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Rectifiable sets in metric and Banach spaces

by

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1 Introduction

In this paper we study $\mathcal{H}^k$-rectifiable sets in metric spaces, i.e. sets S which can be covered, up to $\mathcal{H}^k$-negligible sets, by a countable family of Lipschitz images of subsets of $\mathbf{R}^k$. One of the reasons for our interest in this class of sets is the development, in a forthcoming paper [2], of a general theory of currents in metric spaces, along the lines proposed by E. De Giorgi in [5], [6]. Indeed, in this general setting we will prove the Federer–Fleming closure theorem and the boundary rectifiability theorem for integral currents, which are supported on countably $\mathcal{H}^k$-rectifiable sets.

The theory of rectifiable sets in Euclidean spaces provides on the one hand side a powerful tool for the solution of geometrical problems in the calculus of variations. On the other hand side, it allows to decide if a general set is of this particular type (so being curve- or surface-like) knowing only some of its metric (densities) or geometric properties (size of projections, existence of approximate tangent planes). This theory started in the pioneering work by A.S. Besicovitch in the late 20'ties treating these questions in deep for sets of finite length in the plane. Subsequent contributions by H. Federer, J.M. Marstrand and P. Mattila extended these results to sets of any dimension in general Euclidean spaces and finally D. Preiss established the relation between density and geometry for general measures in such spaces. A nice presentation of the whole subject can be found in [23].

As concerns rectifiable sets in general metric spaces the situation is much less understood. The first results concerning countably $\mathcal{H}^k$-rectifiable sets, in particular for $k > 1$, in this situation were proved by the second author in [15]. Using a new *metric* differentiability theorem for Lipschitz functions $f : \mathbf{R}^k \to E$, in [15] an area formula for these maps was estabilished, and this formula was used to study the k-dimensional density $\lim_{\varrho\to 0} \mathcal{H}^k(S \cap B_\varrho(x))/(\omega_k \varrho^k)$ of rectifiable sets with finite measure. Moreover, it was proved that in a suitable approximate sense the distance function locally behaves on S as a norm (called *local norm*), not necessarily induced by an inner product.

In this paper we use an isometric embedding of E into a Banach space Y (typically l^∞, as in [11]) to gain a linear structure. This structure is necessary if one intends to define an approximate tangent space to rectifiable sets as in the Euclidean case. Our main technical tool is an extension of the Rademacher differentiability theorem for Lipschitz maps $f : \mathbf{R}^k \to Y$, with Y dual of a separable Banach space, saying that for $\mathcal{H}^k$-a.e. $x \in \mathbf{R}^k$ the difference quotients satisfy

$$\begin{cases} w^* - \displaystyle\lim_{y\to x} \frac{f(y) - f(x) - wdf_x(y-x)}{|y-x|} = 0 \\ \displaystyle\lim_{y\to x} \frac{\|f(y) - f(x)\| - \|wdf_x(y-x)\|}{|y-x|} = 0 \end{cases} \tag{1.1}$$

for some linear map $wdf_x : \mathbf{R}^k \to Y$, called w^*-*differential* of f. Simple examples show that this statement is optimal: indeed, if $k = 1$, $Y = L^1(0,1)$ and $f(x) = \chi_{(0,x)}$ the difference quotients are nowhere converging, and this shows the necessity to deal with dual spaces. Moreover, if f is viewed as a map with values in the space $(C[0,1])^*$ of Radon measures in $[0,1]$, then (1.1) holds with $wdf_x(t) = t\delta_x$, but the difference quotients are not strongly converging. Notice that (1.1) implies Frechet differentiability if Y is uniformly convex.

The plan of our paper is the following: in Section 3 we collect the main facts about differentiability of Lipschitz functions, in Section 4 we define a notion of jacobian for linear maps $L : V \to W$, with V, W finite dimensional Banach spaces and we use it in Section 5 to estabilish a general area formula between rectifiable subsets of metric spaces. In the same section we study rectifiable sets, introducing the approximate tangent space to them; it turns out that in the general metric setting the approximate tangent space is uniquely determined up to isometries, and that its norm is exactly the local norm of [15]. Moreover, if $E = Y$ is the dual of a separable Banach space the approximate tangent space can be characterized by the w^*-limits of secant vectors: the geometric counterpart of (1.1) is the w^*-convergence of *unit* secant vectors to *unit* tangent vectors. In Section 6 we see that the above mentioned properties of rectifiable sets are sharp, giving rectifiability criteria for sets and measures. Moreover, revisiting an unpublished work of S. Konyagin [17], we show in Section 7 that rectifiability can not be recovered using Euclidean projections: in fact, for any $s > 0$ we exhibit a compact metric space X_s such that $\mathcal{H}^s(X_s) = 1$ and $\mathcal{H}^s(f(X_s)) = 0$ for any Lipschitz map f into any Euclidean space $\mathbf{R}^p$. This property implies that, for integer s, X_s is purely $\mathcal{H}^s$-unrectifiable, i.e. $\mathcal{H}^s(f(M)) = 0$ for any Lipschitz map $f : M \subset \mathbf{R}^s \to X_s$ (see Theorem 11 in [15]).

The final two sections of the paper are devoted to the area and coarea formula in a general metric setting, i.e. for Lipschitz functions defined on countably $\mathcal{H}^k$-rectifiable subsets of a metric space.

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2 Notations

We denote by $\mathcal{B}(X)$ the σ-algebra of Borel sets in a metric space (X, d) and by $\mathcal{M}(X)$ the class of finite Borel measures in X, i.e. σ-additive functions $\mu : \mathcal{B}(X) \to [0, +\infty)$.

We define the k-dimensional Hausdorff measure in X as in [8], 2.10.2(1), and will denote it by $\mathcal{H}^k$. Since $\mathcal{H}^k_X(B) = \mathcal{H}^k_Y(B)$ whenever $B \subset X$ and X is isometrically embeded in Y, our notation for the Hausdorff measure does not emphasize the ambient space. Even if we will often work in non-separable spaces, sets of finite or σ-finite Hausdorff measure will of course always be separable.

We recall (see for instance [15], Lemma 6(i)) that if X is a k-dimensional vector space and B_1 is its unit ball, then $\mathcal{H}^k(B_1)$ is a dimensional constant independent of the norm of X and equal, in particular, to the Lebesgue measure of the Euclidean unit ball. This constant will be denoted by ω_k, and the Lebesgue measure in $\mathbf{R}^k$ will be denoted by $\mathcal{L}^k$.

The *upper and lower k-dimensional densities* of a finite Borel measure μ at x are respectively defined by

$$\Theta_k^*(\mu,x) := \limsup_{\varrho\downarrow 0} \frac{\mu(B_\varrho(x))}{\omega_k \varrho^k} \qquad \Theta_{*k}(\mu,x) := \liminf_{\varrho\downarrow 0} \frac{\mu(B_\varrho(x))}{\omega_k \varrho^k} \quad .$$

We recall that the implications

$$\Theta_k^*(\mu,x) \geq t \ \ \forall x \in B \quad \Longrightarrow \quad \mu \geq t\mathcal{H}^k \llcorner B \tag{2.1}$$

$$\Theta_k^*(\mu,x) \leq t \ \ \forall x \in B \quad \Longrightarrow \quad \mu \llcorner B \leq 2^k t\mathcal{H}^k \llcorner B \tag{2.2}$$

hold in any metric space X whenever $t \in (0,\infty)$ and $B \in \mathcal{B}(X)$ (see [8], 2.10.19).

Let (X,d_X), (Y,d_Y) be metric spaces; we say that $f : X \to Y$ is a *Lipschitz* function if

$$d_Y(f(x),f(y)) \leq M d_X(x,y) \qquad \forall x,\, y \in X$$

for some constant $M \in [0,\infty)$; the least constant with this property will be denoted by $\mathrm{Lip}(f)$, and the collection of Lipschitz functions will be denoted by $\mathrm{Lip}(X,Y)$ (Y will be omitted if $Y = \mathbf{R}$). Furthermore, we use the notation $\mathrm{Lip}_1(X,Y)$ for the collection of Lipschitz functions with Lipschitz constant less or equal to 1.

Given a Lipschitz functions $f : A \subset X \to Y$, with X, Y Banach spaces, one often needs an extension which is still a Lipschitz function, possibly with the same Lipschitz constant. If $Y = \mathbf{R}$ then f can be extended to the whole of X, preserving the Lipschitz constant, by

$$\tilde{f}(x) := \inf_{y\in A} f(y) + \mathrm{Lip}(f)\|x-y\| \qquad x \in X \quad .$$

A similar result holds if both X and Y are Euclidean (which in this paper always includes finiteness of the dimension) spaces, but the construction of an extension is not elementary (see [8], 2.10.43). If X is an Euclidean space, then without any assumption on Y there is a Lipschitz extension, not necessarily with the same Lipschitz constant (see [14]). If $Y = l^\infty$ an extension preserving the Lipschitz constant can easily be obtained with the same procedure used in the case $Y = \mathbf{R}$, arguing on the single components of f.

We will often use isometric embeddings into l^∞ or, more generally, into duals of separable Banach spaces. To this aim, we recall that any separable metric space can be isometrically embedded into l^∞ by the map

$$j(x) := (\varphi_0(x) - \varphi_0(x_0), \varphi_1(x) - \varphi_1(x_0), \dots) \qquad x \in X$$

where $\varphi_i(x) = d(x,x_i)$ and $(x_i) \subset X$ is a dense sequence.

Finally, if $Y = G^*$ is the dual of a separable Banach space G we define the distance

$$d_w(x,y) := \sum_{n=0}^{\infty} 2^{-n} |\langle x-y, g_n\rangle| \ , \tag{2.3}$$

where (g_n) is a countable dense set in the unit ball of G. It is easy to check that d_w induces the w^* topology on bounded subsets of Y and that (Y,d_w) is separable.

3 Differentiability of Lipschitz functions

In this section we study the differentiability properties of Lipschitz functions $f : \mathbf{R}^k \to Y$, where Y is a metric space or a dual Banach space.

Definition 3.1 (Metric differential) *Let E be a metric space; we say that a function $f : \mathbf{R}^k \to E$ is* metrically differentiable *at $x \in \mathbf{R}^k$ if there exists a seminorm $\| \cdot \|_x$ in $\mathbf{R}^k$ such that*

$$d\,(f(y), f(x)) - \|y - x\|_x = o(|y - x|) \ .$$

This seminorm will be said to be the metric differential and be denoted by $mdf(x)$.

The following differentiability result has first been estabilished in [15] (see also [1] for the case $k = 1$).

