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by

Stefan Müller

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Stefan Müller
Max Planck Institute for
Mathematics in the Sciences
Inselstr. 22-26, 04103 Leipzig, Germany

Abstract

We prove a conjecture of Tartar regarding weak lower semicontinuity of functionals on sequences $u_j, v_j \colon \Omega \subset \mathbf{R}^2 \to \mathbf{R}$ which satisfy $\partial_2 u_j \to 0, \partial_1 v_j \to 0$ in H^{-1}. This is the simplest example in the theory of compensated compactness for which the constant rank condition fails. The proof uses the fact that certain coefficients in the Haar basis expansion can be estimated in terms of the Riesz transform which seems to be of independent interest. Applications to the relation between rank-1 convexity and quasiconvexity are indicated.

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1 Main results

In this note we prove the following result in compensated compactness theory and indicate its consequences for the relation between rank-1 convexity and quasiconvexity.

Theorem 1 *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a separately convex function which satisfies $0 \le f(z) \le C(1 + |z|^2)$. Let $U \subset \mathbf{R}^2$ be open and suppose that*

$$u_j \rightharpoonup u_\infty, \quad v_j \rightharpoonup v_\infty, \qquad \text{in } L^2_{loc}(U), \tag{1.1}$$

$$\partial_2 u_j \to \partial_2 u_\infty, \quad \partial_1 v_j \to \partial_1 v_\infty, \qquad \text{in } H^{-1}_{loc}(U). \tag{1.2}$$

Then for every open set $V \subset U$

$$\int_V f(u_\infty, v_\infty)\, dx \le \liminf_{j\to\infty} \int_V f(u_j, v_j)\, dx. \tag{1.3}$$

The question whether such a result holds was raised in Tartar's seminal paper [Ta 79] where he proved estimate (1.3) under the stronger condition that $\partial_2 u_j$ and $\partial_1 v_j$ are bounded in L^2_{loc}. In this case he even showed that the Young measure generated by the pair (u_j, v_j) (see below for definitions) is a tensor product, and he gave an example that this need no longer be the case if only (1.2) holds ([Ta 79, Ta 93]). The situation in Theorem 1 is the simplest example in the theory of compensated compactness in which the operator of differential constraints $A(D)(u, v) = (\partial_2 u, \partial_1 v)$ does not satisfy the constant rank condition. Hence one cannot use projection in Fourier space to write (u_j, v_j) as a sum of a pair in the kernel of $A(D)$ and a small L^2 perturbation.

The main new idea is to use localization both in real space and in Fourier space. Localization in real space allows one to treat the nonlinearity of f (see Lemma 6 below) while localization in Fourier space is a natural way to exploit (1.2). A key ingredient in the proof is a estimate for certain wavelet coefficients in terms of the Riesz transform, which appears to be new to the best of my knowledge (see Theorem 5 below). More specifically we express u_j and v_j in the Haar basis of L^2 and show that for u_j the basis functions that depend on x_2 can be neglected while for v_j those depending on x_1 are irrelevant (see Theorem 5). The proof can then be finished by a simple induction argument (see Lemma 6). The latter argument was inspired by the work of Ball and Murat [BM 91] who showed that for special sequences with multiple scales (1.3) holds. Pedregal showed [Pe 93b] showed that (1.3) holds if $\partial_2 u_j$ is bounded in L^2 and $\partial_1 v_j$ is compact in H^{-1}, but his argument does not suffices to treat assumption (1.2).

To explain the connection of Theorem 1 with the title let us first recall the relevant definitions. A function $f : M^{m\times n} \to \mathbf{R}$ on the $m \times n$ matrices is called rank-1 convex if it is convex on each rank-1 line, i.e., if all the functions $t \mapsto f(F + a \otimes b)$ are convex. It is quasiconvex if for all open sets U and all $F \in M^{m\times n}$

$$\int_U (f(F + \nabla\varphi) - f(F))\, dx \geq 0, \quad \forall \varphi \in C^1_0(U).$$

Quasiconvexity implies rank-1 convexity. Whether the converse is true for $m = 2$, $n \geq 2$, is one of the major unresolved problems in the calculus of variations with ramification to a number of other areas (see e.g. [As 98], [Iw 98]); for $m \geq 3$ Šverák [Sv 92] found a striking counterexample in 1992, fourty years after the problem was raised in Morrey's fundamental paper [Mo 52].

