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by

Georg Dolzmann, Bernd Kirchheim, Stefan Müller,
and Vladimir Šverák

THE TWO-WELL PROBLEM IN THREE DIMENSIONS

GEORG DOLZMANN, BERND KIRCHHEIM, STEFAN MÜLLER AND VLADIMIR ŠVERÁK

ABSTRACT. We study properties of generalized convex hulls of the set $K = \mathrm{SO}(3) \cup \mathrm{SO}(3)H$ with $\det H > 0$. If K contains no rank-1 one connection we show that the quasiconvex hull of K is trivial if H belongs to a certain (large) neighbourhood of the identity. We also show that the polyconvex hull of K can be nontrivial if H is sufficiently far from the identity, while the (functional) rank-1 convex hull is always trivial. If the second well is replaced by a point then the polyconvex hull is trivial provided that there are no rank-1 connections.

1. INTRODUCTION

Mathematical models of solid-solid phase transitions [BJ1, BJ2, CK] motivate the following questions. Consider a compact subset K of the space of 3×3 matrices $\mathbb{M}^{3\times 3}$, a bounded domain $\Omega \subset \mathbb{R}^3$ and a sequence of maps $u_j : \Omega \to \mathbb{R}^3$ that satisfies

$$(1.1) \qquad \mathrm{dist}(Du_j, K) \to 0 \quad \text{in measure,}$$

$$(1.2) \qquad u_j \overset{*}{\rightharpoonup} u \quad \text{in } W^{1,\infty}(\Omega; \mathbb{R}^3).$$

What is the smallest set K' such that for every such sequence u_j its limit u satisfies $Du \in K'$? For which sets K do (1.1) and (1.2) imply that $Du_j \to Du$ in measure or, equivalently, in all L^p, $p < \infty$ (in a more general context these questions were already raised in the seminal paper [Ta1] of Tartar)? In applications the set K corresponds to the set of energy minimizing affine deformations of a crystal lattice, while K' describes the set of (macroscopic) affine boundary conditions for which global (almost) minimizers exist.

The theory of quasiconvexity [Mo] and (gradient) Young measures [Yo1, Yo2, BL, Ta1, Bd, Ba] (see [Va, Pe2, Mu] for recent reviews) gives an abstract answer to the above questions. Our goal is to verify the abstract conditions for the simplest three-dimensional example with the physical rotation symmetry. To explain the relevant notions let us first consider the abstract setting. A function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ is quasiconvex (in the sense of Morrey) if one has

$$\int_{[0,1]^n} f(F + D\varphi)\, dx \geq f(F) \qquad \forall \varphi \in C_0^\infty([0,1]^n; \mathbb{R}^m),\ \forall F \in \mathbb{M}^{m\times n},$$

and the set K' turns out to be exactly the quasiconvex hull of K given by

$$K^{qc} := \{F \in \mathbb{M}^{m\times n} : f(F) \leq \sup_K f \quad \forall f \text{ quasiconvex}\}$$

(i.e. K^{qc} consists of those points that cannot be separated from K by quasiconvex functions).

The fundamental theorem on Young measures states that for any bounded sequence of functions $v_j : \Omega \to \mathbb{R}^d$ there exists a subsequence and a map $\nu : \Omega \to \mathcal{P}(\mathbb{R}^d)$ into probability measures on $\mathbb{R}^d$

such that for all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ and all $g \in L^1(\Omega)$

$$\int_\Omega g(x)f(v_j(x))\,dx \to \int_\Omega g(x)\langle \nu_x, f\rangle\,dx,$$

where $\langle \nu_x, f\rangle = \int f\,d\nu_x$. A Young measure generated by gradients of a sequence that satisfies (1.2) is called a gradient Young measure, and Kinderlehrer and Pedregal [KP] have shown that such measures are essentially characterized by Jensen's inequality for quasiconvex functions, i.e. by the condition

$$(1.3) \qquad\qquad \langle \nu_x, f\rangle \geq f(\langle \nu_x, id\rangle) \quad \forall f \text{ quasiconvex.}$$

The class of homogeneous (i.e. x-independent) gradient Young measures supported on K is denoted by $\mathcal{M}^{qc}(K)$, and it is not difficult to check (see e.g. [Sv3]; [Mu], Thm. 4.10 and Cor 3.2) that (1.1) and (1.2) imply strong convergence of Du_j if and only if $\mathcal{M}^{qc}(K)$ is trivial, i.e. contains only Dirac masses. Moreover K^{qc} consists exactly of the barycentres of measures in $\mathcal{M}^{qc}(K)$ (see e.g. [Mu], Thm. 4.10).

In application to solid-solid phase transitions in elastic crystals the set K is invariant under (left) rotations and the simplest non-trivial example corresponds to the so-called two-well problem $K = \mathrm{SO}(3)A \cup \mathrm{SO}(3)B$, $\det A, \det B > 0$. After a suitable change of coordinates it suffices to consider the case

$$(1.4) \qquad K = \mathrm{SO}(3) \cup \mathrm{SO}(3)H, \quad H = \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix}, \quad h_1 \geq h_2 \geq h_3 > 0.$$

We say that K contains no rank-1 connection if $\mathrm{rank}(A - B) \neq 1$ for all $A, B \in K$. A short calculation, see for example [Ja], shows that a set of the form (1.4) contains no rank-1 connection if and only if

$$(1.5) \qquad\qquad\qquad\qquad h_2 \neq 1.$$

The following conjecture was raised by D. Kinderlehrer:

Conjecture 1.1. *If K is of the form (1.4) and contains no rank-1 connection then*

$$\mathcal{M}^{qc}(K) \text{ is trivial and in particular } K^{qc} = K.$$

Note that if $A, B \in K$ differ by a matrix of rank 1 then $\lambda\delta_A + (1-\lambda)\delta_B$ is a nontrival element of $\mathcal{M}^{qc}(K)$.

The conjecture was established by Matos [Ma1] (see also [Sv2]) provided that for some i one has $(h_i - 1)(h_{i-1}h_{i+1} - 1) \geq 0$ (here and in the following we count the index i modulo 3, i.e. $h_4 = h_1$ etc.). Using elliptic regularity arguments and in particular work of F. John on BMO estimates for gradients of deformations with finite strain Kohn and Lods [KL] recently proved the conjecture provided that H is sufficiently close to the identity (and satisfies a certain technical condition). Our main result gives a (rather large) explicit neighbourhood of the identity for which the conjecture holds.

