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**Convex integration for Lipschitz  
mappings and counterexamples to  
regularity**

by

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# Convex Integration for Lipschitz Mappings and Counterexamples to Regularity

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# 1 Introduction

In this paper we study Lipschitz solutions of partial differential relations of the form

$$\nabla u(x) \in K \quad \text{a. e. in } \Omega, \quad (1)$$

where  $u$  is a (Lipschitz) mapping of an open set  $\Omega \subset \mathbf{R}^n$  into  $\mathbf{R}^m$ ,  $\nabla u(x)$  is its gradient (i. e. the matrix  $\partial u_i(x)/\partial x_j$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , defined for almost every  $x \in \Omega$ ), and  $K$  is a subset of the set  $M^{m \times n}$  of all real  $m \times n$  matrices. In addition to relation (1), boundary conditions and other conditions on  $u$  will also be considered.

Relation (1) is a special case of partial differential relations which have been extensively studied in connection with certain geometrical problems, such as isometric immersions. For example, the celebrated results of Nash [Na 54] and Kuiper [Ku 55] and their far-reaching generalizations by Gromov [Gr 86] showed striking and completely unexpected features of the behavior of  $C^1$ -isometric immersions of  $\mathbf{R}^n$  to  $\mathbf{R}^{n+1}$ , and Lipschitz isometric immersions of  $\mathbf{R}^n$  to  $\mathbf{R}^n$ . A general result describing a large class of Lipschitz solutions of partial differential relations more general than (1) can be found in the book of Gromov [Gr 86], page 218.

More recently, problems concerning solutions of relations of the form (1) have been studied in connection with the characterization of absolute minimizers of variational integrals describing the elastic energy of crystals exhibiting interesting microstructures ([BJ 87], [CK 88]). An important observation which came from this direction [Ba 90] is that relation (1) can have highly oscillatory solutions even when the difference of any two (non-identical) matrices in  $K$  has rank  $\geq 2$ . This situation, which does occur in some very interesting cases, is not covered by the theorem of Gromov mentioned above. In technical terms to be explained below, the reason is that Gromov's *P-convex hull* of the set  $K$  is again  $K$  in that situation. The main result of this paper, Theorem 3.2, covers many of these cases and shows that in the Lipschitz case it seems to be more natural to work with a different hull, which is defined in terms of rank-one convex functions, and can be significantly larger than the *P-convex hull*.

As an application of the theorem we give a solution of a long-standing problem regarding regularity of weak solutions of elliptic systems. We construct an example of a variational integral  $I(u) = \int_{\Omega} F(\nabla u)$ , where  $\Omega$  is the unit disc in  $\mathbf{R}^2$ ,  $u$  is a mapping of  $\Omega$  into  $\mathbf{R}^2$ , and  $F$  is a smooth, strongly quasi-convex function with bounded second derivatives, such that the Euler-Lagrange equation of  $I$  has a large class of weak solutions which are Lipschitz but not  $C^1$  in any open subset of  $\Omega$ , and have some other "wild" features. This result should be compared with the well-known result of Evans [Ev 86] which says that *minimizers* of  $I$  are smooth outside a closed subset of  $\Omega$  of measure zero. Our method also gives new conditions on  $F$  which are necessary for regularity. The conditions are expressed in terms of geometrical properties of the gradient mapping  $X \rightarrow DF(X)$ . We expect that the method is applicable to other interesting problems.

Our construction is quite different from well-known counterexamples to regularity of solutions of elliptic systems, such as [DG 68], [GM 68], or [HLN 96]. We should emphasize, however, that our method does not apply when  $F$  is convex. Very recently we became aware of the work of Scheffer [Sch 74], in which important partial results, including counterexamples, related to the regularity problem for the elliptic systems described above were obtained. It seems that the work was never published in a journal and has not received the attention it deserves. The point of view taken in that paper is implicitly quite similar to ours and in particular the  $T_4$ -configurations discussed in Section 4.2 play an important role in Scheffer's work. At the same time, the new techniques we develop enable us to answer questions which [Sch 74] left open.

## 2 Preliminaries

Let us first recall the various notions of convexity related to lower-semicontinuity of variational integrals of the form  $I(u) = \int_{\Omega} f(\nabla u)$ , where  $\Omega$  is a bounded domain in  $\mathbf{R}^n$ ,  $u: \Omega \rightarrow \mathbf{R}^m$  is a (sufficiently regular) mapping, and  $f: M^{m \times n} \rightarrow \mathbf{R}$  is a continuous function defined on the set  $M^{m \times n}$  of all real  $m \times n$  matrices.

A function  $f: M^{m \times n} \rightarrow \mathbf{R}$  is *quasi-convex* if  $\int_{\Omega} (f(A + \nabla \varphi) - f(A)) \geq 0$  for each  $A \in M^{m \times n}$  and each smooth, compactly supported  $\varphi: \Omega \rightarrow \mathbf{R}^m$ . This definition was introduced by Morrey (see e. g. [Mo 66]) who also proved that the quasi-convexity of  $f$  is necessary and sufficient for the functional  $I$  to be lower-semicontinuous with respect to the uniform convergence of uniformly Lipschitz functions. It is also necessary and sufficient for the weak sequential lower-semicontinuity of  $I$  on Sobolev spaces  $W^{1,p}(\Omega, \mathbf{R}^m)$ , if natural growth conditions are satisfied, see [Ma 85] and [AF 87]. The definition of quasi-convexity is independent of  $\Omega$ , as can be seen by a simple scaling and covering argument ([Mo 66]). In fact, we have the following simple observation made by many authors:

**Lemma 2.1** *Let  $\mathbf{T}^n$  be a flat  $n$ -dimensional torus. A function  $f: M^{m \times n} \rightarrow \mathbf{R}$  is quasi-convex if and only if  $\int_{\mathbf{T}^n} (f(A + \nabla \varphi) - f(A)) \geq 0$  for each  $A \in M^{m \times n}$  and each smooth  $\varphi: \mathbf{T}^n \rightarrow \mathbf{R}^m$ .*

The reader is referred for example to [Sv 92a] for a proof of this statement.

We also recall that, with the notation above,  $f: M^{m \times n} \rightarrow \mathbf{R}$  is *strongly quasi-convex* if there exists  $\gamma > 0$  such that  $\int_{\Omega} (f(A + \nabla \varphi) - f(A)) \geq \gamma \int_{\Omega} |\nabla \varphi|^2$  for each  $A \in M^{m \times n}$  and each smooth, compactly supported  $\varphi: \Omega \rightarrow \mathbf{R}^m$ . This notion appears naturally in the regularity theory, see for example [Ev 86].

A function  $f: M^{m \times n} \rightarrow \mathbf{R}$  is *rank-one convex* if it is convex along any line whose direction is given by a matrix of rank one, i. e.  $t \rightarrow f(A + tB)$  is convex for each  $A \in M^{m \times n}$  and each  $B \in M^{m \times n}$  with  $\text{rank } B = 1$ . This class of functions will play a particularly important rôle in our analysis. It can be proved that any quasi-convex function is rank-one convex, but the opposite implication fails when  $n \geq 2$ ,  $m \geq 3$  ([Sv 92a]). (The case  $n \geq 2$ ,  $m = 2$  is open.)

We will also deal with functions which are defined only on symmetric matrices. We will denote by  $S^{n \times n}$  the set of all symmetric  $n \times n$  matrices. The notions introduced above for functions on  $M^{m \times n}$  can be modified in the obvious manner to apply to functions on symmetric matrices. For example, a function  $f: S^{n \times n} \rightarrow \mathbf{R}$  is quasi-convex, if  $\int_{\Omega} (f(A + \nabla^2 \phi) - f(A)) \geq 0$  for each  $A \in S^{n \times n}$  and each smooth, compactly supported  $\phi: \Omega \rightarrow \mathbf{R}$ . Again, the definition is independent of  $\Omega$  and, in fact,  $\Omega$  can be replaced by any flat  $n$ -dimensional torus.

In the rest of this section we look in more detail on facts related to rank-one convexity.

Let  $\mathcal{O} \subset M^{m \times n}$  be an open and let  $f: \mathcal{O} \rightarrow \mathbf{R}$  be a function. We say that  $f$  is *rank-one convex* in  $\mathcal{O}$ , if  $f$  is convex on each rank-one segment contained in  $\mathcal{O}$ . It is easy to see that every rank-one convex function  $f: \mathcal{O} \rightarrow \mathbf{R}$  is locally Lipschitz in  $\mathcal{O}$ .

We will use  $\mathcal{P}$  to denote the set of all compactly supported probability measures in  $M^{m \times n}$ . For a compact set  $K \subset M^{m \times n}$  we use  $\mathcal{P}(K)$  to denote the set of all probability measures supported in  $K$ . For  $\nu \in \mathcal{P}$  we denote by  $\bar{\nu}$  the center of mass of  $\nu$ , i. e.  $\bar{\nu} = \int_{M^{m \times n}} X d\nu(X)$ .

Following [Pe 93], we say that a measure  $\nu \in \mathcal{P}$  is a *laminar* if  $\langle \nu, f \rangle \geq f(\bar{\nu})$  for each rank-one convex function  $f: M^{m \times n} \rightarrow \mathbf{R}$ . At the center of our attention will be the sets  $\mathcal{P}^{\text{rc}}(K) = \{\nu \in \mathcal{P}(K), \nu \text{ is a laminar}\}$ , which are defined for any compact set  $K \subset M^{m \times n}$ .

Let  $\mathcal{O}$  be an open subset of  $M^{m \times n}$ . We now define an important subset  $\mathcal{L}(\mathcal{O})$  of laminars, called *laminars of finite order* in  $\mathcal{O}$ . The definition is by induction:

1. For each  $A \in \mathcal{O}$ , the Dirac mass at  $A$ , denoted by  $\delta_A$ , belongs to  $\mathcal{L}(\mathcal{O})$ .
2. Assume  $\lambda_1, \dots, \lambda_m \geq 0$  with  $\sum \lambda_j = 1$ , and that  $\nu = \sum_{j=1}^m \lambda_j \delta_{A_j}$  belongs to  $\mathcal{L}(\mathcal{O})$ . Assume also that  $[B_1, B_2]$  is a rank-one segment contained in  $\mathcal{O}$ , and that there is  $0 \leq s \leq 1$  such that  $(1-s)B_1 + sB_2 = A_m$ . Then the measure  $\mu = \sum_{j=1}^{m-1} \lambda_j \delta_{A_j} + (1-s)\lambda_m \delta_{B_1} + s\lambda_m \delta_{B_2}$  also belongs to  $\mathcal{L}(\mathcal{O})$ .