Theorem 3.2 (Metric differentiability) *Any Lipschitz function $f : \mathbf{R}^k \to E$ is metrically differentiable at $\mathcal{L}^k$-a.e. $x \in \mathbf{R}^k$.*

Using an isometric embedding of $f(\mathbf{R}^k)$ in a dual space we will obtain in Theorem 3.5 a new proof of this differentiability result. In the following theorem we see how the metric differentiability property can be strengthened, taking into account also $d\,(f(y), f(z))$ for y, z close to x. We shall use the natural metric on seminorms given by

$$\delta(s, s') := \sup_{|x| \leq 1} |s(x) - s'(x)| \ .$$

Theorem 3.3 *For any Lipschitz map $f : \mathbf{R}^k \to E$ we have*

$$d\,(f(y), f(z)) - mdf_x(y - z) = o(|y - x| + |z - x|)$$

for $\mathcal{L}^k$-a.e. $x \in \mathbf{R}^k$. Furthermore, there exist a sequence of compact set K_h whose union covers $\mathcal{L}^k$-almost all of $\mathbf{R}^k$ and moduli of continuity ω_h such that $x \mapsto mdf_x$ is δ-continuous in K_h and

$$|d\,(f(y), f(z)) - mdf_z(y - z)| \leq \omega_h(|y - z|)|y - z| \qquad \forall y \in \mathbf{R}^k,\ z \in K_h$$

for any $h \in \mathbf{N}$.

Proof. The first part of the statement is proved in Theorem 2 of [15]. By Lusin theorem we can find a family of compact sets C_h whose union covers $\mathcal{L}^k$-almost all of $\mathbf{R}^k$ and such that $x \mapsto mdf_x$ is δ-continuous in C_h. Analogously, by Egorov theorem, we can find a family of compact sets L_h whose union covers $\mathcal{L}^k$-almost all of $\mathbf{R}^k$ and such that

$$|d\,(f(y), f(z)) - mdf_z(y - z)| \leq \omega_h(|y - z|)|y - z| \qquad \forall y \in \mathbf{R}^k,\ z \in L_h$$

for some modulus of continuity ω_h. By taking the intersections $C_h \cap L_k$ the proof is achieved. $\square$

Now we introduce a natural w^*-differentiability property for Lipschitz maps with values in dual Banach spaces. This concept is of course closely related to other kind of weak-differentials which are around since the foundation of Banach space theory. However, it seems that our particular notion was used for the first time in [12].

Definition 3.4 (Weak* differential) *Let $Y = G^*$ be a dual Banach space and let $f : \mathbf{R}^k \to Y$ be a function; we say that f is w^*-differentiable at x if there exists a linear map $L : \mathbf{R}^k \to Y$ satisfying*

$$w^* - \lim_{y \to x} \frac{f(y) - f(x) - L(y-x)}{|y-x|} = 0 \ .$$

This map L will be said to be the w^*-differential of f at x and it will be denoted by wdf_x.

The metric differential and the w^*-differential at a given point are obviously related by

$$\|wdf_x(v)\| \le mdf_x(v) \qquad \forall v \in \mathbf{R}^k \ ,$$

by the w^*-lower semicontinuity of the norm. However, the following result can be estabilished:

Theorem 3.5 (Weak* differentiability) *Let $Y = G^*$, with G separable. Any Lipschitz function $f : \mathbf{R}^k \to Y$ is w^*-differentiable and metrically differentiable and fulfills*

$$mdf_x(v) = \|wdf_x(v)\| \qquad \forall v \in \mathbf{R}^k \tag{3.1}$$

for $\mathcal{L}^k$-a.e. $x \in \mathbf{R}^k$.

Proof. For the convenience of the reader we repeat the existence proof for the w^* differential, which could also be considered as a kind of folklore.

Let $D \subset G$ be a dense and countable vector space over $\mathbf{Q}$; by the Rademacher theorem we can find a $\mathcal{L}^k$-negligible set $N \subset \mathbf{R}^k$ such that $f_g(x) = \langle f(x), g \rangle$ is differentiable at any $x \in \mathbf{R}^k \setminus N$ for any $g \in D$. By continuity, we can find for any $x \in \mathbf{R}^k \setminus N$ a linear function $\nabla f(x) : \mathbf{R}^k \to Y$ such that $\langle \nabla f(x), g \rangle = \nabla f_g(x)$ for any $x \in \mathbf{R}^k \setminus N$ and any $g \in D$. By a density argument it is easy to check that f is w^*-differentiable at any $x \in \mathbf{R}^k \setminus N$ and $\nabla f(x) = wdf_x$.

Using the lower w^*-semicontinuity of the norm we infer

$$\|wdf_x(v)\| \le \liminf_{t \downarrow 0} \frac{\|f(x+tv) - f(x)\|}{t} \qquad \forall v \in \mathbf{R}^k \ . \tag{3.2}$$

Let $D' \subset \mathbf{S}^{k-1}$ be a countable dense set; setting $\nabla f_g = 0$ in N and $\nabla f = 0$ in N as well, for any $x \in \mathbf{R}^k$ and any $v \in D'$ we define $\nabla_v f(x)$ as the unique element of $y \in Y$ such that $\langle y, g \rangle = \nabla_v f_g(x)$ for any $g \in D$. By a well known theorem about derivatives of functions in Sobolev spaces (see for instance [31], Theorem 2.1.4) there exists a $\mathcal{L}^k$-negligible set $N' \subset \mathbf{R}^k$ such that

$$\langle f(x+tv) - f(x), g \rangle = \int_0^t \nabla_v f_g(x + \tau v)\, d\tau$$

and

$$\lim_{\varrho\downarrow 0}\frac{1}{\varrho}\int_0^{\varrho}\|\nabla_v f(x+\tau v)\|\,d\tau=\|\nabla_v f(x)\|$$

for any $t>0$, $v\in D'$, $g\in D$ and $x\in\mathbf{R}^k\setminus N'$. By density this yields

$$|\langle f(x+tv)-f(x),g\rangle|\leq\int_0^t|\nabla_v f_g(x+\tau v)|\,d\tau=\int_0^t|\langle\nabla_v f(x+\tau v),g\rangle|\,d\tau$$

for any $t>0$, $v\in D'$, $g\in G$ and $x\in\mathbf{R}^k\setminus N'$, hence

$$\|f(x+tv)-f(x)\|\leq\int_0^t\|\nabla_v f(x+\tau v)\|\,d\tau\ .$$

If $x\notin(N\cup N')$ and $v\in D'$ we can divide both sides by t and let $t\downarrow 0$ to get

$$\limsup_{t\downarrow 0}\frac{\|f(x+tv)-f(x)\|}{t}\leq\|wdf_x(v)\|\ .$$

By density again, the inequality above holds for any $v\in\mathbf{S}^{k-1}$ and, in conjunction with (3.2), gives the metric differentiability of f at x and (3.1). □

Remark 3.6 Assuming that $E=Y$ is the dual of a separable Banach space, the conditions on K_h listed in Theorem 3.3 can be, with a similar argument, strengthened: we can require that f is w^*-differentiable at any point of K_h, and that $x\mapsto wdf_x(v)$ is w^*-continuous in K_h for any $v\in\mathbf{R}^k$; we can also require that

$$d_w\left(f(y)-f(z),wdf_z(y-z)\right)\leq\omega_h(|y-z|)|y-z|\qquad\forall y\in\mathbf{R}^k,\ z\in K_h\ .$$

Using the δ-continuity of the metric differential on K_h we obtain also

$$\begin{aligned}|d\left(f(y),f(z)\right)-mdf_x(y-z)|&=o(|y-z|)\\ d_w\left(f(y)-f(z),wdf_x(y-z)\right)&=o(|y-z|)\end{aligned}\tag{3.3}$$

as $z\in K_h$ converges to x.

4 Norms and jacobians

In the framework of an area formula for Lipschitz mappings between rectifiable metric spaces, we will need to generalize the notion of the jacobian of a linear map between Euclidean spaces. Since the metric differential is only a seminorm, not necessarily given by an inner product, we have to consider general finite dimensional linear maps and spaces.

Definition 4.1 (Jacobians) *Let W, V be Banach spaces, $L:W\to V$ linear. If $k=\dim W$ is finite, the "k-jacobian" of L is defined by*

$$\mathbf{J}_k(L):=\frac{\omega_k}{\mathcal{H}^k(\{x:\ \|L(x)\|\leq 1\})}\ .$$

If s is a seminorm in $\mathbf{R}^k$ we define also

$$\mathbf{J}_k(s):=\frac{\omega_k}{\mathcal{H}^k(\{x:\ s(x)\leq 1\})}\ .$$

Notice that the second definition of jacobian could be considered as a particular case of the first one with $W = \mathbf{R}^k$ and $V = l^\infty$: in fact, any convex and symmetric set $C \subset \mathbf{R}^k$ is the intersection of a sequence of strips

$$S_h := \left\{ x \in \mathbf{R}^k : \ |\langle a_h, x\rangle| \leq 1 \right\} ,$$

for a suitable bounded sequence $(a_h) \subset \mathbf{R}^k$. Hence, given a seminorm s and $C = \{x : \ s(x) \leq 1\}$, by setting

$$L(x) := (\langle a_0, x\rangle, \langle a_1, x\rangle, \dots) \qquad\qquad x \in \mathbf{R}^k$$

we obtain $s(x) = \|L(x)\|$, hence $\mathbf{J}_k(s) = \mathbf{J}_k(L)$.

If W, V are Hilbert spaces it is well known that $\mathbf{J}_k(L)$ coincides with $\sqrt{\det(L^* \circ L)}$. In [15] an expression of the jacobian for linear maps from Euclidean into general Banach spaces can be found (compare also Chapter 6 of [26]). We will often need the following simple chain rule for the jacobians.

Lemma 4.2 *If* $\dim U = \dim V = k \leq \dim W$ *and* $K : U \to V$, $L : V \to W$ *are linear maps, then*

$$\mathbf{J}_k\left(L \circ K\right) = \mathbf{J}_k(L)\mathbf{J}_k(K) \ . \tag{4.1}$$

PROOF. The statement relies on the simple observation that any translation invariant and locally finite measure on a k-dimensional normed space is a certain constant multiple of the k-dimensional Hausdorff measure on this space, in fact any linear isomorphism to $\mathbf{R}^k$ reduces the situation to the more familiar one about multiples of Lebesgue measure in Euclidean space. Since we already noticed that $\mathcal{H}^k(\{x : \ \|x\| \leq 1\}) = \omega_k$ for all k and all norms, we conclude that the jacobian $\mathbf{J}_k L$ of any linear map L is just the proportion of the k-dimensional Hausdorff of the L-image of any set to the $\mathcal{H}^k$-measure of the set itself. So, (4.1) becomes obvious. □

5 Area formula and rectifiable sets

The following generalization of the Euclidean area formula to the case of Lipschitz maps f from the Euclidean space $\mathbf{R}^k$ into a metric space E has been proved in [15], Corollary 8.