Theorem 1.2. *Suppose that K is given by (1.4) and contains no rank-1 connection. Assume in addition that one the following conditions holds:*

   i) *there exists an i such that $(h_i - 1)(h_{i-1}h_{i+1} - 1) \geq 0$,*

   ii) *$h_1 \geq h_2 > 1 > h_3 > \frac{1}{3}$ or $3 > h_1 > 1 > h_2 \geq h_3 > 0$.*

Then

$$\mathcal{M}^{qc}(K) \text{ is trivial and in particular } K^{qc} = K.$$

The proof is completely algebraic. In fact, instead of $\mathcal{M}^{qc}(K)$ we consider the larger class of polyconvex Young measures which are obtained by restricting (1.3) to the simplest quasiconvex functions, namely the minors. Specifically let

$$\mathcal{M}^{pc}(K) = \{\nu \in \mathcal{P}(K) : (1.6) \text{ and } (1.7) \text{ hold}\},$$

$$(1.6) \qquad \qquad \operatorname{cof} \langle \nu, id \rangle = \langle \nu, \operatorname{cof} \rangle,$$

$$(1.7) \qquad \qquad \det \langle \nu, id \rangle = \langle \nu, \det \rangle.$$

The polyconvex hull K^{pc} consists of all barycentres of measures in $\mathcal{M}^{pc}(K)$.

Theorem 1.3. *Under the hypotheses of Theorem 1.2 we have*

$$\mathcal{M}^{pc}(K) \text{ is trivial and in particular } K^{pc} = K.$$

The following result shows that the polyconvex version of Conjecture 1.1 does not hold for all H.

Theorem 1.4. *There exists an $H = \operatorname{diag}(h_1, h_2, h_3)$ with $h_1 \geq h_2 \geq h_3 > 0$, $h_2 \neq 1$ and $\det H = 1$ such that $K = \operatorname{SO}(3) \cup \operatorname{SO}(3)H$ has a nontrivial polyconvex hull, i.e. $K^{pc} \neq K$.*

While this result indicates that Conjecture 1.1 might not be true for all H the following two results show that a counterexample might not be so easy to find. We first consider the (functional) rank-1 convex hull given by

$$K^{rc} := \{F \in \mathbb{M}^{m \times n} : f(F) \leq \sup_K f \quad \forall f \text{ rank-1 convex}\}.$$

We recall that a function f is rank-1 convex if all its restrictions to rank-1 lines are convex. An important example of Tartar [Ta2] (similar examples were discovered by [AH], [CT] and [NM1] in other contexts; see also [BFJK]) shows that K^{rc} can be non-trivial even if K has no rank-1 connections (this fact was recently exploited to construct solutions to strongly elliptic 2×2 systems that are nowhere C^1 [MS]).

Theorem 1.5. *Suppose that K is given by (1.4) and that K contains no rank-1 connection. Then $K^{rc} = K$.*

One also obtains that the corresponding class of measures defined by rank-1 convexity (the so-called laminates [Pe1]) is trivial.

If K consists of one well and one point then one has a global result for the polyconvex hull.

Theorem 1.6. *Consider the set*

$$K = \operatorname{SO}(3) \cup \{H\}, \quad H = \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix}$$

with $h_i > 0$. If K contains no rank-1 connection then

$$\mathcal{M}^{pc}(K) \text{ is trivial and in particular } K^{pc} = K.$$

This improves a result of Matos [Ma2] who showed that for each H with $\det(H - I) \neq 0$ there exists an $\varepsilon_0 > 0$ such that the assertion holds for the set $\operatorname{SO}(3) \cup \{I + \varepsilon(H - I)\}$ whenever $0 < \varepsilon < \varepsilon_0$. We follow Matos' proof to establish first that all matrices in K^{pc} must be diagonal. We then conclude by a careful analysis of the three decoupled equations (1.6) and thus avoid the somewhat lengthy asymptotic expansions in [Ma2].

Finally, for the sake of completness let us briefly consider the case that the two-well set K does contain a rank-1 connection. By (1.5) this corresponds to $h_2 = 1$. In this case the quasiconvex hull and the polyconvex hull are clearly nontrivial. The best we can hope for is that they can be

obtained by successively adding rank-1 segments to the set K. We define the lamination convex hull K^{lc} inductively as follows.

$$K^{lc} = \bigcup_i K^{(i)}, \qquad K^{(0)} = K$$

$$K^{(i+1)} = K^i \cup \{\lambda A + (1-\lambda)B : \lambda \in [0,1], \ A, B \in K^{(i)}, \ \text{rank}(A-B) = 1\}.$$

Alternatively K^{lc} can be defined as the smallest set that contains K and is invariant under the operation of adding rank-1 segments.

In the following theorem we compare these hulls in the case when K contains a rank-1 connection. In order to unify our proofs we formulate a result which covers (up to permutation of coordinates) a slightly more general situation.

Theorem 1.7. *Consider the set*

$$K = \text{SO}(3) \cup \text{SO}(3)H, \quad with \quad H = \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix},$$

where the only restrictions on the h_i's are

$$h_2 \neq 1, (h_1 h_2 - 1)(h_3 - 1) = 0 \text{ and } h_1, h_2, h_3 \text{ are positive.}$$

Then K^{pc} is trivial unless $h_3 = 1$ and $(h_1 - 1)(h_2 - 1) \leq 0$. In this case

$$K^{lc} = K^{rc} = K^{qc} = K^{pc} = \left\{ Q \begin{pmatrix} \hat{F} & \\ & 1 \end{pmatrix} \ : \ Q \in \text{SO}(3), \ \hat{F} \in \hat{K}^{lc} \right\},$$

where

$$\hat{K} = \text{SO}(2) \cup \text{SO}(2) \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \subset \mathbb{M}^{2 \times 2}.$$

An explicit formula for $\hat{K}^{lc} = \hat{K}^{rc} = \hat{K}^{qc} = \hat{K}^{pc}$ is given in [Sv2]. Indeed $\hat{K}^{lc} = \hat{K}^{(3)}$. In the special case $\det H = 1$ one has $\hat{K}^{lc} = \{F \in \text{conv} \, \hat{K} : \det F = 1\}$.

2. Sufficient conditions that polyconvex Young measures are trivial

In this section we first prove Theorem 1.3 which obviously also implies Theorem 1.2. We then show that the polyconvex hull is still trivial if almost all the mass of the Young measure is concentrated on one of the two wells.

