \theta(x^0, z, t)\left(1 - H_\varepsilon(x^0, z)\right)$$

is admissible test function in (4.7), satisfying spt $\varphi_\varepsilon \subset \bar{B}(x^0, \varepsilon L)$, and corresponding test function in (4.12) has then the form

$$\xi_\varepsilon(x^0, z, t) = \theta(x^0, z + \tau_\varepsilon, t)\left(1 - H_\varepsilon(x^0, z + \tau_\varepsilon)\right)$$

and one easily obtains

$$\begin{aligned}
&\int_0^s \int_{\mathcal{G}^\kappa} \int_{B_+(L-m)} \sum_{l,k=1}^n a^{kl}_\varepsilon(z-\tau_\varepsilon)\, \frac{\partial v_\varepsilon}{\partial z_k}\frac{\partial \theta_\varepsilon}{\partial z_l}\,(x^0, z, t)\, dz\, dH^{n-1}(x^0)\, dt \\
&= \int_0^s \int_{\mathcal{G}^\kappa} \int_{B'(L-m)} G_\varepsilon(z' - \tau'_\varepsilon)\, \theta_\varepsilon(x^0, z', 0, t) \\
&\qquad \sqrt{1 + |D\zeta_\varepsilon(\Psi'_\varepsilon(z' - \tau'_\varepsilon))|^2}\, J\Psi'_\varepsilon(z' - \tau'_\varepsilon)\, dz'\, dH^{n-1}(x^0)\, dt\,,
\end{aligned} \tag{4.13}$$

where

$$v_\varepsilon(x^0, z, t) \equiv V_\varepsilon(x^0, z - \tau_\varepsilon, t)\ .$$

If we let $\varepsilon \longrightarrow 0$, we arrive at

$$\begin{aligned}
&\int_0^s \int_{\mathcal{G}^\kappa} \int_{B_+(L-m)} \nabla v\, \nabla\theta\ (x^0, z, t)\, dz\, dH^{n-1}(x^0)\, dt \\
&= \int_0^s \int_{\mathcal{G}^\kappa} \int_{B'(L-m)} g(x^0, t)\, \theta(x^0, z', 0, t)\, dz'\, dH^{n-1}(x^0)\, dt\,,
\end{aligned} \tag{4.14}$$

where

$$g(x,t) = \vartheta(x,t) - \partial_\nu u(x,t)$$

and where v is a weak limit of $\{v_\varepsilon\}$. As κ was choosen arbitrarily large, we can conclude that (4.14) holds also for $\mathcal{G}$ in place of $\mathcal{G}^\kappa$, i.e. for $\partial\Omega$.

This step will be verified in the next section. Now let us continue our consideration simply assuming that (4.14) holds for any sufficiently smooth test function θ with compact support (as the function of z) and such that $\theta = 0$ on $\mathbb{D}$, and that the function v satisfies the following a priori estimates:

$$\int_{\partial\Omega_T}\int_{B'(R)\times(0,R)}(1+z_n)|\nabla v|^2\,(x^0,z,t)\,dz\,dH^{n-1}(x^0)\,dt \le CR^{n-1}\,,$$
$$\int_{\partial\Omega_T}\int_{B'(R)}|v|^2\,(x^0,z',0,t)\,dz'\,dH^{n-1}(x^0)\,dt \le CR^{n-1}, \tag{4.15}$$

for any positive numbers R greater than some given positive number R_0 and some positive constant C independent of R.

We identify now the function v determined by (4.14) and fulfilling (4.15).

Theorem 4.2. *Assume $v \in L^2(\partial\Omega_T; W^{1,2}_{loc}(\mathbb{R}^n_+))$, $v(x^0,z,t) = 0$ if*

$$z \in \mathbb{D}_0(x^0) \equiv \{z = (z',0) \in \mathbb{R}^n \mid \chi(x^0,z') = 1\}$$

for a.e. $(x^0,t) \in \partial\Omega_T$, and such that (4.14), (4.15) hold. Then

$$v(x^0,z,t) \equiv g(x^0,t)\,\omega(x^0,z)\,, \tag{4.16}$$

where ω is the only bounded weak solution of the problem

$$\begin{aligned} \Delta_z\omega &= 0 && \text{in } \mathbb{R}^n_+\,,\\ \omega &= 0 && \text{on } \mathbb{D}_0\,,\\ \partial_\nu\omega &= 1 && \text{on } \mathbb{N}_0\,, \end{aligned} \tag{4.17}$$

where $\mathbb{N} \equiv \mathbb{R}^{n-1} \setminus \mathbb{D}$.

Proof. 1. Using Fatou's Lemma and (4.15) we obtain

$$\int_{\partial\Omega_T}\liminf_{R\to\infty} R^{1-n}\left\{\int_{B'(R)\times(0,R)}|\nabla v|^2(1+z_n)\,dz + \int_{B'(R)}|v|^2(z',0)\,dz'\right\}dH^{n-1}(x^0)\,dt \le C\,,$$

i.e. there exists a sequence $\{R_i\}_{i=1}^\infty$, $R_i \to \infty$ such that

$$R_i^{1-n}\left\{\int_{B'(R_i)\times(0,R_i)}|\nabla v|^2(1+z_n)\,dz + \int_{B'(R_i)}|v|^2(z',0)\,dz'\right\} \le F(x^0,t) < \infty \tag{4.18}$$

for a.e. $(x^0,t)\in\partial\Omega_T$, where $F\in L^1(\partial\Omega_T)$. Moreover, for a.e. $(x^0,t)\in\partial\Omega_T$, (4.14) yields

$$\int_{\mathbb{R}^n_+}\nabla v\nabla\psi\,dz=\int_{\mathbb{R}^{n-1}}g(x^0,t)\psi(z',0)\,dz'$$

for any test function $\psi\in W^{1,2}_{loc}(\mathbb{R}^n_+)$ with compact support and such that $\psi=0$ on $\mathbb{D}_0(x^0)$, i.e. v is a weak solution of

$$\begin{aligned}\Delta_z v&=0 && \text{in } \mathbb{R}^n_+\,,\\ v&=0 && \text{on } \mathbb{D}_0\,,\\ \partial_\nu v&=g(x^0,t) && \text{on } \mathbb{N}_0\,.\end{aligned}\tag{4.19}$$

2. *Claim #1: There exists positive constant $C=C(x^0,t)$, independent of $R\geq\max\{1,m\}$, such that*

$$\fint_{B'(R)\times(0,R)}|v(z)|\,dz\;\leq C\left(1+(\ln R)^{1/2}\right)\,.$$

Proof of Claim #1: Note that for $1\leq z_n\leq R$ we have

$$|v(z',z_n)|\;-\;|v(z',1)|\leq\left(\int_1^R\sigma\left|\frac{\partial v}{\partial z_n}(z',\sigma)\right|^2d\sigma\right)^{1/2}(\ln R)^{1/2}$$

by Hölder inequality. Next integrate with respect to z' over $B'(R)$ and with respect to z_n over $(1,R)$ to find

$$\begin{aligned}\fint_{B'(R)\times(1,R)}|v(z)|\,dz\;&\leq\fint_{B'(R)}|v(z',1)|\,dz'\\ &+\frac{1}{\alpha(n-1)}\left(\frac{1}{R^{n-1}}\int_{B'(R)\times(1,R)}z_n\left|\frac{\partial v}{\partial z_n}(z',z_n)\right|^2dz\right)^{1/2}(\ln R)^{1/2}\,.\end{aligned}$$

To finish the proof we need only observe

$$\begin{aligned}\fint_{B'(R)}|v(z',z_n)|\,dz'\leq&\fint_{B'(R)}|v(z',0)|\,dz'\\ &+\frac{1}{\alpha(n-1)R^{n-1}}\int_{B'(R)\times(0,1)}\left|\frac{\partial v}{\partial z_n}(z)\right|dz\end{aligned}$$

for any $0\leq z_n\leq 1$. Indeed, this and (4.18) yield

$$\fint_{B'(R_i)\times(0,R_i)}|v(z)|\,dz\;\leq\;4\sqrt{\frac{F(x^0,t)}{\alpha(n-1)}}\left(1+\sqrt{\ln R_i}\right)\,.$$

This proves *Claim # 1.*

3. We know already that there is a bounded weak solution $v_b \equiv g(x^0,t)\omega$ of Problem (4.19) (see Theorem 3.4), i.e. $V \equiv v - v_b$ satisfies

$$\begin{aligned} \Delta V &= 0 && \text{in } \mathbb{R}^n_+ \,, \\ V &= 0 && \text{on } \mathbb{D}_0 \,, \\ \partial_\nu V &= 0 && \text{on } \mathbb{N}_0 \,, \end{aligned}$$

and

$$\fint_{B^+(R_i)} |V(z)|\, dz \le M \left(1 + \sqrt{\ln R_i}\right)$$

for any i. Hence, Theorem 3.3 in Section 3 implies $V \equiv 0$. This proves Theorem 4.2. ■

5. A priori estimates

We now ascertain the a priori estimates for v_ε ,

$$v_\varepsilon(x^0,z,t) = \frac{u_\varepsilon - u}{\varepsilon}(\Lambda_\varepsilon(x^0,z),t)\,,$$

where

$$\Lambda_\varepsilon(x^0,z) \equiv x^0 + \varepsilon \boldsymbol{O}(x^0)\Psi_\varepsilon(x^0, z - \tau_\varepsilon(x^0))\,,$$

announced in the previous section. Note that for simplicity of notation we shall not always indicate the dependence of $\Lambda_\varepsilon, \boldsymbol{O}, \Psi_\varepsilon, v_\varepsilon$ on x^0.

