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Non-existence of time-periodic solutions of the Dirac equation in an axisymmetric black hole geometry

by

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Non-Existence of Time-Periodic Solutions of the Dirac Equation in an Axisymmetric Black Hole Geometry

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Abstract

We prove that, in the non-extreme Kerr-Newman black hole geometry, the Dirac equation has no normalizable, time-periodic solutions. A key tool is Chandrasekhar's separation of the Dirac equation in this geometry. A similar non-existence theorem is established in a more general class of stationary, axisymmetric metrics in which the Dirac equation is known to be separable. These results indicate that, in contrast with the classical situation of massive particle orbits, a quantum mechanical Dirac particle must either disappear into the black hole or escape to infinity.

1 Introduction

It has recently been proved in [13] that the Dirac equation does not admit normalizable, time-periodic solutions in a non-extreme Reissner-Nordström black hole geometry. This result shows that quantization and the introduction of spin cause a significant qualitative break-down of the classical situation, where it is well-known that there exist special choices of initial conditions for the motion of massive test particles which give rise to closed orbits [6]. Indeed, the above theorem implies that the quantum mechanical Dirac wave function in the gravitational and electromagnetic fields of a static, spherically symmetric black hole describes a particle which must either disappear into the black hole or escape to infinity.

It is quite natural to ask whether this result is stable under perturbations of the background metric; i.e., if the non-existence theorem for normalizable periodic solutions of the Dirac equation remains true if the background metric and electromagnetic field are changed in such a way that the spherical symmetry is destroyed. This is precisely the question that we address in this paper.

We are guided in our choice of a more general background geometry by the uniqueness theorems of Carter, Israel, and Robinson [3, 5], from which we know that the most general charged black hole equilibrium state is given by the Kerr-Newman solution of the Einstein-Maxwell equations. Thus, in order to investigate the non-existence of periodic solutions in the most general black hole geometry, we have to study the Dirac equation in the Kerr-Newman background. The fact that this study is even possible rests on the remarkable discovery made by Chandrasekhar that the Dirac equation is completely separable into ordinary differential equations in the Kerr-Newman background geometry, even though the

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metric is stationary and axisymmetric [7]. In order to state our result, let us first recall that the Kerr-Newman solution is characterized by three parameters, namely its mass M, angular momentum aM and electric charge Q. We prove that the non-existence theorem for normalizable solutions of the Dirac equation remains true in the case of a rotating Kerr-Newman black hole provided that the angular momentum per unit mass and the charge are sufficiently small relative to the total mass of the black hole. Such black holes have both a Cauchy and an event horizon, and are thus referred to as *non-extreme* black holes. Specifically, we prove:

Theorem 1.1 *In a Kerr-Newman black hole for which $a^2 + Q^2 < M^2$, the Dirac equation has no normalizable, time-periodic solutions.*

In Theorem 3.1, we give a similar non-existence result for the most general stationary, axisymmetric metric in which the Dirac equation can be separated by Chandrasekhar's method. The hypotheses of this theorem are stated as much as possible in geometric terms. While the solutions one obtains by imposing the Einstein-Maxwell equations on this metric are of limited physical interest, this theorem indicates that our non-existence result applies in a broader context (e.g. for a more general energy-momentum tensor).

This paper is organized as follows. In Section 2.1, we derive the separability of the Dirac equation in the Kerr-Newman geometry in a form slightly different from the one used by Chandrasekhar so as to recover the equations established in [13] in the spherically symmetric limit, by letting the angular momentum parameter a tend to zero. In Section 2.2, we work out matching conditions for the spinor field across the horizons. This gives rise to a weak solution of the Dirac equation in the physical region of the maximal analytic extension of the Kerr-Newman solution, which is valid across the Cauchy and event horizons. We then proceed in Section 2.3 to establish the non-existence of time-periodic solutions. Just as in the spherically symmetric case, the crucial step consists in exploiting the conservation and positivity of the Dirac current to show that because of the matching conditions, the only way in which a time-periodic solution of the Dirac equation can be normalizable is that each term in the Fourier expansion of the spinor field in time and the angular variable around the axis of symmetry, be identically zero. While the regularity of the angular dependence of the separable solutions is manifest in the spherically symmetric Reissner-Nordström case, this is not so in the axisymmetric case treated in this paper. The regularity is therefore established in the Appendix. Finally, we consider in Section 3 the extension of Theorem 1.1 to more general stationary axisymmetric metrics.

2 The Kerr-Newman Black Hole

2.1 The Dirac Equation in Boyer-Lindquist Coordinates

Recall that, in Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$, the Kerr-Newman metric takes the form [3]

$$\begin{aligned} ds^2 &= g_{jk}\, dx^j x^k \\ &= \frac{\Delta}{U}\,(dt - a\,\sin^2\vartheta\, d\varphi)^2 - U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) - \frac{\sin^2\vartheta}{U}\,(a\,dt - (r^2+a^2)\,d\varphi)^2 \;, \end{aligned} \tag{2.1}$$

where

$$U(r,\vartheta) = r^2 + a^2\cos^2\vartheta \;, \qquad \Delta(r) = r^2 - 2Mr + a^2 + Q^2 \;.$$

and the electromagnetic potential is of the form

$$A_j\, dx^j \;=\; -\frac{Q\, r}{U}\, (dt \;-\; a\, \sin^2\vartheta\; d\varphi) \quad . \tag{2.2}$$

The metric is singular at the origin $r = 0$ and at the zeros of the function Δ. We shall consider the so-called *non-extreme case* $M^2 > a^2 + Q^2$. In this case, Δ has two distinct zeros

$$r_0 \;=\; M \;-\; \sqrt{M^2 - a^2 - Q^2} \qquad \text{and} \qquad r_1 \;=\; M \;+\; \sqrt{M^2 - a^2 - Q^2} \quad .$$

The two radii r_0 and r_1 correspond to the *Cauchy horizon* and the *event horizon* for the non-extreme Kerr-Newman metric, respectively.

We briefly recall some elementary facts about the Dirac operator in curved space-time. The Dirac operator G is a differential operator of first order,

$$G \;=\; i G^j(x) \frac{\partial}{\partial x^j} \;+\; B(x) \quad , \tag{2.3}$$

where B and the Dirac matrices G^j are (4×4)-matrices. The Dirac matrices are related to the Lorentzian metric via the anti-commutation relations

$$g^{jk}(x)\, \mathbb{1} \;=\; \frac{1}{2}\, \{G^j(x),\, G^k(x)\} \;\equiv\; \frac{1}{2} \Big(G^j(x)\, G^k(x) \;+\; G^k(x)\, G^j(x) \Big) \quad . \tag{2.4}$$

The matrix B is determined by the spinor connection and the electromagnetic potential through minimal coupling. As such, it is determined by the Levi-Civita connection of the background Lorentzian metric (2.1) and the potential (2.2). The Dirac matrices are not uniquely determined by the anti-commutation rules (2.4). The ambiguity in the choice of Dirac matrices adapted to a given metric is formulated naturally in terms of the spin and frame bundles [17]. A convenient method for calculating the Dirac operator in this bundle formulation is provided by the Newman-Penrose formalism [6]. More generally, it is shown in [12] that all choices of Dirac matrices satisfying (2.4) yield unitarily equivalent Dirac operators. Furthermore, in [12], explicit formulas for the matrix B in terms of the Dirac matrices G^j are given. In the following, we attempt to combine the advantages of these different approaches; namely, we first choose the Dirac matrices using a Newman-Penrose frame, and then construct the matrix B using the explicit formulas in [12].

We choose the so-called *symmetric frame* $(l, n, m, \overline{m})$ of [4],

$$\begin{aligned}
l &= \frac{1}{\sqrt{2U\,|\Delta|}} \left((r^2 + a^2)\, \frac{\partial}{\partial t} \;+\; \Delta\, \frac{\partial}{\partial r} \;+\; a\, \frac{\partial}{\partial \phi} \right) \\
n &= \frac{\epsilon(\Delta)}{\sqrt{2U\,|\Delta|}} \left((r^2 + a^2)\, \frac{\partial}{\partial t} \;-\; \Delta\, \frac{\partial}{\partial r} \;+\; a\, \frac{\partial}{\partial \phi} \right) \\
m &= \frac{1}{\sqrt{2U}} \left(ia\, \sin\vartheta\, \frac{\partial}{\partial t} \;+\; \frac{\partial}{\partial \vartheta} \;+\; \frac{i}{\sin\vartheta}\, \frac{\partial}{\partial \varphi} \right) \\
\overline{m} &= \frac{1}{\sqrt{2U}} \left(-ia\, \sin\vartheta\, \frac{\partial}{\partial t} \;+\; \frac{\partial}{\partial \vartheta} \;-\; \frac{i}{\sin\vartheta}\, \frac{\partial}{\partial \varphi} \right) \quad ,
\end{aligned}$$

where ϵ is the step function $\epsilon(x) = 1$ for $x \geq 0$ and $\epsilon(x) = -1$ otherwise. (Because of its symmetry properties, this frame is somewhat more convenient than the Kinnersley frame used in [6]. Also, the notation with the step function allows us to give a unified form of the frame both inside and outside the horizons.) The symmetric frame is a Newman-Penrose null frame; i.e.

$$<l,n> \;=\; 1 \quad , \qquad <m,\overline{m}> \;=\; -1 \quad ,$$

and all other scalar products between the elements of the frame vanish. From this complex null frame, we can form a real frame $(u_a)_{a=0,\ldots,3}$ by setting

$$\begin{aligned} u_0 &= \frac{\epsilon(\Delta)}{\sqrt{2}}\,(l+n) \quad , & u_1 &= \frac{1}{\sqrt{2}}\,(l-n) \\ u_2 &= \frac{1}{\sqrt{2}}\,(m+\overline{m}) \quad , & u_3 &= \frac{1}{\sqrt{2}\,i}\,(m-\overline{m}) \quad . \end{aligned}$$

This frame is orthonormal; i.e.