Proof of Theorem 1.3. Assume that ν is a polyconvex Young measure supported on $K = \text{SO}(3) \cup \text{SO}(3)H$. Then ν can be written as

$$\nu = (1 - \lambda)\varrho + \lambda\sigma H,$$

where $\lambda \in [0,1]$ and ϱ and σ are probability measures on $\text{SO}(3)$. Here σH denotes the measure given by $\sigma H(E) = \sigma(EH^{-1})$ where $EH^{-1} = \{F \ : \ FH \in E\}$; in particular $\delta_A H = \delta_{AH}$. Since $\text{cof} \, G = G$ for $G \in \text{SO}(3)$, the minors relations are equivalent to

$$\begin{aligned}
F &= (1-\lambda)R + \lambda SH, \\
\text{cof} \, F &= (1-\lambda)R + \lambda S \, \text{cof} \, H, \\
\det F &= (1-\lambda) + \lambda \det H,
\end{aligned} \tag{2.1}$$

where $F = \langle \nu, id \rangle$, $R = \langle \varrho, id \rangle$ and $S = \langle \sigma, id \rangle$. We will frequently use the following expansions

$$\begin{aligned}
\text{cof}(F - I) &= \text{cof} \, F - (\text{tr} \, F)I + F^T + I, \\
\det(F - I) &= \det F - \text{tr} \, \text{cof} \, F + \text{tr} \, F - \det I,
\end{aligned} \tag{2.2}$$

which hold for all $F \in \mathbb{M}^{3 \times 3}$ and the inequality

$$(2.3) \qquad \operatorname{tr} Q - 2e^T Q e \leq 1 \qquad \forall e \in \mathbb{S}^2, \, \forall Q \in \operatorname{conv} \operatorname{SO}(3),$$

which was proven in [Ja]. A simple calculation shows that the identity $F^T \operatorname{cof} F = (\det F)I$ can be rewritten using (2.1) as

$$
\begin{aligned}
(2.4) \qquad (1 - \lambda)^2 (R^T R - I) &+ \lambda^2 H^T (S^T S - I) \operatorname{cof} H \\
&+ \lambda(1 - \lambda) \big\{ (R^T S - I) \operatorname{cof} H + H^T (S^T R - I) \big\} \\
&= \lambda(1 - \lambda)(H^T - I)(\operatorname{cof} H - I).
\end{aligned}
$$

Multiplication of (2.4) by e_i from the right and by e_i^T from the left shows that

$$
\begin{aligned}
(2.5) \qquad (1 - \lambda)^2 (|Re_i|^2 - 1) &+ \lambda^2 (|Se_i|^2 - 1) \det H \\
&+ \lambda(1 - \lambda)(h_{i-1} h_{i+1} + h_i)\big(\langle Re_i, Se_i \rangle - 1\big) \\
&= \lambda(1 - \lambda)(h_i - 1)(h_{i-1} h_{i+1} - 1)
\end{aligned}
$$

for $i = 1, 2, 3$. Since $R, S \in \operatorname{conv}(\operatorname{SO}(3))$ we deduce

$$|Re_i|^2 - 1 \leq 0, \; |Se_i|^2 - 1 \leq 0, \; \langle Re_i, Se_i \rangle - 1 \leq 0,$$

and in particular the right hand side of (2.5) must be less than or equal to zero:

$$(2.6) \qquad (h_i - 1)(h_{i-1} h_{i+1} - 1) \leq 0, \quad i = 1, 2, 3.$$

Since all terms on the left hand side of (2.5) have the same sign we conclude in addition that

$$(2.7) \qquad \big| \langle Re_i, Se_i \rangle - 1 \big| \leq \frac{|(h_i - 1)(h_{i-1} h_{i+1} - 1)|}{|h_{i-1} h_{i+1} + h_i|}.$$

The proof of Theorem 1.3 with assumption i) follows now from Theorem 1.7.

Regarding assumption ii) we may assume that the first set of inequalities holds as the other case can be reduced to this by replacing H by H^{-1}. Let $K = H - I = \operatorname{diag}(k_1, k_2, k_3)$. Then $k_1, k_2 > 0$, $k_3 < 0$ and (2.6) implies

$$(2.8) \qquad h_1 h_3 < 1, \quad h_2 h_3 < 1.$$

It follows from (2.3) with $e = e_3$ that

$$(2.9) \qquad (R - I)_{11} + (R - I)_{22} - (R - I)_{33} \leq 0 \quad \forall R \in \operatorname{conv} \operatorname{SO}(3).$$

A short calculation using

$$h_i + h_{i-1} h_{i+1} = \operatorname{tr} K + 2 + k_{i-1} k_{i+1} \geq k_{i-1} k_{i+1}$$

and

$$(h_i - 1)(h_{i-1} h_{i+1} - 1) = \det K + k_i (k_{i-1} + k_{i+1})$$

shows that (2.9) applied to $R^T R, \, S^T S, \, R^T S \in \operatorname{conv} \operatorname{SO}(3)$ yields in connection with (2.4)

$$
\begin{aligned}
\lambda(1 - \lambda) \big\{ \big((R^T S)_{11} - 1 \big) k_2 k_3 &+ \big((R^T S)_{22} - 1 \big) k_3 k_1 - \big((R^T S)_{33} - 1 \big) k_1 k_2 \big\} \\
&\geq \lambda(1 - \lambda)(2 k_1 k_2 + k_1 k_2 k_3).
\end{aligned}
$$

This inequality implies with (2.7) and $k_3 < 0$

$$2 k_1 k_2 + k_1 k_2 \Big[k_3 - |k_3| \sum_{i=1}^3 \frac{|h_{i-1} h_{i+1} - 1|}{|h_{i-1} h_{i+1} + h_i|} \Big] \leq 0.$$

Since $k_1 k_2 > 0$ by assumption we obtain the desired contradiction if

$$(2.10) \qquad 2 + k_3 \Big[1 + \sum_{i=1}^3 \frac{|h_{i-1} h_{i+1} - 1|}{|h_{i-1} h_{i+1} + h_i|} \Big] > 0.$$

By (2.8) we may estimate

$$
\begin{aligned}
\frac{|h_2 h_3 - 1|}{h_2 h_3 + h_1} &\leq \frac{1}{h_1} =: a, \\
\frac{|h_3 h_1 - 1|}{h_1 h_1 + h_2} &\leq \frac{1}{h_2} =: b, \\
\frac{|h_1 h_2 - 1|}{h_1 h_2 + h_3} &\leq 1 - \frac{1}{h_1 h_2} = 1 - ab.
\end{aligned}
$$

Since $a + b - ab = 1 - (a-1)(b-1) < 1$ we conclude that the expression in the brackets in (2.10) is estimated by 3 from above and thus (2.10) holds for $k_3 > -\frac{2}{3}$, i.e., $h_3 > \frac{1}{3}$. This concludes the proof of Theorem 1.3. $\qquad\square$

We now turn towards proving an asymptotic result for polyconvex Young measures the mass of which is almost concentrated on one well.

