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by

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Abstract

Uniform Lipschitz estimates are established for stationary points of $\int_\Omega F(Du) + \varepsilon^2(\triangle u)^2\, dx$, where F approaches a strongly elliptic quadratic form at infinity. This generalizes work of Chipot and Evans who considered the case $\varepsilon = 0$ for minimizers. Applications to variational models of solid - solid phase transitions are discussed.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, let $u : \Omega \to \mathbb{R}^m$ and consider the functional

$$I(u) = \int_\Omega F(Du(x))dx.$$

In view of applications e.g. to models of phase transformations (see e.g. [AG 87], [BJ 87], [Fo 89], [Fo 96], [Gu 86], [Mo 87], [KM 94], and the references quoted in these papers) we are interested in nonconvex (or more generally not strongly elliptic) integrands f. Simple examples show that for such integrands minimizers of I need not be C^1 but Chipot and Evans [CE 86] showed that minimizers of I are (locally) Lipschitz provided that f is sufficiently smooth and approaches a quadratic, strongly elliptic, integrand for large arguments. More precisely suppose that

$$F : M^{m\times n} \to \mathbb{R} \qquad \text{is} \quad C^2, \tag{1.1}$$

$$\frac{\partial^2 F}{\partial P^i_\alpha \partial P^i_\beta}(P) \to A^{\alpha\beta}_{ij} \quad \text{as} \quad |P| \to \infty, \tag{1.2}$$

$$A^{\alpha\beta}_{ij} a_\alpha b^i a_\beta b^j \geq \gamma |a|^2 |b|^2, \quad \gamma > 0, \quad \forall a \in \mathbb{R}^n, \quad \forall b \in \mathbb{R}^m. \tag{1.3}$$

Theorem 1.1 [CE 86] *Suppose (1.1) to (1.3) and let u be a minimizer of I (i.e. $I(u+\varphi) \geq I(u) \quad \forall \varphi \in C_0^\infty(\Omega)$). Then $u \in W^{1,\infty}_{loc}(\Omega)$.*

Generalizations of this result have been obtained by Giaquinta-Modica [GM 86], Raymond [Ra 91] and Fuchs [Fu 96]. If A corresponds to a diagonal system, i.e. $A^{\alpha\beta}_{ij} = a^{\alpha\beta}_i \delta_{ij}$ then a similar result was established earlier by Nečas and Stará [NS 72] (see also [Ne 86], Theorem 5.5.3).
One drawback of Theorem 1.1 is that due to the lack of convexity minimizers of I need not exist. One approach to obtain approximate minimizers is to consider the singularly perturbed problem

$$I^\varepsilon(u) = \int_\Omega \quad F(Du) + \varepsilon^2 (\Delta u)^2 dx, \tag{1.4}$$

where $\varepsilon > 0$ is small. The goal of this paper is to show that stationary points (or extremals) of I^ε are (locally) Lipschitz, uniformly with respect to ε.

Theorem 1.2 *Suppose (1.1)–(1.3). Let $\Omega' \subset\subset \Omega$ be open. Then there exists a constant $C(\Omega', \Omega)$ with the following property. If u^ε is a stationary point of I^ε, i.e. a weak solution of*

$$-\text{div } DF(Du^\varepsilon) + \varepsilon^2 \triangle^2 u = 0 \tag{1.5}$$

then

$$\varepsilon|D^2u| + |Du| \leq C(\Omega', \Omega)(1 + \|Du\|_{L^2(\Omega)} + \varepsilon\| \triangle u\|_{L^2(\Omega)}) \quad \text{a.e. in } \Omega'. \tag{1.6}$$

Remarks 1. A similar result holds for more general (e.g. anisotropic) singular perturbations which satisfy suitable smoothness and coercivity conditions. The details are left to the courageous reader.

2. Under suitable assumptions one can establish estimates up to the boundary by the usual means of locally flattening the boundary and extension of the solution by reflection.

3. It would be interesting to have an analogous result for more general integrands in the spirit of [GM 86], [Ra 91], [Fu 96].