$$g_{jk}\,u^j_a\,u^k_b \;=\; \eta_{ab} \quad , \qquad \eta^{ab}\,u^j_a\,u^k_b \;=\; g^{jk} \quad , \tag{2.5}$$

where $\eta_{ab} = \eta^{ab} = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric. We choose the Dirac matrices γ^a, $a = 0,\ldots,3$ of Minkowski space in the Weyl representation

$$\gamma^0 \;=\; \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \quad , \qquad \vec{\gamma} \;=\; \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad ,$$

where $\vec{\sigma}$ are the usual Pauli matrices

$$\sigma^1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \qquad \sigma^2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \qquad \sigma^3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{2.6}$$

The γ^a satisfy the anti-commutation relations

$$\eta^{ab} \;=\; \frac{1}{2}\left\{\gamma^a,\, \gamma^b\right\} \quad . \tag{2.7}$$

We choose as Dirac matrices G^j associated to the Kerr-Newman metric the following linear combinations of the γ^a:

$$G^j(x) \;=\; u^j_a(x)\,\gamma^a \tag{2.8}$$

As an immediate consequence of (2.5) and (2.7), these Dirac matrices satisfy the anti-commutation relations (2.4). Next, we must calculate the matrix B in (2.3). In [12], it is shown that the Dirac matrices induce a spin connection D, which has the general form

$$D_j \;=\; \frac{\partial}{\partial x^j} \;-\; iE_j \;-\; ieA_j \qquad \text{with} \tag{2.9}$$

$$E_j \;=\; \frac{i}{2}\,\rho(\partial_j\rho) \;-\; \frac{i}{16}\,\mathrm{Tr}(G^m\,\nabla_j G^n)\,G_m G_n \;+\; \frac{i}{8}\,\mathrm{Tr}(\rho G_j\,\nabla_m G^m)\,\rho \quad , \tag{2.10}$$

where $\rho = \frac{i}{4!}\epsilon_{jklm}G^jG^kG^lG^m$ is the pseudoscalar matrix, ϵ_{jklm} is the Levi-Civita symbol of curved space-time, and A_j is the electromagnetic potential. Using the spin connection, the Dirac operator (2.3) can be written in the alternative form $G = iG^jD_j$. Thus the matrix B is given by $B = G^j(E_j + eA_j)$. Since, in our context, the Dirac matrices G^j are linear combinations of the γ^a, the matrix ρ is simply the constant $\rho \equiv \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. As a consequence, the first and last summands in (2.10) vanish, and we obtain

$$B \;=\; -\frac{i}{16}\,\mathrm{Tr}(G^m\,\nabla_j G^n)\,G^jG_mG_n \;+\; eG^jA_j \quad . \tag{2.11}$$

Using Ricci's lemma,

$$0 \;=\; 4\nabla_j g^{mn} \;=\; \nabla_j \mathrm{Tr}(G^m\, G^n) \;=\; \mathrm{Tr}((\nabla_j G^m)\, G^n) \;+\; \mathrm{Tr}(G^m\, (\nabla_j G^n)) \quad ,$$

and the commutation rules of the Dirac matrices, we can simplify the trace term as

$$\begin{aligned}
&\mathrm{Tr}(G^m\, \nabla_j G^n)\, G^j G_m G_n \\
&\quad= \;\mathrm{Tr}(G^m\, \nabla_j G^n) \left(\delta^j_m\, G_n \;-\; \delta^j_n\, G_m \;+\; G^j\, g_{mn} \;+\; i\epsilon^j_{\;mnp}\, \gamma^5 G^p\right) \\
&\quad= \;-8\, \nabla_j G^j \;+\; i\epsilon^{jmnp}\, \mathrm{Tr}(G_m\, \nabla_j G_n)\, \gamma^5 G_p \quad .
\end{aligned} \tag{2.12}$$

Since the Levi-Civita connection is torsion-free, we can replace the covariant derivative in the last summand in (2.12) by a partial derivative. We thus conclude that

$$\begin{aligned}
B \;&=\; \frac{i}{2\sqrt{|g|}}\, \partial_j(\sqrt{|g|}\, u^j_a)\, \gamma^a \\
&\quad -\frac{i}{4}\, \epsilon^{jmnp}\, \eta^{ab}\, u_{am}\, (\partial_j u_{bn})\, u_{cn}\, \gamma^5 \gamma^c + eA_j u^j_a\, \gamma^a
\end{aligned}$$

(g denotes as usual the determinant of the Lorentzian metric). This formula for B is particularly convenient, because it only involves partial derivatives, so that it becomes unnecessary to compute the Christoffel symbols of the Levi-Civita connection. Next, we substitute the derived formulas for G^j and B into (2.3) and obtain for the Dirac operator

$$G \;=\; \begin{pmatrix} 0 & 0 & \alpha_+ & \beta_+ \\ 0 & 0 & \beta_- & \epsilon(\Delta)\, \alpha_- \\ \epsilon(\Delta)\, \overline{\alpha}_- & -\overline{\beta}_+ & 0 & 0 \\ -\overline{\beta}_- & \overline{\alpha}_+ & 0 & 0 \end{pmatrix} \qquad \text{with} \tag{2.13}$$

$$\begin{aligned}
\beta_\pm \;&=\; \frac{1}{\sqrt{U}} \left(i\frac{\partial}{\partial\vartheta} + i\,\frac{\cot\vartheta}{2} + \frac{a\,\sin\vartheta}{2U}\,(r - ia\,\cos\vartheta)\right) \pm \frac{1}{\sqrt{U}} \left(a\,\sin\vartheta\, \frac{\partial}{\partial t} + \frac{1}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\right) \\
\overline{\beta}_\pm \;&=\; \frac{1}{\sqrt{U}} \left(i\frac{\partial}{\partial\vartheta} + i\,\frac{\cot\vartheta}{2} - \frac{a\,\sin\vartheta}{2U}\,(r + ia\,\cos\vartheta)\right) \pm \frac{1}{\sqrt{U}} \left(a\,\sin\vartheta\, \frac{\partial}{\partial t} + \frac{1}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\right) \\
\alpha_\pm \;&=\; -\frac{\epsilon(\Delta)}{\sqrt{U\,|\Delta|}} \left(i(r^2+a^2)\,\frac{\partial}{\partial t} \;+\; ia\,\frac{\partial}{\partial\varphi} \;+\; eQr\right) \\
&\qquad\qquad \pm\sqrt{\frac{|\Delta|}{U}} \left(i\frac{\partial}{\partial r} \;+\; i\,\frac{r-M}{2\Delta} \;+\; \frac{i}{2U}\,(r - ia\cos\vartheta)\right) \\
\overline{\alpha}_\pm \;&=\; -\frac{\epsilon(\Delta)}{\sqrt{U\,|\Delta|}} \left(i(r^2+a^2)\,\frac{\partial}{\partial t} \;+\; ia\,\frac{\partial}{\partial\varphi} \;+\; eQr\right) \\
&\qquad\qquad \pm\sqrt{\frac{|\Delta|}{U}} \left(i\frac{\partial}{\partial r} \;+\; i\,\frac{r-M}{2\Delta} \;+\; \frac{i}{2U}\,(r + ia\cos\vartheta)\right) \quad .
\end{aligned}$$

The four-component wave function Ψ of a Dirac particle is a solution of the Dirac equation

$$(G-m)\,\Psi \;=\; \begin{pmatrix} -m & 0 & \alpha_+ & \beta_+ \\ 0 & -m & \beta_- & \epsilon(\Delta)\, \alpha_- \\ \epsilon(\Delta)\, \overline{\alpha}_- & -\overline{\beta}_+ & -m & 0 \\ -\overline{\beta}_- & \overline{\alpha}_+ & 0 & -m \end{pmatrix} \Psi \;=\; 0 \quad . \tag{2.14}$$

It is a remarkable fact that this equation can be completely separated into ordinary differential equations. This was first shown for the Kerr metric by Chandrasekhar [7], and was later generalized to the Kerr-Newman background [16, 18]. We shall now briefly recall how this is done, since we will need the explicit form of the corresponding ordinary differential operators for the proof of Theorem 1.1. We closely follow the procedure in [4]. Let $S(r,\vartheta)$ and $\Gamma(r,\vartheta)$ be the diagonal matrices[1]

$$\begin{aligned} S &= |\Delta|^{\frac{1}{4}} \operatorname{diag}\left((r-ia\cos\vartheta)^{\frac{1}{2}},\ (r-ia\cos\vartheta)^{\frac{1}{2}},\ (r+ia\cos\vartheta)^{\frac{1}{2}},\ (r+ia\cos\vartheta)^{\frac{1}{2}}\right) \\ \Gamma &= -i\operatorname{diag}\left((r+ia\cos\vartheta),\ -(r+ia\cos\vartheta),\ -(r-ia\cos\vartheta),\ (r-ia\cos\vartheta)\right) \quad . \end{aligned}$$

Then the transformed wave function

$$\hat{\Psi} = S\,\Psi \tag{2.15}$$

satisfies the Dirac equation

$$\Gamma S\,(G-m)\,S^{-1}\,\hat{\Psi} = 0 \quad . \tag{2.16}$$

This transformation is useful because the differential operator (2.16) can be written as a sum of an operator $\mathcal{R}$, which depends only on the radius r, and an operator $\mathcal{A}$, which depends only on the angular variable ϑ. More precisely, an explicit calculation gives

$$\Gamma S\,(G-m)\,S^{-1} = \mathcal{R}+\mathcal{A}$$

with

$$\begin{aligned} \mathcal{R} &= \begin{pmatrix} imr & 0 & \sqrt{|\Delta|}\,\mathcal{D}_+ & 0 \\ 0 & -imr & 0 & \epsilon(\Delta)\,\sqrt{|\Delta|}\,\mathcal{D}_- \\ \epsilon(\Delta)\,\sqrt{|\Delta|}\,\mathcal{D}_- & 0 & -imr & 0 \\ 0 & \sqrt{|\Delta|}\,\mathcal{D}_+ & 0 & imr \end{pmatrix} \\ \mathcal{A} &= \begin{pmatrix} -am\cos\vartheta & 0 & 0 & \mathcal{L}_+ \\ 0 & am\cos\vartheta & -\mathcal{L}_- & 0 \\ 0 & \mathcal{L}_+ & -am\cos\vartheta & 0 \\ -\mathcal{L}_- & 0 & 0 & am\cos\vartheta \end{pmatrix} \quad , \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_\pm &= \frac{\partial}{\partial r} \mp \frac{1}{\Delta}\left[(r^2+a^2)\,\frac{\partial}{\partial t} + a\,\frac{\partial}{\partial\varphi} - ieQr\right] \\ \mathcal{L}_\pm &= \frac{\partial}{\partial\vartheta} + \frac{\cot\vartheta}{2} \mp i\left[a\sin\vartheta\,\frac{\partial}{\partial t} + \frac{1}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\right] \quad . \end{aligned}$$