Theorem 2.1. *Assume that $K = \mathrm{SO}(3) \cup \mathrm{SO}(3)H$ with $H = \mathrm{diag}(h_1, h_2, h_3)$, $h_1 \geq h_2 > 1 > h_3 > 0$. Then there exists a $\lambda_0 > 0$ which only depends on H such that the following holds: If $\nu = (1 - \lambda)\varrho + \lambda \sigma H$ is a polyconvex Young measure supported on K with $0 \leq \lambda < \lambda_0$, then ν is trivial, i.e., $\lambda = 0$ and $\nu = \delta_Q$ for some $Q \in \mathrm{SO}(3)$.*

Proof. Assume that there exists a sequence $\lambda_n \to 0$ and corresponding polyconvex Young measures $\nu_n = (1 - \lambda_n)\varrho_n + \lambda_n \sigma_n H$ supported on K. The minors relations imply

$$
\begin{aligned}
(2.11) \qquad F_n &= (1 - \lambda_n)R_n + \lambda_n S_n H, \\
\mathrm{cof}\, F_n &= (1 - \lambda_n)R_n + \lambda_n S_n \,\mathrm{cof}\, H, \\
\det F_n &= (1 - \lambda_n) + \lambda_n \det H,
\end{aligned}
$$

where $F_n = \langle \nu_n, id \rangle$, $R_n = \langle \varrho_n, id \rangle$, $S_n = \langle \sigma_n, id \rangle$. In particular, $R_n, R_n^T, S_n \in \mathrm{conv}\,\mathrm{SO}(3)$ and it follows from (2.3) with $e = e_3$ that

$$
\begin{aligned}
(R_n + R_n^T - 2I)_{11} + (R_n + R_n^T - 2I)_{22} - (R_n + R_n^T - 2I)_{33} &\leq 0, \\
(2.12) \qquad\qquad\qquad\qquad (S_n)_{11} + (S_n)_{22} - (S_n)_{33} &\leq 1.
\end{aligned}
$$

Identity (2.4) applied to R_n yields

$$
\left| (R_n^T R_n - I)_{ii} \right| \leq c_0 \lambda_n,
$$

where c_0 depends only on H. Since $M = I - R_n^T R_n$ is positive semidefinite we have $2|M_{ij}| \leq M_{ii} + M_{jj} \leq 2c_0 \lambda_n$ and hence $R_n^T R_n - I = \mathcal{O}(\lambda_n)$. Therefore the eigenvalues of $R_n^T R_n$ are of order $1 + \mathcal{O}(\lambda_n)$ and since $\sqrt{x} = 1 + \mathcal{O}(x - 1)$ we conclude that

$$
R_n = Q_n(I + \mathcal{O}(\lambda_n)), \text{ where } Q_n \in \mathrm{O}(3).
$$

We also have $\det F_n = (1 - \lambda_n)^3 (\det R_n) + \mathcal{O}(\lambda_n) \to 1$ which implies $\det R_n > 0$ and $Q_n \in \mathrm{SO}(3)$. Premultiplication of ν_n by Q_n^T shows that we may assume $Q_n = I$ and $R_n - I = \mathcal{O}(\lambda_n)$. This allows us to choose a subsequence (again denoted by λ_n) such that $F_n \to F = I$, $S_n \to S$ and

$$
\frac{1}{\lambda_n}(R_n - I) \to \dot{R}, \qquad \frac{1}{\lambda_n}(F_n - I) \to \dot{F}.
$$

Then the minors relations (2.11) and the expansion (2.2) show

$$
\begin{aligned}
\dot{R} + S\operatorname{cof}H - I \quad &\leftarrow \quad \frac{1}{\lambda_n}\left\{(1-\lambda_n)(R_n - I) + \lambda_n(S_n\operatorname{cof}H - I)\right\} \\
&= \quad \frac{1}{\lambda_n}(\operatorname{cof}F_n - \operatorname{cof}I) \\
&= \quad \operatorname{cof}\left(\frac{1}{\sqrt{\lambda_n}}(F_n - I)\right) + \frac{1}{\lambda_n}\left(\operatorname{tr}(F_n - I)I - (F_n^T - I)\right) \\
&\to \quad (\operatorname{tr}\dot{F})I - \dot{F}^T,
\end{aligned}
$$

and similarly $-1 + \det H = \operatorname{tr}\dot{F}$. In the limit we obtain thus the following system of equations

$$
\begin{aligned}
(2.13) && \dot{F} &= \dot{R} - I + SH, \\
(2.14) && (\operatorname{tr}\dot{F})I - \dot{F}^T &= \dot{R} - I + S\operatorname{cof}H, \\
(2.15) && \operatorname{tr}\dot{F} &= \det H - 1,
\end{aligned}
$$

and from (2.12) the inequalities

$$
\begin{aligned}
(2.16) && (\dot{R} + \dot{R}^T)_{11} + (\dot{R} + \dot{R}^T)_{22} - (\dot{R} + \dot{R}^T)_{33} &\leq 0, \\
(2.17) && S_{11} + S_{22} - S_{33} &\leq 1.
\end{aligned}
$$

We will show that an algebraic manipulation of this system leads to a contradiction.
   Substitution of (2.13) and (2.15) into the left hand side of (2.14) gives

$$
(2.18) \qquad \dot{R} + \dot{R}^T = (\det H + 1)I - S\operatorname{cof}H - (SH)^T,
$$

while substracting (2.13) from (2.14) and taking the trace yields

$$
\operatorname{tr}\dot{F} = \operatorname{tr}(S\operatorname{cof}H - SH).
$$

Thus (2.15) implies

$$
(2.19) \qquad \det H - 1 + \operatorname{tr}\big(S(H - \operatorname{cof}H)\big) = 0.
$$

   Inequality (2.16) yields with (2.18)

$$
\det H + 1 - \operatorname{tr}\big(S(\operatorname{cof}H + H)\big) + 2\big(S(\operatorname{cof}H + H)\big)_{33} \leq 0.
$$

If we add this to (2.19) we obtain

$$
\det H - \operatorname{tr}(S\operatorname{cof}H) + \big(S(\operatorname{cof}H + H)\big)_{33} \leq 0.
$$

Since H and $\operatorname{cof}H$ are diagonal, we deduce

$$
h_3\left(h_1 h_2 - S_{11}h_2 - S_{22}h_1 + S_{33}\right) \leq 0.
$$

Because $h_3 > 0$ we obtain from (2.17)

$$
h_1 h_2 - S_{11}h_2 - S_{22}h_1 \leq -S_{33} \leq 1 - S_{11} - S_{22}
$$

and substracting $s_{11}s_{22}$ we conclude that

$$
(h_1 - S_{11})(h_2 - S_{22}) \leq (1 - S_{11})(1 - S_{22}).
$$

This is a contradiction since h_1, $h_2 > 1$. $\qquad\qquad\square$

3. Two incompatible wells with nontrivial polyconvex hull

In this section we construct a matrix H with $\det H = 1$ such that $SO(3)$ and $SO(3)H$ are not rank-one connected, but $K = SO(3) \cup SO(3)H$ supports a nontrivial Young measure ν which satisfies the minors relations (1.6) and (1.7). We also show that $K \subsetneqq K^{pc}$.