Theorem 5.1. *Let (A)-(H) be satisfied. Then for any $R \in (m,\infty)$ there exists a positive $\varepsilon_0 = \varepsilon_0(R)$ such that for any $\varepsilon \in (0,\varepsilon_0)$ the following estimates hold:*

$$I_\varepsilon \equiv \int_{\partial\Omega_T} \int_{\mathcal{B}(R)} |\nabla v_\varepsilon|^2 (1+z_n)\, dz\, dH^{n-1}(x^0)\, dt \le C\, R^{n-1}, \tag{5.1}$$

where ∇ is the gradient in z variable,

$$\mathcal{B}(R) \equiv \bar{B}'(R) \times [0,R]\,,$$

and

$$J_\varepsilon \equiv \int_{\partial\Omega_T} \int_{B'(R)} |v_\varepsilon(z',0)|^2\, dz'\, dH^{n-1}(x^0)\, dt \le C\, R^{n-1}\,, \tag{5.2}$$

wherein the positive constant C does not depend on ε, R.

As a consequence of Theorem 5.1 we have

Corollary 5.2. *There exists a function $v \in L^2(\partial\Omega_T; W^{1,2}_{loc}(\mathbb{R}^n_+))$ and a subsequence of $\{\varepsilon\}$, denoting for simplicity again by ε, such that*

$$v_\varepsilon \rightharpoonup v \quad in \quad L^2(\partial\Omega_T; W^{1,2}(Q)) \quad as \quad \varepsilon \downarrow 0$$

for any relatively compact set $Q \subset \mathbb{R}^n_+$. Moreover, the estimates (4.15) and the integral identity (4.14) hold.

Proof of Theorem 5.1. 1. Note first that

$$\frac{\partial v_\varepsilon}{\partial z_i} = \sum_{j=1}^n a^{ij}_\varepsilon \left(\frac{\partial u_\varepsilon}{\partial x_j} - \frac{\partial u}{\partial x_j} \right)$$

for $i = 1, \dots, n$, where

$$a^{ij}_\varepsilon \equiv \sum_{k=1}^n o_{jk}(x^0) \frac{\partial \Psi^k_\varepsilon}{\partial z_i} (x^0, z - \tau_\varepsilon) .$$

Then

$$I_\varepsilon \le c \int_{\partial\Omega_T} \int_{\mathcal{B}(R)} (1 + z_n) |\nabla(u_\varepsilon - u)|^2 (\Lambda_\varepsilon(x^0, z), t) \, dz \, dH^{n-1}(x^0) \, dt .$$

Put now

$$x = \Lambda_\varepsilon(x^0, z) , \ i.e. \ \ z = \Lambda^{-1}_\varepsilon(x^0, x)$$

and we get

$$I_\varepsilon \le$$
$$C_1 \int_{\partial\Omega_T} \int_{\Lambda_\varepsilon(x^0,\mathcal{B})} |\nabla(u_\varepsilon - u)|^2 (x,t)(1 + \varepsilon^{-1}\phi(x)) \, J\Lambda^{-1}_\varepsilon(x) \, dx \, dH^{n-1}(x^0) \, dt$$
$$\le C_2 \, \varepsilon^{-n} \int_{\partial\Omega_T} \int_{\Lambda_\varepsilon(x^0,\mathcal{B})} |\nabla(u_\varepsilon - u)|^2 (1 + \varepsilon^{-1}\phi) \, dx \, dH^{n-1}(x^0) \, dt ,$$

where

$$\Lambda_\varepsilon(x^0, \mathcal{B}(R)) \equiv \{x \in \bar{\Omega} \mid \ x = \Lambda_\varepsilon(x^0, z) , \ z \in \mathcal{B}(R)\} .$$

Let $\chi^\varepsilon_{\mathcal{B}}(x^0, x)$ be the indicator function of the set $\Lambda_\varepsilon(x^0, \mathcal{B}(R))$, i.e.

$$\chi^\varepsilon_{\mathcal{B}}(x^0, x) = \begin{cases} 1 & : \quad x \in \Lambda_\varepsilon(x^0, \mathcal{B}(R)) \\ 0 & : \quad x \in \bar{\Omega} \setminus \Lambda_\varepsilon(x^0, \mathcal{B}(R)) \end{cases} .$$

Then the above inequality can be rewritten as follows:

$$I_\varepsilon \le C_3 \, \varepsilon^{-n} \int_{\partial\Omega_T} \int_\Omega |\nabla(u_\varepsilon - u)|^2 (1 + \varepsilon^{-1}\phi) \chi^\varepsilon_{\mathcal{B}}(x^0, x) \, dx \, dH^{n-1}(x^0) \, dt$$

$$= C_3\,\varepsilon^{-n}\int_{\Omega_T}|\nabla(u_\varepsilon-u)|^2(1+\varepsilon^{-1}\phi)\int_{\partial\Omega}\chi^\varepsilon_{\mathcal{B}}(x^0,x)\,dH^{n-1}(x^0)\,dx\,dt$$

by Fubini's theorem. It remains to estimate the integral

$$\int_{\partial\Omega}\chi^\varepsilon_{\mathcal{B}}(x^0,x)\,dH^{n-1}(x^0)\quad\text{for fixed}\quad x\in\bar\Omega\,.$$

2. *Claim #1:* There exists a positive constant M, independent of x,ε,R and a positive constant $\varepsilon_0=\varepsilon_0(R)$ such that

$$\int_{\partial\Omega}\chi^\varepsilon_{\mathcal{B}}(x^0,x)\,dH^{n-1}(x^0)\le M(R+m)^{n-1}\varepsilon^{n-1}\,.$$

for any $\varepsilon\in(0,\varepsilon_0]$.

Proof of Claim #1: Note first that $\Phi_\varepsilon(\Psi_\varepsilon(z))=z$ and that $D\Phi_\varepsilon(\Psi_\varepsilon(z))$ $D\Psi_\varepsilon(z)=\boldsymbol{I}$ for $\mathcal{L}^n$ a.e. $z\in\mathcal{R}_\varepsilon(L)$, where we take $L\gg R+m$ and for a moment we do not indicate the dependence of the quantities on x^0. Next, $D\Phi_\varepsilon(\Psi_\varepsilon(z))=\boldsymbol{I}+\boldsymbol{E}_\varepsilon(z)$ for "small" $\boldsymbol{E}_\varepsilon$ (see *Remark (1)* above) and

$$D\Psi_\varepsilon(z)=\boldsymbol{I}+\boldsymbol{F}_\varepsilon(z),\ \boldsymbol{F}_\varepsilon(z)\equiv\boldsymbol{E}_\varepsilon(z)\,(D\Phi_\varepsilon)^{-1}\,(\Psi_\varepsilon(z))\,.$$

Denoting

$$a_\varepsilon(L)\equiv\operatorname{ess}\sup_{x^0\in\partial\Omega}\left(\operatorname{ess}\sup_{z\in\mathcal{R}_\varepsilon(x^0,L)}\left\|\boldsymbol{F}_\varepsilon(x^0,z)\right\|\right)\,,$$
$$\lim_{\varepsilon\to0}a_\varepsilon(L)\;=\;0\,. \tag{5.3}$$