Let  $K$  be a compact subset of  $M^{m \times n}$ . The *rank-one convex hull*  $K^{\text{rc}} \subset M^{m \times n}$  of  $K$  is defined as follows. A matrix  $X$  does not belong to  $K^{\text{rc}}$  if and only if there exists  $f: M^{m \times n} \rightarrow \mathbf{R}$  which is rank-one convex such that  $f \leq 0$  on  $K$  and  $f(X) > 0$ . We emphasize that this definition will be used only when  $K$  is compact. For open sets  $\mathcal{O} \subset M^{m \times n}$ , we define the rank-one convex hull  $\mathcal{O}^{\text{rc}}$  of  $\mathcal{O}$  as  $\mathcal{O}^{\text{rc}} = \cup \{K^{\text{rc}}, K \text{ is a compact subset of } \mathcal{O}\}$ . With this definition we have the property that the rank-one convex hull of an open set is again an open set, which will be useful for our purposes.

We refer the reader to [MP 98] for interesting results about rank-one convex hulls of closed sets. The following lemma, which is a slight generalization a result from [Pe 93], will play an important rôle.

**Theorem 2.1** *Let  $K$  be a compact subset of  $M^{m \times n}$  and let  $\nu \in \mathcal{P}^{\text{rc}}(K)$ . Let  $\mathcal{O} \subset M^{m \times n}$  be an open set such that  $K^{\text{rc}} \subset \mathcal{O}$ . Then there exists a sequence  $\nu_j \in \mathcal{L}(\mathcal{O})$  of laminars of finite order in  $\mathcal{O}$  such that  $\bar{\nu}_j = \bar{\nu}$  for each  $j$  and the  $\nu_j$  converge weakly\* to  $\nu$  in  $\mathcal{P}$ .*

As a preparation for the proof of the theorem, we prove the following lemma.

**Lemma 2.2** *Let  $\mathcal{O}$  be an open subset of  $M^{m \times n}$ . Let  $f: \mathcal{O} \rightarrow \mathbf{R}$  be a continuous function and let  $R_{\mathcal{O}}f: \mathcal{O} \rightarrow \mathbf{R} \cup \{-\infty\}$  be defined by  $R_{\mathcal{O}}f = \sup\{g, g: \mathcal{O} \rightarrow \mathbf{R} \text{ is rank-one convex in } \mathcal{O} \text{ and } g \leq f\}$ . Then for each  $X \in \mathcal{O}$  we have  $R_{\mathcal{O}}f(X) = \inf\{\langle \nu, f \rangle, \nu \in \mathcal{L}(\mathcal{O}) \text{ and } \bar{\nu} = X\}$ .*

**Proof.** Let us denote by  $\tilde{f}$  the function in  $\mathcal{O}$  defined by  $\tilde{f}(X) = \inf\{\langle \nu, f \rangle, \nu \in \mathcal{L}(\mathcal{O}) \text{ and } \bar{\nu} = X\}$ . Clearly  $R_{\mathcal{O}}f \leq \tilde{f}$  in  $\mathcal{O}$ . On the other hand, we see from the definition of the set  $\mathcal{L}(\mathcal{O})$  that it has the following property: if  $\nu_1, \nu_2 \in \mathcal{L}(\mathcal{O})$ , and the segment  $[\bar{\nu}_1, \bar{\nu}_2]$  is a rank-one segment contained in  $\mathcal{O}$ , then any convex combination of  $\nu_1$  and  $\nu_2$  is again in  $\mathcal{O}$ . Using this, we see immediately from the definitions that  $\tilde{f}$  is rank-one convex in  $\mathcal{O}$  and hence  $R_{\mathcal{O}}f = \tilde{f}$ .

**Proof of Theorem 2.1.** Let  $\nu \in \mathcal{P}^{\text{rc}}(K)$  and let  $\bar{\nu} = A$  be its center of mass. From the definitions we see that  $A \in K^{\text{rc}}$ . We choose an open set  $U \subset M^{m \times n}$  satisfying  $K^{\text{rc}} \subset U \subset \bar{U} \subset \mathcal{O}$  and define  $\mathcal{F} = \{\mu \in \mathcal{L}(U), \bar{\mu} = A\}$ . We claim the weak\* closure of  $\mathcal{F}$  contains  $\nu$ . To prove the claim, we argue by contradiction. Assume  $\nu$  does not belong to the weak\* closure of  $\mathcal{F}$ . Since  $\mathcal{F}$  is clearly convex, we see from the Hahn-Banach Theorem that there exists a continuous function  $f: \bar{U} \rightarrow \mathbf{R}$  such that  $\langle \nu, f \rangle < \inf\{\langle \mu, f \rangle, \mu \in \mathcal{L}(U) \text{ and } \bar{\mu} = A\}$ . By Lemma 2.2, we have  $\inf\{\langle \mu, f \rangle, \mu \in \mathcal{L}(U) \text{ and } \bar{\mu} = A\} = R_U f(A)$ . We see that the function  $\tilde{f} = R_U f: U \rightarrow \mathbf{R}$  is rank-one convex in  $U$  and satisfies  $\langle \nu, \tilde{f} \rangle \leq \langle \nu, f \rangle < \tilde{f}(\bar{\nu})$ . By Lemma 2.3 below, there exists a rank-one convex function  $F: M^{m \times n} \rightarrow \mathbf{R}$  such that  $F = \tilde{f}$  on  $K^{\text{rc}}$ . We conclude that  $\nu$  cannot belong to  $\mathcal{P}^{\text{rc}}(K)$ , a contradiction. The proof is finished.

**Lemma 2.3** *Let  $K \subset M^{m \times n}$  be a compact set, let  $\mathcal{O}$  be an open set containing  $K^{\text{rc}}$  (the rank-one convex hull of  $K$ ) and let  $f: \mathcal{O} \rightarrow \mathbf{R}$  be rank-one convex. Then there exists  $F: M^{m \times n} \rightarrow \mathbf{R}$  which is rank-one convex and coincides with  $f$  in a neighborhood of  $K^{\text{rc}}$ .*

**Proof.** We claim there exists a non-negative rank-one convex  $g: M^{m \times n} \rightarrow \mathbf{R}$  such that  $K = \{X, g(X) = 0\}$ . To prove this, we choose  $R > 0$  so that  $K \subset B_{R/2} = \{X, |X| < R/2\}$  and define  $g_1: B_R \rightarrow \mathbf{R}$  by

$$g_1(X) = \sup\{f(X), f: B_R \rightarrow \mathbf{R}, \\ f \text{ is rank-one convex in } B_R \text{ and } f \leq \text{dist}(\cdot, K) \text{ in } B_R\}.$$

The function  $g_1$  is obviously non-negative and rank-one convex in  $B_R$ . Moreover,  $\{X \in B_R, g_1(X) = 0\} \supset K$  and from the definition of  $K^{\text{rc}}$  we see that  $g_1 > 0$  outside  $K^{\text{rc}}$ . We now define

$$g(X) = \begin{cases} \max(g_1(X), 12|X| - 9R) & \text{when } X \in B_R \\ 12|X| - 9R & \text{when } |X| \geq R \end{cases}$$

Clearly  $g$  is rank-one convex in a neighborhood of any point  $X$  with  $|X| \neq R$ . Since  $g_1(X) \leq 2|X|$  when  $|X| = R$ , we see that we have  $g(X) = 12|X| - 9R$  in

a neighborhood of  $\{|X| = R\}$ . We see that  $g$  is non-negative, rank-one convex in  $M^{m \times n}$ ,  $\{X, g(X) = 0\} \supset K$ , and  $\{X, g(X) > 0\} \cap K^{\text{rc}} = \emptyset$ . Therefore  $\{X, g(X) = 0\} = K^{\text{rc}}$

We can now finish the proof of the lemma. Replacing  $f$  by  $f + c$ , if necessary, we can assume that  $f > 0$  in a neighborhood of  $K^{\text{rc}}$ . For  $k > 0$  we let  $U_k = \{X \in \mathcal{O}, f(X) > kg(X)\}$ . We also let  $V_k$  be the union of the connected components of  $U_k$  which have a non-empty intersection with  $K^{\text{rc}}$ . It is easy to see that there exists  $k_0 > 0$  such that  $\bar{V}_{k_0} \subset \mathcal{O}$ . We now let  $F(X) = f(X)$  when  $X \in V_{k_0}$  and  $F(X) = k_0g(X)$  when  $X \in M^{m \times n} \setminus V_{k_0}$ . It is easy to check that the function  $F$  defined in this way is rank-one convex on  $M^{m \times n}$ .

### 3 Constructions

Throughout this section,  $\Omega$  denotes a fixed bounded open subset of  $\mathbf{R}^n$ . We will use the following terminology. A Lipschitz mapping  $u: \Omega \rightarrow \mathbf{R}^m$  is *piecewise affine*, if there exists a countable system of mutually disjoint open sets  $\Omega_j \subset \Omega$  which cover  $\Omega$  up to a set of zero measure, and the restriction of  $u$  to each of the sets  $\Omega_j$  is affine.

Following Gromov ([Gr 86], page 18) we also introduce the following concept. Let  $\mathcal{F}(\Omega, \mathbf{R}^m)$  be a family of continuous mappings of  $\Omega$  into  $\mathbf{R}^m$ . We say that a given continuous mapping  $v_0: \Omega \rightarrow \mathbf{R}^m$  admits a *fine  $C^0$ -approximation* by the family  $\mathcal{F}(\Omega, \mathbf{R}^m)$  if there exists, for every continuous function  $\varepsilon: \Omega \rightarrow (0, \infty)$ , an element  $v$  of the family  $\mathcal{F}(\Omega, \mathbf{R}^m)$  such that  $|v(x) - v_0(x)| < \varepsilon(x)$  for each  $x \in \Omega$ .

#### 3.1 The basic construction

The main building block of all the solutions of relation (1) which we construct in this paper is the following simple lemma.