Assume that $\nu = (1-\lambda)\varrho + \lambda\sigma H$ where ϱ and σ are probability measures on $SO(3)$ and $\lambda \in (0,1)$. Let

$$R = \int_{SO(3)} Q d\varrho(Q), \quad S = \int_{SO(3)} Q d\sigma(Q).$$

Then R, $S \in \text{conv}(SO(3))$, a set which is given in terms of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the symmetric part in the polar decomposition by

$$\text{conv}(SO(3)) = \left\{ QU : Q \in SO(3), \ U = U^T, \ \sum_i \varepsilon_i \lambda_i \leq 1 \text{ for } |\varepsilon_i| = 1 \text{ and } \varepsilon_1 \varepsilon_2 \varepsilon_3 = -1 \right\}$$

(see [Ja]). It follows that $F = \langle \nu, id \rangle = (1-\lambda)R + \lambda SH$, and the minors relations are equivalent to

$$(3.1) \qquad\qquad\qquad \text{cof}\, F - F \ = \ \lambda S(\text{cof}\, H - H),$$
$$(3.2) \qquad\qquad\qquad \det F - 1 \ = \ \lambda(\det H - 1).$$

The idea is to show that for $h > 0$ small enough, $\lambda = \frac{1}{2}$ and $H = \text{diag}(h, h, h^{-2})$ there exists an $s_3 \in [\frac{1}{10}, 1]$ and a solution $F = \text{diag}(f_1, f_2, f_3)$ of (3.1) and (3.2) with $S = \text{diag}(s_1, s_2, s_3) = (\frac{1}{2}, \frac{1}{2}, s_3) \in \text{conv}(SO(3))$ which satisfies

$$(3.3) \qquad\qquad\qquad 0 \leq r_1 = r_2 \leq \frac{1}{2} \text{ and } 0 \leq r_3 \leq 1,$$

where

$$r_i = \frac{f_i - \lambda s_i h_i}{1 - \lambda}.$$

Thus $R = \frac{1}{1-\lambda}(F - \lambda SH) \in \text{conv}(SO(3))$; if then ϱ and σ are arbitrary probability densities supported on $SO(3)$ with $\langle \varrho, id \rangle = R$ and $\langle \sigma, id \rangle = S$, then

$$\nu = \frac{1}{2}(\varrho + \sigma H)$$

is the desired Young measure supported on K. It should also be mentioned that the proof of the description of $\text{conv}(SO(3))$ in [Ja] shows that we can assume the measures ϱ and σ to be supported on the four diagonal matrices in $SO(3)$. On the other hand, since I is the only positive definite diagonal matrix in $SO(3)$ it follows from Proposition 1 in [Sv1] that this ν cannot be a Gradient Young measure provided both ϱ and σ are supported in the diagonal rotations.

We will use the following inequalities:

i) Assume that $f, h > 0$ and $c \in (0, 1)$ satisfy $f - \frac{1}{f} = c(h - \frac{1}{h})$. Then $f \geq c h$. If in addition $c = \frac{1}{4}$ and $h < \frac{1}{\sqrt{33}}$ then $f \in (3h, 4h)$.

ii) Assume that $h > 0$ and that $f > 0$ satisfies $f - \frac{1}{f} = \lambda s(h - \frac{1}{h})$. Then for $c_0 > 0$

$$(3.4) \qquad\qquad\qquad r = \frac{f - \lambda sh}{1 - \lambda} \in [0, c_0]$$

holds if and only if

$$\lambda^2 s^2 \leq 1 \text{ and } 1 \leq \lambda^2 s^2 + c_0 s\lambda(1 - \lambda)\left(h + \frac{1}{h}\right) + c_0^2(1 - \lambda)^2.$$

We choose $h \in (0, \frac{1}{13})$ with $h + \frac{1}{h} = 14$ and $\lambda = s_1 = s_2 = \frac{1}{2}$. By (3.4) we infer $0 \le r_1 = r_2 \le \frac{1}{2}$ and $f_1 = f_2 \in (3h, 4h)$ due to i). Moreover, (3.4) applied for $h_3 = h^{-2}$ also ensures that $0 \le r_3 \le 1$ whenever $s_3 \in [\frac{1}{64}, 1]$. Finally for the same h_3, the estimates in i) give $f_3 \ge \frac{1}{2}h^{-2}$ for $s_3 = 1$, and one easily verifies that $f_3 \le \frac{h^{-2}}{16}$ if $s_3 = \frac{1}{10}$. Consequently, $f_1 f_2 f_3(\frac{1}{10}) \le 1 < f_1 f_2 f_3(1)$. The continuity of f_3 as a function of s_3 now implies the existence of the desired $S = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}, s_3)$ for which the corresponding F fulfills $\det F = 1$. Since $f_1 \in (3h, 4h)$ clearly $F \notin K$.

4. Polyconvex Young measures supported on a point and a well

In this section we prove Theorem 1.6. Assume that ν is given by

$$(4.1) \qquad \nu = (1 - \lambda)\varrho + \lambda\delta_H,$$

where ϱ is a probability measure on $SO(3)$, $\lambda \in (0, 1)$ and $H = \mathrm{diag}(h_1, h_2, h_3)$ with $h_i > 0$. Then the minors relations are equivalent to

$$(4.2) \qquad \mathrm{cof}\, F - F = \lambda(\mathrm{cof}\, H - H),$$

$$(4.3) \qquad \det F - 1 = \lambda(\det H - 1).$$

The fact that $F = \langle \nu, id \rangle$ must be diagonal if $\lambda \notin \{0, 1\}$ was already proven in [Ma2], Proposition 1.7. We include the proof for the convenience of the reader. With different techniques we first show that F is symmetric and then follow the proof in [Ma2] to show that F is diagonal.

Lemma 4.1. *Suppose that $\lambda \in (0, 1)$ and that ν given by (4.1) satisfies the minors relations (4.2) and (4.3). Let $F = \langle \nu, id \rangle$. Then either F is symmetric or one eigenvalue of H is equal to one.*

Proof. Let $K = H - I$, $G = F - I$. If we take the trace in (4.2) and subtract the resulting equation from (4.3) we obtain

$$\det F - \mathrm{tr}\,\mathrm{cof}\, F + \mathrm{tr}\, F - 1 = \lambda(\det H - \mathrm{tr}\,\mathrm{cof}\, H + \mathrm{tr}\, H - 1);$$

from $\det(F - I) = \det F - \mathrm{tr}\,\mathrm{cof}\, F + \mathrm{tr}\, F - 1$ we conclude

$$(4.4) \qquad \det G = \lambda \det K.$$

To prove the lemma it thus suffices to show that G is symmetric assuming that $\det G \ne 0$.