Now for $\hat{\Psi}$ we first employ the ansatz

$$\hat{\Psi}(t,r,\vartheta,\varphi) = e^{-i\omega t}\,e^{-ik\varphi}\,\hat{\Phi}(r,\vartheta) \quad , \qquad \omega\in\mathbb{R}, k\in\mathbf{Z} \tag{2.17}$$

with a function $\hat{\Phi}$, which is composed of radial functions $X_\pm(r)$ and angular functions $Y_\pm(\vartheta)$ in the form

$$\hat{\Phi}(r,\vartheta) = \begin{pmatrix} X_-(r)\,Y_-(\vartheta) \\ X_+(r)\,Y_+(\vartheta) \\ X_+(r)\,Y_-(\vartheta) \\ X_-(r)\,Y_+(\vartheta) \end{pmatrix} \quad . \tag{2.18}$$

[1] We mention for clarity that this transformation of the spinors differs from that in [4] by the factor $|\Delta|^{\frac{1}{4}}$ in the definition of S. Our transformation simplifies the radial Dirac equation; moreover, it will make the form of our matching conditions easier.

By substituting (2.17) and (2.18) into the transformed Dirac equation (2.16), we obtain the eigenvalue problems

$$\mathcal{R}\,\hat{\Psi} \;=\; \lambda\,\hat{\Psi} \quad , \qquad \mathcal{A}\,\hat{\Psi} \;=\; -\lambda\,\hat{\Psi} \quad , \tag{2.19}$$

whereby the Dirac equation (2.16) decouples into the system of ODEs

$$\begin{pmatrix} \sqrt{|\Delta|}\,\mathcal{D}_+ & imr - \lambda \\ -imr - \lambda & \epsilon(\Delta)\,\sqrt{|\Delta|}\,\mathcal{D}_- \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix} \;=\; 0 \tag{2.20}$$

$$\begin{pmatrix} \mathcal{L}_+ & -am\cos\vartheta + \lambda \\ am\cos\vartheta + \lambda & -\mathcal{L}_- \end{pmatrix} \begin{pmatrix} Y_+ \\ Y_- \end{pmatrix} \;=\; 0 \quad , \tag{2.21}$$

where $\mathcal{D}_\pm$ and $\mathcal{L}_\pm$ reduce to the radial and angular operators

$$\mathcal{D}_\pm \;=\; \frac{\partial}{\partial r} \;\pm\; \frac{i}{\Delta}\left[\omega\,(r^2+a^2) \;+\; ka \;+\; eQr\right] \tag{2.22}$$

$$\mathcal{L}_\pm \;=\; \frac{\partial}{\partial\vartheta} \;+\; \frac{\cot\vartheta}{2} \;\mp\; \left[a\omega\,\sin\vartheta \;+\; \frac{k}{\sin\vartheta}\right] \quad . \tag{2.23}$$

For $a \to 0$, the Kerr-Newman metric goes over to the spherically symmetric Reissner-Nordström metric. We now describe how one can recover the Dirac operator of [13] in this limit: First of all, in [13] the Dirac (and not the Weyl) representation of the Dirac matrices is used. Furthermore, instead of working with orthonormal frames, the Dirac matrices are constructed in [13] by multiplying the Dirac matrices of Minkowski space in polar coordinates γ^t, γ^r, γ^ϑ, γ^φ with appropriate scalar functions. Because of these differences, the operator $G^{a=0}$ obtained from (2.13) in the limit $a \to 0$ coincides with the Dirac operators $G^{\text{in/out}}$ in [13] only up to a unitary transformation. More precisely, we have, in the notation of [13],

$$\begin{aligned} G^{\text{in/out}} \;&=\; V\,G^{a=0}\,V^{-1} \qquad \text{with} \\ V(\vartheta,\varphi) \;&=\; \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \frac{1}{2}\Big(1 + i(\sigma^1+\sigma^2+\sigma^3)\Big)\; \exp\!\Big(-i\,\frac{\varphi}{2}\,\sigma^1\Big)\; \exp\!\Big(-i\,\frac{\vartheta}{2}\,\sigma^2\Big) \quad . \end{aligned}$$

In the limit $a \to 0$, Chandrasekhar's separation of variables corresponds to the usual separation of the angular dependence in a spherically symmetric background. The angular operator $\mathcal{A}$ then has the explicit eigenvalues $\lambda = \pm(j+\frac{1}{2})$ with $j = \frac{1}{2}, \frac{3}{2}, \ldots$. The regularity of the eigenfunctions of the angular operator $\mathcal{A}$ for general a is established in the Appendix.

2.2 Matching of the Spinors Across the Horizons

Let us consider the Dirac wave function $\Psi = S^{-1}\hat{\Psi}$ with $\hat{\Psi}$ according to the ansatz (2.17),(2.18). Then the Dirac equation (2.14) separately describes the wave function in the three regions $r < r_0$, $r_0 < r < r_1$, and $r > r_1$; for clarity, we denote Ψ in these three regions by Ψ_I, Ψ_M, and Ψ_O, respectively. Since the ODEs (2.20),(2.21) and the transformation S^{-1} are regular for $r \not\in \{0, r_0, r_1\}$, the functions Ψ_I, Ψ_M, and Ψ_O are smooth. However, the difficulty is that the coefficients in the Dirac equation have poles at $r = r_0$ and $r = r_1$. As a consequence, the Dirac wave function will in general be singular for $r \to r_{0/1}$. Furthermore, is not clear how to treat the Dirac equation across the horizons. In this section, we shall derive *matching conditions* which relate the wave functions inside and outside each horizon. For the derivation, we shall first remove the singularities of the metric on the horizons by transforming to Kerr coordinates. In these coordinates, we can also arrange that the Dirac operator is regular. This will allow us to derive a weak solution of the Dirac equation valid across the Cauchy and event horizons. In the end, we will transform the derived conditions back to Boyer-Lindquist coordinates.

First, we must choose coordinate systems where the metric becomes regular on the horizons. One possibility is to go over to the Kerr coordinates $(u_+, r, \vartheta, \varphi_+)$ given in infinitesimal form by [3]

$$du_+ \;=\; dt \,+\, \frac{r^2+a^2}{\Delta}\, dr \quad , \qquad d\varphi_+ \;=\; d\varphi + \frac{a}{\Delta}\, dr \quad .$$

Alternatively, we can choose the coordinates $(u_-, r, \vartheta, \varphi_-)$ with

$$du_- \;=\; dt \,-\, \frac{r^2+a^2}{\Delta}\, dr \quad , \qquad d\varphi_- \;=\; d\varphi - \frac{a}{\Delta}\, dr \quad .$$

The variables u_+ and u_- are the incoming and outgoing null coordinates, respectively. Along the lines $u_\pm = \text{const}$, the variables t and r are related to each other by

$$dt \;=\; \mp\frac{r^2+a^2}{\Delta}\, dr \quad .$$

By integration, we see that, for $r \to r_0$, we have $t \to \mp\infty$. Similarly, $\lim_{r\to r_1} t = \pm\infty$. The fact that t becomes infinite in the limit $r \to r_{0/1}$ means that the Kerr coordinates describe extensions of the original Boyer-Lindquist space-time, whereby the Cauchy and event horizons have moved to points at infinity. More precisely, the Cauchy horizon corresponds to the points $(r = r_0,\ u_+ = \infty)$ and $(r = r_0,\ u_- = -\infty)$; the event horizon is at $(r = r_1,\ u_+ = -\infty)$ and $(r = r_1,\ u_- = \infty)$. In other words, the chart $(u_+, r, \vartheta, \varphi_+)$ extends the Boyer-Lindquist space-time across the points $(r = r_0,\ t = -\infty)$ and $(r = r_1,\ t = \infty)$, whereas the chart $(u_-, r, \vartheta, \varphi_-)$ gives an extension across $(r = r_0,\ t = \infty)$ and $(r = r_1,\ t = -\infty)$.

We next work out the transformation of the wave functions from Boyer-Lindquist to Kerr coordinates. We first consider the chart $(u_+, r, \vartheta, \varphi_+)$. The transformation of the Dirac equation consists of a transformation of the space-time coordinates and of the spinors. For clarity, we perform these transformations in two separate steps. Changing only the space-time coordinates transforms the Dirac matrices to

$$\begin{aligned}
G^{u_+} &= G^t\,\frac{\partial u_+}{\partial t} \,+\, G^r\,\frac{\partial u_+}{\partial r} \;=\; -\frac{a\sin\vartheta}{\sqrt{U}}\,\gamma^2 \,+\, \frac{r^2+a^2}{\sqrt{U\,|\Delta|}}\,(\gamma^0-\gamma^3) \\
G^r &= -\sqrt{\frac{|\Delta|}{U}}\,\Big(\Theta(\Delta)\,\gamma^3 \,+\, \Theta(-\Delta)\,\gamma^0\Big) \\
G^\vartheta &= -\frac{1}{\sqrt{U}}\,\gamma^1 \\
G^{\varphi_+} &= G^\varphi\,\frac{\partial \varphi_+}{\partial\varphi} \,+\, G^r\,\frac{\partial\varphi_+}{\partial r} \;=\; -\frac{1}{\sin\vartheta\,\sqrt{U}}\,\gamma^2 \,+\, \frac{a}{\sqrt{U\,|\Delta|}}\,(\gamma^0-\gamma^3) \quad ,
\end{aligned}$$

where Θ is the Heaviside function $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ otherwise. The matrices G^{u_+} and G^{φ_+} are singular on the horizons. Therefore we transform the spinors and Dirac matrices according to

$$\Psi \;\to\; \tilde\Psi \;=\; V(r)\,\Psi \quad , \qquad G^j \;\to\; \tilde G^j \;=\; V(r)\,G^j\,V(r)^{-1} \tag{2.24}$$

with

$$V(r) \;=\; \frac{1}{2}\left(|\Delta|^{-\frac{1}{4}} \;+\; |\Delta|^{\frac{1}{4}}\right) \mathbb{1} \;-\; \frac{1}{2}\left(|\Delta|^{-\frac{1}{4}} \;-\; |\Delta|^{\frac{1}{4}}\right) \gamma^0 \gamma^3 \quad . \tag{2.25}$$