A dual version of this question appears in the study of gradient Young measures. A Young measure ν is a (weak* measurable) map from a measurable set $\Omega \subset \mathbf{R}^n$ to the space of probability measures on $\mathbf{R}^d$. The fundamental theorem for Young measures ([Yo 37, Yo 69, BL 73, Ta 79, Ba 89]) implies that every sequence of maps $u_j : \Omega \to \mathbf{R}^d$ which is bounded in L^∞ contains a subsequence (not relabeled) that generates a Young measure ν in the sense that

$$\lim_{j\to\infty} \int_\Omega \eta(x) f(u_j(x))\, dx = \int_\Omega \eta(x) \langle \nu_x, f\rangle\, dx,$$

for all continuous f and all $\eta \in L^1(\Omega)$. Here $\langle \nu_x, f\rangle = \int_{\mathbf{R}^d} f\, d\nu_x$. We say that ν is a ($W^{1,\infty}$) gradient Young measure if Ω is open and ν is generated by a sequence of gradients ∇u_j, where $\{u_j\}$ is bounded in $W^{1,\infty}$. A Young measure is homogeneous if $x \mapsto \nu_x$ is the constant map (a.e.). Kinderlehrer and Pedregal [KP 91] showed that homogeneous Young measures are exactly those probability measures that satisfy Jensen's inequality for all quasiconvex functions:

$$\langle \mu, f\rangle \geq f(\langle \mu, \mathrm{id}\rangle) \quad \forall f \text{ quasiconvex}.$$

Finally μ is called a laminate if Jensen's inequality holds for all rank-1 convex functions (see [Pe 93a, MP 98]). It is well known that the question whether rank-1 convexity implies quasiconvexity can be rephrased as: Is every homogeneous gradient Young measure a laminate (see e.g. [Mu 98]) ? Using standard machinery Theorem 1 can be rephrased as follows.

Theorem 2 *Every gradient Young measure supported on diagonal* 2×2 *matrices is a laminate.*

If we identify the space D of diagonal 2×2 matrices with $\mathbf{R}^2$ via $x \mapsto \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ we obtain the following result (cf. [BM 91]).

Corollary 3 *Suppose that* $f : D \to \mathbf{R}$ *is separately convex. Then for each* $\varepsilon > 0$ *and each compact set* $K \subset D$ *there exists a quasiconvex function* $g : M^{2\times 2} \to \mathbf{R}$ *such that* $|f - g| < \varepsilon$ *on* K.

Using the L^p version of Theorem 5 below in $\mathbf{R}^n$ one can derive the following generalization [Mu 99].

Theorem 4 *Let L be a subspace fo $M^{m\times n}$ which contains only finitely many rank-1 directions $A_1 = a_1 \otimes k_1, \ldots, A_r = a_r \otimes k_r$. Suppose that $A_1, \ldots, A_r$ are linearly independent. Then every gradient Young measure supported on L is a laminate if and only if $k_1, \ldots, k_r$ are linearly independent.*

The 'if' part uses the n-dimensional version of Theorem 1 while the 'only if' part follows from Šverák's construction [Sv 92].

2 Directional wavelet coefficients are controlled by the Riesz transform

Let $u \in L^2(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ with $\int_{\mathbf{R}^2} u\,dx = 0$. Consider the expansion of u into Haar wavelets

$$u = \sum_\lambda a_\lambda h_\lambda,$$

where $\lambda = 2^{-j}(k + \frac{1}{2}\varepsilon)$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}^2$, $\varepsilon \in \{0,1\}^2 \setminus (0,0)$,

$$h_\lambda(x) = h^{(\varepsilon)}_{k,j}(x) = h^{(\varepsilon)}(2^j x - k),$$

$$h^{(1,0)} = h \otimes \chi_{(0,1)}, \quad h^{(0,1)} = \chi_{(0,1)} \otimes h, \quad h^{(1,1)} = h \otimes h,$$

and where $h = 1$ on $(0, 1/2]$, $h = -1$ on $(1/2, 1)$ and $h = 0$ else.

Define the projection operator $P^{(\varepsilon)}$ by

$$P^{(\varepsilon)}u = \sum_{j,k} a^{(\varepsilon)}_{j,k} h^{(\varepsilon)}_{j,k},$$

and consider the Riesz transform $R_j = -i\partial_j(-\Delta)^{-1/2}$ which corresponds to the Fourier multiplier $\xi_j/|\xi|$.