**Lemma 3.1** *Let  $A, B \in M^{m \times n}$  be two matrices with  $\text{rank}(B - A) = 1$ , let  $b \in \mathbf{R}^m$ ,  $0 < \lambda < 1$  and  $C = (1 - \lambda)A + \lambda B$ . Then, for any  $0 < \delta < |A - B|/2$ , the affine mapping  $x \rightarrow Cx + b$  admits a fine  $C^0$ -approximation by piecewise affine mappings  $u: \Omega \rightarrow \mathbf{R}^m$  such that  $\text{dist}(\nabla u(x), \{A, B\}) < \delta$  almost everywhere in  $\Omega$  and  $\text{meas}\{x \in \Omega, |\nabla u(x) - A| < \delta\} = (1 - \lambda)\text{meas}\Omega$ .*

**Proof.** We first note that it is enough to prove the lemma only for a special case when the function  $\varepsilon(x)$  appearing in the definition of a fine  $C^0$ -approximation is constant and the function approximating the function  $u$  satisfies the boundary condition  $u(x) = Cx + b$  for  $x \in \partial\Omega$ . This can be seen by considering a sequence of open sets  $\Omega_j$  which are mutually disjoint, satisfy  $\bar{\Omega}_j \subset \Omega$ , and cover  $\Omega$  up to a set of full measure.

To prove the special case, we note that we can assume without loss of generality that  $A = -\lambda a \otimes e_n$ ,  $B = (1 - \lambda)a \otimes e_n$ , and  $C = 0$ , where  $a \in \mathbf{R}^m$  and  $e_n = (0, \dots, 0, 1) \in \mathbf{R}^n$ . We define  $h: \mathbf{R} \rightarrow \mathbf{R}$  and  $w: \mathbf{R}^n \rightarrow \mathbf{R}^m$  by  $h(s) = (|s| + (2\lambda - 1)s)/2$  and  $w(x) = a \max(0, 1 - |x_1| - \dots - |x_{n-1}| - h(x_n))$ .



We choose a small  $\delta' > 0$ , and set  $v(x) = \delta' w(x_1, \dots, x_{n-1}, x_n/\delta')$ . We also let  $\omega = \{x, v(x) > 0\}$ . We check by a direct calculation that  $\text{dist}(\nabla v(x), \{A, B\}) \leq (n-1)|a|\delta'$  for almost every  $x \in \omega$ . We clearly also have  $v(x) = 0$  when  $x \in \partial\omega$ . By Vitali's theorem we can cover  $\Omega$  up to a set of measure zero by a countable family  $\{\omega_i\}$  of mutually disjoint sets of the form  $\omega_i = y_i + r_i\omega$  (with  $y_i \in \mathbf{R}^n$  and  $r_i \in (0, \epsilon)$ ). We let  $u(x) = r_i v(r_i^{-1}(x - y_i))$  when  $x \in \omega_i$ , and  $u(x) = 0$  if  $x \in \Omega \setminus \cup_i \omega_i$ . It is easy to check that  $u$  satisfies the required conditions, provided  $\delta'$  is sufficiently small.

**Lemma 3.2** *Let  $\nu \in \mathcal{P}(M^{m \times n})$  be a laminate of finite order, let  $A = \bar{\nu}$  be its center of mass. Let us write  $\nu = \sum_{j=1}^r \lambda_j \delta_{A_j}$  with  $\lambda_j > 0$  and  $A_i \neq A_j$  for  $i \neq j$ . Then, for each  $b \in \mathbf{R}^m$ , and each  $\delta, \delta' > 0$ , the mapping  $x \rightarrow Ax + b$  admits a fine  $C^0$ -approximation by piecewise affine mappings  $u$  satisfying  $\text{dist}(\nabla u(x), \{A_1, \dots, A_r\}) < \delta$  a.e. in  $\Omega$  and  $\text{meas}\{x \in \Omega, \text{dist}(\nabla u(x), A_j)\} = \lambda_j' \text{meas} \Omega$ , with  $1 - \delta' < \lambda_j'/\lambda_j < 1 + \delta'$ .*

**Proof.** This can be easily proved by applying iteratively Lemma 3.1 in a way which is naturally suggested by the definition of the laminate of finite order.

### 3.2 Open relations

We recall that the rank-one convex hull  $\mathcal{O}^{\text{rc}}$  of an open set  $\mathcal{O} \subset M^{m \times n}$  is, by definition, the union of the rank-one convex hulls of all compact subsets of  $\mathcal{O}$ . The main result of this subsection is the following.

**Theorem 3.1** *Let  $\mathcal{O} \subset M^{m \times n}$  be open, and let  $P \subset \mathcal{O}^{\text{rc}}$  be compact. Let  $u_0: \Omega \rightarrow \mathbf{R}^m$  be a piecewise affine Lipschitz mapping such that  $\nabla u_0(x) \in P$  for a.e.  $x \in \Omega$ . Then  $u_0$  admits a fine  $C^0$ -approximation by piecewise affine Lipschitz mappings  $u: \Omega \rightarrow \mathbf{R}^m$  satisfying  $\nabla u(x) \in \mathcal{O}$  a.e. in  $\Omega$ .*

**Proof.** As a first step, we prove the following lemma.

**Lemma 3.3** *Let  $K \subset M^{m \times n}$  be a compact set and let  $U \subset M^{m \times n}$  be an open set containing  $K$ . Let  $\nu \in \mathcal{P}^{\text{rc}}(K)$  and denote  $A = \bar{\nu}$ . Then, for any given  $\delta > 0$ , the mapping  $x \rightarrow Ax$  admits a fine  $C^0$ -approximation by piecewise affine mappings  $u$  satisfying  $\nabla u(x) \in U^{\text{rc}}$  a.e. in  $\Omega$  and  $\text{meas}\{x \in \Omega, \nabla u(x) \in U\} > (1 - \delta)\text{meas} \Omega$ .*

**Proof.** We use Theorem 2.1 to approximate  $\nu$  by a laminate of finite order  $\mu$  which is supported in a finite subset of  $U^{\text{rc}}$  and satisfies  $\bar{\mu} = \bar{\nu}$  and  $\mu(U) > (1 - \delta/2)$ . Then we apply Lemma 3.2 to  $\mu$  and the proof is finished.

Theorem 3.1 can now be proved by repeatedly applying Lemma 3.3 in the following way. We first choose a sequence of compact sets  $K_1, K_2, \dots \subset M^{m \times n}$ , a sequence of open sets  $U_1, U_2, \dots \subset M^{m \times n}$ , and a compact set  $Q \subset M^{m \times n}$  such that  $P = K_1 \subset U_1 \subset K_2 \subset U_2 \subset \dots \subset Q \subset \mathcal{O}$ . We also choose  $0 < \delta < 1$ . Let  $\varepsilon = \varepsilon(x) > 0$  be a continuous function on  $\Omega$ . In the first step we apply Lemma 3.3 to approximate  $u_0$  up to  $\varepsilon/2$  by a mapping  $u_1$  satisfying  $\nabla u_1(x) \in U_1^{\text{rc}}$  a.e. in  $\Omega$ ,

together with  $\text{meas}\{x \in \Omega, \nabla u_1(x) \in U_1\} > (1 - \delta)\text{meas}\Omega$ . We now modify  $u_1$  on those subregions of  $\Omega$  where  $\nabla u_1(x)$  does not belong to  $U_1$  by applying Lemma 3.3 again. We obtain a new mapping,  $u_2$ , which approximates  $u_1$  up to  $\varepsilon/4$ , coincides with  $u_1$  a. e. in the set  $\{x \in \Omega, \nabla u_1(x) \in U_1\}$ , and satisfies  $\nabla u_2(x) \in U_2^c$  a.e. in  $\Omega$  together with  $\text{meas}\{x \in \Omega, \nabla u_2(x) \in U_2\} > ((1 - \delta) + \delta(1 - \delta))\text{meas}\Omega$ . By continuing this procedure we get a sequence  $u_k$  of mappings which is easily seen to converge to a mapping  $u$  which gives the required approximation of  $u_0$ .

### 3.3 Closed relations and in-approximations

When considering relation (1) for closed sets  $K$ , it is natural to try to construct solutions by combining Theorem 3.1 and a suitable limit procedure. For simplicity we will assume in this section that  $K$  is compact. Following Gromov ([Gr 86]) we say that a sequence of open sets  $\{U_i\}_{i=1}^\infty$  is an *in-approximation* of  $K$  if  $U_i \subset U_{i+1}^c$  for each  $i$ , and  $\sup_{X \in U_i} \text{dist}(X, K) \rightarrow 0$  as  $i \rightarrow \infty$ .

**Theorem 3.2** *Assume that a compact set  $K \subset M^{m \times n}$  admits an in-approximation by open sets  $U_i$  in the sense of the definition above. Then any  $C^1$ -mapping  $v: \Omega \rightarrow \mathbf{R}^m$  satisfying  $\nabla v(x) \in U_1$  in  $\Omega$  admits a fine  $C^0$ -approximation by Lipschitz mappings  $u: \Omega \rightarrow \mathbf{R}^m$  satisfying  $\nabla u(x) \in K$  a.e. in  $\Omega$ .*

**Proof.** By the same argument as in the proof of Lemma 3.1 it is enough to prove the statement only in the case when the function  $\varepsilon = \varepsilon(x)$  in the definition of a fine  $C^0$ -approximation is constant.

Let  $\rho: \mathbf{R}^n \rightarrow \mathbf{R}$  be the usual mollifying kernel, i.e. we assume that  $\rho$  is smooth, non-negative, supported in  $\{x, |x| < 1\}$ , and  $\int \rho = 1$ . For  $\varepsilon > 0$  we let  $\rho_\varepsilon = \varepsilon^{-n} \rho(x/\varepsilon)$ . For a function  $w \in L^1(\Omega)$  we define  $\rho_\varepsilon * w$  in the usual way, by considering  $w$  as a function on  $\mathbf{R}^n$  with  $w = 0$  outside  $\Omega$ . In other words,  $\rho_\varepsilon * w(x) = \int_\Omega w(y) \rho_\varepsilon(x - y) dy$ .

