

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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 $\mathcal{C}P^1$ lump

by

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Preprint no.: 30

1999



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Abstract

The slow motion of a self-gravitating $\mathbb{C}P^1$ lump is investigated in the approximation of geodesic flow on the moduli space of unit degree static solutions M_1 . It is found that moduli which are frozen in the absence of gravity, parametrizing the lump's width and internal orientation, may vary once gravitational effects are included. If gravitational coupling is sufficiently strong, the presence of the lump shrinks physical space to finite volume, and the moduli determining the boundary value of the $\mathbb{C}P^1$ field thaw also. Explicit formulae for the metric on M_1 are found in both the weak and strong coupling regimes. The geodesic problem for weak coupling is studied in detail, and it is shown that M_1 is geodesically incomplete. This leads to the prediction that self-gravitating lumps are unstable.

1 Introduction

Collective coordinate approximations have proved to be an invaluable tool in the study of topological soliton dynamics. They truncate the infinite dimensional dynamical system of the full field theory to a finite dimensional approximation which (hopefully) captures the important features of soliton motion, but is more amenable to analytic effort. For field theories of Bogomol'nyi type, the truncated system for n -soliton dynamics is obtained simply by restricting the field theoretic Lagrangian to the moduli space of degree n static solutions, M_n [1]. A rather beautiful feature in this case is that the reduced dynamics amounts to geodesic motion on M_n (hence this is often called the geodesic approximation). The metric on M_n descends from the kinetic energy functional of the field theory.

The geodesic approximation has been applied in many contexts (BPS monopoles [2], abelian Higgs vortices [3, 4] and discrete sine-Gordon kinks [5], for example) with considerable success. One system for which the approximation encounters problems is the $\mathbb{C}P^1$ model in $(2+1)$ dimensions [6, 7]. Here, a single static soliton (conventionally called a “lump”) has 6 parameters: 2 give its position, 3 its internal orientation and 1 sets its scale (the static field equation is conformally invariant). Hence M_1 is 6 dimensional. The metric on M_1 is, however, ill defined because at each static solution, 4 of the 6 zero modes are not normalizable, essentially due to the lump's weak (polynomial) spatial localization. The only ones which *are* normalizable generate spatial translations. It follows that 4 of the 6 moduli are frozen in this approximation, since any change in them is impeded by infinite inertia [6]. Of these 4, the 2 which specify the boundary value of the $\mathbb{C}P^1$ valued field should be expected to be frozen. That the other 2, determining the lump's width and its orientation about the boundary value, are frozen is not in good agreement with numerical simulations [8]. These suggest that more general motion than simply constant velocity drift is possible.

So the geodesic approximation for the $\mathbb{C}P^1$ model is unfortunately rather singular. The singularity is removed if physical space \mathbb{R}^2 is replaced by a compact Riemann surface (e.g. S^2 or T^2), yielding well defined

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and mathematically interesting geodesic problems [9, 10]. However, there is no obvious physical justification for such an assumption.

In this note we wish to describe a more natural means of removing the singularity: namely including gravitational self interaction. The moduli space M_1 of static self-gravitating lumps is already explicitly known, due to work of Comtet and Gibbons [11]. The geometric distortion of physical space introduced by gravity is just sufficient to regulate the divergent integrals encountered in [6] when evaluating g , the metric on M_1 . One expects, therefore, that the previously frozen moduli thaw once gravitational effects are taken into account. By restricting the Einstein-Hilbert- $\mathbb{C}P^1$ action to fields which are at all times in M_1 , we shall demonstrate that this is indeed the case. The degree of thawing depends on the strength of the coupling to gravity: for small coupling, physical space has infinite volume and the boundary value of the $\mathbb{C}P^1$ field remains frozen, while above a critical coupling, space has only finite volume and all 4 of the previously frozen moduli thaw. We shall derive explicit expressions for g in both cases, and investigate one-lump dynamics with weak gravitational coupling in detail.