The transformed Dirac matrices are

$$\begin{aligned}
\tilde{G}^{u_+} &= -\frac{a \sin\vartheta}{\sqrt{U}}\,\gamma^2 \;+\; \frac{r^2+a^2}{\sqrt{U}}\,(\gamma^0 - \gamma^3) \\
\tilde{G}^{r} &= -\frac{1}{2\sqrt{U}}\left[(1-\Delta)\,\gamma^0 \;+\; (1+\Delta)\,\gamma^3\right] \\
\tilde{G}^{\vartheta} &= -\frac{1}{\sqrt{U}}\,\gamma^1 \\
\tilde{G}^{\varphi}_{+} &= -\frac{1}{\sin\vartheta\,\sqrt{U}}\,\gamma^2 \;+\; \frac{\sqrt{a}}{U}\,(\gamma^0 - \gamma^3) \quad .
\end{aligned}$$

Now the Dirac matrices are regular except at the coordinate singularities $\vartheta = 0, \pi$ and at the origin $r = 0$. The anti-commutation relations (2.4) allow us to check immediately that the metric is indeed regular across the horizons. The transformed Dirac operator $\tilde{G}$ can be constructed from the Dirac matrices $\tilde{G}^j$ with the explicit formulas (2.11) and (2.3) (these formulas are valid in the same way with an additional tilde, because the matrices $\tilde{G}^j$ are again linear combinations of the Dirac matrices γ^j of Minkowski space). From this, we see that all the coefficients of the Dirac operator $\tilde{G}$ are regular across the horizons. According to the transformation (2.24) of the wave functions, the Dirac operators G and $\tilde{G}$ are related to each other by

$$\tilde{G} \;=\; V\,G\,V^{-1} \quad .$$

Since the operator $\tilde{G}$ is regular across the horizons, we can now study the Dirac equation on the event and Cauchy horizons. We denote our original wave functions transformed to Kerr coordinates by $\tilde{\Psi}_I$, $\tilde{\Psi}_M$, and $\tilde{\Psi}_O$. They are smooth in the regions $r < r_0$, $r_0 < r < r_1$, and $r > r_1$ and satisfy the Dirac equation there. However, they may have singularities at $r = r_0$ and $r = r_1$. Let us assume that $\tilde{\Psi} := \tilde{\Psi}_I + \tilde{\Psi}_M + \tilde{\Psi}_O$ is a generalized solution of the Dirac equation across the horizons. In order to analyze the behavior of the wave function near the Cauchy horizon, we write $\tilde{\Psi}$ in a neighborhood of $r = r_0$ in the form

$$\tilde{\Psi}(u_+, r, \vartheta, \varphi_+) \;=\; \Theta(r_0 - r)\;\tilde{\Psi}_I(u_+, r, \vartheta, \varphi_+) \;+\; \Theta(r - r_0)\;\tilde{\Psi}_M(u_+, r, \vartheta, \varphi_+)$$

and substitute into the Dirac equation $(\tilde{G} - m)\tilde{\Psi} = 0$. Since $\tilde{\Psi}$ is a solution of the Dirac equation for $r \neq r_0$, , we only get a contribution from the derivative of the Heaviside function, i.e. in a formal calculation

$$\begin{aligned}
0 &= i\tilde{G}^r\;\delta(r - r_0)\;(\tilde{\Psi}_M(u_+, r, \vartheta, \varphi_+) - \tilde{\Psi}_I(u_+, r, \vartheta, \varphi_+)) \\
&= -\frac{i}{2\sqrt{U}}\;\delta(r - r_0)\;(\gamma^0 + \gamma^3)\left(\tilde{\Psi}_M(u_+, r, \vartheta, \varphi_+) \;-\; \tilde{\Psi}_I(u_+, r, \vartheta, \varphi_+)\right) \quad .
\end{aligned}$$

To give this distributional equation a precise meaning, we multiply the above formal identity by a test function $\eta(r)$ and integrate,

$$0 \;=\; \int_{r_0-\varepsilon}^{r_0+\varepsilon} \eta(r)\;\delta(r - r_0)\;(\gamma^0 + \gamma^3)\left(\tilde{\Psi}_M(u_+, r, \vartheta, \varphi_+) \;-\; \tilde{\Psi}_I(u_+, r, \vartheta, \varphi_+)\right) \quad . \tag{2.26}$$

By choosing a function $\eta(r)$ which goes to zero sufficiently fast for $r \to r_0$, we can make sense of this integral, even if $(\gamma^0 + \gamma^3)(\tilde{\Psi}_M - \tilde{\Psi}_I)$ is singular in this limit. For example, we can choose η as $\eta = (1 + |(\gamma^0+\gamma^3)(\tilde{\Psi}_B - \tilde{\Psi}_I)|)^{-1} h$, where h is a smooth function. It must be kept in mind, however, that we cannot choose η independently in the two regions $r < r_0$ and $r > r_0$, because η must be smooth at $r = r_0$. As a consequence, we cannot conclude from (2.26) that $\tilde{\Psi}_M$ and $\tilde{\Psi}_I$ must both vanish on the Cauchy horizon. We only get the weaker condition that they have a similar behavior near this horizon; namely, it is necessary that the following "jump condition" holds:

$$\begin{aligned}
&(\gamma^0+\gamma^3)\,(\tilde{\Psi}_M(u_+, r_0+\varepsilon, \vartheta, \varphi_+) \;-\; \tilde{\Psi}_I(u_+, r_0-\varepsilon, \vartheta, \varphi_+) \\
&\quad = \;\; o(1 + |(\gamma^0+\gamma^3)\,\tilde{\Psi}_M(u_+, r_0+\varepsilon, \vartheta, \varphi_+)|) \qquad \text{as } 0<\varepsilon\to 0. \qquad (2.27)
\end{aligned}$$

On the event horizon, we obtain in the same way the condition

$$\begin{aligned}
&(\gamma^0+\gamma^3)\,(\tilde{\Psi}_O(u_+, r_1+\varepsilon, \vartheta, \varphi_+) \;-\; \tilde{\Psi}_M(u_+, r_1-\varepsilon, \vartheta, \varphi_+) \\
&\quad = \;\; o(1 + |(\gamma^0+\gamma^3)\,\tilde{\Psi}_M(u_+, r_1-\varepsilon, \vartheta, \varphi_+)|) \qquad \text{as } 0<\varepsilon\to 0. \qquad (2.28)
\end{aligned}$$

The constructions we just carried out in the chart $(u_+, r, \vartheta, \varphi_+)$ can be repeated similarly in the coordinates $(u_-, r, \vartheta, \varphi_-)$. We list the resulting formulas: The transformation to the chart $(u_-, r, \vartheta, \varphi_-)$ gives for the Dirac matrices G^{u_-} and G^{φ_-}

$$\begin{aligned}
G^{u_-} &= -\frac{a\sin\vartheta}{\sqrt{U}}\,\gamma^2 \;+\; \frac{r^2+a^2}{\sqrt{U}\,|\Delta|}\,\epsilon(\Delta)\,(\gamma^0+\gamma^3) \\
G^{\varphi_-} &= -\frac{1}{\sin\vartheta\,\sqrt{U}}\,\gamma^2 \;+\; \frac{a}{\sqrt{U}\,|\Delta|}\,\epsilon(\Delta)\,(\gamma^0+\gamma^3) \quad .
\end{aligned}$$

To make these matrices regular, we transform the spinors according to (2.24) with

$$V(r) \;=\; \frac{1}{2}\left(|\Delta|^{-\frac{1}{4}} \,+\, \epsilon(\Delta)\,|\Delta|^{\frac{1}{4}}\right) \mathbb{1} \;+\; \frac{1}{2}\left(|\Delta|^{-\frac{1}{4}} \,-\, \epsilon(\Delta)\,|\Delta|^{\frac{1}{4}}\right) \gamma^0\gamma^3 \quad .$$

This gives for the transformed Dirac matrices

$$\begin{aligned}
\tilde{G}^{u_-} &= -\frac{a\sin\vartheta}{\sqrt{U}}\,\gamma^2 \;+\; \frac{r^2+a^2}{\sqrt{U}}\,(\gamma^0+\gamma^3) \\
\tilde{G}^{r} &= -\frac{1}{2\sqrt{U}}\,\Big[(1-\Delta)\,\gamma^0 \;-\; (1+\Delta)\,\gamma^3\Big] \\
\tilde{G}^{\vartheta} &= -\frac{1}{\sqrt{U}}\,\gamma^1 \\
\tilde{G}^{\varphi}_{-} &= -\frac{1}{\sin\vartheta\,\sqrt{U}}\,\gamma^2 \;+\; \frac{a}{\sqrt{U}}\,(\gamma^0+\gamma^3) \quad .
\end{aligned}$$

By evaluating the Dirac equation across the horizons in the weak sense, we obtain the conditions

$$\begin{aligned}
&(\gamma^0-\gamma^3)\,(\tilde{\Psi}_M(u_-, r_0+\varepsilon, \vartheta, \varphi_-) \;-\; \tilde{\Psi}_I(u_-, r_0-\varepsilon, \vartheta, \varphi_-) \\
&\quad = \; o(1 + |(\gamma^0-\gamma^3)\tilde{\Psi}_M(u_-, r_0+\varepsilon, \vartheta, \varphi_-)|) \qquad \text{as } \varepsilon\to 0. \qquad (2.29) \\
&(\gamma^0-\gamma^3)\,(\tilde{\Psi}_O(u_-, r_1+\varepsilon, \vartheta, \varphi_-) \;-\; \tilde{\Psi}_M(u_-, r_1-\varepsilon, \vartheta, \varphi_-) \\
&\quad = \; o(1 + |(\gamma^0-\gamma^3)\tilde{\Psi}_M(u_-, r_1-\varepsilon, \vartheta, \varphi_-)|) \qquad \text{as } \varepsilon\to 0. \qquad (2.30)
\end{aligned}$$