We start the proof by choosing  $\delta_1 > 0$  (the exact value of which will be specified later) and by approximating  $v$  by a piecewise affine  $u_1: \Omega \rightarrow \mathbf{R}^m$  with  $|u_1 - v| < \delta_1$  in  $\Omega$ ,  $u_1 = v$  on  $\partial\Omega$ , and  $\nabla u_1 \in U_1$  a.e. in  $\Omega$ . (We recall that in this paper “piecewise affine” allows for countably many affine pieces.) We also choose  $\varepsilon_1 > 0$  so that  $\|\nabla u_1 * \rho_{\varepsilon_1} - \nabla u_1\|_{L^1(\Omega)} \leq 2^{-1}$ .

Using Theorem 3.1 together with an obvious inductive argument, we construct a sequence of mappings  $u_i: \Omega \rightarrow \mathbf{R}^m$  and numbers  $0 < \varepsilon_i < 2^{-i}$ ,  $\delta_i > 0$  satisfying

$$\begin{aligned} \nabla u_i &\in U_i && \text{a.e. in } \Omega, \\ u_i &= v && \text{on } \partial\Omega, \\ \|\nabla u_i * \rho_{\varepsilon_i} - \nabla u_i\|_{L^1(\Omega)} &\leq 2^{-i}, \\ \delta_{i+1} &= \varepsilon_i \delta_i, \\ |u_{i+1} - u_i| &\leq \delta_{i+1} && \text{in } \Omega. \end{aligned}$$

The mappings  $u_i$  converge uniformly to a Lipschitz function  $u: \Omega \rightarrow \mathbf{R}^m$ . We also have  $|u - v| \leq \sum_i |u_{i+1} - u_i| + |u_1 - v| \leq 2\delta_1$ . It remains to prove that

$\nabla u \in K$  a.e. in  $\Omega$ . This will be clear if we establish that  $\nabla u_i \rightarrow \nabla u$  in  $L^1(\Omega)$ . We can write

$$\begin{aligned} \|\nabla u_i - \nabla u\|_{L^1(\Omega)} &\leq \|\nabla u_i - \nabla u_i * \rho_{\varepsilon_i}\|_{L^1(\Omega)} \\ &+ \|\nabla u * \rho_{\varepsilon_i} - \nabla u\|_{L^1(\Omega)} \\ &+ \|\nabla u_i * \rho_{\varepsilon_i} - \nabla u * \rho_{\varepsilon_i}\|_{L^1(\Omega)}. \end{aligned}$$

The first two terms on the right-hand side of this inequality clearly converge to zero as  $i \rightarrow \infty$ . Defining  $\Omega_i = \{x \in \Omega, \text{dist}(x, \partial\Omega) > 2\varepsilon_i\}$  we can estimate the third term as

$$\|(u_i - u) * \nabla \rho_{\varepsilon_i}\|_{L^1(\Omega)} + \|\nabla u_i - \nabla u\|_{L^1(\Omega \setminus \Omega_i)} \leq \frac{c}{\varepsilon_i} \|u_i - u\|_{\infty} + C \text{meas}(\Omega \setminus \Omega_i),$$

where  $c$  and  $C$  are constants depending only on  $\rho$  and the Lipschitz constant of  $u_i - u$ , respectively.

We have

$$\|u_i - u\|_{\infty} \leq \sum_{j=i}^{\infty} \|u_j - u_{j+1}\|_{\infty} \leq \sum_{j=i+1}^{\infty} \delta_j \leq 2\delta_{i+1}.$$

Hence the third term can be estimated by  $2c\delta_{i+1}/\varepsilon_i + C \text{meas}(\Omega \setminus \Omega_i) \leq 2c\delta_i + C \text{meas}(\Omega \setminus \Omega_i)$  which converges to zero as  $i \rightarrow \infty$ . The proof is finished.

*Remark:* The explanation of the strong convergence of  $\nabla u_i$  is more or less the following. We can achieve a very fast convergence of  $u_i$  in the sup-norm. It may seem that this is not enough to say much about the convergence of  $\nabla u_i$ . However, in the proof we choose the parameters in such a way that  $\|u_i - u\|_{\infty}$  is very small in comparison with a typical length over which  $\nabla u_i$  changes significantly (in an integral sense). This is the main reason we get the strong convergence. The above argument is taken from [MS 96]. A different approach can be found in [DM 97].

## 4 Applications to elliptic systems

Let  $\Omega \subset \mathbf{R}^2$  be a disc. For (sufficiently regular) mappings  $u: \Omega \rightarrow \mathbf{R}^2$  we consider the functional  $I(u) = \int_{\Omega} F(\nabla u(x)) dx$ , where  $F$  is a (smooth) function on the set  $M^{2 \times 2}$  of all real  $2 \times 2$  matrices, which satisfies certain ‘‘ellipticity conditions’’. More precisely, we will require that  $F$  be strongly quasiconvex and that its second derivatives be uniformly bounded in  $M^{2 \times 2}$ .

The purpose of this section is to show how we can apply the results above to construct weak solutions of the Euler-Lagrange equation

$$\text{div } DF(\nabla u) = 0 \tag{2}$$

of the functional  $I$  which are Lipschitz, but not continuously differentiable on any open subset of  $\Omega$ . This is in sharp contrast with regularity properties of minimizers of  $I$ , see, for example [Ev 86]. In fact, we prove the following slightly stronger statement.

**Theorem 4.1** *There exists a smooth strongly quasiconvex function  $F_0: M^{2 \times 2} \rightarrow \mathbf{R}$  with  $|D^2 F_0| \leq c$  in  $M^{2 \times 2}$ , four matrices  $A_1, \dots, A_4 \in M^{2 \times 2}$ ,  $\varepsilon > 0$  and  $\delta > 0$  such that the following is true. Let  $F: M^{2 \times 2} \rightarrow \mathbf{R}$  be a  $C^2$ -function satisfying  $|DF(A_j) - DF_0(A_j)| \leq \delta$  and  $|D^2 F(A_j) - D^2 F_0(A_j)| \leq \delta$  for  $j = 1, 2, 3, 4$ . Then each piecewise  $C^1$ -function  $v: \Omega \rightarrow \mathbf{R}^2$  satisfying  $|\nabla v| < \varepsilon$  a. e. in  $\Omega$  admits a fine  $C^0$ -approximation by Lipschitz mappings  $u: \Omega \rightarrow \mathbf{R}^2$  which are not  $C^1$  on any open subset of  $\Omega$  and are weak solutions of the equation  $\operatorname{div} DF(\nabla u) = 0$  in  $\Omega$ .*

The theorem will be proved in Section 4.4, after we establish some useful facts about quasiconvex functions and rank-one convex hulls. The idea of the construction is the following. We rewrite the equation (2) as a first-order system

$$\nabla U \in K \tag{3}$$

and then show that the strong quasiconvexity does not prevent the rank-one convex hull of  $K$  from being large. (We remark that the strong quasi-convexity does exclude any non-trivial rank-one connections in  $K$ , see [Ba 80].) We can then use the methods developed in the previous sections to construct the desired solutions. Moreover, it turns out the situation is stable under the perturbations of  $F_0$  which are allowed in the theorem.

One way to write equation (2) in the form (3) is the following. We denote by  $J$  the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . The condition that the  $2 \times 2$  tensor  $DF(\nabla u)$  be divergence-free is equivalent to the condition that  $DF(\nabla u)J$  be the gradient of a function  $v: \Omega \rightarrow \mathbf{R}^2$ . We now introduce  $U: \Omega \rightarrow \mathbf{R}^4$  by  $U = \begin{pmatrix} u \\ v \end{pmatrix}$ . We also let  $K$  to be the set of all  $4 \times 2$  matrices of the form  $\begin{pmatrix} X \\ DF(X)J \end{pmatrix}$ , where  $X$  runs through all  $2 \times 2$  matrices. It is clear that, in this notation, system (2) is equivalent to system (3).

## 4.1 Quasiconvex functions

We begin by describing a quasi-convex function which will play an important role in our construction. We will be using notation introduced in Section 2. We define  $f_0: S^{2 \times 2} \rightarrow \mathbf{R}$  by  $f_0(X) = \det X$  when  $X$  is positive definite and by  $f_0(X) = 0$  otherwise.

**Lemma 4.1** *The function  $f_0$  is quasiconvex on  $S^{2 \times 2}$ .*

**Proof.** This result is proved in [Sv 92b]. In that paper the proof is actually carried out for a more general class of functions. We give a simple version of the proof here, for the convenience of the reader. Let  $\Omega = \{x \in \mathbf{R}^2, |x| < 1\}$  and let  $\phi: \Omega \rightarrow \mathbf{R}$  be smooth and compactly supported in  $\Omega$ . We must prove that for each  $A \in S^{2 \times 2}$  we have  $\int_{\Omega} (f_0(A + \nabla^2 \phi) - f_0(A)) \geq 0$ . This is obvious if  $A$  is not positive definite, since then we integrate a non-negative function. If

$A$  is positive definite, we can assume  $A = I$  by a simple change of variables. Let  $u_0(x) = |x|^2/2$  and  $u(x) = u_0(x) + \phi(x)$ . We also set  $\varphi = \nabla u$ , which will be viewed as a map  $\varphi: \Omega \rightarrow \mathbf{R}^2$ . Finally, we let  $E = \{x \in \Omega, \det \nabla \varphi(x) \geq 0\}$ . We must prove that  $\int_E \det \nabla \varphi \geq \text{meas}(\Omega)$ . Since  $\det \varphi \geq 0$  on  $E$ , we can use the area formula ([Fe 69]) to infer that it is enough to prove  $\Omega \subset \varphi(E)$ . Consider an arbitrary  $b \in \Omega$  and let  $a \in \bar{\Omega}$  be a point where the function  $x \rightarrow u(x) - b \cdot x$  attains its minimum in  $\bar{\Omega}$ . It is easy to verify that  $a \in \Omega$  and hence  $\varphi(a) = b$  and  $a \in E$ . We see that  $\Omega \subset \varphi(E)$  and the proof is finished.

In what follows we will use the following notation: for  $X \in M^{2 \times 2}$  we let  $X_{\text{sym}} = (X + X^t)/2$  and  $X_{\text{asym}} = (X - X^t)/2$ .