2 The moduli space

The Einstein-Hilbert- $\mathbb{C}P^1$ action is

$$I[W, \eta] = \int d^3x \sqrt{|\eta|} \left[-\frac{S_\eta}{16\pi G} + 2\mu^2 \eta^{\alpha\beta} \frac{\partial_\alpha W \partial_\beta \bar{W}}{(1 + |W|^2)^2} \right], \quad (1)$$

where η is the Lorentzian metric on spacetime (assumed, for the moment, to be diffeomorphic to $\mathbb{C} \times \mathbb{R}$), $|\eta| = \det(\eta_{\alpha\beta})$, S_η is the scalar curvature of η , W is an inhomogeneous coordinate on $\mathbb{C}P^1$ (or equivalently a complex stereographic coordinate on S^2), G is Newton's constant and μ is a coupling constant. One may obtain static degree 1 (degree here meaning the topological degree of the mapping $W : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C}P^1$) solutions of the corresponding field equations by imposing the metric ansatz

$$\eta = dt^2 - \Omega^2(z, \bar{z}) dz d\bar{z}, \quad (2)$$

z being a complex coordinate on physical space. The static field equation for W is conformally invariant, hence independent of Ω . It follows that all the ‘‘flat space’’ solutions of the non-gravitating model (namely $W =$ rational map in z or \bar{z}) survive. In particular, the most general degree 1 solution is

$$W(z) = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} =: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \odot z = M \odot z \quad (3)$$

with $\det M \neq 0$. Note that the identification of a particular rational map W with a $GL(2, \mathbb{C})$ matrix M is nonunique, since for any $\xi \in \mathbb{C} \setminus \{0\}$, M and ξM yield the same W . One should therefore identify the space of degree 1 rational maps with the projective unimodular group $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2$ [9]. The most convenient parametrization for our purposes is

$$W(z) = U \odot \left(\frac{\beta}{z - \gamma} \right), \quad (4)$$

with $U \in PSU(2) = SU(2)/\mathbb{Z}_2 \cong SO(3)$, $\beta \in (0, \infty)$ and $\gamma \in \mathbb{C}$.

It remains to substitute W back into the Einstein equation for Ω . Since changing U merely alters the global internal orientation of W , Ω must be independent of U , which we may set to \mathbb{I} without loss of generality. The result, as found by Comtet and Gibbons [11] is

$$\Omega(z, \bar{z}) = \frac{1}{(\beta^2 + |z - \gamma|^2)^\lambda}, \quad (5)$$

where $\lambda = 8\pi G\mu^2$. If $\lambda < \frac{1}{2}$, the corresponding spatial metric $h = \Omega^2 dz d\bar{z}$ is asymptotically conical, with a deficit angle $\delta = 4\pi\lambda$, and the singular tip replaced by a region of smooth distortion centred on $z = \gamma$. If $\lambda = \frac{1}{2}$, space is asymptotically cylindrical, with one closed, rounded end. If $\lambda > \frac{1}{2}$, the volume of space is finite (volume = $\pi\beta^2/(2\lambda - 1)$), and except when $\lambda = 1$, the boundary of space $z = \infty$ is replaced by a

conical singularity of deficit angle $4\pi(1-\lambda)$ (or rather “surfeit angle” $4\pi(\lambda-1)$ if $\lambda > 1$) lying at finite proper distance¹. In the special case $\lambda = 1$, space is uniformly spherical, $z - \gamma$ being a stereographic coordinate.

In summary, by M_1 we mean the space of pairs

$$\left(W : z \mapsto U \odot \left(\frac{\beta}{z - \gamma} \right), h : z \mapsto \frac{dz d\bar{z}}{(\beta^2 + |z - \gamma|^2)^{2\lambda}} \right) \quad (6)$$

parametrized by $(U, \beta, \gamma) \in PSU(2) \times (0, \infty) \times \mathbb{C}$, all of which are static solutions of the Einstein- CP^1 equations.