It remains to transform the conditions (2.27)–(2.30) back to Boyer-Lindquist coordinates. Since the t- and φ-dependence of our wave function (2.17) has the form of a plane wave, we immediately conclude that the condition (2.27) is also valid in Boyer-Lindquist coordinates. We do not again write out that we consider the limit $0 < \varepsilon \to 0$, but note that this condition was obtained by extending the Boyer-Lindquist space-time across the point $(r = r_0, t = -\infty)$,

$$\begin{aligned} &(\gamma^0 + \gamma^3)\, (\tilde{\Psi}_M(t, r_0 + \varepsilon, \vartheta, \varphi) \;-\; \tilde{\Psi}_I(t, r_0 - \varepsilon, \vartheta, \varphi) \\ &\quad = \; o(1 + |(\gamma^0 + \gamma^3)\, \tilde{\Psi}_M(t, r_0 + \varepsilon, \vartheta, \varphi)|) \qquad \text{across } t = -\infty. \end{aligned}$$

We next substitute the transformation (2.24) and (2.25) of the spinors. Using the identity

$$\begin{aligned} (\gamma^0 + \gamma^3)\, V^{-1} &= (\gamma^0 + \gamma^3)\, \frac{1}{2} \Big((|\Delta|^{-\frac{1}{4}} + |\Delta|^{\frac{1}{4}}) \;+\; (|\Delta|^{-\frac{1}{4}} - |\Delta|^{\frac{1}{4}})\, \gamma^0 \gamma^3 \Big) \\ &= |\Delta|^{-\frac{1}{4}}\, (\gamma^0 + \gamma^3) \quad , \end{aligned}$$

we obtain the condition

$$\begin{aligned} &|\Delta|^{-\frac{1}{4}}\, (\gamma^0 + \gamma^3)\, (\Psi_M(t, r_0 + \varepsilon, \vartheta, \varphi) \;-\; \Psi_I(t, r_0 - \varepsilon, \vartheta, \varphi) \\ &\quad = \; o(1 + |\Delta|^{-\frac{1}{4}}\, |(\gamma^0 + \gamma^3)\, \Psi_M(t, r_0 + \varepsilon, \vartheta, \varphi)|) \qquad \text{across } t = -\infty. \end{aligned} \tag{2.31}$$

Finally, we consider the transformation (2.15). It also preserves the factor $(\gamma^0 + \gamma^3)$, because

$$\begin{aligned} (\gamma^0 + \gamma^3)\, S \;=\; &\mathrm{diag}\Big((r + ia\, \cos\vartheta)^{\frac{1}{2}},\, (r + ia\, \cos\vartheta)^{\frac{1}{2}},\, (r - ia\, \cos\vartheta)^{\frac{1}{2}},\, (r - ia\, \cos\vartheta)^{\frac{1}{2}} \Big) \\ &\times |\Delta|^{\frac{1}{4}}\, (\gamma^0 + \gamma^3) \quad . \end{aligned} \tag{2.32}$$

The diagonal matrix in this equation is irrelevant because it is regular on the horizons. The factors $|\Delta|^{-\frac{1}{4}}$ in (2.31) are compensated by the factor $|\Delta|^{\frac{1}{4}}$ in (2.32), and we end up with the simple condition

$$\begin{aligned} &(\gamma^0 + \gamma^3)\, (\hat{\Psi}_M(t, r_0 + \varepsilon, \vartheta, \varphi) \;-\; \hat{\Psi}_I(t, r_0 - \varepsilon, \vartheta, \varphi) \\ &\quad = \; o(1 + |(\gamma^0 + \gamma^3)\, \hat{\Psi}_M(t, r_0 + \varepsilon, \vartheta, \varphi)|) \qquad \text{across } t = -\infty. \end{aligned} \tag{2.33}$$

Similarly, the relations (2.28)–(2.30) transform into

$$\begin{aligned} &(\gamma^0 + \gamma^3)\, (\hat{\Psi}_O(t, r_1 + \varepsilon, \vartheta, \varphi) \;-\; \hat{\Psi}_M(t, r_1 - \varepsilon, \vartheta, \varphi) \\ &\quad = \; o(1 + |(\gamma^0 + \gamma^3)\, \hat{\Psi}_M(t, r_1 - \varepsilon, \vartheta, t)|) \qquad \text{across } t = \infty \end{aligned} \tag{2.34}$$

$$\begin{aligned} &(\gamma^0 - \gamma^3)\, (\hat{\Psi}_M(t, r_0 + \varepsilon, \vartheta, \varphi) \;-\; \hat{\Psi}_I(t, r_0 - \varepsilon, \vartheta, \varphi) \\ &\quad = \; o(1 + |(\gamma^0 - \gamma^3) \hat{\Psi}_M(t, r_0 + \varepsilon, \vartheta, \varphi)|) \qquad \text{across } t = \infty \end{aligned} \tag{2.35}$$

$$\begin{aligned} &(\gamma^0 - \gamma^3)\, (\hat{\Psi}_O(t, r_1 + \varepsilon, \vartheta, \varphi) \;-\; \hat{\Psi}_M(t, r_1 - \varepsilon, \vartheta, \varphi) \\ &\quad = \; o(1 + |(\gamma^0 - \gamma^3) \hat{\Psi}_M(t, r_1 - \varepsilon, \vartheta, \varphi)|) \qquad \text{across } t = -\infty. \end{aligned} \tag{2.36}$$

The equations (2.33)–(2.36) are our matching conditions.

2.3 Non-Existence of Time-Periodic Solutions

Before giving the proof of Theorem 1.1, we must specify our assumptions on space-time and on the Dirac wave function in mathematical terms. By patching together the Kerr coordinate charts, one obtains the maximally extended Kerr-Newman space-time [3]. We

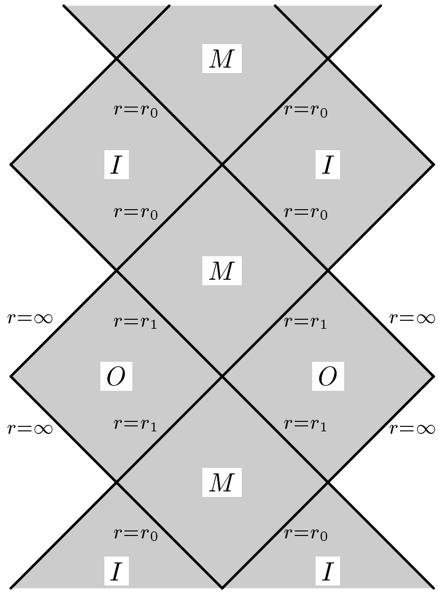

Figure 1: The conformal structure of the non-extreme Kerr-Newman solution

do not need the details of the maximal extension here; it suffices to discuss the Penrose diagram of Figure 1. Abstractly speaking, the time and axial symmetry of the Kerr-Newman solution means that the metric admits two Killing fields. In order to better visualize these symmetries, one can isometrically map the regions of type I, M, and O into the regions $r < r_0$, $r_0 < r < r_1$, and $r > r_1$ of the Boyer-Lindquist coordinates, respectively; then the Killing fields are simply the vector fields ∂_t and ∂_φ. The mappings to Boyer-Lindquist coordinates are unique up to the isometries of the Boyer-Lindquist space-time (i.e. rotations around the symmetry axis, time translations, parity transformations, and, in the case $a = 0$, time reversals). The Kerr coordinates, on the other hand, allow us to describe three adjacent regions of type I, M, and O including the boundaries between them. When we speak of Boyer-Lindquist or Kerr coordinates in the following, we implicitly mean that adjacent regions of the maximal extension are isometrically mapped to these coordinate charts. The maximal extension may be too general for a truly physical situation (if one thinks of a black hole which evolved from a gravitational collapse in the universe, for example, one may want to consider only one asymptotically flat region). Therefore we assume that our *physical space-time* is a given subset of the maximal extension. We call each region of type O which belongs to the physical space-time an *asymptotic end*. We assume that, in each asymptotic end, a *time direction* is given. Thus we can say that each asymptotic end is connected to two regions of type M, one in the future and one in the past. We assume that the physical wave function can be extended to the maximal Kerr-Newman space-time; in the regions which do not belong to the physical space-time, this extension Ψ shall be identically zero. We want to consider a *black hole*, i.e. the situation where particles can disappear into the event horizon, but where no matter can emerge from the interior of a horizon. Therefore we assume that Ψ is set identically zero in all regions of type M which are connected to the asymptotic ends in the past. The remaining assumptions can be stated most easily in Boyer-Lindquist coordinates. Since the phase of a wave function Ψ has no physical significance, when we say that Ψ is *time-periodic with* *period* T we mean that there is a real parameter Ω such that[2]

$$\Psi(t+T, r, \vartheta, \varphi) \;=\; e^{-i\Omega T}\, \Psi(t, r, \vartheta, \varphi) \quad .$$