Theorem 5.1 (Area formula) *Let* $f : \mathbf{R}^k \to E$ *be a Lipschitz function. Then*

$$\int_{\mathbf{R}^k} \theta(x)\mathbf{J}_k(mdf_x)\, dx = \int_E \sum_{x \in f^{-1}(y)} \theta(x)\, d\mathcal{H}^k(y)$$

for any Borel function $\theta : \mathbf{R}^k \to [0, \infty]$ *and*

$$\int_A \theta(f(x))\mathbf{J}_k(mdf_x)\, dx = \int_E \theta(y)\mathcal{H}^0\left(A \cap f^{-1}(y)\right)\, d\mathcal{H}^k(y)$$

for $A \in \mathcal{B}(\mathbf{R}^k)$ *and any Borel function* $\theta : E \to [0, \infty]$.

The proof of Theorem 5.1 is mainly based on the following lemma (see [15], Lemma 4), which is of independent interest.

Lemma 5.2 *Let $f : \mathbf{R}^k \to E$ be a Lipschitz function and let $B \subset \mathbf{R}^k$ be the Borel set of points $x \in \mathbf{R}^k$ such that mdf_x exists and is a norm. Then, for any $\lambda > 1$ there exist a sequence of norms $\|\cdot\|_i$ and a Borel partition (B_i) of B such that*

$$\frac{1}{\lambda}\|x-y\|_i \leq d\left(f(x), f(y)\right) \leq \lambda\|x-y\|_i \qquad \forall x,\, y \in B_i,\ i \in \mathbf{N}\ .$$

Definition 5.3 (Rectifiable sets and measures) *We say that a Borel set $S \subset E$ is countably $\mathcal{H}^k$-rectifiable if there exists a sequence of Lipschitz functions $f_j : A_j \subset \mathbf{R}^k \to E$ such that $\mathcal{H}^k\left(S \setminus \cup_j f_j(A_j)\right) = 0$.*
We say that $\mu \in \mathcal{M}(E)$ is k-rectifiable if $\mu = \theta\mathcal{H}^k \llcorner S$ for some countably $\mathcal{H}^k$-rectifiable set S and some Borel function $\theta : S \to (0,\infty)$.

Countably $\mathcal{H}^k$-rectifiable sets are closed under finite or countable unions, and it is not hard to see that the property of being countably $\mathcal{H}^k$-rectifiable is intrinsic, i.e. if E is isometrically embedded in another metric space F then S is countably $\mathcal{H}^k$-rectifiable in E if and only if S is countably $\mathcal{H}^k$-rectifiable in F. If E is a Banach space, using the Lipschitz extension theorem mentioned in Section 3 it can be easily seen that countably $\mathcal{H}^k$-rectifiability can be restated in an equivalent way by requiring the existence of countably many Lipschitz functions[1] $f_j : \mathbf{R}^k \to E$ whose images cover $\mathcal{H}^k$-almost all of S.

By the Radon–Nikodym theorem, a positive finite Borel measure μ is k-rectifiable if and only if it is absolutely continuous with respect to $\mathcal{H}^k \llcorner S$ for some countably $\mathcal{H}^k$-rectifiable set S. However, the Radon–Nikodym theorem does not provide an explicit formula for θ. Like in the Euclidean spaces, θ can be recovered as a spherical density, as the following theorem shows.

Theorem 5.4 (Spherical density) *Let $\mu = \theta\mathcal{H}^k \llcorner S$ be a k-rectifiable measure in E. Then*

$$\lim_{\varrho\downarrow 0} \frac{\mu(B_\varrho(x))}{\omega_k \varrho^k} = \theta(x) \qquad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in S\ .$$

The above theorem has been proved by the first author (see [15], Theorem 9) when θ is a characteristic function; a simple comparison argument together with (2.1) and (2.2) proves the result in the general case. We also recall that (2.1) easily implies

$$\lim_{\varrho\downarrow 0} \frac{\mu(B_\varrho(x))}{\omega_k \varrho^k} = 0 \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in E \setminus S\ .$$

without any rectifiability assumption on S.

Now we define an approximate tangent space to countably $\mathcal{H}^k$-rectifiable sets in dual Banach spaces; the definition is first given using a Lipschitz parametrization of the set and then it is compared with more intrinsic properties related to w^*-limits of secant vectors to the set. Finally, using an isometric embedding the definition is extended to the general metric case.

[1] or, equivalently, a single Lipschitz map $f : \mathbf{R}^k \to E$

Definition 5.5 (Approximate tangent space) *Let Y be the dual of a separable Banach space, let $S \in \mathcal{B}(Y)$, and assume that $S = f(B)$ for some Lipschitz function $f : \mathbf{R}^k \to Y$, one to one on $B \in \mathcal{B}(\mathbf{R}^k)$. For any $x \in S$ such that f is metrically and w^*-differentiable at $y = f^{-1}(x)$, with $\mathbf{J}_k(wdf_y) > 0$, we define the approximate tangent space $\mathrm{Tan}^{(k)}(S, x)$ as $wdf_y(\mathbf{R}^k)$.*

If $S \subset Y$ is any countably $\mathcal{H}^k$-rectifiable set and $S_{ij} = f_j(B_i)$ are given by Lemma 5.2, we define

$$\mathrm{Tan}^{(k)}(S, x) := \mathrm{Tan}^{(k)}(S_{ij}, x) \qquad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in S_{ij} \cap S \ .$$

Notice that, by the area formula, the S_{ij}'s cover $\mathcal{H}^k$-almost all of S.

Even though the S_{ij}'s above are not disjoint in general, the definition is well posed because of the following result:

Lemma 5.6 (Locality) *Let $S_i = f_i(B_i)$ with $f_i \in \mathrm{Lip}(\mathbf{R}^k, Y)$ one to one on $B_i \in \mathcal{B}(\mathbf{R}^k)$, $i = 1$, 2. Then*

$$\mathrm{Tan}^{(k)}(S_1, x) = \mathrm{Tan}^{(k)}(S_2, x) \qquad \textit{for } \mathcal{H}^k\textit{-a.e. } x \in S_1 \cap S_2 \ .$$

More generally the conclusion above holds for any pair of countably $\mathcal{H}^k$-rectifiable subsets S_1, S_2 of Y.

Proof. Let $K \subset S_1 \cap S_2$ be a closed set and $K_1 = f_1^{-1}(K)$, $K_2 = f_2^{-1}(K)$. We will prove the inclusion $\subset$ for $\mathcal{H}^k$-a.e. $x \in K$ (the other one follows by a symmetric argument).

Let K_1' be the set of points $z \in K_1$ such that f_1 is metrically and w^*-differentiable, $\mathbf{J}_k(wdf_{1z}) > 0$ and K_1 has density 1 at z, and let K_2' be defined analogously with f_2 in place of f_1; we will prove the inclusion at any point $x \in f_1(K_1') \cap f_2(K_2')$. In fact, if $x = f_1(z) = f_2(y)$, since K_1 has density one at z we can find a unitary basis $w_1, \ldots, w_k$ of $\mathbf{R}^k$ such that, for any $i = 1, \ldots, k$, there exists a sequence $(t_k) \downarrow 0$ with $z + t_k w_i \in K_1$ for any $k \in \mathbf{N}$. Setting

$$x_k = f_1(z + t_k w_i) \in K \ , \qquad y_k = f_2^{-1}(x_k) \in K_2$$

we have $(x_k) \to x$ and $(y_k) \to y$. We can assume, possibly extracting a subsequence, that $(y_k - y)/|y_k - y|$ converge to some unit vector v. Hence, using the w^* and the metric differentiability properties of f_i we get

$$\begin{aligned} wdf_{1z}(w_i) &= w^* - \lim_{k \to \infty} \frac{x_k - x}{t_k} = mdf_{1z}(w_i)\, w^* - \lim_{k \to \infty} \frac{x_k - x}{\|x_k - x\|} \\ &= mdf_{1z}(w_i)\, w^* - \lim_{k \to \infty} \frac{f_2(w_k) - f_2(w)}{\|x_k - x\|} = \frac{mdf_{1z}(w_i)}{mdf_{2w}(v)} wdf_{2w}(v) \ . \end{aligned}$$

This proves that $wdf_{1z}(w_i) \in \mathrm{Tan}^{(k)}(S_2, x)$ for any $i = 1, \ldots, k$, whence the inclusion $\subset$ follows.

Finally, the general locality property for any pair of countably $\mathcal{H}^k$-rectifiable sets follows directly by the previous one and by the construction of the approximate tangent space. □

By construction the approximate tangent space is defined only $\mathcal{H}^k$-a.e., and is a k-dimensional subspace of Y. The following proposition shows the intrinsic character of the approximate tangent space: basically we can say that secant vectors generate (taking w^*-limits) the approximate tangent space; the metric counterpart of this statement will be investigated in Proposition 5.8.

Proposition 5.7 (Secant vectors to rectifiable sets) *Let $S \subset Y$ be countably $\mathcal{H}^k$-rectifiable. Then, we can find a countable family of sets S_i whose union covers $\mathcal{H}^k$-almost all of S and such that*

$$\mathrm{Tan}^{(k)}(S_i, x) \cap \partial B_1 = \left\{ p : \ p = w^* - \lim_{y \in S_i \to x} \frac{y - x}{\|y - x\|} \right\} \tag{5.1}$$

for $\mathcal{H}^k$-a.e. $x \in S_i$.

Proof. We assume without loss of generality that $S \subset f(\mathbf{R}^k)$ for some Lipschitz map $f : \mathbf{R}^k \to Y$; let B_i be given by Lemma 5.2 and let B_i' be the set of all points $y \in B_i$ such that f is metrically and w^*-differentiable at y, $\mathbf{J}_k(mdf_y) > 0$ and (3.1) holds. By the area formula, $S_i = f(B_i')$ cover $\mathcal{H}^k$-almost all of S. Moreover, by definition $\mathrm{Tan}^{(k)}(S_i, x) = wdf_y(\mathbf{R}^k)$ for $\mathcal{H}^k$-a.e. $x = f(y) \in S_i$. If $y_h \in B_i' \setminus \{y\}$ and $x_h = f(y_h) \in S_i$ converge to x, then y_h converge to y and we can assume, possibly extracting a subsequence, that $(y_h - y)/|y_h - y|$ converge to some unit vector v. Using both the metric and the w^*-differentiability at y we get

$$w^* - \lim_{h \to \infty} \frac{x_h - x}{\|x_h - x\|} = w^* - \lim_{h \to \infty} \frac{f(y_h) - f(y)}{\|f(y_h) - f(y)\|} = \frac{wdf_y(v)}{mdf_y(v)} \ .$$

This proves the inclusion $\supset$ in (5.1); the opposite inclusion holds, by a similar argument, at $\mathcal{H}^k$-a.e. point $x = f(y)$ such that B_i' has density 1 at y. $\square$

Using (3.3) we can now describe the local metric behaviour of countably $\mathcal{H}^k$-rectifiable sets with finite measure, showing that locally the distance behaves like the norm in the approximate tangent space. A similar property has been proved in Theorem 9 of [15], in a purely metric setting.