Hence,

$$\Psi_\varepsilon(x^0,z)=\left(\boldsymbol{I}+\int_0^1\boldsymbol{F}_\varepsilon(x^0,sz)\,ds\right)\,z \tag{5.4}$$

for any $z\in B_+(L(1-\ell_\varepsilon(L)))$. Observe now that for fixed $x\in\bar\Omega$, $\chi^\varepsilon_{\mathcal{B}}(x^0,x)$ could equal 1 only for those $x^0\in\partial\Omega$ for which

$$x=x^0+\varepsilon\boldsymbol{O}(x^0)\Psi_\varepsilon(x^0,z-\tau_\varepsilon(x^0))$$

for some $z\in\mathcal{B}(R)$. Hence, due to (5.4), $\chi^\varepsilon_{\mathcal{B}}(x^0,x)$ could be 1 only if

$$|x-x^0|\le\varepsilon(\sqrt{2}\,R+m)(1+a_\varepsilon)\,.$$

Then, however,

$$\int_{\partial\Omega}\chi^\varepsilon_{\mathcal{B}}(x^0,x)\,dH^{n-1}(x^0)\;\le\;\mathcal{H}^{n-1}\left(\partial\Omega\cap B(x,\varepsilon L_\varepsilon)\right)$$
$$\le\int_{B'(\varepsilon L_\varepsilon)}\sqrt{1+|D\gamma\,(x,y')|^2}\;dy'\;\le\;(1+c_\varepsilon(L))\,\alpha(n-1)\,\varepsilon^{n-1}L_\varepsilon^{n-1}\,,$$

where $L_\varepsilon \equiv (\sqrt{2}R + m)(1 + a_\varepsilon)$ and

$$c_\varepsilon(L) \equiv \operatorname{ess\,sup}_{x\in\partial\Omega} \left(\operatorname{ess} \sup_{y'\in B'(\varepsilon L)} |D\gamma(x, y')|^2 \right) .$$

As $\gamma \in C^2$ and $D\gamma(x, 0) = 0$,

$$\lim_{\varepsilon\to 0} c_\varepsilon(L) \;=\; 0 \tag{5.5}$$

and the required assertion follows.

3. To derive estimate (5.2) we shall proceed similarly as above. Note first

$$J_\varepsilon = \int_{\partial\Omega_T} \int_{B'(R)} \left| \frac{u_\varepsilon - u}{\varepsilon} \left(x^0 + \varepsilon \boldsymbol{O}(x^0) \hat{\Psi}'_\varepsilon(z' - \tau'_\varepsilon, t \right) \right|^2 dz'\, dH^{n-1}(x^0)\, dt$$

where

$$\varepsilon\hat{\Psi}'_\varepsilon(z' - \tau'_\varepsilon) = (\varepsilon\Psi'_\varepsilon(x^0, z' - \tau'_\varepsilon, 0), \gamma(\varepsilon\Psi'_\varepsilon(x^0, z' - \tau'_\varepsilon, 0))).$$

Put now

$$y' = \varepsilon\Psi'_\varepsilon(z' - \tau'_\varepsilon, 0)\,, \text{ i.e. } \quad z' = \tau'_\varepsilon + \Phi'_\varepsilon \left(x^0, \varepsilon^{-1}y', \gamma\left(\varepsilon^{-1}y'\right)\right)$$

and we arrive, after some computations, at

$$J_\varepsilon \le C_1 \varepsilon^{1-n} \int_{\partial\Omega_T} \int_{\partial\Omega} \left| \frac{u_\varepsilon - u}{\varepsilon} \right|^2 (x, t)\, \chi^\varepsilon_U(x^0, x)\, dH^{n-1}(x)\, dH^{n-1}(x^0)\, dt\ ,$$

where

$$\chi^\varepsilon_U(x^0, x) = \begin{cases} 1 & : \quad x \in U(x^0, \varepsilon R) \\ 0 & : \quad x \in \partial\Omega \setminus U(x^0, \varepsilon R) \end{cases}$$

and

$$U(x^0, \varepsilon R) \equiv \left\{ x \in \partial\Omega \mid \ x = \Lambda_\varepsilon(x^0, z) \ \text{ for } \ z \in B'(R) \times \{z_n = 0\} \right\} . \tag{5.6}$$

Thus, by Fubini's theorem,

$$J_\varepsilon \le C_1 \varepsilon^{1-n} \int_{\partial\Omega_T} \left| \frac{u_\varepsilon - u}{\varepsilon} \right|^2 (x, t) \int_{\partial\Omega} \chi^\varepsilon_U(x^0, x)\, dH^{n-1}(x^0)\, dH^{n-1}(x)\, dt\ ,$$

Now, let $x \in \partial\Omega$ be fixed. Then

$$\chi^\varepsilon_U(x^0, x) = 0 \quad \text{ if } \ x^0 \in \partial\Omega \setminus B(x, \varepsilon(R + m)(1 + a_\varepsilon))\,,$$

i.e.

$$\int_{\partial\Omega} \chi^\varepsilon_U(x^0, x)\, dH^{n-1}(x^0) \;\le\; (1 + c_\varepsilon(L))\alpha(n - 1)\, (\varepsilon(R + m)(1 + a_\varepsilon(L)))^{n-1}$$

(see the proof of Claim #1 above). Hence,

$$J_\varepsilon \le C_3\, R^{n-1} \int_{\partial\Omega_T} \left| \frac{u_\varepsilon - u}{\varepsilon} \right|^2 (x,t)\, dH^{n-1}(x)\, dt\,,$$

which together with the corresponding estimate of Theorem 2.1 gives the desired estimate. ■

Proof of Corrollary 5.2. Observe first that for any $R \in \mathbb{R}_+$ (5.1) and (5.2) yields

$$\int_{\partial\Omega_T} \int_{B_+(R)} \left(|\nabla v_\varepsilon|^2 + |v_\varepsilon|^2 \right)\, dz\, dH^{n-1}(x^0)\, dt \le C(R)\,,$$

provided ε is sufficiently small. To be more precise, for any R there exists $\varepsilon_0 = \varepsilon_0(R)$ such that

$$\|v_\varepsilon\|_{X(R)} \le C(R)\,, \qquad X(R) \equiv L^2(\partial\Omega_T; W^{1,2}(B_+(R)))$$

for any $0 < \varepsilon \le \varepsilon_0$. Now let $\{R_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $R_n \to \infty$ as $n \to \infty$. By the relative weak compactness of the sequence $\{v_\varepsilon\}$ in $X(R_1)$ we can pick a subsequence $\{v_{1\varepsilon_n}\}_{n=1}^\infty$ such that $v_{1\varepsilon_n}$ weakly converges to a function $v \in X(R_1)$. We can now pick a subsequence $\{v_{2\varepsilon_n}\}_{n=1}^\infty$ of $\{v_{1\varepsilon_n}\}_{n=1}^\infty$ such that $v_{2\varepsilon_n}$ weakly converges to v' in $X(R_2)$. Clearly,

$$v' \equiv v \quad \text{on} \quad B_+(R_1)$$

and therefore, we shall write v instead of $v' \in B_+(R_2)$. Continuing in this fashion we obtain a subsequence $\{v_{j\varepsilon_n}\}_{n=1}^\infty$ weakly convergent to v in $X(R_j)$.