For time-periodic wave functions, we can separate out the time dependence in a discrete Fourier series. More precisely, we can write the wave function as a superposition of the form

$$\Psi(t, r, \vartheta, \varphi) \;=\; e^{-i\Omega t} \sum_{n,k \in \mathbf{Z}} \; \sum_{\lambda \in \sigma_k^n(\mathcal{A})} e^{-2\pi i n \frac{t}{T}}\, e^{-ik\varphi}\, \Phi^{\lambda n k} \tag{2.37}$$

with

$$\Phi^{\lambda nk}(r, \vartheta) \;=\; S^{-1}(r, \vartheta)\, \hat{\Phi}^{\lambda nk}(r, \vartheta) \quad , \qquad \hat{\Phi}^{\lambda nk}(r, \vartheta) \;=\; \begin{pmatrix} X_-^{\lambda nk}\, Y_-^{\lambda nk} \\ X_+^{\lambda nk}\, Y_+^{\lambda nk} \\ X_+^{\lambda nk}\, Y_-^{\lambda nk} \\ X_-^{\lambda nk}\, Y_+^{\lambda nk} \end{pmatrix} \quad ,$$

where the radial and angular functions $X^{\lambda nk}(r)$ and $Y^{\lambda nk}(\vartheta)$ satisfy the equations (2.20) and (2.21) with

$$\omega(n) \;=\; \Omega + \frac{2\pi}{T}\, n \quad .$$

The index λ in (2.37) labels the eigenvalue of the operator $\mathcal{A}$ in (2.19); the set $\sigma_k^n(\mathcal{A})$ denotes (for fixed n and k) all the possible values of λ. As shown in the Appendix, the set $\sigma_k^n(\mathcal{A})$ is discrete. Finally, we specify our *normalization condition*: The Dirac wave functions are endowed with a positive scalar product $(.\,|\,.)$. For this, one chooses a space-like hypersurface $\mathcal{H}$ together with a normal vector field ν and considers for two wave functions Ψ and Φ the integral

$$(\Psi \mid \Phi)_{\mathcal{H}} \;:=\; \int_{\mathcal{H}} \overline{\Psi} G^j \Phi\, \nu_j\, d\mu \quad , \tag{2.38}$$

where $\overline{\Psi} = \Psi^* \gamma^0$ is the adjoint spinor, and where $d\mu = \sqrt{g}\, d^3x$ is the invariant measure on $\mathcal{H}$ (g now denotes the determinant of the induced Riemannian metric). In a regular space-time, current conservation $\nabla_j \overline{\Psi} G^j \Phi = 0$ implies that the scalar product (2.38) is independent of the choice of the hypersurface. The integrand of $(\Psi|\Psi)_{\mathcal{H}}$ has the physical interpretation as the probability density of the Dirac particle. Therefore one usually normalizes the wave functions in such a way that $(\Psi \mid \Psi) = 1$. In our context, the singularities of the metric make the situation more difficult. Indeed, the normalization integral for a hypersurface crossing a horizon is problematic since such a hypersurface will necessarily fail to be space-like on a set of positive measure. We will therefore only consider the normalization integral in each asymptotic end away from the event horizon. More precisely, we choose for given $r_2 > r_1$ the one-parameter family of hypersurfaces

$$\mathcal{H}_{t_2} \;=\; \{(t, r, \vartheta, \varphi) \text{ with } t = t_2,\; r > r_2\} \quad .$$

For a normalized solution Ψ of the Dirac equation, the integral $(\Psi \mid \Psi)_{\mathcal{H}_{t_2}}$ gives the probability of the particle to be at time t_2 in the region outside the ball of the radius r_2 around the origin; this probability must clearly be smaller than one. Therefore we impose for a normalizable solution the condition that, in each asymptotic end,

$$(\Psi \mid \Psi)_{\mathcal{H}_{t_2}} \;<\; \infty \qquad \text{for all } t_2. \tag{2.39}$$

[2] We remark that the condition of time-periodicity inside the horizon can be weakened to local uniform boundedness in t. This is proved by an "averaging argument" identical to the one given in [13, Appendix A].

We now begin the non-existence proof by analyzing the wave function in each asymptotic end in Boyer-Lindquist coordinates. The following positivity argument shows that each component of the Dirac wave function must be normalizable: We average the normalization condition (2.39) one period and use the infinite series (2.37) to obtain

$$\begin{aligned} \infty &> \frac{1}{T}\int_t^{t+T} d\tau\; (\Psi \mid \Psi)_{\mathcal{H}_\tau} \\ &= \sum_{n,n'}\sum_{k,k'}\sum_{\lambda,\lambda'} \frac{1}{T}\int_t^{t+T} d\tau\; e^{-2\pi i (n'-n)\,\frac{\tau}{T}} \\ &\qquad\qquad \times \int_{\mathcal{H}_\tau} e^{-i(k'-k)\varphi}\; \overline{\Phi^{\lambda n k}(r,\vartheta)}\; \Phi^{\lambda' n' k'}(r,\vartheta)\; d\mu_{\mathcal{H}} \quad . \end{aligned} \tag{2.40}$$

Since the plane waves in this formula are integrated over a whole period, we only get a contribution if $k=k'$ and $n=n'$. As is shown in the Appendix, the ϑ-integration gives zero unless $\lambda=\lambda'$, and thus the right hand-side of (2.40) reduces to

$$\sum_{n,k\in\mathbf{Z}}\;\sum_{\lambda\in\sigma_k^n(\mathcal{A})} \int_{\mathcal{H}_{t,r_2}} \overline{\Phi^{\lambda n k}(r,\vartheta)}\; \Phi^{\lambda n k}(r,\vartheta)\; d\mu_{\mathcal{H}} \quad . \tag{2.41}$$

Since the scalar product in (2.41) is positive, we conclude that the normalization integral must be finite for each $\Phi^{\lambda n k}$,

$$(\Phi^{\lambda n k} \mid \Phi^{\lambda n k})_{\mathcal{H}_t} \;<\; \infty \qquad \text{for all } n,k\in\mathbf{Z},\ \lambda\in\sigma_k^n(\mathcal{A}),\ t\in\mathrm{I\!R}.$$

In this way, we have reduced our problem to the analysis of static solutions of the Dirac equation. This simplification is especially useful because it enables us to work with the matching conditions of the previous section. In the following, we again use the notation (2.17),(2.18).

Lemma 2.1 *The function $|X|^2$ has finite boundary values on the event horizon. If it is zero at $r=r_1$, then X vanishes identically for $r>r_1$.*

Proof: The radial Dirac equation (2.20) gives for $r>r_1$

$$\begin{aligned} \sqrt{|\Delta|}\,\frac{d}{dr}|X|^2 &= \;<\sqrt{|\Delta|}\,\frac{d}{dr}X,\; X> \;+\; <X,\; \sqrt{|\Delta|}\,\frac{d}{dr}X> \\ &= 2\lambda\,\mathrm{Re}(\overline{X^+}X^-) \;+\; 2mr\,\mathrm{Im}(\overline{X^+}X^-) \end{aligned} \tag{2.42}$$

and thus

$$-(|\lambda|+mr)\,|X|^2 \;\leq\; \sqrt{|\Delta|}\,\frac{d}{dr}|X|^2 \;\leq\; (|\lambda|+mr)\,|X|^2 \quad . \tag{2.43}$$

In the case that $|X|^2(r)$ has a zero for $r>r_1$, the uniqueness theorem for solutions of ODEs yields that X vanishes identically. In the opposite case $|X|^2(r)>0$ for all $r>r_1$, we divide (2.43) by $\sqrt{|\Delta|}\,|X|^2$ to get and integrate,

$$-\int_r^{r'} (|\lambda|+mr)\,|\Delta|^{-\frac{1}{2}} \;\leq\; \log|X|^2\Big|_r^{r'} \;\leq\; \int_r^{r'} (|\lambda|+mr)\,|\Delta|^{-\frac{1}{2}} \quad , \qquad r_1<r<r'.$$

Using that the singularity of $|\Delta|^{-\frac{1}{2}}$ at $r=r_1$ is integrable, we conclude that $\log|X|^2$ has a finite limit at $r=r_1$. ■

Combining this lemma with our normalization and matching conditions, we can show now that the Dirac wave function is identically zero outside the event horizon:

Lemma 2.2 *$\hat{\Psi}_O$ vanishes identically in each asymptotic end.*

Proof: According to our black hole assumption, we can apply the matching condition (2.36) with $\hat{\Psi}_M \equiv 0$. Expressed in the radial functions X, this implies that

$$\lim_{r_1 < r \to r_1} X^-(r) \;=\; 0 \quad . \tag{2.44}$$

Using asymptotic flatness and the transformation

$$\overline{\Psi}\gamma^0\Psi \;=\; |\Delta|^{-\frac{1}{2}}\, U^{-\frac{1}{2}}\, \overline{\hat{\Psi}}\gamma^0\hat{\Psi} \quad ,$$

the normalization condition (2.39) is equivalent to the integral condition

$$\int_{r_2}^{\infty} |X|^2\, dr \;<\; \infty \qquad \text{for } r_2 > r_1. \tag{2.45}$$

As an immediate consequence of the radial Dirac equation (2.20),

$$\frac{d}{dr}\left(|X^+|^2 - |X^-|^2\right) \;=\; 0 \quad . \tag{2.46}$$

Thus the function $|X^+|^2 - |X^-|^2$ is a constant; the normalization condition (2.46) implies that this constant must be zero,

$$|X^+|^2 - |X^-|^2 \;\equiv\; 0 \quad .$$

Together with the condition (2.44), we obtain that $\lim_{r_1<r\to r_1} |X|^2 = 0$. Lemma 2.1 yields that X, and consequently $\hat{\Psi}_O$, must vanish identically. ■

We remark that equation (2.46) can be interpreted physically as the conservation of the Dirac current in radial direction (notice that $\overline{\Psi}G^r\Psi = U^{-\frac{1}{2}}\,(|X^+|^2 - |X^-|^2)\,|Y|^2$).

It remains to show that the wave function also vanishes in the interior of the horizons. For this, we use the matching conditions and estimates similar to those in Lemma 2.1.

Proof of Theorem 1.1: According to Lemma 2.2, Ψ is identically zero in all regions of type O. We first consider a region of type M in Boyer-Lindquist coordinates. Crossing the event horizon in the past and in the future brings us to regions of type O where $\hat{\Psi}$ vanishes. Thus the matching conditions (2.34) and (2.36) yield that

$$\lim_{r_1>r\to r_1} (\gamma^0+\gamma^3)\,\hat{\Psi}_M(t,r,\vartheta,\varphi) \;=\; 0 \;=\; \lim_{r_1>r\to r_1} (\gamma^0-\gamma^3)\,\hat{\Psi}_M(t,r,\vartheta,\varphi) \quad ,$$

and thus

$$\lim_{r_1>r\to r_1} \hat{\Psi}_M(t,r,\vartheta,\varphi) \;=\; 0 \quad . \tag{2.47}$$

The radial Dirac equation (2.20) in the region M implies that

$$\sqrt{|\Delta|}\,\frac{d}{dr}|X|^2 \;=\; 0$$

(this again corresponds to the conservation of the radial Dirac flux). Thus $|X|^2(r)$ can only go to zero for $r_1 > r \to r_1$ if it is identically zero. We conclude that Ψ must be identically zero in all regions of type M.