Expanding both sides of (4.2) we obtain

$$
\begin{aligned}
\mathrm{cof}(I + G) - (I + G) &= I + \mathrm{cof}\, G + (\mathrm{tr}\, G)I - G^T - (I + G) \\
&= \mathrm{cof}\, G + (\mathrm{tr}\, G)I - (G + G^T), \text{ and equals} \\
\lambda\Big(\mathrm{cof}(I + K) - (I + K)\Big) &= \lambda\Big(\mathrm{cof}\, K + (\mathrm{tr}\, K)I - (K + K^T)\Big).
\end{aligned}
$$

Hence, $\mathrm{cof}\, G$ is symmetric. Since $\det G \ne 0$, both

$$G^{-1} = (\det G)^{-1}(\mathrm{cof}\, G)^T$$

and G are also symmetric. $\qquad\square$

Lemma 4.2. *Suppose that ν given by (4.1) satisfies the minors relations (4.2) and (4.3). If $F = \langle \nu, id \rangle$ is symmetric, then F is diagonal.*

Proof. Let $R = \langle \varrho, id \rangle$. Then $F = (1 - \lambda)R + \lambda H$ and the minors relations can be written as

$$
\begin{aligned}
\mathrm{cof}\, F &= (1 - \lambda)R + \lambda \,\mathrm{cof}\, H, \\
\det F &= (1 - \lambda) + \lambda \det H.
\end{aligned}
$$

The identity $(\det F)I = F^T(\mathrm{cof}\, F)$ thus implies

$$\big((1 - \lambda) + \lambda \det H\big)I = (1 - \lambda)^2 R^T R + \lambda(1 - \lambda)\big(R^T H + (\mathrm{cof}\, H)^T R\big) + \lambda^2(\mathrm{cof}\, H)^T H$$

and by the symmetry of R and H

$$(4.5) \qquad (1 + \lambda \det H)I = (1 - \lambda)R^T R + \lambda\big(RH + (\operatorname{cof} H)R\big).$$

It follows that $RH + (\operatorname{cof} H)R$ is symmetric; therefore (with the convention $e_4 = e_1$ and $e_0 = e_3$)

$$\langle (RH + (\operatorname{cof} H)R)e_i, e_k \rangle = \langle e_i, (RH + (\operatorname{cof} H)R)e_k \rangle$$

or

$$h_i \langle Re_i, e_k \rangle + h_{k-1}h_{k+1}\langle Re_i, e_k \rangle = h_k \langle e_i, Re_k \rangle + h_{i-1}h_{i+1}\langle e_i, Re_k \rangle.$$

This is for $k = i + 1$ equivalent to

$$(4.6) \qquad (h_{i+1} - h_i)(h_{i-1} + 1)\langle Re_i, e_{i+1} \rangle = 0.$$

Case 1: All eigenvalues of H are distinct. Then R must be diagonal.

Case 2: Assume that two eigenvalues of H are equal, i.e., $H = \operatorname{diag}(h, h, h_3)$. Then (4.6) implies $R_{23} = R_{31} = 0$ while we deduce from (4.5)

$$(1 - \lambda)\langle R^T Re_1, e_2 \rangle + \lambda\langle (RH + (\operatorname{cof} H)R)e_1, e_2 \rangle = 0$$
$$\Leftrightarrow R_{12}\big((1 - \lambda)(R_{11} + R_{22}) + \lambda h(1 + h_3)\big) = 0.$$

We claim that R_{11}, $R_{22} > 0$ and thus $R_{12} = 0$, i.e. R is diagonal. Indeed, it follows from (4.5) and $(R^T R)_{11} \le 1$ that

$$\begin{aligned}1 + \lambda \det H \;&=\; (1 - \lambda)(R^T R)_{11} + R_{11}\lambda h(1 + h_3) \\ &\le\; (1 - \lambda) + R_{11}\lambda h(1 + h_3).\end{aligned}$$

Thus $0 < \lambda(1 + \det H) \le \lambda h(1 + h_3)R_{11}$ and we conclude similarly that $R_{22} > 0$.

Case 3: All eigenvalues of H are equal. In this case $(h_2 - 1)(h_1 h_3 - 1) > 0$ and hence by Theorem 1.3 there exists no solution of (4.2) and (4.3) with $\lambda \in (0, 1)$. $\qquad\square$

Proof of Theorem 1.6. Assume that $\lambda \in (0, 1)$ and that $\nu = (1 - \lambda)\varrho + \lambda\delta_H$ is a (nontrivial) Young measure which satisfies the minors relations (4.2) and (4.3). Since due to Theorem 1.3 (or see also [Ma1]) the conclusion of Theorem 1.6 holds whenever one of the h_i's equals one, we can exclude this case from our considerations. By Lemma 4.1 and Lemma 4.2 we infer that $F = \operatorname{diag}(f_1, f_2, f_3)$. Thus (4.2) reduces to the system of three equations

$$f_i - \frac{\det F}{f_i} = \lambda\Big(h_i - \frac{\det H}{h_i}\Big).$$

We claim that this system has a solution only if at least one of the h_i's is equal to one, which is the desired contradiction. To prove our claim, note that (4.3) implies that $\det F > 0$ and thus we may rewrite (4.2) as

$$\frac{f_i}{\sqrt{\det F}} - \frac{\sqrt{\det F}}{f_i} = \frac{\lambda\sqrt{\det H}}{\sqrt{\det F}}\Big(\frac{h_i}{\sqrt{\det H}} - \frac{\sqrt{\det H}}{h_i}\Big).$$

Let

$$\alpha = \frac{\lambda\sqrt{\det H}}{\sqrt{\det F}}, \qquad h(x) = \operatorname{arcsinh}(\alpha \sinh(x)).$$

By (4.3) $\alpha < 1$ and it is easy to see that h' is strictly increasing on $[0, \infty)$. Thus h is strictly convex on $[0, \infty)$ and satisfies $h(x) = -h(-x)$. Let

$$x_i = \ln\frac{h_i}{\sqrt{\det H}}, \quad y_i = \left\{ \begin{array}{ll} \ln(f_i/\sqrt{\det F}) & \text{if} \quad f_i > 0, \\ \ln(-\sqrt{\det F}/f_i) & \text{if} \quad f_i < 0, \end{array} \right. , \quad \sigma_i = \left\{ \begin{array}{ll} 1 & \text{if} \quad f_i > 0, \\ -1 & \text{if} \quad f_i < 0. \end{array} \right.$$