3 The restricted action

It was proved in [11] that the static solutions in M_1 saturate a lower bound on deficit angle δ analogous to the Bogomol’nyi bound in the non-gravitating system (δ is an effective, albeit nonlocal, measure of the total gravitational energy of the system. In our case it coincides with Thorne’s C-energy [12] since the metric $\eta = dt^2 - h$ has circular symmetry). In the absence of gravity, it is obvious that time dependence can only increase energy, since kinetic energy is manifestly non-negative. It is not clear that the same statement holds for δ in our case, although this is conjectured in [11]. The geodesic approximation is usually justified by observing that energy conservation forces the field to stay close to M_n for low soliton speeds [1]. Rigorous analysis backs up this physical intuition [13, 14]. In applying the same approximation in the presence of gravity, we are being somewhat more speculative. Nevertheless, given the conjecture above, and previous applications of the approximation to gravitating systems [15], the collective coordinate approximation seems sensible, and worthy of investigation.

To obtain the restricted action $I|$, we substitute into I a field and metric whose time dependence is of the form

$$W(t, z) = U(t) \odot \frac{\beta(t)}{z - \gamma(t)}, \quad \eta(t, z) = dt^2 - \frac{dz d\bar{z}}{(\beta(t)^2 + |z - \gamma(t)|^2)^{2\lambda}}. \quad (7)$$

One finds that

$$I| = \int dt \int dz d\bar{z} \Omega^2 \left[\frac{S_h}{16\pi G} + 2\mu^2 \frac{|\dot{W}|^2}{(1 + |W|^2)^2} - \frac{2\mu^2}{\Omega^2} \frac{\partial_i W \partial_i \bar{W}}{(1 + |W|^2)^2} \right] =: \int dt [L_1 + L_2 + L_3] \quad (8)$$

where $S_\eta = S_{dt \otimes dt} - S_h = -S_h$ has been used. Now L_3 is (up to a constant factor) the potential energy of the non-gravitating system, so this is manifestly constant on M_1 (in fact $L_3 = -4\pi\mu^2$). An appeal to the local Gauss-Bonnet Theorem [16], or a straightforward calculation using the formulae

$$\begin{aligned} r e^{i\theta} &= z - \gamma \\ S_h &= -\frac{1}{r\Omega^2} \frac{d}{dr} \left(\frac{r}{\Omega} \frac{d\Omega}{dr} \right) \end{aligned} \quad (9)$$

establishes that L_1 is also constant ($L_1 = \lambda/4G$). Hence, both L_1 and L_3 may be discarded from $I|$, leaving a reduced action equivalent to a geodesic problem on M_1 . Denoting the 6 moduli collectively by q^a , so that $\dot{W} = \dot{q}^a \partial W / \partial q^a$ one finds that

$$I| = 2\mu^2 \int dt g_q(\dot{q}, \dot{q}) \quad (10)$$

where g is the following Riemannian metric on M_1 :

$$g = \left[\text{Re} \int dz d\bar{z} \frac{|z - \gamma|^4}{(\beta^2 + |z - \gamma|^2)^{2(1+\lambda)}} \frac{\partial W}{\partial q^a} \frac{\partial \bar{W}}{\partial q^b} \right] dq^a dq^b. \quad (11)$$

It remains to compute the metric coefficients g_{ab} . Recall that M_1 is diffeomorphic to $PSU(2) \times (0, \infty) \times \mathbb{C}$. The metric g is invariant under changes of U (global internal rotations), so it suffices to evaluate g in the special case $U = \mathbb{I}$. Elsewhere, g will follow from the action of the isometry group. A convenient coordinate chart on $PSU(2)$ may be defined as follows: use the canonical identification of $SU(2)$ with the unit 3-sphere

¹The authors are grateful to G.W. Gibbons for pointing this out.

$S^3 \subset \mathbb{R}^4$, and identify $PSU(2)$ with the upper hemisphere; project the upper hemisphere (minus boundary) vertically onto the open equatorial 3-ball, to obtain a 3-vector \mathbf{v} :

$$U = \begin{pmatrix} \sqrt{1-|\mathbf{v}|^2} + iv_3 & v_2 + iv_1 \\ v_2 - iv_1 & \sqrt{1-|\mathbf{v}|^2} - iv_3 \end{pmatrix}, \quad |\mathbf{v}| < 1. \quad (12)$$