In some applications f is only nonconvex in certain directions. In this case one might expect similar estimates if one only adds a regularizing term with respect to those directions. For the sake of illustration we consider the following functional which is discussed in [KM 94] in connection with microstructure at solid–solid phase boundaries.
Let

$$h : \mathbb{R} \quad \to \quad \mathbb{R} \quad \text{be} \quad C^2,$$

with

$$h''(p) \quad \to \quad a > 0, \quad \text{as} \quad |p| \to \infty,$$

and consider the functional

$$J^\varepsilon(u) = \int\limits_{(0,1)^2} u_x^2 + h(u_y) + \varepsilon^2 u_{yy}^2 \quad dxdy.$$

Theorem 1.3 *There exists a constant C such that for all stationary points u^ε of J^ε that satisfy the boundary conditions*

$$\begin{aligned} u &= 0 \quad \text{on} \quad \partial(0,1)^2, \\ u_{yy} &= 0 \quad \text{on} \quad \partial(0,1)^2 \end{aligned}$$

one has

$$|Du| + \varepsilon|u_{yy}| \leq C(1 + \|Du\|_{L^2} + \|\varepsilon u_{yy}\|_{L^2}) \quad \text{a.e. in} \quad (0,1)^2.$$

Note that the boundary condition u_{yy} arises as a natural boundary condition at the top and bottom boundaries $y = 0, y = 1$ if one considers all variations that vanish on $\partial(0,1)^2$. At $x = 0, x = 1$ the condition $u_{yy} = 0$ follows from $u = 0$.

One can also consider other boundary conditions as long as they permit the use of reflection arguments.

2 Preliminaries

In the following we will always assume (1.1) to (1.3). For the blow-up argument in the following section consider sequences $\lambda_m \in \mathbb{R} \setminus \{0\}, A_m \in M^{m\times n}$ and the rescaled functions

$$F^m(P) = \frac{F(A_m + \lambda_m P) - F(A_m) - \lambda_m DF(A_m)P}{\lambda_m^2} \tag{2.1}$$

Proposition 2.1 *Suppose that $\lambda_m^2 + |A_m|^2 \to \infty$. After passage to a subsequence (not relabelled) there exist $\alpha_m, \alpha_m', \beta_m, \beta_m' \to 0$ such that*

$$|F^m(P) - \frac{1}{2}P^T AP| \quad \leq \quad \alpha_m |P|^2 + \beta_m, \tag{2.2}$$

$$|DF^m(P) - AP| \quad \leq \quad \alpha_m' |P| + \beta_m', \tag{2.3}$$

where $A = A_{ij}^{\alpha\beta}$ is given by (1.2).

Proof. Estimate (2.2) is just (2.9) in [CE 86]. Estimate (2.3) follows by interpolation from (2.2) and the fact that $\sup |D^2 F^m| = \sup |D^2 F| \leq C$ by (1.1) and (1.2).

For an L^1 function we denote the average over the open ball $B(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\}$ by

$$(f)_{a,r} = \fint_{B(a,r)} f dy = \frac{1}{|B(a,r)|} \int_{B(a,r)} f\, dy.$$

We will frequently use the identity

$$\fint_{B(a,r)} |f|^2\, dy = |(f)_{a,r}|^2 + \fint_{B(a,r)} |f - (f)_{a,r}|^2\, dy \tag{2.4}$$

which holds for all $f \in L^2(B(a,r))$.

Proposition 2.2 *There exists a constant C such that for f in the Sobolev-space $W^{1,2}(B(0,1))$*

$$|(Df)_{0,1}|^2 \leq C \left\{ \int\limits_{B(0,1)} |f|^2 + |Df - (Df)_{0,1}|^2 dy \right\} \tag{2.5}$$

$$\int\limits_{B(0,1)} |Df|^2 \leq C \left\{ \int\limits_{B(0,1)} |f|^2 + |Df - (Df)_{0,1}|^2 dy \right\}. \tag{2.6}$$

Proof. In view of (2.4) it suffices to prove the first estimate. Suppose it failed. Then there exists a sequence f_m such that $A_m := |(Df_m)_{0,1}| = 1, \quad f_m \to 0 \quad \text{in} \quad L^2, \quad Df_m - A_m \to 0$ in L^2. Passing to a subsequence we may assume that $A_m \to A, \quad |A| = 1$. It follows that $Df_m \to 0$ in distributions since $f_m \to 0 \quad \text{in} \quad L^2$ and $Df_m \to A \quad \text{in} \quad L^2$, which is a contradiction. $\square$

Remark. Similarly one shows that for any positively semidefinite quadratic form Q

$$Q\left((Df)_{0,1}\right) \leq \fint_{B(0,1)} Q(Df)\, dy \leq C \fint_{B(0,1)} |f|^2 + Q(Df - (Df)_{0,1})\, dy,$$

whenever the right hand side is finite.