Consider now the "diagonal" sequence $\{v_{n\varepsilon_n}\}_{n=1}^\infty$. We have $\{v_{n\varepsilon_n}\}_{n=j}^\infty$ a subsequence of $\{v_{j\varepsilon_n}\}_{n=1}^\infty$ and so $\{v_{n\varepsilon_n}\}$ weakly converges to v in $X(R_j)$ for any fixed j. This establishes the first assertion of Corollary 5.2, and the second one follows easily. ■

6. The second term in ε–expansion

As noted earlier, we are interested in specifying the second term in ε–expansion of u_ε. Assume for a moment $\zeta \in C_2^{2,1}(\overline{\Omega}_T)$, $\zeta = 0$ on $\partial\Omega_T$, and $\zeta = 0$ on $\Omega \times \{t = T\}$, and insert it as a test function into (4.7). Consequently, we have

$$\begin{aligned} \int_{\partial\Omega_T} W_\varepsilon\, \partial_\nu \zeta\, (x^0, t)\, dH^{n-1}(x^0)\, dt &= \int_{\Omega_T} W_\varepsilon (\partial_t \zeta + \Delta\zeta + F_\varepsilon)\, dx\, dt \\ &\quad - \int_\Omega \mathcal{P}_\varepsilon \zeta(0)\, dx\,. \end{aligned} \tag{6.1}$$

Send $\varepsilon \to 0$ – through a subsequence if necessary – to find

$$\int_{\partial\Omega_T} W_0\, \partial_\nu \zeta\, dH^{n-1}(x^0)\, dt = \int_{\Omega_T} u^1(\partial_t\zeta + \Delta\zeta + f^1)\, dx\, dt \\ - \int_\Omega u_0^1\, \zeta(0)\, dx \tag{6.2}$$

where our hypotheses (C)–(F) have been applied and

$$W_0\,,\ \ u^1 \quad \text{are weak limits of} \quad W_\varepsilon$$

in $L^2(\partial\Omega_T)$, $L^2(\Omega_T)$, respectively. Note that $u^1 \in L^2(\Omega_T) \cap W_2^{2,1}(\Omega'_T)$ for any $\Omega' \subset \Omega$ with a positive distance from $\partial\Omega$ and such that

$$\int_{\Omega_T} |u^1|^2 dx\, dt \ + \max_{0\le t\le T} \int_\Omega |u^1(t)|^2 \phi\, dx \ + \int_{\Omega_T} |\nabla u^1|^2 \phi\, dx\, dt \ \le C\,,$$

$$\operatorname{ess\,sup}_{0\le t\le T} \int_\Omega |\nabla u^1(t)|^2 \phi^3 dx \ + \int_{\Omega_T} |\partial_t u^1|^2 \phi^3\, dx\, dt \ \le \ C\,.$$

Moreover, u^1 is the unique very weak solution of the problem

$$\begin{aligned} &\partial_t u^1 = \Delta u^1 + f^1 && \text{in} \quad \Omega_T\,, \\ &u^1 = W_0 && \text{on} \quad \partial\Omega_T\,, \\ &u^1 = u_0^1 && \text{on} \quad \Omega \times \{t=0\} \end{aligned} \tag{6.3}$$

in the sense of the validity of the integral identity (6.2) for any test function $\zeta \in W_2^{2,1}(\Omega_T)$ such that $\zeta = 0$ on $\partial\Omega_T$. It is not difficult to show that if $\hat{u}^1$ is another very weak solution of (6.3) with $\hat{W}_0$ and $\hat{u}_0^1$, then

$$\int_{\Omega_T} |u^1 - \hat{u}^1|\, dx\, dt \ \le \ C\left(\|W_0 - \hat{W}_0\|_{L^2(\partial\Omega_T)} + \|u_0^1 - \hat{u}_0^1\|_{L^2(\Omega)}\right)\,.$$

Since our goal is to determine the function W_0, we shall focus the first term of (6.1).

Theorem 6.1. *Let Hypotheses (H) be satisfied and assume that*

$$\lim_{R\to\infty} \fint_{B'(R)} \omega(x^0, z', 0)\, dz' \tag{6.4}$$

exists for a.e. $x^0 \in \partial\Omega$. *We shall denote it by* $M(x^0)$ *and suppose that* $M \in L^\infty(\partial\Omega)$. *Then*

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega_T} W_\varepsilon \partial_\nu \zeta = \int_{\partial\Omega_T} M(x^0)\, g(x^0,t)\, \partial_\nu\zeta(x^0,t)\, dH^{n-1}(x^0)\, dt$$

(through a subsequence if necessary) for any $\zeta \in W_2^{2,1}(\Omega_T)$.

Proof of Theorem 6.1. We prove that for $\mathcal{H}^{n-1}$ a.e. $x \in \partial\Omega$,

$$W_0(x,t) \;=\; M(x)\, g(x,t)\,, \quad g(x,t) = \vartheta(x,t) - \partial_\nu u(x,t)\,.$$

To this goal, fix arbitrarily choosen $X \in \mathcal{G}$ and note that there exist $\iota > 0$ and $\kappa \in \mathbb{Z}_+$ such that

$$B(X, 2\iota) \cap \partial\Omega \subset \mathcal{G}^\kappa\,.$$

Recall that $\mathcal{G}, \mathcal{G}^\kappa$ are introduced in in Hypotheses (H) in Section 4. We take now $\varphi \in C(\bar{\Omega}_T)$ such that $\operatorname{spt}\varphi \cap \partial\Omega \subset B(X, 2\iota)$, but arbitrary otherwise, and show that for any $\eta > 0$ there exists $\varepsilon_0(\eta) > 0$ such that

$$\int_{\partial\Omega_T} \big(W_\varepsilon(x^0,t) - M(x^0)\, g(x^0,t)\big)\, \varphi(x^0,t)\, dH^{n-1}(x^0)\, dt \;<\; \eta \tag{6.5}$$

for any $0 < \varepsilon \le \varepsilon_0$.
Denote

$$\begin{aligned} I_\varepsilon + II_\varepsilon \equiv \int_{\partial\Omega_T} \Big(&W_\varepsilon(x^0,t) \\ &- \fint_{U(x^0,\varepsilon R)} W_\varepsilon(x,t)\, dH^{n-1}(x)\Big)\, \varphi(x^0,t)\, dH^{n-1}(x^0)\, dt \\ + \int_{\partial\Omega_T} &\fint_{U(x^0,\varepsilon R)} W_\varepsilon(x,t)\, dH^{n-1}(x)\, \varphi(x^0,t)\, dH^{n-1}(x^0)\, dt\,, \end{aligned}$$

where the set $U(x^0, \varepsilon R)$ is given by (5.6) above. Denoting the characteristic function of the set $U(x^0, \varepsilon R)$ by $\chi_U^\varepsilon(x^0, x)$ (see (5.6) above), using Fubini's theorem and relabeling the integration variable in the first term of I_ε we arrive at

$$I_\varepsilon = \int_{\partial\Omega_T} W_\varepsilon(x,t)\, (\varphi(x,t) - \varphi_\varepsilon(x,t))\, dH^{n-1}(x)\, dt\,,$$

where

$$\varphi_\varepsilon(x,t) \equiv \int_{\partial\Omega} \frac{\chi_U^\varepsilon(x^0,x)\varphi(x^0,t)}{\mathcal{H}^{n-1}(U(x^0,\varepsilon R))}\, dH^{n-1}(x^0)\,.$$

Claim #1: For any $R \in \mathbb{R}_+$ and $\varepsilon \in (0,1)$ there exists a positive constant $\varpi_\varepsilon(R)$ such that

$$\lim_{\varepsilon\to 0} \varpi_\varepsilon(R) \;=\; 0$$

and

$$\left| \frac{\mathcal{H}^{n-1}(U(x^0,\varepsilon R))}{\mathcal{L}^{n-1}(B'(\varepsilon R))} \;-\; 1 \right| \le \frac{m}{R} + \varpi_\varepsilon(R)$$

for any $x^0 \in \mathcal{G}$.

Proof of Claim #1: As $|\tau_\varepsilon| \le m$,

$$U(R-m) \subset U(x^0, \varepsilon R) \subset U(R+m)\,,$$

where

$$U(N) \equiv \Big\{ x \in \partial\Omega \mid \ x = x^0 + \varepsilon \boldsymbol{O}(x^0)\hat{\Psi}'_\varepsilon(x^0, z') \ \text{ for } \ z' \in B'(N) \Big\}\ .$$