Note that $\mathbf{v} = \mathbf{0} \Leftrightarrow U = \mathbb{I}$. One then finds that

$$\begin{aligned} \frac{\partial W}{\partial v_1} \Big|_{\mathbf{v}=\mathbf{0}} &= i \left[1 - \frac{\beta^2}{(z-\gamma)^2} \right], & \frac{\partial W}{\partial v_2} \Big|_{\mathbf{v}=\mathbf{0}} &= 1 + \frac{\beta^2}{(z-\gamma)^2}, & \frac{\partial W}{\partial v_3} \Big|_{\mathbf{v}=\mathbf{0}} &= \frac{2i\beta}{z-\gamma}, \\ \frac{\partial W}{\partial \beta} \Big|_{\mathbf{v}=\mathbf{0}} &= \frac{1}{z-\gamma}, & \frac{\partial W}{\partial \gamma} \Big|_{\mathbf{v}=\mathbf{0}} &= \frac{\beta}{(z-\gamma)^2}. \end{aligned} \quad (13)$$

Substituting these expressions into (11), it is immediately apparent that all g_{ab} are independent of γ , which may thus be set to 0. The integrals are easily computed. If $\lambda \leq \frac{1}{2}$ neither $g_{v_1 v_1}$ nor $g_{v_2 v_2}$ exists, the corresponding integrals being divergent. It follows that v_1 and v_2 are frozen, and the metric (at $v_3 = 0$) is

$$g|_{\mathbf{v}=\mathbf{0}} = \frac{\pi}{(2\lambda+1)\beta^{4\lambda}} \left[d\gamma d\bar{\gamma} + \frac{1}{2\lambda} (d\beta^2 + 4\beta^2 dv_3^2) \right]. \quad (14)$$

If $\lambda > \frac{1}{2}$, all g_{ab} are finite, so all 6 moduli are free to change with time. In this case,

$$g|_{\mathbf{v}=\mathbf{0}} = \frac{\pi}{(2\lambda+1)\beta^{4\lambda}} \left[d\gamma d\bar{\gamma} + \frac{1}{2\lambda} (d\beta^2 + 4\beta^2 (C(\lambda)dv_1^2 + C(\lambda)dv_2^2 + dv_3^2)) + 2\beta(d\gamma_1 dv_2 - d\gamma_2 dv_1) \right] \quad (15)$$

where $C(\lambda) = (2\lambda^2 - \lambda + 1)/(4\lambda - 2)$ and $\gamma = \gamma_1 + i\gamma_2$. To globalize these formulae (remove the condition $U = \mathbb{I}$) we may define a left invariant basis $\{\sigma_a : a = 1, 2, 3\}$ on $T^*SU(2)$ by expanding the left invariant one form $U^{-1}dU$ in a basis for the Lie algebra $su(2)$. We choose

$$U^{-1}dU = \boldsymbol{\sigma} \cdot \left(\frac{i}{2} \boldsymbol{\tau} \right), \quad (16)$$

$\{\tau_a : a = 1, 2, 3\}$ being the Pauli spin matrices. The connexion with the previous basis is simple:

$$\sigma_a|_{\mathbf{v}=\mathbf{0}} = 2dv_a|_{\mathbf{v}=\mathbf{0}}. \quad (17)$$

The formulae (14), (15) are easily re-written using σ_a , and then hold equally true away from $\mathbf{v} = \mathbf{0}$ by $PSU(2)$ left translation invariance of g . Hence,

$$g = \frac{\pi}{(2\lambda+1)\beta^{4\lambda}} \left[d\gamma d\bar{\gamma} + \frac{1}{2\lambda} (d\beta^2 + \beta^2 \sigma_3^2) \right] \quad (18)$$

if $0 < \lambda \leq \frac{1}{2}$, while

$$g = \frac{\pi}{(2\lambda+1)\beta^{4\lambda}} \left[d\gamma d\bar{\gamma} + \frac{1}{2\lambda} (d\beta^2 + \beta^2 (C(\lambda)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2)) + \beta(d\gamma_1 \sigma_2 - d\gamma_2 \sigma_1) \right] \quad (19)$$

if $\lambda > \frac{1}{2}$. In the case of weak coupling, once the boundary value of W is specified, U must remain in the stabilizer of this boundary value, a $SO(2)$ subgroup of $PSU(2) \cong SO(3)$. So the lump may spin, but only about its own axis. Geodesic motion takes place on a single leaf $M_1^{W(\infty)}$ of a foliation of M_1 by 4-manifolds parametrized by the boundary value $W(\infty) \in \mathbb{C}P^1$. More general rotational dynamics is possible in the regime of strong coupling. Note that $C(1) = 1$, so in the case of spherical spatial geometry, g is exceptionally symmetric, as one would expect.