**Lemma 4.2** *Let  $f: S^{2 \times 2} \rightarrow \mathbf{R}$  be a smooth function such that  $|D^2 f| \leq c$  in  $S^{2 \times 2}$ . Assume that  $f$  is strongly quasi-convex in the sense that for some  $\gamma > 0$  we have  $\int_{\mathbf{R}^2} (f(A + \nabla^2 \phi) - f(A)) \geq \gamma \int_{\mathbf{R}^2} |\nabla^2 \phi|^2$  for all smooth, compactly supported  $\phi: \mathbf{R}^2 \rightarrow \mathbf{R}$ . Then for sufficiently large  $\kappa > 0$  the function  $\tilde{f}: M^{2 \times 2} \rightarrow \mathbf{R}$  defined by  $\tilde{f}(X) = f(X_{\text{sym}}) + \kappa |X_{\text{asym}}|^2$  is strongly quasi-convex.*

**Proof.** Let  $\mathbf{T}^2$  be the two-dimensional torus  $\mathbf{R}^2/\mathbf{Z}^2$ . Let  $\varphi: \mathbf{T}^2 \rightarrow \mathbf{R}^2$  be a smooth function and let  $A \in M^{2 \times 2}$ . We want to prove that  $\int_{\mathbf{T}^2} (\tilde{f}(A + \nabla \varphi) - \tilde{f}(A)) \geq \gamma/2 \int_{\mathbf{T}^2} |\nabla \varphi|^2$ . Let us consider the Helmholtz decomposition  $\varphi = \nabla \phi + \nabla^\perp \eta + a$  of  $\varphi$ , where  $\phi$  and  $\eta$  are scalar functions,  $\nabla^\perp \eta = J \nabla \eta$  (with  $J$  as above), and  $a$  a constant vector. We have  $\nabla \varphi = \nabla^2 \phi + \nabla \nabla^\perp \eta$ . Set  $Y = (\nabla \nabla^\perp \eta)_{\text{sym}}$ . A standard calculation (involving integration by parts and the use of the identity  $\int_{\mathbf{T}^2} \det \nabla^2 \eta = 0$ ) gives  $\int_{\mathbf{T}^2} |Y|^2 = |\nabla^2 \eta|^2/2 = \int_{\mathbf{T}^2} (\Delta \eta)^2/2 = \int_{\mathbf{T}^2} |(\nabla \nabla^\perp \eta)_{\text{asym}}|^2$ . We can write

$$\begin{aligned} & \int_{\mathbf{T}^2} (\tilde{f}(A + \nabla \varphi) - \tilde{f}(A)) \\ &= \int_{\mathbf{T}^2} (f(A_{\text{sym}} + \nabla^2 \phi) - f(A_{\text{sym}})) \\ &+ \int_{\mathbf{T}^2} (\kappa |A_{\text{asym}} + (\nabla \nabla^\perp \eta)_{\text{asym}}|^2 - \kappa |A_{\text{asym}}|^2) \\ &+ \int_{\mathbf{T}^2} (f(A_{\text{sym}} + \nabla^2 \phi + Y) - f(A_{\text{sym}} + \nabla^2 \phi)) \\ &= I + II + III. \end{aligned}$$

We have  $I \geq \gamma \int_{\mathbf{T}^2} |\nabla^2 \phi|^2$  by our assumptions and Lemma 2.1. The second term can be evaluated as  $II = \int_{\mathbf{T}^2} \kappa |Y|^2$  by using the calculation above and the fact that  $\int_{\mathbf{T}^2} \nabla^2 \eta = 0$ . Finally, the third term can be written as

$$\begin{aligned} III &= \int_{\mathbf{T}^2} (f(A_{\text{sym}} + \nabla^2 \phi + Y) - f(A_{\text{sym}} + \nabla^2 \phi) - Df(A_{\text{sym}} + \nabla^2 \phi)Y) \\ &+ \int_{\mathbf{T}^2} (Df(A_{\text{sym}} + \nabla^2 \phi) - Df(A_{\text{sym}}))Y \end{aligned}$$

$$\begin{aligned}
&\geq - \int_{\mathbf{T}^2} (c/2|Y|^2 + c|\nabla^2\phi||Y|) \\
&\geq - \int_{\mathbf{T}^2} (\gamma/2|\nabla^2\phi|^2 + c/2|Y|^2 + c^2/(2\gamma)|Y|^2).
\end{aligned}$$

We see that we get the right inequality when  $\kappa \geq \gamma/2 + c/2 + c^2/(2\gamma)$ . The proof is finished.

Lemma 4.2 cannot be directly applied to the function  $f_0$  from Lemma 4.1. However, we can modify  $f_0$  in the following way. We consider a smooth mollifier  $\omega$  on  $S^{2 \times 2}$  which is supported in the ball of radius  $1/8$  centered at 0 and satisfies  $\int_{S^{2 \times 2}} \omega = 1$ ,  $\int_{S^{2 \times 2}} X\omega(X) dX = 0$ , and  $\int_{S^{2 \times 2}} \det(X)\omega(X) dX = 0$ . We let  $f_1(X) = \max(f_0(X), |X|^2 - 25)$  and  $f_2 = f_1 * \omega$ . We note that  $f_2(X) = f_0(X)$  when  $|X| \leq 5$  and the open ball  $B_{X, \frac{1}{8}}$  is contained in the set of the positive definite matrices. We choose a small  $\gamma > 0$  (to be specified later) and set  $f_3(X) = f_2(X) + \gamma|X|^2$ . We denote by  $\tilde{f}_3$  the strongly quasi-convex extension of  $f_3$  to  $M^{2 \times 2}$  obtained in Lemma 4.2 (for a suitable  $\kappa$ ).

Let  $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . We define  $\theta: M^{2 \times 2} \rightarrow M^{2 \times 2}$  by  $\theta \cdot X = TXJ^t$ , where  $J$  is the rotation by  $\pi/2$  introduced above. We note that the diagonal matrices are invariant under  $\theta$  and that  $\theta$  restricted to the diagonal matrices can be thought of as a rotation by  $\pi/2$ .

Let  $H = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & -\frac{5}{4} \end{pmatrix}$ . We define  $f_4: M^{2 \times 2} \rightarrow \mathbf{R}$  by

$$f_4(X) = \sum_{k=0}^3 \tilde{f}_3(\theta^{-k} \cdot X - H).$$

It is easy to see that  $f_4$  satisfies  $f_4(\theta \cdot X) = f_4(X)$  for each  $X \in M^{2 \times 2}$  and therefore  $Df_4(\theta \cdot X) = \theta \cdot Df_4(X)$  for each  $X \in M^{2 \times 2}$ .

We now let  $A_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$  and  $A_k = \theta^k \cdot A_1$ . By a direct calculation we get  $Df_4(A_1) = \begin{pmatrix} \frac{1}{4} + 14\gamma & 0 \\ 0 & \frac{7}{4} + 2\gamma \end{pmatrix}$ . By considering functions of the form  $\frac{1}{2}\alpha|X|^2 + \beta f_4(X)$  we can easily obtain the following lemma, by choosing suitable  $\alpha, \beta$ , and  $\gamma$ .

**Lemma 4.3** *There exist a smooth, strongly quasiconvex function  $F_1: M^{2 \times 2} \rightarrow \mathbf{R}$  with uniformly bounded  $D^2F_1$  which satisfies (in the notation introduced above)  $F_1(\theta \cdot X) = F_1(X)$  for each  $X$  and  $DF_1(A_1) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ .*

**Proof.** See above.

The set  $K$  corresponding to the function  $F = F_1$  (see the beginning of the section) contains the matrices  $\begin{pmatrix} A_k \\ DF_1(A_k)J \end{pmatrix}$ ,  $k = 1, \dots, 4$ . These are the

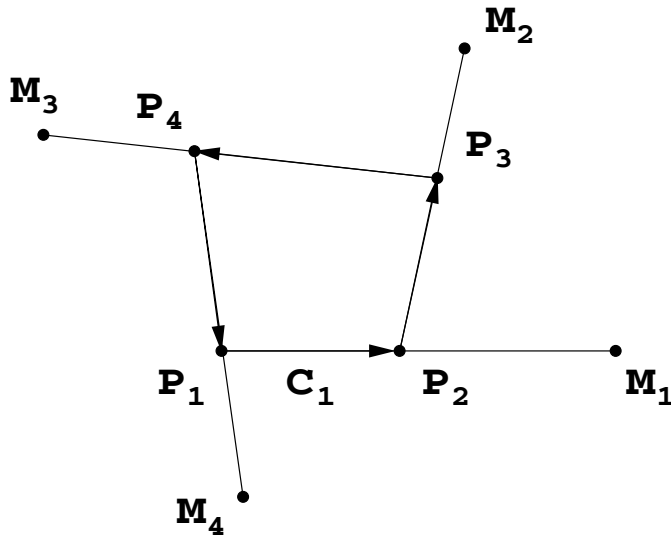


Figure 1:  $T_4$  configuration with  $P_1 = P$ ,  $P_2 = P + C_1$ ,  $P_3 = P + C_1 + C_2$ ,  $P_4 = P + C_1 + C_2 + C_3$ . The lines indicate rank-1 connections. Note that the figure need not be planar

matrices

$$M_1^0 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \\ 0 & -1 \\ 3 & 0 \end{pmatrix}, M_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 3 \\ 1 & 0 \end{pmatrix}, M_3^0 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \\ 0 & 1 \\ -3 & 0 \end{pmatrix}, M_4^0 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \\ 0 & -3 \\ -1 & 0 \end{pmatrix}.$$

## 4.2 Deformations of $T_4$ -configurations

Let us consider four  $m \times n$  matrices  $M_1, \dots, M_4$ . We say that  $M_1, \dots, M_4$  are in  $T_4$ -configuration (see Figure 1) if  $\text{rank}(M_i - M_j) \neq 1$  for all  $i, j$ , and if there exist rank-one matrices  $C_1, \dots, C_4$  with  $\sum_k C_k = 0$ , real numbers  $\kappa_1, \dots, \kappa_4 > 1$ , and a matrix  $P \in M^{m \times n}$  such that

$$\begin{aligned} M_1 &= P + \kappa_1 C_1, \\ M_2 &= P + C_1 + \kappa_2 C_2, \\ M_3 &= P + C_1 + C_2 + \kappa_3 C_3, \\ M_4 &= P + C_1 + C_2 + C_3 + \kappa_4 C_4. \end{aligned}$$

This configuration was discovered independently by several authors. We are aware of [Sch 74], where it is used in a similar context as below, [AH 86], and [Ta 93], where it is used in a different context. Slightly different examples exhibiting similar features were also independently discovered in [NM 91]

and [CT 93]. The paper [BFJK 94] contains an interesting example using a  $T_4$ -configuration. The following observation appears in [AH 86], [Ta 93] and implicitly also in the other papers.