Proposition 5.8 (Local metric behaviour) *Let $S \subset Y$ be a countably $\mathcal{H}^k$-rectifiable set with $\mathcal{H}^k(S) < \infty$. Then, for $\mathcal{H}^k$-a.e. $x \in S$ there exist a Borel set S_x and a linear and w^*-continuous map $\pi_x : Y \to \mathrm{Tan}^{(k)}(S, x)$ equal to the identity on $\mathrm{Tan}^{(k)}(S, x)$, such that $\Theta^*_k(S \setminus S_x, x) = 0$ and*

$$\limsup_{\varrho \downarrow 0} \left\{ \left| \frac{\|\pi_x(y) - \pi_x(z)\|}{\|y - z\|} - 1 \right| : \ y, \, z \in S_x \cap B_\varrho(x), \ y \neq z \right\} = 0 \ .$$

Proof. It is not restrictive to assume that $S \subset f(\mathbf{R}^k)$ for some Lipschitz map $f : \mathbf{R}^k \to Y$. Let K_h be given by Remark 3.6 and B_i given by Lemma 5.2. Let h, i be fixed and $S_{ih} = f(K_h \cap B_i)$; let $x' \in K_h \cap B_i$, $x = f(x')$, assume $S_x = wdf_{x'}(\mathbf{R}^k)$ to be k-dimensional and let π_x be a w^*-continuous linear projection of Y onto S_x.

Since $\mathcal{H}^k$-almost any point of S is a point of density 0 for one of the sets $S \setminus S_{ih}$ the conclusion will be achieved with $S_x = S_{ih}$ if we show that (using a selfexplaining notation)

$$\lim_{\varrho\downarrow 0} \left\{ \frac{\|y-z\|}{\|\pi_x(y-z)\|} \;:\; y,\, z \in B_\varrho(x) \cap S_{ih},\; y \neq z \right\} = 1 \ .$$

Writing $y = f(y')$, $z = f(z')$ with $y',\, z' \in K_h \cap B_i$, the claimed equality is implied by

$$\lim_{\varrho\downarrow 0} \left\{ \frac{\|f(y') - f(z')\|}{md f_{x'}(y'-z')} \;:\; y',\, z' \in B_\varrho(x') \cap K_h \cap B_i,\; y' \neq z' \right\} = 1$$

and

$$\lim_{\varrho\downarrow 0} \left\{ \frac{\|wdf_{x'}(y'-z')\|}{\|\pi_x(f(y') - f(z'))\|} \;:\; y',\, z' \in B_\varrho(x') \cap K_h \cap B_i,\; y' \neq z' \right\} = 1 \ .$$

The first identity follows at once from the first one in (3.3); the prove the second one, consider sequences (y'_l), (z'_l) in $K_h \cap B_i$ both converging to x' and assume with no loss of generality that $v_l = (y'_l - z'_l)/|y'_l - z'_l|$ converge to some unit vector v. Then, the second equality in (3.3) and the w^*-continuity of π_x imply

$$\begin{aligned}
& \lim_{l\to\infty} \frac{\|wdf_{x'}(y'_l - z'_l)\|}{\|\pi_x(f(y'_l) - f(z'_l))\|} \\
= \; & \lim_{l\to\infty} \frac{\|wdf_{x'}(y'_l - z'_l)\|}{|y'_l - z'_l|} \cdot \lim_{l\to\infty} \frac{|y'_l - z'_l|}{\|\pi_x(f(y'_l) - f(z'_l))\|} \\
= \; & \|wdf_{x'}(v)\| \cdot \frac{1}{\|\pi_x(wdf_{x'}(v))\|} = \frac{\|wdf_{x'}(v)\|}{\|wdf_{x'}(v)\|} = 1 \ .
\end{aligned}$$

□

Finally, we conclude this section pointing out how the definition of approximate tangent space can be given for countably $\mathcal{H}^k$-rectifiable subsets S of a general metric space E.

Definition 5.9 *Let S, E as above and let $j : S \to Y$ be an isometric embedding, with $Y = G^*$, G separable (for instance $G = l^1$, $Y = l^\infty$). We define*

$$\mathrm{Tan}^{(k)}(S,x) := \mathrm{Tan}^{(k)}\left(j(S), j(x)\right) \qquad\qquad \forall x \in S \ .$$

Of course, the approximate tangent space is defined $\mathcal{H}^k$-a.e. on S and depends on the choice of the space Y and of the embedding j. However, since j is an isometry, Proposition 5.8 shows that different choices of Y and j simply produce approximate tangent spaces which are isometric for $\mathcal{H}^k$-a.e. $x \in S$. In this sense the definition above is well posed, and will be used to estabilish general area and coarea formulas for Lipschitz maps between rectifiable subsets of metric spaces.

6 Rectifiability criterions

In this section we find some rectifiability criterions for sets and measures in dual Banach spaces. We will see that the condition stated in Proposition 5.7, namely the w^*-convergence of unit secant vectors to nonzero (actually, unit) vectors in a suitable k-dimensional subspace actually provides a characterization of k-rectifiable sets.

For any pair of Banach spaces Y, M, with Y dual space, we define $\Pi_k(Y,M)$ as the collection of all w^*-continuous linear maps $\pi : Y \to M$ such that $\dim(\pi(Y)) = k$. In $\Pi_k(Y,M)$ we define a pseudometric γ as follows:

$$\gamma(\pi,\pi') := \sup_{\|x\|\leq 1} \left|\|\pi(x)\| - \|\pi'(x)\|\right| \ .$$

In general γ is not a metric: for instance, if $Y = M = (C[0,1])^*$, $k=1$ and $\pi_t(\mu) = \mu([0,1])\delta_t$, then $\gamma(\pi_t,\pi_t') = 0$ whenever t, $t' \in [0,1]$. The advantage of γ is that it makes (the quotient space of) $\Pi_k(Y,M)$ separable even though Y is not separable, as the following lemma shows.

Lemma 6.1 *If Y is the dual of a separable Banach space, the set $\Pi_k(Y,M)$, endowed with the pseudometric γ, is separable.*

PROOF. Any $\pi \in \Pi_k(Y,M)$ can be factored (not uniquely) as $\varphi(\lambda)$, where $\lambda \in \Pi_k(Y,\mathbf{R}^k)$ and $\varphi \in \Pi_k(\mathbf{R}^k,M)$.

Let $D \subset G$ be a countable dense set and let

$$\mathcal{F} := \left\{\sum_{i=1}^k \langle x, g_i\rangle \bar{e}_i : \ g_i \in D\right\} \subset \Pi_k(Y,\mathbf{R}^k)$$

where $(\bar{e}_1,\ldots,\bar{e}_k)$ is the canonical basis of $\mathbf{R}^k$. Let $(\varphi_i) \subset \Pi_k(\mathbf{R}^k,M)$ be a sequence such that the sets $C_i = \{v : \ \|\varphi_i(v)\| \leq 1\}$ are dense in

$$\mathcal{C} := \left\{\{v : \ \|\varphi(v)\| \leq 1\} : \ \varphi \in \Pi_k(\mathbf{R}^k,M)\right\}$$

with respect to the Hausdorff topology on compact, convex, symmetric sets.

We will prove that the class $\varphi_i(\bar{\lambda})$ with $i \in \mathbf{N}$ and $\bar{\lambda} \in \mathcal{F}$ is dense. In fact, if $\varphi(\lambda) \in \Pi_k(Y,M)$ and $\varepsilon > 0$ are given, we can find $\bar{\lambda} \in \mathcal{F}$ such that $\|\varphi\|\|\lambda - \bar{\lambda}\| < \varepsilon/2$, hence

$$\gamma\left(\varphi(\lambda),\varphi(\bar{\lambda})\right) \leq \|\varphi\|\|\lambda-\bar{\lambda}\| < \frac{\varepsilon}{2} \ .$$

On the other hand, by the density of the associated convex sets in the Hausdorff topology, we can find $i \in \mathbf{N}$ such that

$$\|\bar{\lambda}\| \left|\|\varphi(y)\| - \|\varphi_i(y)\|\right| \leq \frac{\varepsilon}{2} \qquad \forall y \in \overline{B}_1 \ ,$$

hence $\gamma\left(\varphi(\bar{\lambda}),\varphi_i(\bar{\lambda})\right) < \varepsilon/2$. By the triangle inequality the conclusion follows. $\square$

Using the previous lemma and a standard argument in finite dimensional spaces (see for instance [28], Theorem 11.8) we can establish the following rectifiability result.

Theorem 6.2 (Rectifiability criterion for sets) *Let $Y = G^*$, with G separable, let $S \subset Y$ and assume that for any $x \in S$ there exist $\varepsilon(x) > 0$, $\varrho(x) > 0$ and $\pi_x \in \Pi_k(Y, Y)$ such that*

$$\|\pi_x(y - x)\| \geq \varepsilon(x)\|y - x\| \qquad \forall y \in S \cap B_{\varrho(x)}(x) \ .$$

Then, there exists a sequence of Lipschitz functions $f_h : \mathbf{R}^k \to Y$ such that $S \subset \bigcup_h f_h(\mathbf{R}^k)$.

PROOF. Possibly splitting S in a countable union of sets we can assume with no loss of generality the existence of an integer $j \geq 1$ such that $\varepsilon(x) \geq 1/j$ and $\varrho(x) \geq 1/j$ for any $x \in S$. Let $\{\pi_i\}_{i \in \mathbf{N}}$ be given by Lemma 6.1 and let

$$S_i := \left\{x \in S : \ \gamma(\pi_x, \pi_i) < \frac{1}{2j}\right\} \ , \qquad V_i := \pi_i(Y) \ .$$

We will prove that any subset A of S_i with diameter less than $1/j$ is contained in $f(V_i)$ for some Lipschitz function $f : V_i \to Y$. In fact, if x_1, $x_2 \in A$ we can apply the hypothesis with $x = x_2$ to get

$$\|\pi_i(x_1 - x_2)\| \geq \|\pi_{x_2}(x_1 - x_2)\| - \frac{1}{2j}\|x_1 - x_2\| \geq \frac{1}{2j}\|x_1 - x_2\| \ .$$

This shows that $\pi_i : A \to V_i$ is one to one and that its inverse function has Lipschitz constant less than $2j$. □

Theorem 6.3 (Rectifiability criterion for measures) *Assume $\mu \in \mathcal{M}(Y)$. Then, μ is k-rectifiable if and only if for μ-a.e. $x \in Y$ the following two conditions hold: $0 < \Theta_{k*}(\mu, x) \leq \Theta^*_k(\mu, x) < \infty$ and there exist $\pi_x \in \Pi_k(Y, Y)$ and $\varepsilon(x) > 0$ such that*

$$C_x := \{y \in Y : \|\pi_x(y - x)\| \leq \varepsilon(x)\|y - x\|\}$$

has μ-density 0 at x.