Then

$$(4.7) \qquad \sum_{i=1}^{3} x_i = -\ln\sqrt{\det H}, \quad e^{\sigma_i y_i} = \frac{|f_i|}{\sqrt{\det F}}, \quad e^{\sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3} = \frac{1}{\sqrt{\det F}}$$

and we conclude that

$$(4.8) \qquad \sum_{i=1}^{3} \sigma_i y_i = -\ln\sqrt{\det F}.$$

Since $\sinh(x) = -\sinh(-x)$ and $h(\ln z) = \operatorname{arcsinh}(\frac{\alpha}{2}(z - \frac{1}{z}))$ we have

$$(4.9) \qquad \sum_{i=1}^{3} h(\sigma_i x_i) = \sum_{i=1}^{3} \sigma_i y_i.$$

On the other hand by (4.3)

$$(4.10) \qquad h(-\ln\sqrt{\det H}) = -\ln\sqrt{\det F}$$

and combining (4.7) - (4.10) we obtain

$$\sum_{i=1}^{3} h(\sigma_i x_i) = -\ln\sqrt{\det F} = h(-\ln\sqrt{\det H}) = h(\sum_{i=1}^{3} x_i).$$

Moreover, $\sigma_1\sigma_2\sigma_3 = 1$ since $\det F > 0$. It follows from Lemma 4.3 below that there exists an $i \in \{1, 2, 3\}$ such that

$$-\ln\sqrt{\det H} = \sum_{j=1}^{3} x_j = x_i = \ln\frac{h_i}{\sqrt{\det H}} = \ln h_i - \ln\sqrt{\det H}.$$

So indeed $h_i = 1$. $\qquad\square$

In the proof of the theorem we used the following observation:

Lemma 4.3. *Let* $g : \mathbb{R} \to \mathbb{R}$ *satisfy* $g(-x) = -g(x)$ *for all* x *and suppose* g *to be continuous, strictly convex and nonnegative on* $[0, \infty)$. *Assume that* $x_i \in \mathbb{R}$ *and that there exist* $\sigma_i \in \{-1, 1\}$ *with* $\sigma_1\sigma_2\sigma_3 = 1$ *such that*

$$(4.11) \qquad g\left(\sum_{i=1}^{3} x_i\right) = \sum_{i=1}^{3} g(\sigma_i x_i).$$

Then $x = \sum_{i=1}^{3} x_i \in \{x_1, x_2, x_3\}$.

Proof. In the proof we will use the following two facts:

i) Since g is strictly convex on $[0, \infty)$ and $g(0) = 0$ we conclude

$$g(y_1 + y_2) > g(y_1) + g(y_2) \quad \text{for } y_1, y_2 > 0.$$

Thus $g(y_1 + y_2 + y_3) = g(y_1) + g(y_2) + g(y_3)$, and $y_1, y_2, y_3 \geq 0$ implies that at least two of the y_i are equal to zero.

ii) Assume that $y_i \geq 0$, $i = 1, \ldots, 4$ and that $y_1 + y_2 = y_3 + y_4$, $g(y_1) + g(y_2) = g(y_3) + g(y_4)$. Then $\{y_1, y_2\} = \{y_3, y_4\}$. This follows from the fact that the function $h : [0, y_3 + y_4] \to \mathbb{R}$, $h(x) = g(y_3 + y_4 - x) + g(x)$ is strictly decreasing on $[0, \frac{1}{2}(y_3 + y_4)]$ and strictly increasing on $[\frac{1}{2}(y_3 + y_4), y_3 + y_4]$. Thus $h(x) = g(y_1) + g(y_2) = g(y_3) + g(y_4)$ implies $x \in \{y_3, y_4\}$. The assertion follows now from $h(y_1) = g(y_1) + g(y_2)$.

Now, suppose first that $(\sigma_1, \sigma_2, \sigma_3) \neq (1,1,1)$. Then after a permutation of indices $\sigma_1 = \sigma_2 = -1$ and $\sigma_3 = 1$. So (4.11) implies

$$(4.12) \qquad g(x_1 + x_2 + x_3) + g(x_1) = g(-x_2) + g(x_3).$$

Let $h(x) = g(x + x_2 + x_3) + g(x)$. Since g is strictly monotone on $\mathbb{R}$ the same holds for h. It follows from (4.12) that

$$h(-x_2) = g(x_3) + g(-x_2) = g(x_1 + x_2 + x_3) + g(x_1) = h(x_1)$$

and thus $x_1 = -x_2$ and $x_3 = x$.

It remains to consider the situation that $\sigma_i = 1$, $i = 1,2,3$. We assume that $x_1 + x_2 + x_3 \geq 0$ (otherwise replace x_i by $-x_i$) and, after permutation if necessary, that $x_1 \leq x_2 \leq x_3$. By hypothesis $g(x_1 + x_2 + x_3) = g(x_1) + g(x_2) + g(x_3)$. We have to distinguish three cases:

Case 1: If $x_1 \geq 0$ we conclude from i) that $x_1 = x_2 = 0$ and $x_3 = x$.

Case 2: If $x_1 < 0 \leq x_2$, then

$$g(x_1 + x_2 + x_3) + g(-x_1) = g(x_2) + g(x_3)$$

and it follows from ii) that $\{-x_1, x_1 + x_2 + x_3\} = \{x_2, x_3\}$.

Case 3: If $x_1 \leq x_2 < 0 \leq x_3$ then

$$g(x_1 + x_2 + x_3) + g(-x_1) + g(-x_2) = g(x_3) = g(x_1 + x_2 + x_3 - x_1 - x_2)$$

and it follows again from i) that at least two of the numbers $\{-x_1, -x_2, x_1 + x_2 + x_3\}$ are equal to zero, a contradiction.

This proves the lemma. $\qquad\square$

5. The rank-1 convex hull of two wells

Here we derive Theorem 1.5 and Theorem 1.7. These two results easily yield a complete description of the rank-1 convex hull of two arbitrary SO(3)-wells.

Proof of Theorem 1.5. All we need to do here is to combine two ingredients. The first one is the asymptotic result for the polyconvex hull obtained in Theorem 2.1. The other one is a certain connectedness property of the rank-1 convex hull (see [Pe1] or [MP], a more detailed consideration can be found in [Ki]). It says in particular that if K^{rc} consists of two metrically separated components C_1, C_2 then the generation of the rank-1 convex hull is done in each of the C_i locally and therefore independent of the situation in the other component. In short, $K^{rc} \cap C_i = (K \cap C_i)^{rc}$. While this statement is also trivially true for the ordinary convex hull K^c, well-known examples show that it can fail for K^{pc}. A counterexample for the quasiconvex case (situated in $\mathbb{M}^{6\times 2}$) is due to Šverák, see e.g. [Mu], Section 4.7.

So, let K be given by (1.4) and suppose that K does not contain any rank-1 connection. If all h_i are bigger than one then we are done by Theorem 1.2 i). Since $(K \cdot H^{-1})^{rc} = K^{rc} \cdot H^{-1}$, it suffices to consider the case $h_1 \geq h_2 > 1 > h_3 > 0$. Hence the assumptions of Theorem 2.1 are satisfied.