We claim now that

$$\mathcal{H}^{n-1}(U(N)) = \mathcal{L}^{n-1}(B'(\varepsilon N))(1 + h_\varepsilon(x^0, N))\,, \quad \lim_{\varepsilon\to 0} h_\varepsilon(x^0, N) = 0$$

uniformly in $x^0 \in \mathcal{G}$ for any fixed N. Indeed,

$$\begin{aligned}\mathcal{H}^{n-1}(U(N)) &= \int_{\partial\Omega\cap B(x^0,r)} \chi_{U(N)}(x)\, dH^{n-1}(x) \\ &= \varepsilon^{n-1} \int_{B'(N)} \sqrt{1 + |D\gamma(x^0, \varepsilon\Psi'_\varepsilon(x^0, z')|^2}\, J\Psi'_\varepsilon(x^0, z')\, dz'\end{aligned}$$

and the assertion follows easily due to (5.3) and the fact that $D\gamma(x^0, 0) = 0$ for any $x^0 \in \partial\Omega$. Thus,

$$\begin{aligned}\Big(1 - \frac{m}{R}\Big)^{n-1} \big(1 + h_\varepsilon(x^0, R-m)\big) \ &\le \ \frac{\mathcal{H}^{n-1}(U(x^0, \varepsilon R))}{\mathcal{L}^{n-1}(B'(\varepsilon R))} \\ &\le \Big(1 + \frac{m}{R}\Big)^{n-1} \big(1 + h_\varepsilon(x^0, R+m)\big)\ ,\end{aligned}$$

which proves (6.7) after some manipulations. This proves *Claim # 1.*

Hence,

$$\begin{aligned}\varphi_\varepsilon(x,t) \ &= \frac{\varphi(x,t)}{\mathcal{L}^{n-1}(B'(\varepsilon R))} \int_{\partial\Omega} \chi^\varepsilon_U(x^0, x)\, dH^{n-1}(x^0) \ + \\ &\frac{1}{\mathcal{L}^{n-1}(B'(\varepsilon R))} \int_{\partial\Omega} \chi^\varepsilon_U(x^0, x)\, \frac{\varphi(x^0,t) - \varphi(x,t) - \theta_\varepsilon(x^0, R)}{1 + \theta_\varepsilon(x^0, R)}\, dH^{n-1}(x^0)\,,\end{aligned} \tag{6.6}$$

where

$$\theta_\varepsilon(x^0, R) \equiv \frac{\mathcal{H}^{n-1}(U(x^0, \varepsilon R))}{\mathcal{L}^{n-1}(B'(\varepsilon R))} \ - \ 1$$

and due to *Claim # 1*,

$$|\theta_\varepsilon(x^0, R)| \le \frac{m}{R} + \varpi_\varepsilon(R)\ . \tag{6.7}$$

Claim #2: For any $R \in \mathbb{R}_+$ and $\varepsilon \in (0,1)$ there exists a positive constant $\mu_\varepsilon(R)$ such that

$$\lim_{\varepsilon \to 0} \mu_\varepsilon(R) \;=\; 0$$

and

$$\left| \frac{\mathcal{H}^{n-1}(\mathcal{F}(x))}{\mathcal{L}^{n-1}(B'(\varepsilon R))} \;-\; 1 \right| \leq \frac{m}{R} + \mu_\varepsilon(R) \tag{6.8}$$

for any $x \in \mathcal{G}$, where

$$\mathcal{F}(x) \equiv \{x^0 \in \partial\Omega \mid \; x \in U(x^0, \varepsilon R)\}\,.$$

Proof of Claim #2: Note first that

$$\mathcal{H}^{n-1}(\mathcal{F}(x)) = \int_{\partial\Omega \cap B(x^0,r)} \chi_U^\varepsilon(x^0, x)\, dH^{n-1}(x^0)$$

for small enough $\varepsilon > 0$ and denote for a moment

$$\mathcal{A}(x) \equiv \{x^0 \in \partial\Omega \mid \; |x - x^0| \leq r_\varepsilon\}\,, \qquad r_\varepsilon \equiv \varepsilon(R-m)/(1+\ell_\varepsilon(L))\,,$$

$$\mathcal{Z}(x) \equiv \{x^0 \in \partial\Omega \mid \; |x - x^0| \leq R_\varepsilon\}\,, \qquad R_\varepsilon \equiv \varepsilon(R+m)(1+a_\varepsilon(L))\,,$$

where ℓ_ε is given in (4.5) a_ε in (5.3), $L > R + m$ being given.
Then $\mathcal{A}(x) \subset \mathcal{F}(x) \subset \mathcal{Z}(x)$ for any $x \in \mathcal{G}$ and thus,

$$\mathcal{H}^{n-1}(\mathcal{A}(x)) \leq \mathcal{H}^{n-1}(\mathcal{F}(x)) \leq \mathcal{H}^{n-1}(\mathcal{Z}(x))\,.$$

Indeed, let $x^0 \in \mathcal{A}(x)$. Setting $z = \tau_\varepsilon(x^0) + \Phi_\varepsilon(x^0, \varepsilon^{-1}\boldsymbol{O}^{-1}(x^0)(x - x^0)))$ one can see that

$$|z| \;\leq\; m + \varepsilon^{-1}(1+\ell_\varepsilon(L))|x - x^0| \;\leq\; R\,, \quad \text{i.e.} \quad x^0 \in \mathcal{F}(x)\,.$$

Now, let $x^0 \in \mathcal{F}(x)$, i.e. there exists $z \in B'(R) \times \{z_n = 0\}$ such that $x = \Lambda_\varepsilon(x^0, z)$. Then, however,

$$|x - x^0| \leq R_\varepsilon\,, \quad \text{i.e.} \quad x^0 \in \mathcal{Z}(x)\,.$$

Next, it is not difficult to see that

$$\mathcal{H}^{n-1}(\mathcal{Z}(x)) \leq (1 + c_\varepsilon(L))\mathcal{L}^{n-1}(B'(R_\varepsilon))$$

and

$$\mathcal{L}^{n-1}\left(B'\left(r_\varepsilon / \sqrt{1 + \mathrm{Lip}^2(\gamma\,|_{B'(r_\varepsilon)})}\right)\right) \leq \mathcal{H}^{n-1}(\mathcal{A}(x))\,.$$

(6.8) follows again after some manipulations and *Claim # 2* is proved.

Note that *Claim # 2* yields

$$\int_{\partial\Omega} \chi_U^{\varepsilon}(x^0, x)\, dH^{n-1}(x^0) = \mathcal{L}^{n-1}(B'(\varepsilon R))(1 + \xi_\varepsilon(x, R))\,, \tag{6.9}$$

where

$$|\xi_\varepsilon(x, R)| \le \frac{m}{R} + \mu_\varepsilon(R)$$

uniformly in $x \in \mathcal{G}$.

Now, with the assistance of the estimates from Theorem 2.1, (6.7) and (6.9) in (6.6), we arrive at

Claim #3: There exist a constant C independent of ε, R and for any $R \in \mathbb{R}_+$, $0 < \varepsilon < 0$ a positive constant $\sigma_\varepsilon(R)$ such that

$$\lim_{\varepsilon \to 0} \sigma_\varepsilon(R) = 0$$

for any fixed R, and

$$|I_\varepsilon| \le C \left(\frac{m}{R} + \sigma_\varepsilon(R) + \max_{|x-x^0| \le R_\varepsilon, 0 \le t \le T} |\varphi(x^0, t) - \varphi(x, t)| \right) .$$

Proof of Claim #3: Indeed, as

$$|\varphi_\varepsilon(x,t) - \varphi_\varepsilon(x,t)| \le |\varphi(x,t)|\, |\xi_\varepsilon(x,R)| + \frac{1 + |\xi_\varepsilon(x,R)|}{1 - \operatorname{ess\,sup}_{x^0 \in \partial\Omega} |\theta_\varepsilon(x,R)|}$$
$$\times \left(\max_{|x-x^0| \le R_\varepsilon} |\varphi(x^0,t) - \varphi(x,t)| + \operatorname{ess\,sup}_{x^0 \in \partial\Omega} |\theta_\varepsilon(x^0,R)| \right),$$

the assertion of *Claim # 3* follows easily due to (6.7) and (6.9).

Claim #4: There exists a constant C_m independent of ε, R and for any $R \in \mathbb{R}_+$ and $0 < \varepsilon < 1$ there exists a positive constant $s_\varepsilon(R)$ such that

$$II_\varepsilon = \int_{\partial\Omega_T} \fint_{B'(R)} \omega(x^0, z', 0)\, dz'\; g(x^0, t)\, \varphi(x^0, t)\; dH^{n-1}(x^0)\, dt + s_\varepsilon(R) + \frac{C_m}{R}$$

and

$$\lim_{\varepsilon \to 0} s_\varepsilon(R) = 0$$

for any fixed R.

Proof of Claim #4: Note that due to the notation prior to (6.7),

$$II_\varepsilon = \int_{\mathcal{G}_T^\kappa} \frac{1}{1 + \theta_\varepsilon(x^0, R)} \fint_{B'(R)} v_\varepsilon(x^0, z', 0, t)\, \sqrt{1 + |D\gamma(x^0, \varepsilon\Psi_\varepsilon')|^2}$$
$$\times J\Psi_\varepsilon'(x^0, z' - \tau_\varepsilon'(x^0))\, dz'\, \varphi(x^0, t)\, dH^{n-1}(x^0)\, dt$$

and the assertion follows due to (6.7), (5.3) and Corollary 5.2.