Theorem 5 *The operator $P^{(\varepsilon)}$ can be extended to a bounded operator on L^2 and one has*

$$\begin{aligned} ||P^{(\varepsilon)}u||_2 &\le C \quad ||u||_2^{1/2} ||R_2 u||_2^{1/2}, \quad \textit{for } \varepsilon \ne (1,0) \\ ||P^{(\varepsilon)}u||_2 &\le C \quad ||u||_2^{1/2} ||R_1 u||_2^{1/2}, \quad \textit{for } \varepsilon \ne (0,1), \end{aligned}$$

and these estimates are optimal.

Remarks. 1. Similar estimates hold in L^p, also in the n-dimensional setting, and will appear elsewhere [Mu 99].

2. A powerful theory of generalized Calderon-Zygmund operators based on Haar basis expansions was developed by Figiel [Fi 88, Fi 91]. I do not know whether his approach can be used to prove Theorem 5. At any rate for the situation at hand we may freely exploit (almost) orthogonality, which is exactly what Figiel tries to avoid in view of applications to general UMD spaces.

Proof. It suffices to prove the first estimate. Let θ be a smooth function with support in the annulus $1/2 < |\xi| < 3/2$ that satisfies $\sum_{j=-\infty}^{\infty} \theta(2^{-j}\xi) \equiv 1$. Consider the Paley-Littlewood decomposition

$$u = \sum_j u_j, \quad u_j = \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot)\mathcal{F}u),$$

where $\mathcal{F}u(\xi) = \int_{\mathbf{R}^2} e^{-i\xi\cdot x} u(x)\, dx$ denotes the Fourier transform. Let $P_j^{(\varepsilon)}$ denote the orthogonal projection onto the subspace spanned by $h^{(\varepsilon)}(2^j x - k)$, $k \in \mathbf{Z}^2$ and define

$$T_l = \sum_j P_j^{(\varepsilon)} \Delta_{j+l}.$$

Then $P^{(\varepsilon)} = \sum_l T_l$ and we will show that

$$||T_l^{(\varepsilon)}|| \le C2^{-l/2}, \quad \text{for } l \ge 0, \tag{2.1}$$

$$||T_l^{(\varepsilon)} R_2^{-1}|| \le C2^{l/2}, \quad \text{for } l \in \mathbf{Z}, \varepsilon \neq (1,0), \tag{2.2}$$

where $||\cdot||$ denotes the operator norm with respect to L^2. The assertion follows from this by using the first estimate for $l \ge M$ and the second for $l < M$, with $2^{-M} \approx ||R_2 u||_2/||u||_2$.

To estimate $||T_l^{(\varepsilon)}||$ we eploit the fact that the $P_j^{(\varepsilon)}$ project onto mutually orthogonal subspaces and that the Δ_j are almost orthogonal, i.e. $\Delta_j \Delta_m = 0$ if $|m-j| \ge 2$. Fix l and let

$$S_j = P_j^{(\varepsilon)} \Delta_{j+l},$$

$A = \sup_j ||S_j||$. If $A < \infty$ we have $S_i^* S_j = 0$ if $i \neq j$, $S_i S_j^* = 0$ if $|i-j| \ge 2$ and $||S_i S_j^*|| \le A^2$ if $|i-j| \le 1$. Hence Cotlar's lemma (see e.g. [St 93], p. 279) yields $||T_l|| \le 3A$. It thus suffices to estimate S_j. Using the isometry $\tau_j f(\cdot) = 2^j f(2^j \cdot)$ we see that $S_j = \tau_j S_0 \tau_j^{-1}$ so we only need to estimate S_0.

We can write

$$S_0 = \sum Q_k, \quad \text{where } Q_k u = h_k(h_k, \Delta_l u).$$

Here we wrote for brevity $h_k(x) = h^{(\varepsilon)}(x-k)$. Then $Q_k^* v = (h_k, v)\Delta_l h_k$. Using the orthogonality of the h_k and the translation invariance of the Δ_j we find

$$||Q_{k'}^* Q_k|| \le \delta_{kk'} ||\Delta_l h_0||^2,$$
$$||Q_{k'} Q_k^*|| \le |(\Delta_l h_{k'-k}, \Delta_l h_0)| =: \gamma^2(k'-k).$$