Finally, we consider a region of type I in Boyer-Lindquist coordinates. We can cross the Cauchy horizon at $t = \infty$ or $t = -\infty$; this brings us to regions of type M, where $\hat{\Psi}_M \equiv 0$. Thus the matching conditions (2.33) and (2.35) imply that

$$\lim_{r_0 < r \to r_0} \hat{\Psi}_I(t, r, \vartheta, \varphi) \;=\; 0 \quad . \tag{2.48}$$

The radial Dirac equation (2.20) inside the Cauchy horizon is the same as in the asymptotic end. Thus the estimate (2.43) again holds, and we conclude from (2.48) that $\hat{\Psi}_I$ vanishes identically. ■

3 The General Separable Case

Our goal in the present section is to prove an analogue of Theorem 1.1 for the most general stationary axisymmetric and orthogonally transitive metric in which the Dirac equation can be separated into ordinary differential equations by Chandrasekhar's procedure[3]. An expression for this metric was determined in [14]. When the Einstein-Maxwell equations are imposed, this metric gives rise to all the generalizations of the Kerr-Newman solution discovered by Carter [1], as well as to a family of exact solutions for which the orbits of the two-parameter Abelian isometry group are null surfaces [9].

Our aim will be to state as many of the hypotheses of our theorem as possible in purely geometric terms. We will thus begin by giving a geometric characterization of the metric which constitutes the starting point of [9]. We will limit ourselves to the case when the orbits of the isometry group are time-like 2-surfaces, since the procedure is quite similar in the case of space-like or null orbits. We wish to point out that the assumption of orthogonal transitivity is a very natural one to make in a general relativistic context. Indeed, it is a classical theorem of Carter and Papapetrou [2] that if the energy-momentum tensor satisfies some mild invariance conditions, then every stationary axisymmetric solution of the Einstein equations has the property of admitting through every point a 2-surface which is orthogonal to the orbit of the isometry group through that point, i.e. the metric is orthogonally transitive.

Let us first recall from [8] that any four-dimensional metric of Lorentzian signature admitting a two-parameter Abelian group of isometries acting orthogonally transitively on time-like orbits admits local coordinates (u, v, w, x) in which

$$ds^2 \;=\; T^{-2} \left[(L\,du \;+\; M\,dv)^2 \;-\; (N\,du \;+\; P\,dv)^2 \;-\; \frac{dw^2}{W} \;-\; \frac{dx^2}{X} \right] \;, \tag{3.1}$$

where the metric coefficients L, M, N, P and the conformal factor T^{-1} are functions of x and w only, and where $X = X(x)$, $W = W(w)$. Furthermore, we have $LP - MN \neq 0$, $T > 0$, $W > 0$, $X > 0$ in order for the metric to have the required Lorentzian signature. It is manifest that the action on the isometry group generated by the Killing vectors $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ is orthogonally transitive since the orbits, which are given by the time-like

[3]We should point out that there exist metrics admitting only one Killing vector, in which the Dirac equation is completely separable into ordinary differential equations [11]. These metrics are not covered by Chandrasekhar's approach since the symmetry operators underlying this different type of separability are necessarily of order 2 or higher. In contrast, Chandrasekhar's method always gives rise to symmetry operators of order 1 since it fits into Miller's theory of factorizable separable systems [15].

2-surfaces $x = c_1$, $w = c_2$, admit the orthogonal 2-surfaces given by $u = c'_1$, $v = c'_2$. Carter [2] proved furthermore that the four-dimensional Lorentzian metrics admitting an Abelian isometry group acting orthogonally transitively on non-null orbits have the following remarkable property: there exists in the isotropy subgroup at every point an element of order two whose differential $\mathcal{L}$ maps every vector X in the tangent space to the orbit through that point to its opposite, and maps every vector Y in the normal space to the orbit to itself. In the local coordinates (u, v, w, x), this involutive isometry is given by mapping $(u, v, w, x) \to (-u, -v, w, x)$. Recall now that a Newman-Penrose null frame for the metric (3.1) is said to be *symmetric* [8] if under the involution $\mathcal{L}$, we have

$$\mathcal{L}(l) = -n \ ,; \ \mathcal{L}(n) = -l \ , \ \mathcal{L}(m) = -\overline{m} \ , \ \mathcal{L}(\overline{m}) = -m \ . \tag{3.2}$$

A symmetric null frame $(l, n, m, \overline{m})$ for the metric (3.1) is given by

$$\begin{aligned}
l &= \frac{T}{\sqrt{2}}\left(\frac{1}{LP-MN}\left(P\frac{\partial}{\partial u} - N\frac{\partial}{\partial v}\right) + \sqrt{W}\frac{\partial}{\partial w}\right)\\
n &= \frac{T}{\sqrt{2}}\left(\frac{1}{LP-MN}\left(P\frac{\partial}{\partial u} - N\frac{\partial}{\partial v}\right) - \sqrt{W}\frac{\partial}{\partial w}\right)\\
m &= \frac{T}{\sqrt{2}}\left(\frac{1}{LP-MN}\left(-M\frac{\partial}{\partial u} + L\frac{\partial}{\partial v}\right) - i\sqrt{X}\frac{\partial}{\partial x}\right)\\
\overline{m} &= \frac{T}{\sqrt{2}}\left(\frac{1}{LP-MN}\left(-M\frac{\partial}{\partial u} + L\frac{\partial}{\partial v}\right) + i\sqrt{X}\frac{\partial}{\partial x}\right) \quad ,
\end{aligned}$$

Besides the metric (3.1), we will also consider the electromagnetic vector potential given by

$$\mathcal{A} = A_j dx^j = T\,(H\,(L\,du \,+\, M\,dv) + \,K\,(N\,du \,+\, P\,dv)) \quad , \tag{3.3}$$

where $H = H(w)$ and $K = K(x)$. The 1-form (3.3) is manifestly invariant under the action of the isometry group of the metric (3.1). Note that the Maxwell field 2-form $F = d\mathcal{A}$ is skew-invariant under the involution $\mathcal{L}$, so that the electromagnetic energy-momentum tensor is invariant under $\mathcal{L}$, in accordance with the hypotheses of the Carter-Papapetrou theorem [2].

It was shown in [14] that a necessary condition for the Dirac equation $(G - m)\,\Psi = 0$ to be separable in the above symmetric null frame for the metric (3.1) and in the electromagnetic vector potential (3.3) is that the metric functions L, M, N and P satisfy the constraints

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{L}{LP-MN}\right) &= 0 \ , \quad \frac{\partial}{\partial x}\left(\frac{M}{LP-MN}\right) = 0 \quad ,\\
\frac{\partial}{\partial w}\left(\frac{N}{LP-MN}\right) &= 0 \ , \quad \frac{\partial}{\partial w}\left(\frac{P}{LP-MN}\right) = 0 \quad .
\end{aligned}$$

In geometric terms, these conformally invariant conditions are equivalent to the requirement that the flows of the null vectors fields l and n of the symmetric frame be *geodesic* and *shear-free*,

$$\begin{aligned}
l^j\nabla_j l^k &= 0 \ , \quad 2(\nabla_j l_k + \nabla_k l_j) - (\nabla^p l_p) g_{jk} = 0 \quad ,\\
n^j\nabla_j n^k &= 0 \ , \quad 2(\nabla_j n_k + \nabla_k n_j) - (\nabla^p n_p) g_{jk} = 0 \quad .
\end{aligned}$$

It is noteworthy that these conditions are necessary and sufficient for the Stäckel separability in the metric (3.1) of the Hamilton-Jacobi equation for the null geodesic flow,

$$g^{jk}\frac{\partial S}{\partial x^j}\frac{\partial S}{\partial x^k} = 0 \quad .$$

By implementing these separability conditions and doing some relabeling of the metric functions and the coordinates we see (see [9, Section 3]) that the metric can be put in the form

$$ds^2 \;=\; T^{-2}\left[\frac{W}{Z}\,(e_1\,du\;+\;m\,dv)^2\;-\;\frac{Z}{W}\,dw^2\;-\;\frac{X}{Z}\,(e_2\,du\;+\;p\,dv)^2\;-\;\frac{Z}{X}\,dx^2\right]\quad, \tag{3.4}$$

where e_1 and e_2 are constants and where

$$m \;=\; m(x)\quad,\quad p\;=\;p(w)\quad,\quad Z\;=\;e_1 p - e_2 m\quad .$$

The next necessary condition for the Dirac equation to be solvable by separation of variables in the symmetric frame for the metric (3.1) and the electromagnetic potential (3.3) is is that the algebraic structure of the Weyl conformal curvature tensor be of *type D* in the Petrov-Penrose classification [14]. This conformally invariant condition is necessary and sufficient for the separability of the massless Dirac or Weyl neutrino equation $G\Psi = 0$ (see [14, Theorem 2]). The type D condition for the metric (3.4) can be expressed by the condition that the 1-form ω given by

$$\omega \;:=\; (4Z)^{-1}\,(e_1 m'(x)\,dw\;+\;e_2\,p'(w)\,dx)$$

be closed,

$$d\omega = 0. \tag{3.5}$$

We therefore have locally

$$\omega \;=\; d\mathcal{B}$$

for some real-valued function $\mathcal{B}(w,x)$.

Finally, it was shown in [14, Theorem 3] that the massive Dirac equation $(G-m)\,\Psi = 0$ is separable in the symmetric frame for the metric (3.4) if in addition to the conditions stated above, there exist real-valued functions $g(x)$ and $h(w)$ such that

$$Z^{1/2}\,T^{-1}\,\exp(2i\mathcal{B}) \;=\; h(w) + ig(x)\quad . \tag{3.6}$$

This is the only separability condition for the Dirac equation which is not conformally invariant.