First, we note that

$$\text{dist}(K^{pc} \setminus SO(3), SO(3)) > 0.$$

Indeed, otherwise we find a sequence $F_n = (1 - \lambda_n)R_n + \lambda_n S_n H \notin SO(3)$ such that $R_n, S_n \in \text{conv } SO(3)$ and $\text{dist}(F_n, SO(3)) \to 0$. Since we know that $\lambda_n \geq \lambda_{0,H} > 0$ for all n, we infer the existence of $\lambda_\infty > 0, R_\infty, S_\infty \in \text{conv } SO(3)$ which satisfy

$$F_\infty = (1 - \lambda_\infty)R_\infty + \lambda_\infty S_\infty H \in SO(3).$$

This implies that

$$1 = \|F_\infty e_3\| \leq (1 - \lambda_\infty)\|R_\infty e_3\| + \lambda_\infty h_3 \|S_\infty e_3\| \leq 1 - \lambda_\infty(1 - h_3),$$

which is a contradiction.

Because $K^{rc} \subset K^{pc}$, we find disjoint compact sets $C_1 \supset \mathrm{SO}(3)$ and $C_2 \supset \mathrm{SO}(3)H$ such that $C_1 \cup C_2 = K^{rc}$. By the already mentioned topological considerations, see e.g. Corollary 2.9 in [MP], we conclude that

$$K^{rc} = (K \cap C_1)^{rc} \cup (K \cap C_2)^{rc} = \mathrm{SO}(3)^{rc} \cup (\mathrm{SO}(3)H)^{rc} = K. \qquad \square$$

Proof of Theorem 1.7. Let us fix an arbitrary matrix in $K^{pc} \setminus K$ of the form $F = (1-\lambda)R + \lambda SH$ such that $\lambda \in (0,1)$ and the minor relations (2.1) hold. Now (2.5) ensures that

$$(R^T R - I)_{33} = (S^T S - I)_{33} = (R^T S - I)_{33} = 0.$$

This implies that $Re_3 = Se_3 \in \mathbb{S}^2$. Since $M = I - R^T R$ is positive semidefinite and $M_{33} = 0$ we deduce that $0 = M_{13} = M_{23} = Re_3 \cdot Re_1 = Re_3 \cdot Re_2$. We choose $Q \in \mathrm{SO}(3)$ such that $QRe_3 = QSe_3 = e_3$. Hence QR, and similarly QS, is of block structure. We postmultiply the identity for F by Q to obtain

$$QF = (1-\lambda)\begin{pmatrix} \hat{R} & \\ & 1 \end{pmatrix} + \lambda \begin{pmatrix} \hat{S} & \\ & 1 \end{pmatrix} \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix}.$$

It is easily checked that the minor relations (2.1) hold true also for this new representation. We have

$$QR = \int_{\mathrm{SO}(3)} QX d\varrho(X).$$

Since $QX \in \mathrm{SO}(3)$ and hence $(QX)_{33} \leq 1$ ϱ-a.e while $(QR)_{33} = 1$ we deduce

$$\mathrm{supp}(\varrho) = \{X \in \mathrm{SO}(3) \ : \ (QX) = \begin{pmatrix} \hat{Y} & \\ & 1 \end{pmatrix}, Y \in \mathrm{SO}(2)\}.$$

Thus $\hat{R} \in \mathrm{conv}(\mathrm{SO}(2))$ and similarly $\hat{S} \in \mathrm{conv}(\mathrm{SO}(2))$. Of course, QF is also of block structure. Next we will show that $h_3 = 1$. If $h_3 \neq 1$ then $h_1 h_2 = 1$ and the assumption $h_2 \neq 1$ yields $h_1 \neq h_2$. Since $\hat{R}$ and $\hat{S}$ are conformal and $\hat{H} = \mathrm{diag}(h_1, h_2)$ is not conformal, we either have $\hat{S}\hat{H} - \hat{R} \neq 0$ or $\hat{S} = \hat{R} = 0$. The latter situation cannot arise since it would yield the contradiction $0 = \det(QF) = (1-\lambda) + \lambda \det H$. Thus there exist $i, j \in \{1, 2\}$ such that $\hat{S}_{ij} h_j - \hat{R}_{ij} \neq 0$. Choosing $i' = 3 - i$, $j' = 3 - j$, the cofactor relation implies

$$\begin{aligned} (\mathrm{cof}(QF))_{i'j'} &= (-1)^{i+j}(\lambda(\hat{S}_{ij}h_j - \hat{R}_{ij}) + \hat{R}_{ij})(\lambda(h_3 - 1) + 1) \\ &= (1-\lambda)\hat{R}_{i'j'} + \lambda \hat{S}_{i'j'}h_j h_3. \end{aligned}$$

Since equality in this nondegenerate quadratic equation occurs for both $\lambda = 1$ and $\lambda = 0$, we obtain a contradiction with $\lambda \in (0,1)$. Therefore, we conclude $h_3 = 1$.

Moreover, the minor relation for $(\mathrm{cof}(QF))_{33}$ ensures that the first block $\hat{F} = ((QF)_{ij})_{i,j \leq 2}$ satisfies

$$\det \hat{F} = (1-\lambda) + \lambda \det \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}.$$

Thus $\hat{F} \in \hat{K}^{pc}$. Moreover $\hat{F} \notin \hat{K}$ because otherwise $F \in K$ as $h_3 = 1$. By Šverák's result we conclude $(h_1 - 1)(h_2 - 1) \leq 0$ and

$$F = Q^{-1}\begin{pmatrix} \hat{F} & \\ & 1 \end{pmatrix} \in \left\{ \tilde{Q}\begin{pmatrix} \hat{F} & \\ & 1 \end{pmatrix} \ : \ \tilde{Q} \in \mathrm{SO}(3), \ \hat{F} \in \hat{K}^{lc} \right\}.$$

On the other hand, if the h_i's fulfill the conditions mentioned above then it is known that $\hat{K}^{pc} = \hat{K}^{lc}$ is nontrivial. Since the set of rank-1 segments is invariant under rotations and the same

holds true for K, we obtain the remaining inclusion

$$K^{lc} \supset \left\{ Q \begin{pmatrix} \hat{F} & \\ & 1 \end{pmatrix} \ : \ Q \in \mathrm{SO}(3), \ \hat{F} \in \hat{K}^{lc} \right\}.$$

This finishes our proof.   □

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Georg Dolzmann *, California Institute of Technology, Mail-Stop 104-44, Pasadena, CA 91125
  E-mail address: dolzmann@cco.caltech.edu

Bernd Kirchheim *, Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK,
  E-mail address: bkir@math.ucl.ac.uk

Stefan Müller, Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany
  E-mail address: sm@mis.mpg.de

Vladimir Šverák, School of Mathematics, University of Minnesota, 206 Church Street S.E., Minneapolis, MN 55455
  E-mail address: sverak@math.umn.edu

*on leave from Max Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, D-04103 Leipzig, Germany