Let $\eta = \mathcal{F}^{-1}\theta$, $\eta_l(x) = 2^{2l}\eta(2^l x)$. Then η belongs to the Schwartz class $\mathcal{S}$ and $||\eta_l||_1 = ||\eta||_1$. Moreover $\Delta_l v = \eta_l * v$. Hence $||\Delta_l h_0||_\infty \le C$,

$$\begin{aligned} ||\Delta_l h_0||_1 &= ||\nabla(-\Delta)^{-1}\eta_l * \nabla h_0||_1 \\ &\le ||\nabla(-\Delta)^{-1}\eta_l||_1 \, ||\nabla h_0||_{\mathcal{M}(\mathbf{R}^2)} \le C 2^{-l}. \end{aligned}$$

Thus $\gamma(0) \le C2^{-l/2}$, and taking into account the rapid decay of η one easily checks that $\sum_{k\in\mathbf{Z}^2} \gamma(k) \le C2^{-l/2}$, for $l \ge 0$. Cotlar's lemma yields the desired bound for S_0 and hence for T_l.

The estimate for the operator $T_l^{(\varepsilon)} R_2^{-1}$ (originally defined on the dense subspace $D = \{R_2 u : u \in L^2 \cap L^1, \int u = 0\}$) is similar. The main observation is that the primitive $H^{(\varepsilon)}(x) = \int_0^{x_2} h^{(\varepsilon)}(x_1, s) ds$ has compact support in $[0,1]^2$ (and is bounded in L^∞ and BV). Thus we can write

$$Q_k R_2^{-1} u = h_k(\partial_2 H_k, \Delta_l R_2^{-1} u) = h_k(H_k, -\partial_2 \Delta_l R_2^{-1} u).$$

Let $\tilde{\Delta}_l = -2^{-l}\partial_2 \Delta_l R_2^{-1} = -2^{-l} i(-\Delta)^{1/2}\Delta_l$. This operator has the symbol $\tilde{\theta}(2^{-l}\xi)$ where $\tilde{\theta}(\xi) = -i|\xi|\theta(\xi)$. The estimate for $T_l R_2^{-1}$ thus reduces to an estimate for

$$\tilde{\gamma}(k) = 2^l |(\tilde{\Delta}_l H_k, \tilde{\Delta}_l H_0)|^{1/2}.$$

For $l \ge 0$ this is obtained in the same way as the estimate for $\gamma(k)$. Hence, for those l, $T_l^{(\varepsilon)} R_2^{-1}$ is bounded on the dense set D and can therefore be extended to a bounded operator on L^2.

For $l < 0$ we directly estimate $S_0 u = P_0^{(\varepsilon)} \Delta_l u$. Let v denote the projection of $\Delta_l u$ onto functions which are piecewise constant in x_2 on the intervals $(k, k+1)$, $k \in \mathbf{Z}$, i.e.,

$$v(x_1, x_2) = \int_k^{k+1} (\Delta_l u)(x_1, s)\, ds, \quad x_2 \in (k, k+1).$$

Then $P_0^{(\varepsilon)} v = 0$ and by Poincaré's inequality

$$||\Delta_l u - v||_2 \le ||\partial_2 u||_2 = 2^l ||\tilde{\Delta}_l R_2 u||_2 \le C 2^l ||R_2 u||_2.$$

Hence $||S_0 u||_2 \le C 2^l ||R_2 u||_2$ and $S_0 R_2^{-1}$ can be extended to a bounded operator on L^2 with $||S_0|| \le C 2^l \le C 2^{l/2}$, for $l \le 0$. This finishes the proof of (2.2) and of Theorem 5.

3 Proof of Theorem 1

Theorem 1 is an easy consequence of Theorem 5 and the following result.

Lemma 6 *Suppose that $u, v \in L^2(\mathbf{R}^2)$ have the finite expansions*

$$u = \sum_{j=J}^{K} \sum_{k \in \mathbf{Z}^2} a_{j,k} h_{j,k}^{(1,0)}, \quad v = \sum_{j=J}^{K} \sum_{k \in \mathbf{Z}^2} b_{j,k} h_{j,k}^{(0,1)}$$

in the Haar basis. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a separately convex function. Then

$$\int_{\mathbf{R}}^{2} (f(u,v) - f(0,0))\, dx \ge 0.$$

Proof. We argue by induction over K. Let

$$\tilde{u} = \sum_{j=J}^{K-1} \sum_{k \in \mathbf{Z}^2} a_{j,k} h_{j,k}^{(1,0)}, \quad \tilde{v} = \sum_{j=J}^{K-1} \sum_{k \in \mathbf{Z}^2} b_{j,k} h_{j,k}^{(0,1)}.$$