4 Geodesics in the case of weak coupling

For the sake of simplicity, we henceforth assume that $0 < \lambda < \frac{1}{2}$. One may, without loss of generality, assume that the frozen boundary value is $W(\infty) = 0$. Then $U = \exp(i\psi\tau_3/2)$, and

$$W(z) = \frac{\beta e^{i\psi}}{z - \gamma}. \quad (20)$$

This leaf M_1^0 of M_1 is, like every other, diffeomorphic to $\mathbb{R}^4 \setminus \mathbb{R}^2$. It has metric

$$g = \frac{\pi}{(2\lambda + 1)\beta^{4\lambda}} \left(d\gamma d\bar{\gamma} + \frac{d\beta^2 + \beta^2 d\psi^2}{2\lambda} \right). \quad (21)$$

We remark in passing that (M_1^0, g) is conformally flat.

The two-dimensional submanifold on which $\gamma = 0$, call it Σ , is isometric to a cone of deficit angle $4\pi\lambda$ (with its tip, $\beta = 0$, missing). This cone is a totally geodesic submanifold since it is the fixed point set of the discrete group of isometries $\{\text{Id}, \gamma \mapsto -\gamma\}$. It immediately follows that (M_1^0, g) is geodesically incomplete, a property found to be universal in the non-gravitating model on compact physical space [17]. Generically, geodesics on Σ miss the singularity $\beta = 0$ – they emerge from and return to the asymptotic region $\beta \rightarrow \infty$ – and may be extended forever in the future and past. In interpreting such a geodesic one must take care to account for the β dependence of the metric on physical space, h . Given a fixed pair of points in $\mathbb{C}P^1$ (not coincident), the proper distance between their preimages under W scales as $\beta^{1-2\lambda}$, so for $0 < \lambda < \frac{1}{2}$ it is correct to regard β as a measure of the lump’s width. Hence, the generic geodesic on Σ corresponds to a spinning lump shrinking to some minimum width, then spreading out indefinitely, spinning ever more slowly as it expands. There are also completely irrotational geodesics which hit $\beta = 0$ in finite time, corresponding to the lump collapsing to zero width – the geodesic approximation predicts that self-gravitating lumps are unstable.

Returning to the full four-dimensional geodesic problem, the geodesic equation for β is

$$\ddot{\beta} - \frac{2\lambda}{\beta} \dot{\beta}^2 = -4\lambda^2 \beta^{8\lambda-1} \left[|p|^2 - \frac{(1-2\lambda)}{\beta^2} J^2 \right], \quad (22)$$

$p = \beta^{-4\lambda}\dot{\gamma}$ and $J = \beta^{2(1-2\lambda)}\dot{\psi}/2\lambda$ being the constant momenta conjugate to the cyclic variables $\bar{\gamma}$ and ψ respectively. Clearly, the lump maintains constant “velocity” $\dot{\gamma}$ if and only if β remains constant, and in this case $\dot{\psi}$ remains constant too. Hence we may obtain translating-spinning solutions with constant β (shape), $\dot{\gamma}$ (velocity) and $\dot{\psi}$ (angular velocity) by setting the right hand side of (22) to zero. Given an initial velocity $\dot{\gamma}(0) = v \in \mathbb{C}$, the lump translates undistorted with $\beta = \beta_0$, $\dot{\psi} = \omega$, constant, if and only if β_0 and ω satisfy

$$(\beta_0\omega)^2 = \frac{4\lambda^2}{1-2\lambda}|v|^2. \quad (23)$$

The narrower the lump, the faster it must spin in order to travel undistorted at a given speed.