**Lemma 4.4** *If  $M_1, \dots, M_4$  are in  $T_4$ -configuration, the rank-one convex hull of the set  $\{M_1, \dots, M_4\}$  contains the points  $P_1 = P, P_2 = P + C_1, P_3 = P + C_1 + C_2, P_4 = P + C_1 + C_2 + C_3$ .*

**Proof.** To see this, let us consider a rank-one convex function  $f: M^{m \times n} \rightarrow \mathbf{R}$  which vanishes at the points  $M_1, \dots, M_4$ . We have  $f(P_{i+1}) \leq 1/\kappa_i f(M_i) + (1 - 1/\kappa_i)f(P_i) = (1 - 1/\kappa_i)f(P_i)$  for each  $i$ , where the indices are considered modulo 4. Applying this recursively, we get that  $f(P_i) \leq 0$  for each  $i$ .

The matrices  $M_k^0$  at the end of subsection 4.1 are in  $T_4$ -configuration and they also lie on the set

$$K_1 = \left\{ \begin{pmatrix} X \\ DF_1(X)J \end{pmatrix}; X \in M^{2 \times 2} \right\} \subset M^{4 \times 2}$$

given by the quasi-convex function  $F_1$  constructed in Lemma 4.3. This shows that the rank-one convex hull  $K_1^{\text{rc}}$  of  $K_1$  is non-trivial. We now wish to establish that  $K_1^{\text{rc}}$  is sufficiently large, so that we can apply Theorem 3.2. We will see later that rather than trying to work with the specific function  $F_1$ , it is more convenient to work with a small perturbation  $F = F_1 + \varepsilon V$  of  $F_1$ , where  $V$  is a compactly supported smooth function, the properties of which will be specified later. For the moment we will only assume that  $F$  satisfies  $DF(A_k) = DF_1(A_k)$  for  $k = 1, 2, 3, 4$ , where the matrices  $A_k$  are the same as in Subsection 4.1. We also denote by  $K \subset M^{4 \times 2}$  the set corresponding to  $F$ . By our assumptions we know that  $K$  contains a  $T_4$ -configuration given by the matrices  $M_k^0$ ,  $k = 1, 2, 3, 4$  defined above. It is natural to investigate deformations of this  $T_4$ -configuration. In other words, we will investigate four-tuples  $M_1, \dots, M_4$  such that, for  $k = 1, \dots, 4$ ,  $M_k$  is close to  $M_k^0$ ,  $M_k \in K$ , and  $M_1, \dots, M_4$  are in  $T_4$ -configuration.

We introduce the following notation.

$$\begin{aligned} e_1 &= (1, 0) & e_2 &= (0, 1) \\ f_1 &= (2, 0, 0, 2) & f_2 &= (0, 2, 2, 0) \\ C_1^0 &= f_1 \otimes e_1 & C_2^0 &= f_2 \otimes e_2 \\ C_3^0 &= -C_1^0 & C_4^0 &= -C_2^0 \\ P^0 &= -(C_1^0 + C_2^0)/2 \\ \kappa_1^0 &= 2 & \kappa_2^0 &= 2 \\ \kappa_3^0 &= 2 & \kappa_4^0 &= 2 \end{aligned}$$

We parameterize the rank-one matrices  $C_k$  in a small neighborhood of  $C_k^0$  as follows.

$$C_1 = (f_1 + a_1) \otimes (e_1 + \beta_1 e_2)$$



$$\begin{aligned}
C_2 &= (f_2 + a_2) \otimes (e_2 - \beta_2 e_1) \\
C_3 &= (-f_1 + a_3) \otimes (e_1 + \beta_3 e_2) \\
C_4 &= (-f_2 + a_4) \otimes (e_2 - \beta_4 e_1),
\end{aligned}$$

where  $a_1, \dots, a_4$  are (small) vectors in  $\mathbf{R}^4$ , and  $\beta_1, \dots, \beta_4$  are (small) real numbers. We linearize the equation  $\sum_k C_k = 0$  around the solution  $C_k^0$ . The linearized equation is equivalent to

$$\begin{aligned}
a_1 + a_3 + (\beta_4 - \beta_2)f_2 &= 0 \\
a_2 + a_4 + (\beta_1 - \beta_3)f_1 &= 0.
\end{aligned}$$

Using these formulae and the above expressions for  $M_k$ , we easily check (with the help of the implicit-function theorem) that the four-tuples  $(M_1, \dots, M_4)$  of the  $4 \times 2$  matrices which are close to  $(M_1^0, \dots, M_4^0)$  and form  $T_4$ -configuration such that the parameters  $P, C_j, \kappa_j$  are close to  $P^0, C_j^0, \kappa_j^0$  form a 24-dimensional manifold  $\mathcal{M}$ . The tangent space  $L_{\mathcal{M}}$  of  $\mathcal{M}$  at the point  $(M_1^0, \dots, M_4^0)$  can be identified with four-tuples  $(Z_1, \dots, Z_4)$  of  $4 \times 2$  matrices of the form

$$\begin{aligned}
Z_1 &= \begin{pmatrix} p_{11} + 2a_{11} + \kappa'_1 & p_{12} + 2\beta'_1 \\ p_{21} + 2a_{21} & p_{22} \\ p_{31} + 2a_{31} & p_{32} \\ p_{41} + 2a_{41} + \kappa'_1 & p_{42} + 2\beta'_1 \end{pmatrix}, \\
Z_2 &= \begin{pmatrix} p_{11} + a_{11} & p_{12} + 2a_{12} + \beta'_1 \\ p_{21} + a_{21} - 2\beta'_2 & p_{22} + 2a_{22} + \kappa'_2 \\ p_{31} + a_{31} - 2\beta'_2 & p_{32} + 2a_{32} + \kappa'_2 \\ p_{41} + a_{41} & p_{42} + 2a_{42} + \beta'_1 \end{pmatrix}, \\
Z_3 &= \begin{pmatrix} p_{11} - a_{11} - \kappa'_3 & p_{12} + a_{12} - 2\beta'_3 + \beta'_1 \\ p_{21} - a_{21} + \beta'_2 - 2\beta'_4 & p_{22} + a_{22} \\ p_{31} - a_{31} + \beta'_2 - 2\beta'_4 & p_{32} + a_{32} \\ p_{41} - a_{41} - \kappa'_3 & p_{42} + a_{42} - 2\beta'_3 + \beta'_1 \end{pmatrix}, \\
Z_4 &= \begin{pmatrix} p_{11} & p_{12} - a_{12} + \beta'_3 - \beta'_1 \\ p_{21} + \beta'_4 & p_{22} - a_{22} - \kappa'_4 \\ p_{31} + \beta'_4 & p_{32} - a_{32} - \kappa'_4 \\ p_{41} & p_{42} - a_{42} + \beta'_3 - \beta'_1 \end{pmatrix},
\end{aligned}$$

where the values of all the 24 parameters run through all real numbers. Moreover, there is a well-defined mapping  $(M_1, \dots, M_4) \rightarrow (P_1, \dots, P_4)$  from  $\mathcal{M}$  to the four-tuples of  $4 \times 2$  matrices, where (in the notation introduced in the definition of  $T_4$ -configuration)  $P_1 = P$ ,  $P_2 = P_1 + C_1$ ,  $P_3 = P_2 + C_2$ ,  $P_4 = P_3 + C_3$  as above.

We now consider the additional constraint  $M_k \in K$ , where  $K$  is the set determined by  $F$ . The four-tuples  $(M_1, \dots, M_4)$  satisfying  $M_k \in K$  clearly form a 16-dimensional manifold  $\mathcal{K} = K \times K \times K \times K$ . The tangent space  $L_{\mathcal{K}}$  of  $\mathcal{K}$  at  $(M_1^0, \dots, M_4^0)$  can be identified with the four-tuples

$$\left( \begin{pmatrix} X_1 \\ D^2F(A_1)X_1J \end{pmatrix} \right), \left( \begin{pmatrix} X_2 \\ D^2F(A_2)X_2J \end{pmatrix} \right), \left( \begin{pmatrix} X_3 \\ D^2F(A_3)X_3J \end{pmatrix} \right), \left( \begin{pmatrix} X_4 \\ D^2F(A_4)X_4J \end{pmatrix} \right)$$

where  $X_1, \dots, X_4$  run through all  $2 \times 2$  matrices.

We now consider the maps  $(M_1, \dots, M_4) \rightarrow (M_k, P'_k)$ , where  $P_k$  is defined as above and where we denote (with a slight abuse of notation) by  $P'_k$  the orthogonal projection of the point  $P_k$  into the space  $(T_{A_k}K)^\perp$ , the normal space of  $K$  at  $A_k$ . We would like to establish the following non-degeneracy conditions, which will be important later when we construct in-approximations.

**Condition (C):**  $\mathcal{M}$  and  $\mathcal{K}$  intersect transversely at  $(M_1^0, \dots, M_4^0)$  and, (after perhaps replacing  $\mathcal{M}$  by a sufficiently small neighborhood of  $(M_1^0, \dots, M_4^0)$  in  $\mathcal{M}$ ) the map  $(M_1, \dots, M_4) \rightarrow (M_k, P'_k)$  is, for each  $k$ , a non-degenerate diffeomorphism of  $\mathcal{M} \cap \mathcal{K}$  and a neighborhood of  $(M_k^0, (P_k^0)')$  in  $K \times (T_{A_k}K)^\perp$ .

Rather than trying to decide whether these non-degeneracy conditions are satisfied for an explicitly given function  $F$ , it seems to be more natural to verify that the conditions are satisfied in the generic case. More specifically, we note that  $F = F_1 + \varepsilon V$  is strongly quasi-convex for sufficiently small  $\varepsilon$ . (We recall that  $V$  is assumed to be smooth and compactly supported.) By choosing  $V$  in a suitable way, we can perturb  $D^2F(A_1), \dots, D^2F(A_4)$  to any prescribed values which are close enough to the original values, without changing the values of  $DF(A_1), \dots, DF(A_4)$ , and without affecting the strong quasi-convexity. For the purpose of the construction of the counter-example announced at the beginning of this section, we can therefore restrict our considerations to the generic case.