3 Main Lemma

Heuristically we expect(uniform) L^∞ bounds for stationary points of I^ε since for small values of Du^ε there is nothing to show while for large values of Du^ε the equation is close to a linear strongly elliptic equation for which good estimates are known. The following alternative makes this reasoning precise. Theorem 1.2 is easily deduced from this alternative, see [CE 86], pp. 300–301 for the details.

Main Lemma *Let $\Omega' \subset\subset \Omega, r_0 = dist\,(\Omega', \partial\Omega)/2$. Then there exist constants $C^* > 0$ and $\tau \in (0, 1/4)$ such that for each $x \in \Omega'$ each $r \in (0, r_0)$ each $\varepsilon > 0$ and each stationary point u of I^ε one has*

i)

$$\fint_{B(x,r)} |Du|^2 + \varepsilon^2 |D^2u|^2 dx \leq C^*$$

or

ii)

$$\begin{aligned}
&\fint_{B(x,\tau r)} |Du - (Du)_{x,\tau r}|^2 + \varepsilon^2 |D^2 u - (D^2u)_{x,\tau r}|^2 dy \\
&\leq \frac{1}{2} \fint_{B(x,r)} |Du - (Du)_{x,r}|^2 + \varepsilon^2 |D^2 u - (D^2u)_{x,r}|^2 dy.
\end{aligned}$$

Proof. Following Chipot and Evans [CE 86] we use the blow-up method to obtain a contradiction.

Step 1 Assume the lemma was false and fix $\tau \in (0, 1/4)$. Later τ will be chosen sufficiently small to obtain a contradiction. Then there exist sequences $x_m, r_m\tau_m, \varepsilon_m$ and u_m such that

$$\fint_{B(x_m, r_m\tau_m)} |Du_m|^2 + \varepsilon_m^2 |D^2u_m|^2 dy \to \infty \tag{3.1}$$

and

$$
\begin{aligned}
&\fint_{B(x_m,\tau r_m)} |Du-(Du)_{(x_m,\tau r_m)}|^2+\varepsilon^2|D^2u-(D^2u)_{x_m,\tau r_m}|^2 dy \\
&> \frac{1}{2}\fint_{B(x_m,r_m)} |Du-(Du)_{x_m,r_m}|^2+\varepsilon^2|D^2u-(D^2u)_{x_m,r_m}|^2 dy. \qquad (3.2)
\end{aligned}
$$

Define rescaled functions as follows. Let

$$
\begin{aligned}
a_m &= (u_m)_{x_m,\tau_m r_m}, \quad A_m=(Du_m)_{x_m,\tau_m r_m}, \\
\lambda_m &= \left\{\fint_{B(x_m,\tau_m r_m)} |Du_m-A_m|^2+\varepsilon_m^2|D^2u_m|^2 dy\right\}^{1/2} \\
v_m(x) &= \frac{u_m(x_m+r_m x)-a_m-r_m A_m x}{\lambda_m r_m}, \quad x\in B(0,1), \qquad (3.3)
\end{aligned}
$$

and let

$$
\bar{\varepsilon}_m=\frac{\varepsilon_m}{r_m}.
$$

Then

$$
\begin{aligned}
(v_m)_{0,1} &= (Dv_m)_{0,1}=0, \\
Dv_m(x) &= \frac{Du_m(x_m+r_m x)-A_m}{\lambda_m}, \\
D^2v_m(x) &= \frac{r_m D^2u_m(x_m+\lambda_m x)}{\lambda_m},
\end{aligned}
$$

and v_m is an extremal of the rescaled functional

$$
J^m(v)=\int_{B(0,1)} F_m(Dv)+\bar{\varepsilon}_m^2(\Delta v)_m^2\, dx, \qquad (3.4)
$$

where

$$
F_m(P)=\frac{F(A_m+\lambda_m P)-F(A_m)-\lambda_m DF(A_m)P}{\lambda_m^2}.
$$

It follows from the definition of λ_m that

$$
\fint_{B(0,1)} |Dv_m|^2+\bar{\varepsilon}_m^2|D^2v_m|^2\, dx=1. \qquad (3.5)
$$