The general geodesic motion is most readily understood within the Hamiltonian formalism, wherein $(\gamma, \beta, \psi, p, \xi, J)$, $\xi = \beta^{-4\lambda}\dot{\beta}/2\lambda$ being the momentum conjugate to β , are thought of as local coordinates on the cotangent bundle $T^*M_1^{W(\infty)}$, and geodesic flow is generated by the Hamiltonian

$$H = \frac{1}{2}\beta^{4\lambda} \left[|p|^2 + 2\lambda \left(\xi^2 + \frac{J^2}{\beta^2} \right) \right]. \quad (24)$$

From the independence of H on γ , ψ and t , it follows that p , J and H itself are conserved under the flow, so the phase space trajectories are simply the intersection curves of the level sets of these 4 conserved quantities (recall $p \in \mathbb{C}$). Since geodesic trajectories scale trivially with initial speed, one may without loss of generality consider only the case $H = 1$ (the case $H = 0$ is trivial, consisting only of static solutions). Then the projection of a trajectory onto the (β, ξ) phase space satisfies

$$\xi^2 = \frac{1}{2\lambda} \left(\frac{2}{\beta^{4\lambda}} - |p|^2 \right) - \frac{J^2}{\beta^2}. \quad (25)$$

Note that $0 < 4\lambda < 2$. The nature of this curve depends critically on p and J (but not on λ), as depicted in figure 1. For generic initial data, $p \neq 0$ and $J \neq 0$, and the (projected) trajectory is a closed curve, indicating periodic (projected) motion: the lump travels in a straight line with speed, width and angular velocity oscillating in phase. One may imagine fixing J^2 while reducing $|p|^2$. The result is that the closed curve depicted shrinks smaller and smaller until, at some critical $|p|^2$, it degenerates to a single point on the β axis. This represents one of the constant speed translating-spinning solutions described above. Note that for generic initial data, the trajectory does not encounter the singularity $\beta = 0$, nor does β grow unbounded, that is, the lump neither collapses, nor spreads out indefinitely.

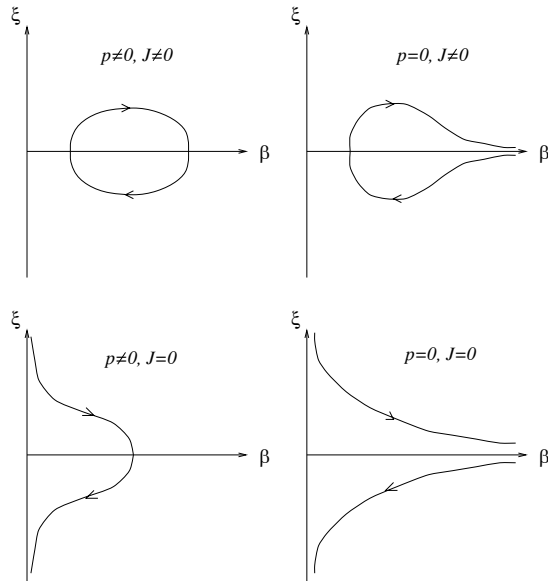


Figure 1: Geodesic trajectories projected onto the (β, ξ) phase space.

Various other possibilities arise if one restricts attention to measure zero subsets of the space of initial data. If $p = 0$ one obtains the geodesics on the embedded cone Σ already described (spinning if $J \neq 0$, irrotational if $J = 0$). If $p \neq 0$ and $J = 0$, the (projected) trajectory tends to $\beta = 0$ both forwards and backwards, reaching the singularity in finite time. This corresponds to an infinitely narrow, stationary lump being “born” at time $t = -t_s > -\infty$, spreading out and accelerating in a straight line, so that at $t = 0$ both its width and its speed achieve their maxima, then decelerating and collapsing to zero width again at time $t = t_s$.

In summary, one predicts that, although lumps are unstable and may collapse to zero width (in finite time), or spread out indefinitely, neither type of instability occurs generically. Spinning prevents collapse, while translating prevents indefinite spread.