Now we are ready to prove (6.5). Employing *Claim # 3 and Claim # 4* we can write

$$\begin{aligned}
&\left|\int_{\partial\Omega_T} \big(W_\varepsilon(x^0,t) - M(x^0)\, g(x^0,t)\big)\, \varphi(x^0,t)\, dH^{n-1}(x^0)\, dt\right| \\
&\qquad \le \frac{C}{R} + \mathcal{O}_\varepsilon(R) + C \max_{|x-x^0|\le R_\varepsilon, 0\le t\le T} |\varphi(x^0,t) - \varphi(x,t)| + \qquad (6.10)\\
&\int_{\partial\Omega_T} \left| \fint_{B'(R)} \omega(x^0,z',0)\, dz' \; - \; M(x^0)\, g(x^0,t)\right| |\varphi(x^0,t)|\, dH^{n-1}(x^0)\, dt\,,
\end{aligned}$$

where

$$\lim_{\varepsilon\to 0} \mathcal{O}_\varepsilon(R) \;=\; 0 \quad \text{for fixed} \quad R\,.$$

Finally, we choose R large enough in order to have the first term and the last one on the right hand side of (6.10) less than $\eta/2$ and then ε_0 small enough to have the additional two terms less than $\eta/2$ for each $0 < \varepsilon < \varepsilon_0$.

We have so far proved that for any $X \in \mathcal{G}$ there exists a ball $B(X,\iota), \iota > 0$ such that for $\mathcal{H}^{n-1}$ a.e. $x \in \partial\Omega \cap B(X,\iota)$, $W_0(x,t) = M(x)\, g(x,t)$. Therefore, we can conclude that this equality holds for $\mathcal{H}^{n-1}$ a.e. $x \in \partial\Omega$ and the theorem follows. ■

7. Capacity

In this section we introduce capacity as a way to find sufficiently general structural conditions on the set $\mathbb{D}$ in order to garantee the existence of the mean value of ω for $x_n = 0$, see (6.4) above. As the dependence of ω on $x^0 \in \partial\Omega$ does not play any role in this considerations, let us omit to write it down here. Recall

$$\begin{aligned}
\Delta\omega &= 0 && \text{in} \quad \mathbb{R}^n_+\,,\\
\omega &= 0 && \text{on} \quad \mathbb{D}_0\,,\\
\partial_\nu\omega &= 1 && \text{on} \quad \mathbb{N}_0\,,
\end{aligned} \qquad (7.1)$$

$\mathbb{D}_0 \equiv \mathbb{D} \times \{x_n = 0\}$, and till now we have supposed that $\mathbb{D}$ is asymptotically dense in $\mathbb{R}^{n-1}$ (see Definition 3.1). In order to be able to formulate our result, it is convenient to introduce some preliminairies, developed in the theory of almost periodic functions (see e.g. [2], [10]).

Definition 7.1. *Given a function $v \in L^\infty(\mathbb{R}^{n-1})$ and numbers $0 < \eta < 1$, $0 < R_\eta < \infty$, a vector $\lambda \in \mathbb{R}^{n-1}$ is said to be a translation vector belonging*

to η if

$$\sup_{R \geq R_\eta} \fint_{C'(R)} |v(x'+\lambda) - v(x')|\, dx' \;\leq\; \eta\,. \tag{7.2}$$

Recall, $C'(R) \equiv \{y' \in \mathbb{R}^{n-1} \mid |y_i| < R\,,\ i = 1, 2, \ldots, n-1\}$.

A set $E \subset \mathbb{R}^{n-1}$ is said to be ℓ–relatively dense if there is a number $\ell > 0$ such that any cube of side ℓ in $\mathbb{R}^{n-1}$ contains at least one point of E.

Theorem 7.2. *Assume that for any $\eta > 0$ correspond numbers $R_\eta > 0$ and $\ell_\eta > 0$ such that the set of all translation vectors of $v \in L^\infty(\mathbb{R}^{n-1})$ is ℓ_η–relatively dense in $\mathbb{R}^{n-1}$. Then the mean value of v exists, i.e. the limit of*

$$\fint_{B'(R)} v(x')\, dx' \qquad \text{as } R \to \infty\,,$$

exists.

Proof. Let $k \in \{1, 2, \ldots\}$ and denote by $C'(x^i, R)$ for $x^i \in B'(kR)$ open disjoint cubes in $\mathbb{R}^{n-1}$, i.e.

$$C'(x^i, R) \equiv \{y' \in \mathbb{R}^{n-1} \mid \quad |x^i_j - y_j| < R\,,\ j = 1, \ldots, n-1\}$$

such that

$$B'(kR) \;\subset\; C_k \equiv \bigcup_{i=1}^{N_k} \bar{C}'(x^i, R) \;\subset\; B'((k+2)R)\,.$$

As for $A \subset C \subset D$,

$$\left| \fint_A v - \fint_C v \right| \;\leq\; 2\|v\|_{L^\infty(\mathbb{R}^{n-1})} \frac{\mathcal{L}^{n-1}(D) - \mathcal{L}^{n-1}(A)}{\mathcal{L}^{n-1}(A)}\,,$$

we have

$$\left| \fint_{B'(kR)} v(x')\, dx' - \fint_{C_k} v(x')\, dx' \right| \leq \frac{c}{k}$$

with a constant $c \in \mathbb{R}_+$ independent of k, R. Compute now

$$\begin{aligned}\int_{C_k} v(x')\, dx' = \sum_{i=1}^{N_k} \Bigg[&\int_{C'(R)} v(y'+\lambda^i)\, dy' + \int_{C'(x^i-\lambda^i,R)\setminus C'(R)} v(y'+\lambda^i)\, dy' \\ &- \int_{C'(R)\setminus C'(x^i-\lambda^i,R)} v(y'+\lambda^i)\, dy' \Bigg] \\ = N_k \int_{C'(R)} v(y')\, dy' + \sum_{i=1}^{N_k} \int_{C'(R)} &\left| v(y'+\lambda^i) - v(y') \right| dy' + R^{n-1} N_k \theta_R\,,\end{aligned}$$

where

$$|\theta_R| \le \frac{c}{R}\,,$$

i.e.

$$\fint_{C_k} v(x')\,dx' = \fint_{C'(R)} v(y')\,dy' + \vartheta_\eta\,\eta + \theta_R\ ,$$

for some $|\vartheta_\eta| \le 1$. In summary,

$$\left|\fint_{B'(kR)} v(x')\,dx' - \fint_{B'(\ell R)} v(x')\,dx'\right| \le\ \eta\ +\ c\left(\frac{1}{R}+\frac{1}{k}+\frac{1}{\ell}\right)\ .$$

Thus, for arbitrarily large numbers X_j, $j=1,2$ let k_j be such that $k_jR \le X_j < k_jR+1$ and we arrive at

$$\left|\fint_{B'(X_1)} v(x')\,dx' - \fint_{B'(X_2)} v(x')\,dx'\right| \le\ c\left(\eta\ +\frac{1}{R}+\frac{1}{k_1}+\frac{1}{k_2}\right)$$

and the assertion of Theorem 7.2. follows. ■

Our plan is hereafter to find circumstances under which bounded weak solution ω of Problem (7.1) satisfies hypotheses of Theorem 7.2 for $x_n = 0$. We propose here the following assumptions. Denote

$$\mathbb{D}^\lambda \equiv \mathbb{D}+\lambda\,,\quad \mathbb{D}_0^\lambda \equiv \mathbb{D}^\lambda \times \{x_n=0\}$$

for $\lambda \in \mathbb{R}^{n-1}$, recall $\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda \equiv \mathbb{D}_0 \cup \mathbb{D}_0^\lambda \setminus \mathbb{D}_0 \cap \mathbb{D}_0^\lambda$ and define

$$\mathrm{Cap}_L(\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda)\ \equiv\ \sup_{R\ge L} R^{1-n} \inf_{\mathcal{H}\in V(R)} \int_{B_+(R)} |\nabla\mathcal{H}|^2\,, \tag{7.3}$$

where

$$V(R) \equiv \{\mathcal{H}\in W^{1,2}(B_+(R))\ |\quad \mathcal{H} = \left\{\begin{array}{lcl} 1 & : & \text{on } (\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda)\cap B(R) \\ 0 & : & \text{for } x_n > 1 \end{array}\right.\ \}$$

and $L \in \mathbb{R}_+$, $\lambda \in \mathbb{R}^{n-1}$ given.