PROOF. By (2.1) the measure μ is concentrated on a Borel set S σ-finite with respect to $\mathcal{H}^k$, and (2.2) implies that μ is absolutely continuous with respect to $\mathcal{H}^k$. Assuming with no loss of generality that both conditions in the statement of the theorem are satisfied for any $x \in S$, we will prove that the sets

$$S_\alpha := \left\{x \in S : \frac{\mu(B_\varrho(x))}{\varrho^k} \geq \alpha \quad \forall \varrho \in (0, \alpha)\right\} \qquad \alpha > 0$$

satisfy the assumptions of Theorem 6.2, and hence are countably $\mathcal{H}^k$-rectifiable. In fact, let $x \in S_\alpha$ and let $\gamma \in (0, 1)$ such that

$$\frac{\varepsilon(x)/2 + \gamma\|\pi_x\|}{1 - \gamma} \leq \varepsilon(x) \ .$$

We claim that $2\|\pi_x(y-x)\| \geq \varepsilon(x)\|y-x\|$ if $y \in S_\alpha$ and $\|y-x\|$ is small enough; in fact, by the triangle inequality we have

$$\|y-y'\| \leq \frac{\gamma}{1-\gamma}\|y'-x\| \ , \quad \|y-x\| \leq \left(1+\frac{\gamma}{1-\gamma}\right)\|y'-x\| = \frac{1}{1-\gamma}\|y'-x\|$$

for any $y' \in B_{\gamma\|y-x\|}(y)$. Setting $r = \|y-x\|$, $2\|\pi_x(y-x)\| \leq \varepsilon(x)\|y-x\|$ implies

$$\begin{aligned} \|\pi_x(y'-x)\| &\leq \frac{\varepsilon(x)}{2}r + \gamma\|\pi_x\|r \leq \frac{\varepsilon(x)/2+\gamma\|\pi_x\|}{1-\gamma}\|y'-x\| \\ &\leq \varepsilon(x)\|y'-x\| \qquad\qquad \forall y' \in B_{\gamma r}(y) \ . \end{aligned}$$

This proves that $B_{\gamma r}(y)$ is contained in C_x; as $\mu(B_{2r}(x)\cap C_x) = o(r^k)$ and

$$\alpha\gamma^k r^k \leq \mu(B_{\gamma r}(y)) \leq \mu(B_{2r}(x)\cap C_x)$$

the claim follows. $\square$

The density condition on C_x is implied by the w^*-convergence of unit secant vectors to a k-dimensional subspace, with a lower bound on the norms of the w^*-limits. The following example shows that the only w^*-convergence of secant vectors to a k-dimensional subspace is not sufficient for rectifiability, not even if supplemented with a uniform density lower bound.

Example 6.4 Let $E = [0,1]$, endowed with the distance $d(x,y) = \sqrt{|x-y|}$. Then, E isometrically embeds in $L^2([0,1])$ with the mapping $t \mapsto \chi_{(0,t)}$. It is easy to check that $\mathcal{H}^2(E) = \pi/4$ and, more generally, that

$$\frac{\mathcal{H}^2(B_\varrho(t))}{\pi\varrho^2} = \frac{1}{2} \qquad \text{for any ball } B_\varrho(t) \subset [0,1] \ .$$

In particular, by Theorem 5.4, E is purely $\mathcal{H}^2$-unrectifiable, i.e. no subset of E with strictly positive $\mathcal{H}^2$-measure is countably $\mathcal{H}^2$-rectifiable. On the other hand, the secant vectors to $\chi_{(0,t)}$

$$\frac{\chi_{(0,s)} - \chi_{(0,t)}}{\sqrt{|s-t|}}$$

weakly converge to 0 in $L^2([0,1])$ as $s \to t$. The same is true if we embed, using l^2 coordinates, $L^2([0,1])$ in l^∞: in this case the secant vectors w^*-converge to 0 in l^∞.

To our knowledge, the problem whether

$$\lim_{\varrho\downarrow 0}\frac{\mathcal{H}^k(E\cap B_\varrho(x))}{\omega_k\varrho^k} = 1 \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in E$$

implies rectifiability for a general metric space E is open. This is known to be true in Euclidean spaces (see [21], [23]) or in case one dimensional measures are considered (see [27]). For two dimensional measures, first promising nonEuclidean results have been obtained in [20]. Finally, the results in [3] indicate that the implication might be true for spaces which can be isometrically embedded in Hilbert spaces.

7 Unrectifiable metric spaces

In this section we deal with examples of purely k-unrectifiable metric spaces, i.e. metric spaces E such that $\mathcal{H}^k(S) = 0$ for any countably $\mathcal{H}^k$-rectifiable set $S \subset E$.

The first example is the Heisenberg group H; for simplicity we consider the lowest dimensional one, made of all pairs (z, t) with $z \in \mathbf{C}$ and $t \in \mathbf{R}$. The noncommutative group operation is

$$(z, t)(z', t') := (z + z', t + t' + 2\mathrm{Im}(z\bar{z}'))$$

so that $(0, 0)$ is the identity and $(z, t)^{-1} = (-z, -t)$. The Heisenberg group becomes a metric space (see [19]) when endowed with the homogeneous norm $\|(z, t)\| = (|z|^4 + t^2)^{1/4}$ and with the distance

$$d(x, y) := \|x^{-1}y\| \ .$$

It is easy to check that H has Hausdorff dimension 4, strictly larger than the topological dimension. The group law, the norm and the distance are well behaved with respect to the dilations $\delta_r(z, t) = (rz, r^2t)$; these dilations can be used to prove the following differentiability theorem, proved by P. Pansu in the more general framework of Lipschitz maps between Carnot–Carathéodory spaces.

Theorem 7.1 *Let $A \subset \mathbf{R}^k$ be a Borel set and let $f : A \to H$ be a Lipschitz function. Then for $\mathcal{L}^k$-a.e. $x \in A$ there exists a group homomorphism $df_x : \mathbf{R}^k \to H$ such that*

$$\lim_{t \to 0} \delta_{1/t} \left([f(x)]^{-1} f(x + tv) \right) = df_x(v) \qquad \forall v \in \mathbf{R}^k \ .$$

Notice that the result is stated in [25] under the assumption that A is an open set, but its proof works with minor modifications also in the general case. Using the Pansu and the metric differentiability theorems and following basically the argument in §11.5 of [4] we can obtain the following result:

Theorem 7.2 *The Heisenberg group is purely k-unrectifiable for $k = 2$, 3, 4.*

PROOF. Let $f : A \subset \mathbf{R}^k \to H$ be a Lipschitz map and let us prove that $\mathcal{H}^k(f(A)) = 0$. Since H is complete we can assume with no loss of generality that A is closed. By the area formula we need only to check that $\mathbf{J}_k(mdf_x) = 0$ at any metric differentiability point where the Pansu differential df_x is defined. Since $df_x(\mathbf{R}^k)$ is a commutative subgroup of H, it must be contained in $\mathbf{R}z_0 \times \mathbf{R}$ for some $z_0 \in \mathbf{C}$; on the other hand, writing $df_x(v) = (z(v), t(v))$, the inequality

$$|t(v) - t(v')| \leq [\mathrm{Lip}(df_x)]^2 \, |v - v'|^2 \qquad \forall v,\, v' \in \mathbf{R}^k$$

implies that t is constant, hence the image of df_x is contained in $\mathbf{R}z_0 \times \{0\}$ and the kernel of df_x has dimension at least $k - 1 \geq 1$. Since

$$mdf_x(v) = \lim_{t \downarrow 0} \frac{d\left(f(x + tv), f(x)\right)}{t} = \lim_{t \to 0} \|\delta_{1/t}\left([f(x)]^{-1} f(x + tv) \right)\| = \|df_x(v)\|$$

for any $v \in \mathbf{R}^k$ we conclude that $\mathcal{H}^k\left(\{v \in \mathbf{R}^k : \ mdf_x(v) \leq 1\}\right) = \infty$ and hence that $\mathbf{J}_k(mdf_x) = 0$. $\square$

The statement is false for $k = 1$; indeed, it can be proved (see for instance [29], Section III.4) that any pair of points in H can be connected by a curve with finite length. The lack of rectifiable sets in the Heisenberg group suggests that more intrinsic definitions of rectifiability could be useful in this space, related for instance to level sets of regular functions. Some ideas in this direction can be found in [9].

In the following definition we introduce a property stronger than pure k-unrectifiability.

Definition 7.3 (Strongly k-unrectifiable spaces) *Let (E, d) be a metric space with $\mathcal{H}^k(E) < \infty$. We say that E is strongly k-unrectifiable if $\mathcal{H}^k(f(E)) = 0$ for any Lipschitz map f with values into an Euclidean space.*

By Lemma 5.2 we infer that any strongly k-unrectifiable space is purely k-unrectifiable, but the opposite implication does not hold: in fact, there are simple examples of purely 1-unrectifiable sets in the Euclidean plane having linear projections with strictly positive $\mathcal{H}^1$-measure, see for instance Lemma 18.12 in [23]. An example of purely 1-unrectifiable set E in the Euclidean plane such that $\mathcal{H}^1(E) = 1$ and $\mathcal{H}^1(f(E)) = 0$ for any $f \in \mathrm{Lip}(E, \mathbf{R})$ was constructed by A.G. Vituškin, L.D. Ivanov and M.S. Melnikov in [30] (see also [16] for a simplified and rigorous presentation); this property is close to strong k-unrectifiability, but of course no subset of any Euclidean space can be strongly k-unrectifiable.

We conclude this section with a remarkable example of strongly k-unrectifiable space; this shows that that rectifiability can not be deduced by the rectifiability of the projections, not even if nonlinear projections on Euclidean spaces of arbitrary dimension are allowed. The construction is a modification of an unpublished idea of S. Konyagin [17] which answers the more special question posed in [10] for the case of one dimensional measure and real Lipschitz functions.

Theorem 7.4 *For any dimension $\sigma > 0$ there exists a compact metric space (X, σ) such that $\mathcal{H}^s_\sigma(X) = 1$ but any Lipschitz image of X in any Euclidean space is $\mathcal{H}^s$-negligible.*

Proof. For any $j \geq 1$ we consider the space

$$X(j) := \{0, 1\}^j \cong \{A : \ A \subset \{1, \ldots, j\}\}$$

equipped with the normalized l^1-metric

$$\rho_j(x, y) := \frac{1}{j} \sum_{i=1}^{j} |x_i - y_i| \cong \frac{1}{j} \mathrm{card}(x \Delta y) \quad .$$

In the sequel we will use the following two observations.

Fact 1. If $1 \leq k \leq j$ then

$$\sum_{l=0}^{k} \binom{j}{l} \leq 2^{j+1} \binom{k}{j} \quad .$$

Indeed, this inequality obviously holds if $k = 1$ or $k \geq j/2$. Moreover, the left hand side is convex on $\{1, \ldots, [(j+1)/2]\}$ whereas the right one is linear in k.