Then $\tilde{u}$ and $\tilde{v}$ are constant on each square $Q_{K,k} = 2^{-K}((0,1)^2 + k)$, $k \in \mathbf{Z}^2$. Jensen's inequality applied to the integration in x_1 and x_2 yields

$$\int_{Q_{K,k}} f(u,v)\, dx = \int_{Q_{K,k}} f(\tilde{u}(x) + a_{K,k} h_{K,k}^{(1,0)}(x_1), \tilde{v}(x) + b_{K,k} h_{K,k}^{(0,1)}(x_2))\, dx_1 dx_2$$

$$\ge \int_{Q_{K,k}} f(\tilde{u}(x), \tilde{v}(x) + b_{K,k} h_{K,k}^{(0,1)}(x_2))\, dx_1 dx_2 \ge \int_{Q_{K,k}} f(\tilde{u}, \tilde{v})\, dx.$$

Hence $\int_{\mathbf{R}^2} f(u,v) - f(\tilde{u}, \tilde{v})\, dx \ge 0$, and the assertion follows by induction.

Proof of Theorem 1. For abbreviation we write $w_j = (u_j, v_j)$. We only show estimate (1.3) holds if V is a dyadic cube Q and w_∞ is constant on Q.

The general case can be recovered in the usual way through approximation by piecewise constant functions and exhaustion of V by dyadic cubes.

We may assume that $w_\infty = 0$, since otherwise we can consider the integrand $\tilde{f}(z) = f(w_\infty + z)$. Choose subsequences such that the limit inferior in (1.3) is a limit and $|w_j|^2 \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(Q)$, the space of Radon measures supported in Q. Let $\varphi \in C_0^\infty(Q)$, $\varphi \geq 0$, $\varphi \not\equiv 0$. We first prove the inequality for the modified sequence

$$\tilde{w}_j = \varphi(w_j - \frac{\int \varphi w_j}{\int \varphi}).$$

We have $\tilde{w}_j \rightharpoonup 0$ in L^2, $\partial_2 \tilde{u}_j \to 0$ in $H^{-1}(\mathbf{R}^2)$, $\partial_1 \tilde{v}_j \to 0$ in $H^{-1}(\mathbf{R}^2)$. Thus Theorem 5 yields $||P^{(1,0)}\tilde{u}_j - \tilde{u}_j||_2 + ||P^{(0,1)}\tilde{v}_j - \tilde{v}_j||_2 \to 0$. Since $\mathrm{supp}\tilde{w}_j \subset Q$ and $\int_Q \tilde{w}_j = 0$ the projections $P^{(1,0)}\tilde{u}_j$ and $P^{(0,1)}\tilde{v}_j$ are also supported in Q.

Now f is separately convex and of quadratic growth and one easily checks that this implies the Lipschitz condition

$$|f(z) - f(z')| \leq C(1 + |z| + |z'|)|z - z'|. \tag{3.1}$$

Hence passage to the limit $K \to \infty$ in Lemma 6 yields

$$\begin{aligned} &\liminf_{j\to\infty} \int_Q f(\tilde{w}_j) - f(0)\, dx \\ &= \liminf_{j\to\infty} \int_{\mathbf{R}^2} \left\{ f(P^{(1,0)}\tilde{u}_j, P^{(0,1)}\tilde{v}_j) - f(0) \right\} dx \\ &\geq 0. \end{aligned} \tag{3.2}$$

Moreover

$$\begin{aligned} \int_Q |f(w_j) - f(\tilde{w}_j)|\, dx &\leq C \limsup_{j\to\infty} ||w_j - \tilde{w}_j||_2 \\ &\leq C \left\{ \int_Q (1-\varphi)^2\, d\mu \right\}^{1/2}. \end{aligned}$$

Now for every $\varepsilon > 0$ there exists a $\varphi \in C_0^\infty(Q)$ such that $\int (1-\varphi)^2\, d\mu \leq \varepsilon^2$. Hence

$$\lim_{j\to\infty} \int_Q f(w_j) \geq \int_Q f(0) - C\varepsilon, \quad \forall \varepsilon > 0,$$

which is the desired assertion.

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