Moreover (3.1) is equivalent to

$$
\lambda_m^2+|A_m|^2\to\infty, \qquad (3.6)
$$

while (3.2) is equivalent to

$$
\begin{aligned}
& \fint_{B(0,\tau)} |Dv_m - (Dv_m)_{0,\tau}|^2 + \bar{\varepsilon}_m^2 |D^2 v_m - (D^2 v_m)_{0,\tau}|^2 \, dx \\
> \; & \frac{1}{2} \fint_{B(0,1)} |Dv_m|^2 + \bar{\varepsilon}_m^2 |D^2 v_m - (D^2 v_m)_{0,1}|^2 dx. \qquad (3.7)
\end{aligned}
$$

In view of (2.6) and (3.5) we deduce that

$$
\begin{aligned}
\fint_{B(0,\tau)} |Dv_m - (Dv_m)_{0,\tau}|^2 + \bar{\varepsilon}_m^2 |D^2 v_m - (D^2 v_m)_{0,\tau}|^2 \, dx &\geq c > 0, \text{ if} \\
\bar{\varepsilon}_m &\leq 2. \qquad (3.8)
\end{aligned}
$$

Step 2 By (3.5) the functions v_m have a weak limit (modulo passage to a subsequence which will not be relabelled). The idea is to show that they converge *strongly* to a solution v of a linear equation. Elliptic regularity will then yield a contradiction with (3.7) or (3.8).

Again passing to a subsequence we may suppose that $\bar{\varepsilon}_m = \frac{\varepsilon_m}{r_m}$ has a limit (possibly $+\infty$). We distinguish three cases corresponding to the limit ∞, a finite limit and the limit 0.

If

$$\lim_{m\to\infty} \bar{\varepsilon}_m \geq 1$$

then the higher order perturbation dominates and it is useful to consider the following further rescaling. This corresponds to a second order approximation of u_m near x_m while (3.3) corresponds to a first order approximation. Let

$$
\begin{aligned}
B_m &= (D^2 v_m)_{0,1}, \quad b_m = \fint_{B(0,1)} \frac{1}{2} x^T B_m x \, dx \\
\mu_m &= \left\{ \fint_{B(0,1)} |Dv_m|^2 + \bar{\varepsilon}_m^2 |D^2 v_m - B_m|^2 \, dx \right\}^{1/2} \\
w_m(x) &= \bar{\varepsilon}_m \frac{v_m(x) - \frac{1}{2} x^T B_m x + b_m}{\mu_m}. \qquad (3.9)
\end{aligned}
$$

Thus

$$
\begin{aligned}
(w_m)_{0,1} = (Dw_m)_{0,1} = (D^2 w_m)_{0,1} &= 0, \\
\fint_{B(0,1)} |\frac{B_m x}{\mu_m} + \frac{1}{\bar{\varepsilon}_m} Dw_m|^2 + |D^2 w_m|^2 dx &= 1.
\end{aligned}
$$

Hence

$$\|w_m\|_{W^{2,2}(B(0,1))} \leq C. \qquad (3.10)$$

By (3.5) and (2.5)

$$\mu_m^2 + \bar{\varepsilon}_m^2 |B_m|^2 = 1, \tag{3.11}$$

$$\mu_m \geq c \min(1, 1/\bar{\varepsilon}_m), \tag{3.12}$$

and division of (3.7) by μ_m yields

$$\fint_{B(0,\tau)} |\frac{B_m x}{\mu_m} + \frac{1}{\bar{\varepsilon}_m}(Dw_m - (Dw_m)_{0,\tau})|^2 + |D^2 w_m - (D^2 w_m)_{0,\tau}|^2 > \frac{1}{2}.$$