Fact 2. For each $\delta > 0$ there is an integer j_δ such that for any $j \geq j_\delta$ and any $A, B \subset X(j)$ fulfilling $\mathrm{card}(A), \mathrm{card}(B) \geq \delta \cdot \mathrm{card}(X(j))$ the ρ_j-distance between A and B is necessarily less than δ. The best estimate of this kind can be obtained using Harper's inequality based on combinatorial considerations, see [13]. Alternatively, the claim is also a consequence of the isoperimetric results for the Hamming metric proved using martingale techniques, see §7.9 of [24]. It states that (here $\rho_j(x, M)$ denotes the ρ_j distance of x from M)

$$\mathrm{card}\left\{x \in X(j) : \ \rho_j(x, M) \geq \frac{\delta}{2}\right\} \leq 2\exp(-j\delta^2/64)2^j$$

for any $M \subset \{0,1\}^j$ with $\mathrm{card}(M) \geq 2^{j-1}$ and positive δ. Our statement follows now by the usual argument applied e.g. in the local theory of Banach spaces to prove the phenomenon of concentration of measure. Indeed, we choose

$$T = \inf\left\{t \geq 0 : \ \mathrm{card}\{x : \ \rho_j(x, A) \leq t\} \geq 2^{j-1}\right\} \ .$$

Then T is the median of $x \to \rho_j(A, x)$, which means that both sets $M_+ = \{\rho_j(\cdot, A) \geq T\}$, $M_- = \{\rho_j(\cdot, A) \leq T\}$ have cardinality at least $\mathrm{card}(X(j))/2$. Since for j large enough $2\exp(-j\delta^2/64) < \delta$, we conclude that $\mathrm{dist}_{\rho_j}(A, M_+) \leq \delta/2$, hence $T \leq \delta/2$. Analogously, we infer that $\mathrm{dist}_{\rho_j}(B, M_-) \leq \delta/2$ and so the definition of M_- gives that $\mathrm{dist}_{\rho_j}(A, B) \leq T + \delta/2 \leq \delta$.

For $k \geq 0$ we set $m_k = \sum_{l=0}^{k} l = k(k+1)/2$ and choose the set $I_{k+1} = \{m_k + 1, \ldots, m_{k+1}\}$ of cardinality $k+1$. Our space X will be the set $\{0,1\}^{\mathbf{N}_+}$ of all sequences $(\gamma_1, \gamma_2, \ldots)$ with $\gamma_i \in \{0,1\}$. So, for $\gamma \in X$ the restriction $\gamma \llcorner I_k$ can be understood as an element of $X(k)$. Given $n \geq 0$ we define the "tail" of $\gamma \in X$ by

$$T_n(\gamma) = \{\bar{\gamma} : \ \gamma_i = \bar{\gamma}_i \text{ for } i \leq n\} \ ,$$

so that in particular $T_0(\gamma) = X$. Finally, for different $\gamma, \bar{\gamma} \in X$ define the distance of γ to $\bar{\gamma}$ to be

$$\sigma_s(\gamma, \bar{\gamma}) = 2^{-m_{j-1}/s} \left[\rho_j(\gamma \llcorner I_j, \bar{\gamma} \llcorner I_j)\right]^{\min(1,1/s)}$$

where

$$j := j(\gamma, \bar{\gamma}) = \min\{j \geq 1 : \ \gamma \llcorner I_j \neq \bar{\gamma} \llcorner I_j\} \ .$$

Since $2^{j/s} \geq j^{\min(1,1/s)}$ for all $s > 0$, it follows that

$$2^{-m_j/s} \leq \sigma_s(\gamma, \bar{\gamma}) \leq 2^{-m_{j-1}/s}$$

and using these inequalities it is easily checked that σ_s is indeed a metric which induces on $\{0,1\}^{\mathbf{N}_+}$ the canonical product topology.

Now we show that $\omega_s/2^{s+1} \leq \mathcal{H}^s_{\sigma_s}(X) \leq 1$. The upper estimate readly follows using covers of the type $\{T_{m_k}(\gamma) \ ; \ \gamma \in X\}$ (for a fixed $k \geq 1$) which consists of 2^{m_k} sets of diameter $2^{-m_k/s}$. To obtain the lower estimate we consider the canonical product probability measure μ on X such that $\mu(\{\gamma:\ \gamma_i = 1\}) = 1/2$ for all i. Obviously, it suffices to show that

$$\mu(A) \le 2 \cdot \mathrm{diam}_{\sigma_s}(A)^s \text{ for all } A \in \mathcal{B}(X) \ . \tag{7.1}$$

Of course, we can suppose that A contains at least two points and set $j_0 = \min\{j(\gamma, \bar\gamma):\ \gamma, \bar\gamma \in A \text{ different}\}$. We also fix $\gamma_0 \in A$, set

$$\tilde A := \{\gamma \llcorner I_{j_0}:\ \gamma \in A\} \subset X(j_0)$$

and notice that $\tilde A$ contains at least two points. Since $\mu(T_n(\gamma)) = 2^{-n}$ for all $n \ge 0$, $\gamma \in X$, we have

$$\mu(A) \le \mu\left(\{\gamma:\ \gamma \llcorner I_j = \gamma_0 \llcorner I_j \text{ if } j < j_0 \text{ and } \gamma \llcorner I_{j_0} \in \tilde A\}\right) = 2^{-m_{j_0}} \mathrm{card}(\tilde A) \ .$$

Choosing the maximal k which satisfies $\mathrm{card}(\tilde A) > \sum_{l=0}^{k-1} \binom{j_0}{l}$, we obtain

$$1 \le k \le j_0 \text{ and } \mathrm{diam}_{\sigma_s}(A) \ge 2^{-m_{j_0-1}/s} \cdot \left(\frac{k}{j_0}\right)^{\min(1,1/s)} \ .$$

Consequently, due to Fact 1

$$\begin{aligned} \mathrm{diam}_{\sigma_s}(A)^s &\ge 2^{-m_{j_0-1}} \left(\frac{k}{j_0}\right) \ge 2^{-m_{j_0-1}-j_0-1} \sum_{l=0}^{k} \binom{j_0}{l} \\ &\ge \frac{1}{2} 2^{-m_{j_0}} \mathrm{card}(\tilde A) \ge \mu(A)/2 \end{aligned}$$

which establishes (7.1).

Moreover, the natural isomorphism between the tails $T_k(\gamma)$ and $T_k(\bar\gamma)$ ensures that

$$\mathcal{H}^s_{\sigma_s}(T_k(\gamma)) = 2^{-k} \cdot \mathcal{H}^s_{\sigma_s}(X) \text{ for all } \gamma \in X,\ k \ge 0 \ . \tag{7.2}$$

To finish the proof of the theorem, we assume now by contradiction that we are given an integer $d \ge s$ and a 1-Lipschitz map $f : (X, \sigma_s) \to \mathbf{R}^d$ such that $\mathcal{H}^s(f(X)) > 0$.

Since $\mathcal{H}^s(f(X)) \le \mathcal{H}^s_{\sigma_s}(X) < \infty$, we have $\Theta^*_s(f(X), x) \le 1$ at $\mathcal{H}^s$-a.e. $x \in \mathbf{R}^d$. Therefore, we find a set $Y \subset f(X)$ with $\mathcal{H}^s(Y) > \varepsilon > 0$ and a j_0 such that

$$\mathcal{H}^s(Z) \le 2^{1+s} \mathrm{diam}(Z)^s \text{ if } Z \subset Y \text{ and } \mathrm{diam}(Z) \le 2^{-m_{j_0}/s}.$$

Next, we choose an integer N and a positive δ such that

$$N > \sqrt[s]{2^{2+s} \cdot 3^d/\varepsilon} \cdot \sqrt{d} \text{ and } \delta < \min\{\sqrt[s]{\varepsilon/2^{1+s}}/(N\sqrt{d}), \varepsilon/(2N^d)\} \tag{7.3}$$

and we select $j_1 \ge \max\{j_0, j_\delta\}$ where j_δ was introduced in Fact 2.

Obviously, there is a $\gamma_0 \in X$ such that

$$B = T_{m_{j_1}}(\gamma_0) \text{ satisfies } \mathcal{H}^s(f(B) \cap Y) \ge \varepsilon 2^{-m_{j_1}} \ .$$

Since $\mathrm{diam}(f(B)) \le \mathrm{diam}(B) = 2^{-m_{j_1}/s}$, we find a cube Q of size $q \le 2^{-m_{j_1}/s}$ containing $f(B) \cap Y$. Let $\mathcal{F}$ be the family of N^d disjoint subcubes of Q of size q/N. We denote by $\mathcal{F}_s$ the subfamily of those $\tilde{Q} \in \mathcal{F}$ fulfilling $\mathcal{H}^s(\tilde{Q} \cap f(B) \cap Y) < \frac{\varepsilon}{2} 2^{-m_{j_1}} N^{-d}$. Obviously $\mathcal{H}^s(\bigcup \mathcal{F}_s \cap f(B) \cap Y) \le \frac{\varepsilon}{2} 2^{-m_{j_1}}$, so

$$\mathcal{H}^s(\bigcup(\mathcal{F} \setminus \mathcal{F}_s) \cap f(B) \cap Y) \ge \frac{\varepsilon}{2} 2^{-m_{j_1}} \quad .$$

We also know that for each $\tilde{Q} \in \mathcal{F}$

$$\mathcal{H}^s(\tilde{Q} \cap f(B) \cap Y) \le 2^{1+s} \left(\frac{q}{N}\right)^s d^{s/2} \le 2^{1+s} d^{s/2} 2^{-m_{j_1}} N^{-s} \quad .$$

Consequently, based on our choice of N we conclude

$$\mathrm{card}(\mathcal{F} \setminus \mathcal{F}_s) \ge \frac{\varepsilon}{2} 2^{-m_{j_1}} / (2^{1+s} d^{s/2} 2^{-m_{j_1}} N^{-s}) = \frac{\varepsilon}{2^{2+s} d^{s/2}} N^s > 3^d \quad .$$

In particular, there are Q_1, $Q_2 \in \mathcal{F} \setminus \mathcal{F}_s$ with

$$\mathrm{dist}(Q_1, Q_2) \ge \frac{q}{N} \ge \frac{\sqrt[s]{\varepsilon 2^{-m_{j_1}} / 2^{1+s}}}{N\sqrt{d}} > \delta \cdot 2^{-m_{j_1}/s} \quad , \tag{7.4}$$

since $q\sqrt{d} \ge \mathrm{diam}(f(B) \cap Y) \ge \sqrt[s]{\mathcal{H}^s(f(B) \cap Y)/2^{1+s}}$.