Assumption (P). *For any $\eta > 0$ there exist positive numbers ℓ_η, L_η and a ℓ_η– relatively dense set E_η in $\mathbb{R}^{n-1}$ such that*

$$Cap_{L_\eta}\left(\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda\right)\ \le\ \eta \tag{7.4}$$

for any $\lambda \in E_\eta$.

Theorem 7.3. *Assume that ω is a bounded weak solution of Problem (7.1), where the set $\mathbb{D}$ satisfies Assumption (P).*

Then the function $\omega(x',0)$ for $x' \in \mathbb{R}^{n-1}$ satisfies all hypotheses required in Theorem 7.2 above.

Proof. 1. Fix $\lambda \in E_\eta$ given in **Assumption (P)** and define

$$u(x) \equiv \omega(x'+\lambda, x_n) - \omega(x', x_n)\,, \quad u^D(x') \equiv u(x',0)\,.$$

Then

$$\begin{aligned} \Delta u &= 0 && \text{in} \quad \mathbb{R}^n_+\,, \\ u &= 0 && \text{on} \quad \mathbb{D}_0 \cap \mathbb{D}_0^\lambda\,, \\ u &= u^D && \text{on} \quad \mathbb{D}_0 \triangle \mathbb{D}_0^\lambda\,, \\ \partial_\nu u &= 0 && \text{on} \quad \mathbb{N}_0 \cap \mathbb{N}_0^\lambda\,, \end{aligned}$$

in the sense that $u \in W^{1,2}_{loc}(\mathbb{R}^n_+)$, $u = u^D$ on $\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda$ and

$$\int_{\mathbb{R}^n_+} \nabla u\, \nabla \psi \;=\; 0 \tag{7.5}$$

for any $\psi \in W^{1,2}_{loc}(\mathbb{R}^n_+)$, $\psi = 0$ on $\mathbb{D}_0 \cup \mathbb{D}_0^\lambda$, with a compact support in $\mathbb{R}^n_+$.
2. Take next sufficiently large R and insert

$$\psi = u\,\phi_R^2\,(1-\mathcal{H}_R)^2$$

as a test function in (7.5), where functions $\mathcal{H}_R$ and ϕ_R are defined as follows: $\mathcal{H}_R$ is the minimizer of Dirichlet's integral

$$J(\mathcal{H}) = \int_{B_+(3R)} |\nabla \mathcal{H}|^2$$

on $V(3R)$ (see (7.3) above), i.e.

$$\begin{aligned} \Delta \mathcal{H}_R &= 0 && \text{in} \quad U_{3R} \equiv B_+(3R) \cap \{x \mid\ 0 < x_n < 1\}\,, \\ \mathcal{H}_R &= 0 && \text{on} \quad B_+(3R) \setminus U_{3R}\,, \\ \mathcal{H}_R &= 1 && \text{on} \quad \big(\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda\big) \cap B(3R)\,, \\ \partial_\nu \mathcal{H}_R &= 0 && \text{on} \quad S_{3R}\,, \end{aligned}$$

where

$$S_{3R} \equiv \partial U_{3R} \setminus \big(\{x \in \bar{B}_+(3R) \mid\ x_n = 1\} \cup (\mathbb{D}_0 \cap \mathbb{D}_0^\lambda)\big)\,,$$

and

$$\begin{aligned} \Delta \phi_R &= 0 && \text{in} \quad B_+(3R)\,, \\ \phi_R &= 0 && \text{on} \quad \partial B_+(3R) \cap \mathbb{R}^n_+\,, \\ \phi_R &= 1 && \text{on} \quad B(2R) \cap \{x \mid\ x_n = 0\}\,, \\ \partial_\nu \phi_R &= 0 && \text{on} \quad (B(3R) \setminus B(2R)) \cap \{x \mid\ x_n = 0\}\,. \end{aligned}$$

Thus,

$$\int_{B^+(3R)} |\nabla u|^2 (1-\mathcal{H}_R)^2 \phi_R^2$$
$$\leq C_1 \left(\int_{B^+(3R)} |\nabla \mathcal{H}_R|^2 |u|^2 \phi_R^2 + \int_{B^+(3R)} |\nabla \phi_R|^2 (1-\mathcal{H}_R)^2 |u|^2 \right) .$$

Since,

$$\int_{B^+(3R)} |\nabla \phi_R|^2 \leq C_2 R^{n-2} ,$$

$0 \leq \phi_R$, $\mathcal{H}_R \leq 1$ on $B_+(3R)$, $0 < \nu \leq \phi_R$ on $B_+(2R)$ and $u \in L^\infty(\mathbb{R}^n_+)$, the inequality above implies

$$\int_{B_+(2R)} |\nabla(u(1-\mathcal{H}_R))|^2 \leq C_3 \left(R^{n-2} + \int_{B_+(3R)} |\nabla \mathcal{H}_R|^2 \right) . \tag{7.6}$$

Note that the facts that $\mathbb{D}$ is asymptotically dense in $\mathbb{R}^{n-1}$, $u(1-\mathcal{H}_R) = 0$ on $\mathbb{D}_0 \cup \mathbb{D}_0^\lambda$ and (7.6) yield

$$\int_{B_+(2R)\cap\{x|\, x_n=1\}} |u|^2 \leq C_4 \left(R^{n-2} + \int_{B_+(3R)} |\nabla \mathcal{H}_R|^2 \right) . \tag{7.7}$$

3. Assume now that u_h is a solution of

$$\begin{aligned} \Delta u_h &= 0 && \text{in} \quad U_{2R} , \\ u_h &= |u| && \text{on} \quad B_+(2R) \cap \{x \mid x_n = 1\} , \\ u_h &= 0 && \text{on} \quad \left(\mathbb{D}_0 \cup \mathbb{D}_0^\lambda\right) \cap B(2R) , \\ \partial_\nu u_h &= 0 && \text{on} \quad S_{2R} , \end{aligned}$$

(S_{2R} is the rest part of ∂U_{2R}). Note that u_h is the minimizer of the Dirichlet's integral $J(w) = \int_{U_{2R}} |\nabla w|^2$ over the set

$$X \equiv \left\{ w \in W^{1,2}(U_{2R}) \mid \quad w\,|_{x_n=1} = |u| \ \text{ and } \ w = 0 \ \text{ on } \ \mathbb{D}_0 \cup \mathbb{D}_0^\lambda \right\} .$$

As $|u|(1-\mathcal{H}_R) \in X$, (7.6) yields

$$\int_{U_{2R}} |\nabla u_h|^2 \leq C \left(R^{n-2} + \int_{B_+(3R)} |\nabla \mathcal{H}_R|^2 \right) . \tag{7.8}$$

4. *Claim #1: Let* $M \equiv \|\omega\|_{L^\infty(\mathbb{R}^n_+)}$, *then*

$$|u| \leq u_h + M\mathcal{H}_R + 2M\varphi_R$$

on U_{2R}, where

$$\begin{aligned}
\Delta\varphi_R &= 0 && \text{in} \quad B_+(2R)\,,\\
\varphi_R &= 0 && \text{on} \quad (\mathbb{D}_0 \cup \mathbb{D}_0^\lambda) \cap B(2R)\,,\\
\varphi_R &= 1 && \text{if} \quad |x| = 2R\,,\\
\partial_\nu \varphi_R &= 0 && \text{on} \quad S_{2R}\,,
\end{aligned}$$

where S_{2R} denotes the rest of $\partial B_+(2R)$.

Proof of Claim #1. Note first that $|u|$ is a subsolution of the problem

$$\begin{aligned}
\Delta v &= 0 && \text{in} \quad U_{2R}\,,\\
v &= 0 && \text{on} \quad (\mathbb{D}_0 \cap \mathbb{D}_0^\lambda) \cap B(2R)\,,\\
v &= |u^D| && \text{on} \quad (\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda) \cap B(2R)\,,\\
v &= |u| && \text{on} \quad B_+(2R) \cap \{x \mid x_n = 1 \text{ or } |x| = 2R\}\,,\\
\partial_\nu v &= 0 && \text{on} \quad (\mathbb{N}_0 \cap \mathbb{N}_0^\lambda) \cap B(2R)\,,
\end{aligned} \tag{7.9}$$

i.e.

$$\int_{U_{2R}} \nabla|u|\nabla\psi\, dx \le 0$$

for any $\psi \in W^{1,2}(U_{2R})$, $\psi \ge 0$ and $\psi = 0$ on $\mathbb{D}_0 \cap \mathbb{D}_0^\lambda$ and on $\{x \mid x_n = 1 \text{ or } |x| = 2R\} \cap \bar{U}_{2R}$.