**Lemma 4.5** *Assume that  $DF(A_k) = DF_1(A_k)$  for  $k = 1, 2, 3, 4$ . Then condition (C) above is satisfied for the generic values of  $D^2F(A_k)$ ,  $k = 1, \dots, 4$ .*

**Proof.** The condition that  $\mathcal{M}$  and  $\mathcal{K}$  intersect transversely at  $(M_1^0, \dots, M_4^0)$  and that the map  $(M_1, \dots, M_4) \rightarrow (M_1, P'_1)$  is a non-degenerate diffeomorphism of a small neighborhood of  $(M_1^0, \dots, M_4^0)$  in  $\mathcal{M} \cap \mathcal{K}$  and a neighborhood of  $(M_1^0, (P_1^0)')$  in  $K \times (T_{A_1}K)^\perp$  is easily seen to be equivalent to the condition that the following linear homogeneous system of 40 equations for 40 unknowns has no non-trivial solutions.

$$\begin{aligned} Z_j &= \begin{pmatrix} X_j \\ D^2F(A_j)X_jJ \end{pmatrix}, \quad j = 1, 2, 3, 4 \\ \begin{pmatrix} p_{31} & p_{32} \\ p_{41} & p_{42} \end{pmatrix} &= D^2F(A_1) \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} J \\ X_1 &= 0, \end{aligned}$$

where  $Z_j = Z_j(p_{kl}, a_{kl}, \beta'_k, \kappa'_k)$  (with  $k = 1, 2, 3, 4, l = 1, 2$ ) are the  $4 \times 2$  matrices introduced above and  $X_1, X_2, X_3, X_4$  are  $2 \times 2$  matrices. The determinant of the corresponding  $40 \times 40$  matrix is a polynomial expression in the entries of the matrices  $D^2F(A_j)$  (which are now considered as parameters), and will be denoted by  $Q_1$ . The polynomial  $Q_1$  is not identically zero, since for

$$D^2F(A_1) = I, \quad D^2F(A_2) = I, \quad D^2F(A_3) = 0, \quad D^2F(A_4) = I$$

we can check by a straightforward calculation that the system has no non-trivial solutions.

By using symmetry we see that, for each  $k = 1, 2, 3, 4$ , the condition that  $\mathcal{M}$  and  $\mathcal{K}$  intersect transversely at  $(M_1^0, \dots, M_4^0)$  and that the map  $(M_1, \dots, M_4) \rightarrow (M_k, P_k')$  is a non-degenerate diffeomorphism of a small neighborhood of  $(M_1^0, \dots, M_4^0)$  in  $\mathcal{M} \cap \mathcal{K}$  and a neighborhood of  $(M_k^0, (P_k^0)')$  in  $K \times (T_{A_k}K)^\perp$  can be expressed as  $Q_k \neq 0$ , where  $Q_k$  is a suitable non-zero polynomial in the entries of the matrices  $D^2F(A_j)$ . Hence all of our non-degeneracy conditions will be satisfied at all values of  $D^2F(A_j)$  where the polynomial  $Q = Q_1Q_2Q_3Q_4$  does not vanish. Since  $Q$  is not identically zero, the result follows.

### 4.3 In-approximation

To be able to use Theorem 3.2, we need to have a suitable in-approximations.

**Lemma 4.6** *Using the notation above, assume that condition (C) is satisfied. Let  $r > 0$ . Then there exists an in-approximation  $\{U_i\}_{i=1}^\infty$  of  $K_r = \cup_{j=1}^4 \{X \in M^{4 \times 2}, |X - M_j^0| \leq r\} \cap K$  such that  $U_1$  contains a (small) neighborhood of the rank-one convex hull of the points  $P_1^0, \dots, P_4^0$ .*

**Proof.** We consider a sequence  $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2 \dots$  of open neighborhoods of  $(M_1^0, \dots, M_4^0)$  in  $\mathcal{M} \cap \mathcal{K}$ , such that each  $\mathcal{O}_j$  is diffeomorphic to the eight-dimensional unit ball and that, for each  $j = 0, 1, 2, \dots$  we have  $\bar{\mathcal{O}}_j \subset \mathcal{O}_{j+1}$ . We also consider a sequence of numbers  $0 = \lambda_0, 1/2 < \lambda_1 < \dots < \lambda_j < \dots < 1$  converging to 1 as  $j \rightarrow \infty$ . For  $j = 0, 1, 2, \dots$  we let  $\mathcal{U}_{k,j} = \{(1 - \lambda_j)P_k + \lambda_j M_k, (M_1, \dots, M_4) \in \mathcal{O}_j\}$ , where  $P_k = P_k(M_1, \dots, M_4)$  is the map considered in subsection 4.2. We also let  $\mathcal{U}_j = \cup_{k=1}^4 \mathcal{U}_{k,j}$ . Condition (C) implies that there exists  $j_0$  such that the sets  $\mathcal{U}_j$  are open when  $j \geq j_0$  and the sets  $\mathcal{O}_j$  are contained in a sufficiently small neighborhood  $\mathcal{O}$  of  $(M_1^0, \dots, M_4^0)$ . To see this, consider for example  $k = 1$  and let us write points  $M_1 \in K$  which are close to  $M_1^0$  as  $M_1 = M_1^0 + X + \xi(X)$ , with  $X \in T_{A_1}K$  and  $\xi(X) \in (T_{A_1}K)^\perp$ . We can also write  $P_1 = P_1^0 + Y + \eta$  with  $Y \in (T_{A_1}K)^\perp$  and  $\eta \in T_{A_1}K$ . If Condition (C) is satisfied, we know that, in a small neighborhood of  $(M_1^0, \dots, M_4^0)$ , we can take  $X$  and  $Y$  as local coordinates in  $\mathcal{M} \cap \mathcal{K}$ . For  $(M_1, \dots, M_4) \in \mathcal{M} \cap \mathcal{K}$  which is close to  $(M_1^0, \dots, M_4^0)$  and  $P_1 = P_1(M_1, \dots, M_4)$ , we can therefore write the  $\eta$ -component of  $P_1$  in the above decomposition as  $\eta = \eta(X, Y)$ , where  $\eta$  is a smooth function of  $X$  and  $Y$  with  $\eta(0, 0) = 0$ . In the coordinates  $(X, Y)$ , the derivative of the map  $(X, Y) \rightarrow (1 - \lambda)P_1 + \lambda M_1$  is given by the block matrix

$$\begin{pmatrix} \lambda I + (1 - \lambda)\partial_X \eta & (1 - \lambda)\partial_Y \eta \\ \lambda \partial_X \xi & (1 - \lambda)I \end{pmatrix}.$$

Since  $\partial_X \xi(0) = 0$ , we see that the matrix is regular when  $X$  is small and  $\lambda$  is close to 1. The openness of  $\mathcal{U}_{1,j}$  for large  $j$ ,  $\lambda$  close (but not equal) to 1, and small  $\mathcal{O}$  follows.

We can see from the definitions that, for each  $j = 0, 1, \dots$  the closure of the rank-one convex hull of  $\mathcal{U}_j$  is contained in the rank-one convex hull of  $\mathcal{U}_{j+1}$ .

Moreover, the rank-one convex hull of  $\mathcal{U}_0$  contains a neighborhood of the square given by the convex hull of the points  $P_1^0, \dots, P_4^0$  (which coincides with the rank-one convex hull of these points, since the points lie in a two-dimensional plane). The required in-approximation has therefore been established.

#### 4.4 Solutions with nowhere continuous gradients

**Proof of Theorem 4.1.** The main idea of the proof is described in heuristic terms in the remarks immediately following the theorem. In the proof below we will be freely using the notation introduced earlier in Section 4.

The matrices  $A_1, \dots, A_4$  are the matrices

$$A_1 = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, A_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

as in Section 4.1. We let  $F_0$  be a suitable small perturbation of the quasiconvex function  $F_1$  from Lemma 4.3 such that  $DF_0(A_k) = DF_1(A_k)$  for  $k = 1, \dots, 4$  and condition (C) is satisfied. Since the transversality and the other non-degeneracy conditions are stable under small perturbations, a version of (C) with  $M_1^0, \dots, M_4^0$  replaced by close-by matrices  $\tilde{M}_1^0, \dots, \tilde{M}_4^0$  will also be satisfied for any  $F$  as in the statement of the theorem, provided  $\delta$  is sufficiently small. Moreover, we see easily that by choosing  $\delta$  sufficiently small we can also achieve that Lemma 4.6 can be applied (with  $M_1^0, \dots, M_4^0$  replaced by close-by matrices  $\tilde{M}_1^0, \dots, \tilde{M}_4^0$ ) with a fixed small  $r > 0$  to any set  $K$  arising from a function  $F$  satisfying the assumptions of the theorem. In addition, we see easily that the in-approximations can be constructed so that  $U_1$  contains a fixed small neighborhood of the zero matrix for any  $F$  satisfying the assumptions. We see that the assumptions of Theorem 3.2 are satisfied in our situation. However, it does not seem to be immediately clear that the solutions obtained from Theorem 3.2 are not continuously differentiable on any open subset of  $\Omega$ . To obtain such solutions, we will analyze the construction more closely.

We first look in more detail at the in-approximation introduced in Lemma 4.6. Since the map  $(M_1, \dots, M_4) \rightarrow (1 - \lambda_j)P_k + \lambda_j M_k$  considered there is (for a good choice of parameters) a diffeomorphism of  $\tilde{\mathcal{O}}_j$  and  $\tilde{\mathcal{U}}_{k,j}$ , we can define  $\phi_j: \mathcal{P}(\tilde{\mathcal{U}}_j) \rightarrow \mathcal{P}(\tilde{\mathcal{U}}_{j+1})$  as follows. The map  $\phi_j$  is first defined on Dirac masses: given  $X = (1 - \lambda_j)P_k + \lambda_j M_k \in \tilde{\mathcal{U}}_{k,j}$ , we let  $\phi_j(\delta_X) = \sum_{l=1}^4 \mu_l Y_l$ , where  $Y_l = (1 - \lambda_{j+1})P_l + \lambda_{j+1} M_l \in \mathcal{U}_{l,j+1}$ , and  $\mu_1, \dots, \mu_4$  are determined by requiring that the measure  $\phi_j(X)$  be a laminate with center of mass  $X$ . (These conditions determine  $\mu_1, \dots, \mu_4$  uniquely. This is obvious if the affine span of  $Y_1, \dots, Y_4$  is three-dimensional. If it is two-dimensional, we must use the condition that  $\phi_j(X)$  is a laminate to get the uniqueness.) We can now extend  $\phi_j$  to  $\mathcal{P}(\tilde{\mathcal{U}}_j)$  by requiring that  $\phi_j$  be affine and continuous in the  $w^*$ -topologies on  $\mathcal{P}(\tilde{\mathcal{U}}_j)$  and  $\mathcal{P}(\tilde{\mathcal{U}}_{j+1})$ . The measure  $\phi_j(\delta_X)$  can be thought of as a result of splitting the Dirac mass  $\delta_X$  into a convex combination of the four Dirac masses  $\delta_{Y_1}, \dots, \delta_{Y_4}$ .