5 Concluding remarks

We have derived a collective coordinate approximation for the slow motion of a single self-gravitating $\mathbb{C}P^1$ lump by restricting the Einstein-Hilbert- $\mathbb{C}P^1$ action to the degree 1 moduli space of Comtet and Gibbons, M_1 . This approximation takes the form of a geodesic problem either on a single leaf of a foliation of M_1 by 4-manifolds (weak coupling) or on all M_1 (strong coupling). The physical interpretation of this is that when gravitational coupling is weak, the boundary of space lies at infinite proper distance, so the boundary value of the field remains frozen. Every leaf of the foliation is geodesically incomplete, leading us to predict that self-gravitating lumps are unstable to collapse.

It would be interesting to investigate the general n -lump case along similar lines. Here it will be impossible to find closed formulae for g analogous to (18) and (19). However, one should still be able to obtain qualitative information (on completeness of (M_n, g) , for example) and identify especially symmetric geodesics. One could

also investigate single lump motion at strong coupling, using the more complicated metric (19), although this regime is presumably far from real cosmological applications (the lump may be regarded as a slice through an infinitely long, straight cosmic string). A rigorous justification for the approximation similar to the work of Stuart [13, 14], seems well beyond reach. Numerical tests of the approximation would therefore be useful.

References

- [1] N.S. Manton, “A remark on the scattering of BPS monopoles” *Phys. Lett.* **110B** (1982) 54.
- [2] G.W. Gibbons and N.S. Manton, “Classical and quantum dynamics of BPS monopoles” *Nucl. Phys.* **B274** (1986) 183.
- [3] T.M. Samols, “Vortex scattering” *Commun. Math. Phys.* **145** (1992) 149.
- [4] I.A.B. Strachan, “Low-velocity scattering of vortices in a modified Abelian Higgs model” *J. Math. Phys.* **33** (1992) 102.
- [5] J.M. Speight and R.S. Ward, “Kink dynamics in a novel discrete sine-Gordon system” *Nonlinearity* **7** (1994) 475.
- [6] R.S. Ward, “Slowly-moving lumps in the $\mathbb{C}P^1$ model in $(2+1)$ dimensions” *Phys. Lett.* **158B** (1985) 424.
- [7] R.A. Leese, “Low-energy scattering of solitons in the $\mathbb{C}P^1$ model” *Nucl. Phys.* **B344** (1990) 33.
- [8] W.J. Zakrzewski and B. Piette, “Shrinking of solitons in the $(2+1)$ -dimensional S^2 sigma model” *Nonlinearity* **9** (1996) 897.
- [9] J.M. Speight, “Low-energy dynamics of a $\mathbb{C}P^1$ lump on the sphere” *J. Math. Phys.* **36** (1995) 796.
- [10] J.M. Speight, “Lump dynamics in the $\mathbb{C}P^1$ model on the torus” *Commun. Math. Phys.* **194** (1998) 513.
- [11] A. Comtet and G.W. Gibbons, “Bogomol’nyi bounds for cosmic strings” *Nucl. Phys.* **B299** (1988) 719.
- [12] K.S. Thorne, “Energy of infinitely long, cylindrically symmetric systems in General Relativity” *Phys. Rev.* **138B** (1965) 251.
- [13] D. Stuart, “Dynamics of abelian Higgs vortices in the near Bogomolny regime” *Commun. Math. Phys.* **159** (1994) 51.
- [14] D. Stuart, “The Geodesic approximation for the Yang-Mills Higgs equations” *Commun. Math. Phys.* **166** (1994) 149.
- [15] G.W. Gibbons and P.J. Ruback, “Motion of extreme Reissner-Nordstrom black holes in the low-velocity limit” *Phys. Rev. Lett.* **57** (1986) 1492.
- [16] T.J. Willmore, *Riemannian Geometry* (Clarendon Press, Oxford, UK, 1996) p. 146.
- [17] L.A. Sadun and J.M. Speight, “Geodesic incompleteness of the $\mathbb{C}P^1$ model on a compact Riemann surface” *Lett. Math. Phys.* **43** (1998) 329.