For $\bar{\varepsilon}_m \geq 1/2$ we have $|\frac{B_m}{\mu_m}| \leq C\varepsilon_m |B_m| \leq C$ and thus

$$\fint_{B(0,\tau)} \frac{1}{\bar{\varepsilon}_m^2} |(Dw_m - Dw_m)_{0,\tau}|^2 + |D^2 w_m - (D^2 w_m)_{0,\tau}|^2 dx > \frac{1}{2} - C\tau^2, \quad \text{for} \quad \bar{\varepsilon}_m \geq \frac{1}{2}. \tag{3.13}$$

Step 3 Suppose that $\lim_{m\to\infty} \bar{\varepsilon}_m = 0$. Since v^m is an extremal of the functional J^m in (3.4) it satisfies the Euler-Langrage equation

$$\bar{\varepsilon}_m^2 \triangle^2 v_m - \operatorname{div} DF^m(Dv_m) = 0. \tag{3.14}$$

By (3.5) we may assure (for a subsequence)

$$v_m \rightharpoonup v \quad \text{in} \quad W^{1,2}, \qquad \bar{\varepsilon}_m D^2 v_m \rightharpoonup 0 \quad \text{in} \quad L^2. \tag{3.15}$$

Recall that $A = \lim_{P\to\infty} D^2 F(P)$ and write the nonlinear terms as

$$DF^m(Dv_m) = ADv_m + g_m.$$

Then, by (2.3),

$$g_m \to 0 \quad \text{in} \quad L^2 \tag{3.16}$$

and thus

$$-\operatorname{div} A\, Dv = 0. \tag{3.17}$$

Testing the difference of (3.14) and (3.17) with $\varphi^2(v_m - v)$, where $\varphi \in C_0^\infty(B(0,1))$, we deduce

$$v_m \to v \quad \text{in} \quad W^{1,2}_{loc}, \quad \bar{\varepsilon}_m D^2 v_m \to 0 \quad \text{in} \quad L^2_{loc}.$$

Now standard regularity results for the linear elliptic system (3.17) yield

$$\sup_{B(0,1/2)} |D^2 v| \leq C \|Dv\|_{L^2} \leq C.$$

Hence

$$\int_{B(0,\tau)} |Dv - (Dv)_{0,\tau}|^2 \leq C\tau^2,$$

and passing to the limit in (3.8) we obtain

$$C\tau^2 \geq c > 0.$$

This leads to a contradiction if τ is chosen sufficiently small.
Step 4 Suppose that $\lim_{m\to\infty} \bar{\varepsilon}_m = \bar{\varepsilon}, \quad 0 < \bar{\varepsilon} < \infty$. In this case we may assume (for a subsequence)

$$v_m \rightharpoonup v \qquad \text{in} \quad W^{2,2}$$

and using (3.16) we deduce that

$$\bar{\varepsilon}^2 \triangle v \; - \; \operatorname{div} A\, Dv \; = \; 0. \tag{3.18}$$

Testing the difference of (3.18) and (3.14) by $\varphi^2(v_m - v)$ we obtain

$$v_m \to v \qquad \text{in} \quad W^{2,2}.$$

Standard regularity results yield

$$\sup_{B(0,1/2)} (\bar{\varepsilon}|D^3 v| + |D^2 v|) \leq C(\|\bar{\varepsilon} D^2 v\|_{L^2} + \|Dv\|_{L^2}) \leq C.$$

If $\bar{\varepsilon} \leq 1$, then passage to the limit in (3.8) yields again a contradiction if τ is chosen sufficiently small.

If $\bar{\varepsilon} > 1$ we rewrite (3.14) in terms of w_m and obtain

$$\triangle^2 w_m - \frac{1}{\bar{\varepsilon}_m^2} \operatorname{div} A\, Dw_m = \frac{1}{\bar{\varepsilon}_m \mu_m} \operatorname{div}(AB_m x + g_m). \tag{3.19}$$

In view of (3.10) – (3.11) we may assume $w_m \rightharpoonup w$ in $W^{2,2}$, $B_m \to B$, $\mu_m \to \mu$, and using (3.16) we easily pass to the limit and deduce

$$\triangle^2 w - \frac{1}{\bar{\varepsilon}^2} \operatorname{div} A\, Dw = \frac{1}{\bar{\varepsilon}\mu} \operatorname{div} A\, Bx = const \tag{3.20}$$

$$w_m \to w \quad \text{in} \quad W^{2,2}_{loc}.$$

Applying standard regularity theory to de derivative of (3.20) and taking into account that $\bar{\varepsilon} > 1$ we find that

$$\sup_{B(0,1/2)} \left(|D^3 w| + \frac{1}{\bar{\varepsilon}} |D^2 w| \right) \leq C.$$

Passage to the limit in (3.13) yields again an contradiction if τ is chosen sufficiently small.