The function $V = u_h + M\mathcal{H}_R + 2M\varphi_R$ is, however, a supersolution of Problem (7.9), i.e.

$$|u| \le V$$

on U_{2R} and the assertion follows.

5. Finally, *Claim #1* yields

$$\begin{aligned}
&\int_{B'(R)} |u(x',0)|^2\, dx'\\
&\le C_5 \left(\|u_h\|^2_{W^{1,2}(U_{2R})} + \|\mathcal{H}_R\|^2_{W^{1,2}(U_{2R})} + \int_{B'(R)} |\varphi_R(x',0)|^2\, dx' \right) \qquad (7.10)\\
&\le C_6 \left(R^{n-2} + \int_{B_+(3R)} |\nabla\mathcal{H}_R|^2 + \int_{B'(R)} |\varphi_R(x',0)|^2 dx' \right)\,,
\end{aligned}$$

where (7.7), (7.8) have been taking account. It remains to estimate the last integral on the right hand side of (7.10). Here we apply arguments of the proof of Theorem 3.3 and we can conclude that

$$|\varphi_R| \le C\,/\,R$$

on $B'(R) \times \{x \mid 0 \le x_n \le 1\}$ for some positive constants C independent of R. Hence,

$$\fint_{B'(R)} |\omega(x'+\lambda,0) - \omega(x',0)|\, dx' \le C_7 \left(\frac{1}{R} + \fint_{B_+(3R)} |\nabla \mathcal{H}_R|^2 \right)^{1/2}$$

and this completes the proof of the theorem. ■

8. Applications

In this section we identify conditions ensuring the existence of a nontrivial structure of the set $\mathbb{D}$ satisfying Assumpption (P) of the previous section.

By the trivial structure of the set $\mathbb{D}$ we mean its periodic structure. To be more precise, let $F : \mathbb{R}^{n-1} \to \{0,1\}$ be a periodic function in each of its variable , say, with period 1 and let there are $y' \in \mathbb{R}^{n-1}$ and $\iota > 0$ such that $B(y',\iota) \subset \{x' \in \mathbb{R}^{n-1} \mid F(x') = 1\}$. In addition to, let $\boldsymbol{A} \in L(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ be a regular matrix, i.e. $\det \boldsymbol{A} \neq 0$. If we define

$$\chi(x') \equiv F(\boldsymbol{A}x') \qquad \text{for} \quad x' \in \mathbb{R}^{n-1} \tag{8.1}$$

and

$$\mathbb{D} \equiv \{x' \in \mathbb{R}^{n-1} \mid \quad \chi(x') = 1\},$$

then $\mathbb{D}$ satisfies Assumption (P) trivially and it is asymptotically dense as well.

Definition 8.1. *We shall call a function $\chi : \mathbb{R}^{n-1} \to \{0,1\}$ quasiperiodic if for any $\eta > 0$ there exists a continuous quasiperiodic function φ_η such that the set*

$$\Lambda_\eta \equiv \{\, x' \in \mathbb{R}^{n-1} \mid \quad |\chi(x') - \varphi_\eta(x')| > 0 \,\}$$

has a small capacity in the following sense: Let $\mathcal{H}_\eta$ satisfy

$$\begin{aligned} \Delta \mathcal{H}_\eta &= 0 && \text{in} \quad \mathbb{R}^{n-1} \times (0,1), \\ \mathcal{H}_\eta(x',0) &= 1 && \text{for} \quad x' \in \Lambda_\eta, \\ \mathcal{H}_\eta(x',1) &= 0 && \text{for} \quad x' \in \mathbb{R}^{n-1}, \\ \partial_\nu \mathcal{H}_\eta(x',0) &= 0 && \text{for} \quad x' \in \mathbb{R}^{n-1} \setminus \Lambda_\eta, \end{aligned}$$

then

$$\mathcal{C}(\Lambda_\eta) \equiv \sup_{\hat{x} \in \mathbb{R}^{n-1} \times \{x_n = 0\}} \sup_{R \ge 1} \fint_{B(\hat{x},R) \cap \{x \mid 0 < x_n < 1\}} |\nabla \mathcal{H}_\varepsilon|^2\, dx \le \eta. \tag{8.2}$$

Proposition 8.2. *Let $\chi : \mathbb{R}^{n-1} \to \{0,1\}$ be a quasiperiodic function and assume that there are $y' \in \mathbb{R}^{n-1}$ and $\iota > 0$ such that*

$$B(y', \iota) \subset \mathbb{D} \equiv \{x' \in \mathbb{R}^{n-1} \mid \chi(x') = 1\} .$$

Then the set $\mathbb{D}$ satisfies Assumption (P) (see (7.4) above) and it is the asymptotically dense set in the sense of Definition 3.1.

Proof. Note first that by a continuous quasiperiodic function in $\mathbb{R}^{n-1}$ we mean a function $\varphi \in C(\mathbb{R}^{n-1})$ such that

$$\varphi(y') \equiv f(\boldsymbol{B}y') , \tag{8.3}$$

where $f \in C(\mathbb{R}^k)$, $k \geq 1$ is periodic in each of its variable with period 1 and $\boldsymbol{B} \in L(\mathbb{R}^{n-1}, \mathbb{R}^k)$. Moreover, without loss of generality one may assume that

$$\boldsymbol{B} \cdot \mathbb{R}^{n-1} + \mathbb{Z}^k$$

is dense in $\mathbb{R}^k$ (see [11]).

Our aim is to show that for any $\eta > 0$ there exist positive numbers ℓ_η, L and ℓ_η– relative dense set $E_\eta \subset \mathbb{R}^{n-1}$ such that

$$\mathrm{Cap}_L(\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda) \leq \eta$$

for any $\lambda \in E_\eta$.

1. As χ is supposed to be quasiperiodic, for any $\eta > 0$ there exists a continuous quasiperiodic function φ_η such that

$$\mathcal{C}(\Lambda_\eta) \leq \eta .$$

Fix $\eta > 0$. It follows from Kronecker–Weyl's Theorem (see e.g. [10], [11]) that for any $\delta > 0$ there exists ℓ_δ and a ℓ_δ–relative dense set E_δ in $\mathbb{R}^{n-1}$ such that for any $\lambda \in E_\delta$,

$$\varphi_\eta(x' + \lambda) - \varphi_\eta(x') = \varphi_\eta(x' + \lambda_\delta(x')) - \varphi_\eta(x') \tag{8.4}$$

and

$$|\lambda_\delta(x')| < \delta ,$$

$x' \in \mathbb{R}^{n-1}$ being arbitrary. As φ_η is uniformly continuous on $\mathbb{R}^{n-1}$ (see (8.3)), (8.4) yields that $|\varphi_\eta(x' + \lambda) - \varphi_\eta(x')| < 1$, say,

$$\max_{x' \in \mathbb{R}^{n-1}} |\varphi_\eta(x' + \lambda) - \varphi_\eta(x')| \leq 1/2 \tag{8.5}$$

if δ is choosen sufficiently small.
2. If $\lambda \in E_\delta$,

$$|\chi(x'+\lambda) - \chi(x')| \;\le\; |\chi(x'+\lambda) - \varphi_\eta(x'+\lambda)| \;+\; 1/2 \\ + |\varphi_\eta(x') - \chi(x')|\,,$$

wherein the left hand side can attain only two values, zero and one. If $|\chi(x'+\lambda) - \varphi_\eta(x'+\lambda)| = 0$ and at the same time $|\varphi_\eta(x') - \chi(x')| = 0$, then also $|\chi(x'+\lambda) - \chi(x')| = 0$.

Thus, $|\chi(x'+\lambda) - \chi(x')|$ could equal one only if $x' \in \Lambda_\eta$ or $x'+\lambda \in \Lambda_\eta$. This yields

$$\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda \;\subset\; \Lambda_\eta\,,$$

hence

$$\mathrm{Cap}_1\left(\mathbb{D}_0 \triangle \mathbb{D}_0^\lambda\right) \;\le\; \mathcal{C}(\Lambda_\eta) \;\le\; \eta$$

for any $\lambda \in E_\eta \equiv E_\delta$. This proves our statement. ■

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