An easy calculation shows that when  $X \in \mathcal{U}_{k,j}$  and  $\phi_j(\delta_X) = \sum_{l=1}^{l=4} \mu_l Y_l$ , then, for sufficiently large  $j$ , we have  $\mu_k > 1 - (\lambda_{j+1} - \lambda_j)$  and  $\mu_l > 0$  for all  $l = 1, 2, 3, 4$ .

Let us now go back to the construction in the proof of Theorem 3.2.

Let us consider  $A \in \mathcal{U}_j^c$ , and assume that  $u_j$  is a countably piecewise affine function on  $\Omega$  with  $\nabla u_j = A$  in an open set  $U \subset \Omega$ . Let us write  $\phi_j(A) = \sum_{k=1}^{k=4} \mu_k \delta_{A_k}$ , where we use the map  $\phi_j$  defined above. The inductive step in the construction described in Section 3 is to replace  $u_j$  by  $u_{j+1}$  which satisfies  $u_{j+1} = u_j$  on  $\partial U$ ,  $\sup_U |u_{j+1} - u_j|$  small,  $\nabla u_{j+1}(x)$  close to the set  $A_1, \dots, A_4$  for a. e.  $x \in U$ , and the ratio  $\text{meas}\{x \in U, \nabla u(x) \text{ is close to } A_k\} / \text{meas } U$  close to  $\mu_k$ . (More precisely,  $U$  should be thought of as one set of a countable disjoint family which covers  $\Omega$  up to the set of measure zero - see Section 3 for details.) Because  $\mu_k > 1 - (\lambda_{j+1} - \lambda_j)$ , we see that we can do the construction so that  $\text{meas}\{x \in U, \nabla u_{j+1} \notin \mathcal{U}_{k,j+1}\} < (\lambda_{j+1} - \lambda_j) \text{meas } U$ . Following the same procedure at each inductive step, we conclude that  $\text{meas}\{x \in U, \nabla u_{j+p} \notin \mathcal{U}_{k,j+p}\} < (\lambda_{j+p} - \lambda_j) \text{meas } U$ .

On the other hand, in the construction of  $u_{j+1}$  we have to use a matrix  $B \in \mathcal{U}_{l,j+1}$  with  $l \neq k$ . Moreover, we can carry out the construction in such a way that any ball of radius  $1/j$  has a non-empty intersection with an open set where a matrix from  $\mathcal{U}_{l,j+1}$  is used. Let  $V \subset U$  be a maximal open connected subset of  $U$  on which  $u_{j+1}$  is affine with  $\nabla u_{j+1} = B$ . Then the same argument as above gives  $\text{meas}\{x \in V, \nabla u_{j+p} \notin \mathcal{U}_{l,j+p}\} < (\lambda_{j+p} - \lambda_{j+1}) \text{meas } V$ .

Using this we see easily that the limit function  $u_\infty = \lim u_j$  has the property that the essential oscillation of  $\nabla u_\infty$  over any open subset of  $\Omega$  is bounded from below by a fixed strictly positive constant. This finishes the proof of Theorem 4.1.

*Remark:* The above construction is quite similar to the following simpler example. Let us consider a sequence  $0 < \lambda_0 < \lambda_1 < \dots < \lambda_j < \dots < 1$ , with  $\lim_{j \rightarrow \infty} \lambda_j = 1$ . Let  $X \subset L^\infty(0,1)$  be the space of all piecewise constant functions. For a function  $f \in X$  with  $|f| \leq \lambda_j$  we define  $T_j f \in X$  in the following way. Let  $(a,b)$  be a maximal open interval on which  $f$  is constant. Let  $c = (a+b)/2$ . We find  $d \in (a,c)$  and  $e \in (c,b)$  such that the function  $g: (a,b) \rightarrow \mathbf{R}$  defined by  $g(x) = -\lambda_j$  when  $x \in (a,d)$ ,  $g(x) = \lambda_j$  when  $x \in (d,c)$ ,  $g(x) = -\lambda_j$  when  $x \in (c,e)$ , and  $g(x) = \lambda_j$  when  $x \in (e,b)$  has the same average as  $f$  over the intervals  $(a,c)$  and  $(c,b)$ . We then set  $T_j f(x) = g(x)$  for  $x \in (a,b)$ , and repeat the same construction on the other maximal intervals on which  $f$  is constant. Let  $0 < A < \lambda_0$  and let  $f_0 \equiv A$  in  $(0,1)$ . Set  $f_{j+1} = T_{j+1} f_j$ . It is not difficult to see that the sequence  $f_j$  converges in  $L^1(0,1)$  to a function  $f_\infty$ . Moreover, the essential oscillation of  $f_\infty$  over any open set is 2.

## 4.5 Linear Systems

The examples above can be used to answer open questions (raised in [GS 85]) concerning solutions of linear  $2 \times 2$  systems of the form

$$\partial_\alpha a_{ij}^{\alpha\beta}(x) \partial_\beta v_j = 0, \quad i = 1, 2 \quad (4)$$

where the coefficients are in  $L^\infty$  and satisfy the strong Legendre-Hadamard condition

$$a_{ij}^{\alpha\beta}(x)\xi_\alpha\xi_\beta\bar{u}^i\bar{u}^j \geq \nu|\xi|^2|\bar{u}|^2$$

for each  $\xi, \bar{u} \in \mathbf{R}^2$  and almost every  $x$ . (As usual,  $\nu > 0$ .) In what follows we will write the system (4) as  $\operatorname{div} A(x)\nabla v = 0$ .

There is a well known procedure for passing from solutions of non-linear equations to solutions of linear equations with measurable coefficients (see e. g. [Mo 66]). We will use it to construct our examples. These examples will be based on the following proposition.

**Proposition 4.1** *There exists a smooth strictly quasiconvex function  $F: M^{2 \times 2} \rightarrow \mathbf{R}$  with uniformly bounded  $D^2F$  and a non-trivial Lipschitz function  $u: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  which vanishes for  $|x| > 1$  and satisfies (weakly) the equation  $\operatorname{div} DF(\nabla u) = 0$  in  $\mathbf{R}^2$ .*

**Proof.** We will use the notation introduced earlier in Section 4. We note that the function  $F_1$  from Lemma 4.3 satisfies  $DF_1(0) = 0$  and therefore the zero matrix belongs to the set  $K_1 \subset M^{4 \times 2}$  corresponding to  $F_1$ . Therefore we see that the function  $F_0$  in Theorem 4.1 can be taken so that  $DF_0(0) = 0$ . Hence the set  $K$  corresponding to  $F = F_0$  in Theorem 4.1 can be taken so that it contains the zero matrix. We know that there are non-trivial solutions of  $DU \in K$  a. e. in  $\Omega$  which vanish at  $\partial\Omega$ . Extending  $U$  by zero outside  $\Omega$ , we get solutions with the required properties.

**Proposition 4.2** *There exist  $L^\infty$ -coefficients  $A(x)$  defined in  $\mathbf{R}^2$  which satisfy the strong Legendre-Hadamard condition such that weak solutions of the linear system  $\operatorname{div} A(x)\nabla v = 0$  exhibit the following behavior.*

(i) *There exists a compactly supported solution  $v$  belonging to the Sobolev space  $W^{1,2}$  but not to  $W^{1,2+\delta}$  for any  $\delta > 0$ .*

(ii) *There exists a sequence  $v_j, j = 1, 2, \dots$  of Lipschitz solutions which are supported in  $\{x, |x| < 1\}$ , and converge to zero weakly but not strongly in  $W^{1,2}$ .*

**Proof.** Let  $F$  and  $u$  be as in Proposition 4.1 and let

$$\tilde{A}(x) = \int_0^1 D^2F(t\nabla u(x)) dt.$$

Since  $F$  is smooth and  $|D^2F| \leq c$ ,  $\tilde{A}(x)$  is a well-defined  $L^\infty$ -function. Since  $F$  is strongly quasiconvex, it is also strongly rank-one convex, and therefore  $\tilde{A}(X)$  satisfies the Legendre-Hadamard condition. Moreover, we have

$$\operatorname{div} \tilde{A}(x)\nabla u = \operatorname{div} (DF(\nabla u(x)) - DF(0)) = 0 \quad \text{in } \mathbf{R}^2$$

in the weak sense.

Let us consider a sequence  $B_{a_j, r_j} \subset \{x \in \mathbf{R}^2, |x| < 1\}$  of mutually disjoint balls centered at  $a_j$  with radius  $r_j > 0$  so that  $a_j \rightarrow 0$  in  $\mathbf{R}^2$  and  $r_j \rightarrow 0$ . We let

$$\begin{aligned} A(x) &= D^2F(0) + \sum_{j=1}^{\infty} \left( \tilde{A}(r_j^{-1}(x - a_j)) - D^2F(0) \right) \quad \text{and} \\ v_j(x) &= u(r_j^{-1}(x - a_j)), \quad j = 1, 2, \dots \end{aligned}$$

The coefficients  $A(x)$  are again bounded and satisfy the strong Legendre-Hadamard condition. We also have  $\operatorname{div} A(x) \nabla v_j = 0$ ,  $j = 1, 2, \dots$ . The sequence  $v_1, v_2, \dots$  gives (ii). To obtain (i), we consider a sequence  $c_1, c_2, \dots$  satisfying  $\sum_{j=1}^{\infty} c_j^2 < \infty$  and  $\sum_{j=1}^{\infty} c_j^{2+\delta} = \infty$  for each  $\delta > 0$ . Then  $v = \sum_{j=1}^{\infty} c_j v_j$  has the required properties.

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