Step 5 Finally consider the case $\lim_{m\to\infty} \bar{\varepsilon}_m = \infty$. We start from (3.19) and use (3.11) and (3.12) to deduce that $B_m \to 0$ while $\bar{\varepsilon}_m \mu_m$ remains bounded from below. Hence standard compactness and convergence results for the biharmonic equation yield $w_m \to w$ in $W^{2,2}_{loc}$, $\triangle^2 w = 0$ and $\sup_{B(0,1/2)} |D^3 w| \leq C$. Once again this contradicts (3.13) as $m \to \infty$. This finishes the proof of the main lemma. $\square$

4 Degenerate perturbations

In this section we establish the Lipschitz estimates for stationary points of

$$J^3(u) = \int_\Omega \qquad u_x^2 + h(u_y) + \varepsilon^2 u_{yy}^2 \, dxdy,$$

where Ω is a bounded domain in $\mathbb{R}^2$, where h is C^2 and $h''(p) \to a > 0$ as $|p| \to \infty$. We first show that the previous reasoning for nondegenerate perturbations still applies to balls whose radius is larger than ε. For smaller balls we estimate the L^∞ norm of larger $|Du| + \varepsilon |u_{yy}|$ directly (see Lemma 4.2 below). Global regularity for $\Omega = (0,1)^2$ under the boundary conditions $u = u_{yy} = 0$ on $\partial\Omega$ is then established by a simple reflection argument.

Lemma 4.1 *Let $\Omega' \subset\subset \Omega$ and let $r_0 = dist(\Omega', \partial\Omega)/2$. Then there exist constants C^*, $C' > 0$ and $\tau \in (0, 1/4)$ such that for each $x \in \Omega'$, each $\varepsilon > 0$ and each $r \in [\varepsilon, r_0)$ and each solution of*

$$\varepsilon^2 u_{yyyy} - u_{xx} - h(u_y)_y = 0 \tag{4.1}$$

one has

(i)

$$\fint_{B(x,\Gamma)} |Du|^2 + \varepsilon^2 u_{yy}^2 \, dx \leq C^*$$

or

(ii)

$$\begin{aligned} & \fint_{B(x,\tau r)} |Du - (Du)_{x,\tau r}|^2 + \varepsilon^2 |u_{yy} - (u_{yy})_{x,\tau r}|^2 \, dy \\ \leq \; & \frac{1}{2} \fint_{B(x,r)} |Du - (Du)_{x,r}|^2 + \varepsilon |u_{yy} - (u_{yy})_{x,r}|^2 \, dy. \end{aligned}$$

Moreover, for each $x \in \Omega'$

$$\fint_{B(x,\varepsilon)} |Du|^2 + \varepsilon^2 u_{yy}^2 \, dy \leq C'(1 + \|Du\|^2_{L^2(\Omega)} + \varepsilon^2 \|u_{yy}\|^2_{L^2(\Omega)}). \tag{4.2}$$

Proof. To show that (i) or (ii) holds we proceed by contradiction exactly as in the previous section. In particular we define v_m as before with

$$\lambda_m^2 = \fint_{B(x_m, r_m)} |Du_m - A_m|^2 + \bar{\varepsilon}_m^2 u_{myy}^2 \, dxdy.$$

Since we only need to consider $r_m \geq \varepsilon_m$ we have $\bar{\varepsilon}_m \leq 1$, and it follows from the remark after Proposition 2.2 that the counterpart of (3.8) holds, i.e.

$$\fint_{B(0,\tau)} |Dv_m - (Dv_m)_{0,\tau}|^2 + \bar{\varepsilon}_m^2 |v_{myy} - (v_{myy})_{0,\tau}|^2 \, dxdy \geq c > 0. \quad (4.3)$$