Denote $M_i = f^{-1}(Q_i \cap Y) \cap B$, $A_i = \{\gamma \llcorner I_{j_1+1} \ ; \ \gamma \in M_i\} \subset X(j_1+1)$. Since $Q_i \notin \mathcal{F}_s$, we see that $\mathcal{H}^s(M_i) \ge \mathcal{H}^s\left(Q_i \cap Y \cap f(B)\right) \ge \varepsilon 2^{-m_{j_1}-1} N^{-d}$. Moreover, the definition of B and (7.2) ensure that $\mathcal{H}^s_{\sigma_s}(M_i) \le \mathrm{card}(A_i) \cdot 2^{-m_{j_1+1}}$. Therefore,

$$\mathrm{card}(A_i) \ge \varepsilon 2^{j_1} N^{-d} \ge \delta \cdot \mathrm{card}(X(j_1+1)) \text{ for } i = 1,\ 2 \quad .$$

Since $j_1 \ge j_\delta$, we conclude from Fact 2 that $\mathrm{dist}_{\rho_{j_1+1}}(A_1, A_2) \le \delta$, which in turn implies

$$\mathrm{dist}_{\sigma_s}(M_1, M_2) \le 2^{-m_{j_1}/s} \cdot \delta \quad .$$

However, this combined with $\mathrm{Lip}(f) \le 1$ obviously contradicts (7.4). $\square$

8 Tangential differentiability and general area formula

In this section we prove a general area formula for Lipschitz maps defined on general countably $\mathcal{H}^k$-rectifiable subsets S of a metric space E. We consider first the case when $E = Y$ is the dual of a separable Banach space and then we recover the general case using an isometric embedding.

The jacobian appearing in the general area formula depends on a "tangential differential", seen as a linear map defined on $\mathrm{Tan}^{(k)}(S, x)$, whose existence is ensured by the following theorem.

Theorem 8.1 (Tangential differential on rectifiable sets) *Let $Y = G^*$, $Z = H^*$ be duals of a separable Banach spaces G, H, let $S \subset Y$ be countably $\mathcal{H}^k$-rectifiable and let $g \in \mathrm{Lip}(S, Z)$. Let $\theta : S \to (0, \infty)$ be integrable with respect to $\mathcal{H}^k \llcorner S$ and set $\mu = \theta \mathcal{H}^k \llcorner S$.*

Then, for $\mathcal{H}^k$-a.e. $x \in S$ there exist a linear and w^*-continuous map $L : Y \to Z$ and a Borel set $S^x \subset S$ such that $\Theta^*_k(\mu \llcorner S^x, x) = 0$ and

$$\lim_{y \in S \setminus S^x \to x} \frac{d_w\left(g(y), g(x) + L(y - x)\right)}{|y - x|} = 0 \ . \tag{8.1}$$

The map L is uniquely determined on $\mathrm{Tan}^{(k)}(S, x)$, and its restriction to this space, denoted by $d^S g_x$, satisfies the chain rule

$$wd(g \circ h)_y = d^S g_{h(y)} \circ wdh_y \qquad \textit{for } \mathcal{L}^k\textit{-a.e. } y \in A \tag{8.2}$$

for any Lipschitz function $h : A \subset \mathbf{R}^k \to S$.

Proof. We first assume $Z = \mathbf{R}$ and, without loss of generality, $S \subset f(\mathbf{R}^k)$ for some Lipschitz map $f : \mathbf{R}^k \to Y$. Let $S_i = f(B_i)$ be as in the proof of Proposition 5.7; we will prove that (8.1) holds, for a suitable w^*-continuous map $L : Y \to \mathbf{R}$ and with $S^x = S \setminus S_i$, at any point $x \in S \cap S_i$ where the following conditions hold:

(a) f is metrically and w^*-differentiable at $z = f^{-1}(x) \in B_i$, $\mathbf{J}_k(mdf_z) > 0$ and $\mathrm{Tan}^{(k)}(S_i, x) = wdf_z(\mathbf{R}^k)$;

(b) $g \circ f$ is differentiable at y and

$$\lim_{\varrho \downarrow 0} \frac{\mu(B_\varrho(x) \cap S_i)}{\omega_k \varrho^k} = \theta(x) > 0 \ , \qquad \lim_{\varrho \downarrow 0} \frac{\mu(B_\varrho(x) \setminus S_i)}{\omega_k \varrho^k} = 0 \ .$$

We define $L(v) = d(g \circ f)_z \left(wdf_z\right)^{-1}(v)$ for any $v \in \mathrm{Tan}^k(S_i, x)$, and extend L to a linear and w^*-continuous map on the whole of Y. Under the above density assumptions, (8.1) is implied by the pointwise limit

$$\lim_{y \to x, y \in S_i} \frac{g(y) - g(x) - L(y - x)}{|y - x|} = 0 \ .$$

Writing $y = f(w)$, the limit above is equivalent to

$$\lim_{k \to \infty} \frac{g(f(z_k)) - g(f(z)) - L(f(z_k) - f(z))}{|f(z_k) - f(z)|} = 0 \tag{8.3}$$

for any sequence $(z_k) \subset B_i \setminus \{z\}$ converging to z. Assuming with no loss of generality that $(z_k - z)/|z_k - z|$ converge to some unit vector ν, we infer that the limit above is equal to

$$\frac{d(g \circ f)_y(v) - L(wdf_y(v))}{mdf_z(v)}$$

which is 0 by the construction of L. A similar argument based on (5.1) also proves that the restriction of L to $\mathrm{Tan}^{(k)}(S_i, x)$ is uniquely determined by (8.2) with $A = B_i$, $h = f$ and $S = S_i$.

In the general case, let $Z = H^*$, with H separable, and let $D \subset H$ be a dense and countable vector space over $\mathbf{Q}$; by using the tangential differentiability of the real valued functions $g^d(x) = \langle g(x), d\rangle$ we can recover, arguing as in Theorem 3.5, the existence for μ-a.e. $x \in Y$ of a linear and w^*-continuous map $L_x : Y \to (Z, d_w)$ such that

$$\langle L_x(v), g\rangle = d^S g^d_x(v) \qquad\qquad \forall d \in D,\ v \in Y \ .$$

By construction it can be easily checked that L_x satisfies (8.1) and (8.2) with $h = f$.

Finally (8.2) with a generic function h follows by the uniqueness of L, repeating the argument above with h in place of f. □

Consider now a Lipschitz map $g : S \to F$, with E, F separable metric spaces and $S \subset E$ countably $\mathcal{H}^k$-rectifiable. We can embed isometrically E, F respectively in duals of separable Banach spaces Y, Z with maps j_E, j_F and define the “lifted” map

$$\bar g := j_F \circ g \circ j_E^{-1} : j_E(S) \subset Y \to Z \ .$$

Then, we can define

$$\mathbf{J}_k(d^S g_x) := \mathbf{J}_k\left(d^{j_E(S)}\bar g_{j_E(x)}\right) \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in S \ . \tag{8.4}$$

Notice that $\bar g$ and its tangential differential depend of course on the choice of the spaces Y, Z and the embeddings j_E, j_F. However, the following result shows that $\mathbf{J}_k(d^S g_x)$ has an intrinsinc character.

Theorem 8.2 (General area formula) *Let $g : E \to F$ be a Lipschitz function and let $S \subset E$ be a countably $\mathcal{H}^k$-rectifiable set. Then $\mathbf{J}_k(d^S g)$ is well defined $\mathcal{H}^k$-a.e. by (8.4). Moreover*

$$\int_S \theta(x)\mathbf{J}_k(d^S g_x)\, d\mathcal{H}^k(x) = \int_F \sum_{x \in S\cap g^{-1}(y)} \theta(x)\, d\mathcal{H}^k(y)$$

for any Borel function $\theta : S \to [0, \infty]$ and

$$\int_A \theta(g(x))\mathbf{J}_k(d^S g_x)\, d\mathcal{H}^k(x) = \int_F \theta(y)\mathcal{H}^0\left(A \cap g^{-1}(y)\right)\, d\mathcal{H}^k(y)$$

for any $A \in \mathcal{B}(E)$ and any Borel function $\theta : F \to [0, \infty]$.

Proof. We first assume $E \subset Y$ and $F \subset Z$, so that $\bar g = g$. By the locality properties of the approximate tangent space (and of $x \mapsto d^S g_x$ as well) we can assume $S \subset f(\mathbf{R}^k)$ for some $f \in \mathrm{Lip}(\mathbf{R}^k, E)$. Arguing as in Theorem 3.2.3 of [8], the proof follows by the area formula for Lipschitz maps defined in $\mathbf{R}^k$ once we prove the chain rule

$$\mathbf{J}_k\left(wd(g \circ f)_y\right) = \mathbf{J}_k(d^S g_{f(y)})\mathbf{J}_k(wdf_y) \tag{8.5}$$

for jacobians. The identity (8.5) follows by (8.2) (with $h = f$) and (4.1).

In the general case, if we choose different embeddings j_{E1}, j_{E2} to define $\bar{g}_1$ and $\bar{g}_2$, by applying the formula just proved we get

$$\begin{aligned}\int_{S_1} \theta(j_{1E}^{-1}(y))\mathbf{J}_k(d^{S_1}\bar{g}_{1y})\, d\mathcal{H}^k(y) &= \int_F \sum_{x\in S\cap g^{-1}(w)} \theta(x)\, d\mathcal{H}^k(w) \\ &= \int_{S_2} \theta(j_{2E}^{-1}(z))\mathbf{J}_k(d^{S_2}\bar{g}_{2z})\, d\mathcal{H}^k(z)\end{aligned}$$

for any Borel function $\theta : S \to [0,\infty]$, with $S_1 = j_{1E}(S)$ and $S_2 = j_{2E}(S)$. Being θ arbitrary, we get

$$\mathbf{J}_k(d^{S_1}\bar{g}_{1y}) = \mathbf{J}_k(d^{S_2}\bar{g}_{2z}) \qquad \text{with } y = j_{1E}(x),\ z = j_{2E}(x)$$

for $\mathcal{H}^k$-a.e. $x \in S$. This proves that $\mathbf{J}_k(d^S g)$ is well defined $\mathcal{H}^k$-a.e. and that the area formula holds. □

9 Coarea formula

In this section we prove a coarea formula for $\mathbf{R}^k$-valued Lipschitz maps defined on countably $\mathcal{H}^n$-rectifiable subsets S of a metric space E.