Now if $\bar{\varepsilon}_m \to 0$ we reach a contradiction exactly as in Step 3 in Section 3. If $\bar{\varepsilon}_m \to \bar{\varepsilon} > 0$, we have $\bar{\varepsilon} \leq 1$, and as in Step 4 in Section 3 we deduce that $v_m \to v \quad \text{in} \quad W^{1,2}_{loc}$ and $v_{myy} \to v_{yy}$ in L^2_{loc} and

$$\bar{\varepsilon}^2 v_{yyyy} - v_{xx} - a v_{yy} = 0. \quad (4.4)$$

To reach a contradiction with (4.3) (for sufficiently small τ) we only need to show that, for $\bar{\varepsilon} \leq 1$,

$$\sup_{B(0,1/2)} |\bar{\varepsilon} v_{yyy}| + |D^2 v| \leq C \|v\|_{W^{1,2}(B(0,1))}. \quad (4.5)$$

This is easily established by the standard Campanato technique. Indeed, testing (4.4) by $\varphi^4 v$, where $\varphi \in C_0^\infty(B(0,t))$, $\varphi \equiv 1$ on $B(0,s)$, $0 \leq \varphi \leq 1$, we obtain

$$\int_{B(0,t)} \varphi^4 (\bar{\varepsilon}^2 v_{yy}^2 + |Dv|^2) \, dxdy \leq C(s,t) \int_{B(0,t)} \varphi^2 \bar{\varepsilon}^2 v_y^2 + v^2 \, dxdy. \quad (4.6)$$

Integration by parts yields, for $\delta < 1$,

$$\begin{aligned} \int \varphi^2 v_y^2 \, dxdy &= \int -\varphi^2 v_{yy} v - 2\varphi \, \varphi_y \, v \, v_y \, dxdy \\ &\leq \delta \int \varphi^4 v_{yy}^2 \, dxdy + \frac{C}{\delta} \int v^2 \, dxdy + \frac{1}{2} \int \varphi^2 v_y^2 \, dxdy. \end{aligned}$$

Absorbing the last term on the right hand side and inserting the resulting estimate in (4.6) we obtain, for a suitable choice of δ,

$$\int_{B(0,s)} \bar{\varepsilon}^2 v_{yy}^2 + |Dv|^2 \, dxdy \leq C(s,t) \int_{B(0,t)} v^2 \, dxdy.$$

Application of this inequality to the first, second and third derivatives of (4.4) and use of the Sobolev embedding theorem yield (4.5).

This shows that estimate (i) or (ii) in Lemma 4.1 must hold. Finally (4.2) follows from this alternative by a simple iteration argument (see [CE 86], pp. 300–301 for the details).

□

To obtain the estimate for balls of radius ε consider $u : B(a, \varepsilon) \to \mathbb{R}$ and the rescaling

$$v(z) = \frac{u(a + \varepsilon z) - u(a)}{\varepsilon}.$$

Then (4.1) becomes

$$v_{yyyy} - v_{xx} - (h(v_y))_y = 0 \quad \text{in} \quad B(0,1), \tag{4.7}$$

and we have

$$\begin{aligned} \fint_{B(0,1)} v_{yy}^2 + |Dv|^2 \, dxdy &= \fint_{B(a,\varepsilon)} \varepsilon^2 u_{yy}^2 + |Du|^2 \, dxdy, & (4.8)\\ \sup_{B(0,1/2)} |v_{yy}| + |Dv| &= \sup_{B(a,\varepsilon/2)} \varepsilon |u_{yy}| + |Du|. & (4.9) \end{aligned}$$

Lemma 4.2 *Suppose that v satisfies (4.7). Then*

$$\sup_{B(0,1/2)} (|v_{yy}| + |Dv|) \leq C \left\{ \int_{B(0,1)} |v_{yy}|^2 + |Dv|^2 \, dxdy \right\}^{1/2}. \tag{4.10}$$

To prove this result we first consider the corresponding linear equation.