Definition 9.1 (Coarea factor) *Let X, Y be finite dimensional linear spaces with $\dim X = n \geq \dim Y = k$ and $f : X \to Y$ linear. We define $\mathbf{C}_k(f)$ as the unique constant such that*

$$\mathbf{C}_k(f)\mathcal{H}^n(A) = \int_Y \mathcal{H}^{n-k}\left(A \cap f^{-1}(y)\right)\, d\mathcal{H}^k(y) \qquad \forall A \in \mathcal{B}(X)\ . \tag{9.1}$$

The definition is well posed because the right side in (9.1) is shift invariant, hence coincides with a constant multiple of $\mathcal{H}^n(A)$. Notice that, by the area formula, $\mathbf{C}_k(f) = \mathbf{J}_n(f)$ if $n = k$. By Theorem 2.10.25 of [8] the coarea factor can be estimated from above with $\omega_k\omega_{n-k}\left[\mathrm{Lip}(f)\right]^k/\omega_n$. By applying Fubini theorem and a polar decomposition it is not hard to see (see for instance [7]) that in the case $Y = \mathbf{R}^n$ the coarea factor can be computed by

$$\mathbf{C}_k(f) = \left[\det(f \circ f^*)\right]^{1/2}\ . \tag{9.2}$$

We will prove the coarea formula using the Euclidean one and a parametrization of the rectifiable set; the following general chain rule will be useful.

Lemma 9.2 *Let $f : X \to Y$, $g : Y \to Z$ be linear maps, with $\dim X = \dim Y = n \geq \dim Z = k$. Then*

$$\mathbf{C}_k(g)\cdot\mathbf{J}_n(f) = \mathbf{C}_k(g\circ f)\cdot\mathbf{J}_{n-k}\left(f|_{\mathrm{Ker}(g\circ f)}\right)\ . \tag{9.3}$$

PROOF. Let $K = \mathrm{Ker}(g \circ f)$; using the identity

$$\begin{aligned} \dim f(X) &= \dim \mathrm{Ker}(g|_{f(X)}) + \dim \mathrm{Im}(g|_{f(X)}) \\ &= \dim f(K) + \dim g \circ f(X) \end{aligned}$$

it can be easily checked that if either f is not injective or g is not surjective then both sides in (9.3) are zero. Hence, in the following we assume that f is bijective and g is surjective, and thus that $\dim K = n - k$. We fix a vector space $X' \subset X$ such that $X = K \oplus X'$ and choose Borel sets $B \subset K$, $C \subset X'$ such that $\mathcal{H}^{n-k}(B) = \mathcal{H}^n(A) = 1$, with $A = B + C$. The definition of $\mathbf{C}_k$ gives

$$\begin{aligned} \mathbf{C}_k(g \circ f) &= \int_Z \mathcal{H}^{n-k}\left(A \cap (g \circ f)^{-1}(z)\right) d\mathcal{H}^k(z) \\ &= \mathcal{H}^k(g \circ f(C)) \cdot \mathcal{H}^{n-k}(B) = \mathcal{H}^k(g \circ f(A)) \end{aligned}$$

because $A \cap (g \circ f)^{-1}(g(f(x))) = B + x$ for any $x \in g \circ f(C)$, and is empty otherwise. On the other hand

$$\begin{aligned} \mathbf{C}_k(g) \cdot \mathbf{J}_n(f) &= \mathbf{C}_k(g) \cdot \mathcal{H}^n(f(A)) \\ &= \int_Z \mathcal{H}^{n-k}\left(f(A) \cap g^{-1}(z)\right) d\mathcal{H}^k(z) \\ &= \int_Z \mathcal{H}^{n-k}\left(f(A \cap (g \circ f)^{-1}(z))\right) d\mathcal{H}^k(z) \\ &= \mathcal{H}^{n-k}(f(B)) \cdot \mathcal{H}^k(g \circ f(A)) \\ &= \mathbf{J}_{n-k}(f|_{\mathrm{Ker}(g \circ f)}) \cdot \mathbf{C}_k(g \circ f) \ . \end{aligned}$$

□

By using a similar decomposition argument we can also obtain a different representation of $\mathbf{C}_k(f)$.

Lemma 9.3 *Let X, Y and f be as in Definition 9.1, let K be the kernel of f, assume that $\dim K = n - k$. Let $p : X \to \mathbf{R}^{n-k}$ be a linear map injective on K. Then*

$$\mathbf{C}_k(f) = \frac{\mathbf{J}_n(q)}{\mathbf{J}_{n-k}(p|_K)} \ ,$$

where $q : X \to \mathbf{R}^n$ is given by $q(x) = (p(x), f(x))$.

PROOF. Choosing $K' = \mathrm{Ker}\, \mathrm{p}$, we have $X = K \oplus K'$. Again we fix $B \subset K$ and $C \subset K'$ compact such that for $A = B + C$ the normalization condition $\mathcal{H}^n(A) = \mathcal{H}^{n-k}(B) = 1$ holds. Since we have the orthogonal sum

$$q(A) = q(B) + q(C) = (p(B) \times \{0\}) \oplus (\{0\} \times f(C)),$$

we conclude

$$\mathbf{J}_n(q) = \mathcal{L}^n\left(q(A)\right) = \mathcal{H}^{n-k}\left(p(B)\right) \mathcal{H}^k\left(f(C)\right) = \mathbf{J}_{n-k}(p|_K) \mathbf{C}_k(f) \ .$$

□

Let $S \subset E$ be a countably $\mathcal{H}^n$-rectifiable set and let $g \in \mathrm{Lip}(E, \mathbf{R}^k)$, with $k \le n$. Arguing as in the previous section we can define $\mathbf{C}_k(d^S g)$ first in the case when E is contained in the dual of a separable Banach space (using Theorem 8.1) and then in the general case, using an isometric embedding.

Theorem 9.4 (General coarea formula) *Under the above assumptions the following properties hold:*

(a) for $\mathcal{H}^k$-a.e. $y \in \mathbf{R}^k$ the set $g^{-1}(y) \cap S$ is countably $\mathcal{H}^{n-k}$-rectifiable;

(b) for $\mathcal{H}^k$-a.e. $y \in \mathbf{R}^k$ and $\mathcal{H}^{n-k}$-a.e. $x \in g^{-1}(y) \cap S$ is $\mathrm{Tan}^{(n-k)}(g^{-1}(y), x) = \mathrm{Ker} d^S g_x$.

(c) for every Borel function $\theta : S \to [0, \infty]$ we have

$$\int_S \theta(x) \mathbf{C}_k(d^S g_x) \, d\mathcal{H}^n(x) = \int_{\mathbf{R}^k} \left(\int_{g^{-1}(y)} \theta(x) \, d\mathcal{H}^{n-k}(x) \right) d\mathcal{H}^k(y) \quad .$$

PROOF. We assume first that E is the dual of a separable Banach space. We know that S can be written as a disjoint union $S_0 \cup \bigcup_i S_i$ where $\mathcal{H}^n(S_0) = 0$ and each S_i is a bilipschitz image of a compact set in $\mathbf{R}^n$. Due to Theorem 2.10.25 of [8] we have

$$\mathcal{H}^{n-k}\left(S_0 \cap g^{-1}(y)\right) = 0 \qquad \text{for } \mathcal{H}^k\text{-a.e. } y \in \mathbf{R}^k \quad .$$

Consequently, by the σ-additivity of the integral and the locality properties of approximate tangent spaces we can restrict our attention the case $S = f(P)$, with $P \subset \mathbf{R}^n$ compact and $f : P \to S$ bilipschitz. We set $h = g \circ f : P \to \mathbf{R}^k$. Theorem 3.5 and Theorem 8.1 ensure that the conditions

(i) the differential $dh_x : \mathbf{R}^n \to \mathbf{R}^k$ exists

(ii) the w^*-differential $wdf_x : \mathbf{R}^n \to \hat{E}$ exists and is injective

(iii) the approximate tangential differential $d^S g_{f(x)} : \mathrm{Tan}^{(n)}(S, f(x)) \to \mathbf{R}^k$ exists

(iv) $\mathrm{Tan}^{(n)}(S, f(x)) = wdf_x(\mathbf{R}^n)$ and $dh_x = d^S g_{f(x)} \circ wdf_x$

are satisfied for $\mathcal{H}^n$-a.e. $x \in P$. Consequently, Lemma 9.2 gives

$$\mathbf{C}_k(dh_x) \cdot \mathbf{J}_{n-k}\left(wdf_x|_{\mathrm{Ker}(dh_x)}\right) = \mathbf{C}_k(d^S g_{f(x)}) \cdot \mathbf{J}_n(wdf_x) \qquad \text{for } \mathcal{H}^n\text{-a.e. } x \in P \quad .$$

The Euclidean coarea formula (see Theorem 3.2.22 of [8]) ensures that for $\mathcal{H}^k$-a.e. $y \in \mathbf{R}^k$ the level set $h^{-1}(y)$ is compact and countably $\mathcal{H}^{n-k}$-rectifiable. So the same holds true for $g^{-1}(y) = f(h^{-1}(y))$. This estabilishes statement (a). Moreover, the same theorem implies that for $\mathcal{H}^k$-a.e. $y \in \mathbf{R}^k$ we have

$$dh_x(\mathbf{R}^n) = \mathbf{R}^k \qquad \text{and} \qquad \mathrm{Tan}^{(n-k)}(h^{-1}(y), x) = \mathrm{Ker}(dh_x)$$

for $\mathcal{H}^{n-k}$-a.e. $x \in h^{-1}(y)$. Hence, $wdf_x|_{\mathrm{Ker}(dh_x)} = d^{h^{-1}(y)} f_x$, which ensures (b), and the area formula gives

$$\int_{g^{-1}(y)} \theta(x) \, d\mathcal{H}^{n-k}(x) = \int_{h^{-1}(y)} \theta(f(x)) \mathbf{J}_{n-k}(d^{h^{-1}(y)} f_x) \, d\mathcal{H}^{n-k}(x)$$

for $\mathcal{H}^k$-a.e. $y \in \mathbf{R}^k$. Finally, we apply the Euclidean coarea formula once more and Lemma 9.2 to find

$$
\begin{aligned}
&\int_{\mathbf{R}^k} \int_{g^{-1}(y)} \theta(x)\, d\mathcal{H}^{n-k}(x)\, d\mathcal{H}^k(y) \\
= \; & \int_{\mathbf{R}^k} \int_{h^{-1}(y)} \theta(f(x)) \mathbf{J}_{n-k}(wdf_x|_{\mathrm{Ker}(dh_x)})\, d\mathcal{H}^{n-k}(x)\, d\mathcal{H}^k(y) \\
= \; & \int_P \theta(f(x)) \mathbf{J}_{n-k}(wdf_x|_{\mathrm{Ker}(dh_x)}) \mathbf{C}_k(dh_x)\, d\mathcal{H}^n(x) \\
= \; & \int_P \theta(f(x)) \mathbf{C}_k(d^S g_{f(x)}) \mathbf{J}_n(wdf_x)\, d\mathcal{H}^n(x) \\
= \; & \int_S \theta(x) \mathbf{C}_k(d^S g_x)\, d\mathcal{H}^n(x) \quad .
\end{aligned}
$$

In the general metric case we argue exactly as in Theorem 8.2. □

It should be noted that a construction in [18] shows that even for $\mathcal{C}^1$-functions $f : [0,1]^2 \to [0,1]$ the level sets $f^{-1}(t)$ can in general be covered by countably many lipschitz curves only up to $\mathcal{H}^1$-zero sets.

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