Lemma 4.3 *Let $t > s > 0$, $B = B(0,t)$, $B' = B(0,s)$. Suppose that $p \geq 2$, $f \in L^p(B)$ and that v satisfies*

$$v_{yyyy} - v_{xx} = f \quad \textit{in} \quad B$$

in the sense of distributions. Then

$$\begin{aligned} & \|v_{yyyy}\|_{L^p(B')} + \|v_{yyx}\|_{L^p(B')} + \|v_{xx}\|_{L^p(B')} \\ \leq \; & C_{p,s,t} \left(\|f\|_{L^p(B)} + \|v_{yy}\|_{L^p(B)} + \|Dv\|_{L^p(B)} \right) \end{aligned} \tag{4.11}$$

and the same estimate holds for $\|v_{yyy}\|_{L^p(B')}$ and $\|v_{xy}\|_{L^p(B')}$.

Proof. We may assume that v is smooth (otherwise apply convolution first) and that $\int_B v \, dx = 0$. If v has compact support the left hand side of (4.11) is bounded by $\|f\|_{L^p(B)}$ in view of the Marcinkiewicz multiplier theorem (see e.g. [St 70], Chapter IV, Theorem 6'). In the general case we apply the same reasoning to $\varphi^2 v$, where $\varphi \in C_0^\infty(B(0,1))$, $\varphi \equiv 1$ in B'. This yields L^p bounds for $\varphi^2(|v_{yyyy}| + |v_{yyx}| + |v_{xx}|)$ in terms of the right hand side of (4.11) and the terms $\|v\|_p, \|\varphi v_{yx}\|_p$ and $\|\varphi v_{yyy}\|_p$. The term in $\|v\|_p$ can be estimated by $\|Dv\|_p$ since $\int_B v \, dx = 0$. To absorb the term φv_{yyy} note first that for all smooth w one has

$$\|(\varphi w)_y\|_p \leq C_p \|\varphi(\varphi w)_{yy}\|_p^{1/2} \|w\|_p^{1/2}. \tag{4.12}$$

This is easily verified by integrating the term

$$|(\varphi w)_y|^p = \left\{ |(\varphi w)_y|^{p-1} \operatorname{sgn} \, (\varphi w)_y \right\} (\varphi w)_y$$

by parts and applying Hölder's inequality. Application of (4.12) with $w = v_{yy}$ and $w = v_x$ yields (4.11) as well as the estimates for v_{yyy} and v_{yx}. $\square$

Proof of Lemma 4.2 Since $|h'(v_y)_y| \le |h''(v_y)v_{yy}| \le C|v_{yy}|$ application of Lemma 4.3 with $s = 3/4,\ t = 1$ and $p = 2$ shows that $D^2 v$ and Dv_{yy} are in $L^2(B(0, 3/4))$. By the Sobolev embedding theorem Dv and v_{yy} belong to $L^p(B(0, 3/4))$, for all $p < \infty$. Hence another application of Lemma 4.3 and the Sobolev embedding theorem yield (4.10). $\square$

Proof of Theorem 1.3. Combining the estimate (4.2) for all balls of radius larger than ε with Lemma 4.2 and using (4.8) and (4.9) we see that

$$\sup_{\Omega'}(|Du| + \varepsilon|u_{yy}|) \le C(\Omega')(1 + \|Du\|_{L^2(\Omega)}) + \varepsilon\|u_{yy}\|_{L^2(\Omega)},$$

whenever $\Omega' \subset\subset \Omega$ and $\mathrm{dist}(\Omega, 2\Omega') \ge 2\varepsilon$. To obtain the global estimate we extend u by reflection. First extend u in the y direction by

$$\begin{array}{lll} u(x,y) = -u(x,-y) & \text{if} & -1 < y < 0, \\ u(x,y) = -u(x, 2-y) & \text{if} & 1 < y < 2. \end{array}$$

By definition u_y and u_{yyy} do not jump at $y = 0$ and $y = 1$ while continuity of u and u_{yy} follows from the boundary conditions. Hence we have

$$u_{yyyy} - u_{xx} + h'(u_y) = 0 \quad \text{on} \quad (0,1) \times (-1, 2).$$

Similarly we can extend u to $x \in (-1, 2)$. Now the desired global estimate follows from the interior estimate in $(-1, 2)^2$. $\square$

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