We begin by analyzing the type D condition 3.5. It was proved in [9, Section 4] that,if this condition is satisfied, then a set of coordinates (u,v,w,x) for the metric 3.4 can be chosen in such a way that the metric functions $p(w), m(x)$, the constants e_1, e_2 and the function $\mathcal{B}$ take one of the following four forms:

Case A:

$$\begin{aligned} e_1 \;&=\; 1 \;=\; e_2\;,\quad p(w)\;=\;w^2\;,\quad m(x)\;=\;-x^2\;,\\ \mathcal{B}\;&=\;\frac{i}{4}\,\log\left(\frac{w-ix}{w+ix}\right)\quad . \end{aligned}$$

Case B_-:

$$\begin{aligned} e_1 \;&=\; 0\;,\quad e_2\;=\;1\;,\quad p(w)\;=\;2kw\;,\quad m(x)\;=\;-x^2-k^2\;,\\ \mathcal{B}\;&=\;\frac{i}{4}\,\log\left(\frac{k-ix}{k+ix}\right)\quad . \end{aligned}$$

Case B_+:

$$\begin{aligned} e_1 &= 1 \ , \quad e_2 \;=\; 0 \ , \quad p(w) \;=\; w^2 + l^2 \ , \quad m(x) \;=\; -2lx \ , \\ \mathcal{B} &= \frac{i}{4}\log\left(\frac{w-il}{w+il}\right) \quad . \end{aligned}$$

Case C^{00}:

$$\begin{aligned} e_1 &= 1 \ , \quad e_2 \;=\; 0 \ , \quad p(w) \;=\; 1 \ , \quad m(x) \;=\; 0 \ , \\ \mathcal{B} &= -\pi/4 \quad . \end{aligned}$$

We are using the same labelling for the cases given above as the one used in [1, 9]. Each of these cases admits an invariant characterization which we will not recall here but which is given in [10]. Next, by imposing the separability condition (3.6), we conclude that the conformal factor T^{-1} must be constant in each of the cases A, B_-, B_+, and C^{00}. This constant can be normalized to one by a rescaling of the coordinates.

We will restrict our attention for the rest of this section to Case A, which can be thought of as the generic case. Indeed, we will see below that the solutions one obtains when imposing the Einstein-Maxwell equations in Case A contain the Kerr-Newman solution as a special case. On the other hand, it is shown in [1, 9] that the solutions one obtains when imposing the Einstein-Maxwell equations in the remaining cases have isometry groups of dimension 4 in Cases B_-, B_+ and 6 in Case C^{00}. These solutions notably include the Taub-NUT and Robinson-Bertotti metrics. Furthermore, it is shown in the above references that all these solutions can be obtained from Carter's A solution by a suitable limiting process.

The metric in Case A is thus given by

$$ds^2 \;=\; \frac{W(w)}{w^2+x^2}(du - x^2\,dv)^2 - \frac{w^2+x^2}{W(w)}dw^2 - \frac{X(x)}{w^2+x^2}(du + w^2\,dv)^2 - \frac{w^2+x^2}{X(x)}dx^2 \ . \quad (3.7)$$

The Dirac equation for the metric (3.7) can be separated by a procedure analogous to the one described in Section 2 for the Kerr-Newman metric. It is proved in [14] that the spinor field given by [4]

$$\Psi(u,v,w,x) \;=\; e^{-i(\omega u + kv)}\,(W(w))^{\frac{1}{4}} \begin{pmatrix} (w-ix)^{1/2}\,X_-(w)\,Y_-(x) \\ (w-ix)^{1/2}\,X_+(w)\,Y_+(x) \\ (w+ix)^{1/2}\,X_+(w)\,Y_-(x) \\ (w+ix)^{1/2}\,X_-(w)\,Y_+(x) \end{pmatrix} \ , \quad (3.8)$$

where ω and k are constants, will be a solution of the Dirac equation expressed in the symmetric frame for the metric (3.7) and the Weyl representation of the Dirac matrices, if and only if the transformed spinor

$$\Phi(w,x) \;=\; \begin{pmatrix} X_-(w)\,Y_-(x) \\ X_+(w)\,Y_+(x) \\ X_+(w)\,Y_-(x) \\ X_-(w)\,Y_+(x) \end{pmatrix} \quad . \quad (3.9)$$

[4] To facilitate the comparison with the expression given in (2.15) for the Kerr-Newman case, we have chosen to include the factor $W^{\frac{1}{4}}$ in the transformation of the spinor field. Just as in the Kerr-Newman case, this transformation has the effect of producing a slightly simpler eigenvalue equation in the variable w than the one obtained in [14].

satisfies the eigenvalue problems given by

$$\mathcal{W}\,\Phi \;=\; \lambda\,\Phi \quad , \qquad \mathcal{X}\,\Phi \;=\; -\lambda\,\Phi \quad , \tag{3.10}$$

where

$$\mathcal{W} \;=\; \begin{pmatrix} im\,w & 0 & \sqrt{W}\,\mathcal{D}_{w+} & 0 \\ 0 & -im\,w & 0 & \sqrt{W}\,\mathcal{D}_{w-} \\ \sqrt{W}\,\mathcal{D}_{w-} & 0 & -im\,w & 0 \\ 0 & \sqrt{W}\,\mathcal{D}_{w+} & 0 & im\,w \end{pmatrix}$$

$$\mathcal{X} \;=\; \begin{pmatrix} -m\,x & 0 & 0 & \sqrt{X}\,\mathcal{L}_{x+} \\ 0 & m\,x & -\sqrt{X}\,\mathcal{L}_{x-} & 0 \\ 0 & \sqrt{X}\,\mathcal{L}_{x+} & -m\,x & 0 \\ -\sqrt{X}\,\mathcal{L}_{x-} & 0 & 0 & m\,x \end{pmatrix} \quad ,$$

and $\mathcal{D}_{w\pm}$ and $\mathcal{L}_{x\pm}$ are the ordinary differential operators defined by

$$\begin{aligned} \mathcal{D}_{w\pm} \;&=\; \frac{\partial}{\partial w} \mp \frac{1}{W(w)}\left[-i\omega w^2 \;+ ik \;-\; ieH(w)\right] \\ \mathcal{L}_{x\pm} \;&=\; \frac{\partial}{\partial x} \;\mp i\frac{1}{X(x)}\left[i\omega x^2 \;-\; ik - \frac{1}{4}X'(x) - ieK(x)\right] \quad . \end{aligned}$$

As a motivation for our generalization of Theorem 1.1, we recall from [3] how the Kerr-Newman solution arises as a special case of the metric (3.7) and vector potential (3.3). For this, we first impose the Einstein-Maxwell equations

$$R_{ij} - \frac{1}{2}R\,g_{ij} - \Lambda\,g_{ij} = F_{ik}F_j^k - \frac{1}{4}g_{ij}F_{kl}F^{kl} \quad , \tag{3.11}$$

which determine the remaining functions X, W, and the electromagnetic field $\mathcal{F} = d\mathcal{A}$. The general solution is given by

$$W \;=\; \frac{\Lambda}{3}\,w^4 + f_2 w^2 + f_1 w + f_0 + Q^2 + P^2 \tag{3.12}$$

$$X \;=\; \frac{\Lambda}{3}\,x^4 - f_2 x^2 + g_1 x + f_0 \tag{3.13}$$

$$\mathcal{A} \;=\; \frac{1}{w^2+x^2}\,\Big(Q\,w\,(du \;-\; x^2\,dv) + P\,x\,(du \;+\; w^2\,dv)\Big) \quad , \tag{3.14}$$

where Q and P denote the electric and magnetic monopole moments and f_0, f_1, f_2, g_1 are arbitrary parameters. These solutions were first discovered by Carter [1]. The Kerr-Newman solution is obtained as a special case by putting suitable restrictions on the parameters $\Lambda, f_0, f_1, f_2, g_1$ appearing in the metric and by implementing through the choice of coordinates (u, v, w, x) the additional hypothesis that the metric is stationary axisymmetric (as opposed to simply admitting a two-dimensional Abelian isometry group). This procedure will serve us as a guide in formulating the conditions under which the analogue of Theorem 1.1 holds for the Dirac equation in the metric (3.7).

We first re-label the coordinates (u, v, w, x) as (t, φ, r, μ) and rescale the ignorable coordinates t and φ so as to have $e_1 = 1,\ e_2 = a$, where a is a constant. Note that while a is freely normalizable at this stage, it will become an essential non-normalizable parameter once our freedom to scale the coordinates will have been exhausted. The expressions of the metric functions $p(r)$ and $m(\mu)$ will thus be slightly different in this normalization of the coordinates. We have

$$p(r) \;=\; r^2 \;\;,\quad m(\mu) \;=\; -a\mu^2 \quad .$$

Taking the cosmological constant Λ to be zero, the metric functions $W^2(r)$ and $X^2(\mu)$ appearing in Carter's A solution will take the form

$$W(r) \;=\; f_2\, r^2 - 2M\, r + f_0\, a^2 + Q^2 + P^2 \;\;,\quad X(\mu) = -f_2\, \mu^2 + g_1\, \mu + f_0 \quad .$$

We now implement the hypothesis of axisymmetry of the metric by letting φ be an angular polar coordinate adapted to the axis of symmetry of the metric and taking μ to be proportional to the cosine of the polar angle measured from the axis. The range of φ will thus be the interval $(0, 2\pi)$ and the range of μ will be a bounded open interval. In order for the metric to be of hyperbolic signature, we must require that $X(\mu)$ be positive for μ varying in that bounded interval. It follows that the roots of $X(\mu)$ must be distinct and that $X(\mu)$ must be positive as μ varies in the interval bounded by these roots. This gives the constraints

$$f_2 \;>\; 0 \quad , \qquad g_1^2 + 4 f_2 f_0 \;>\; 0 \quad .$$

Next, we can ensure that the singular behavior of the metric at the roots of $X(\mu)$ is caused by nothing more than the usual angular coordinate singularity at the polar axis by choosing the ignorable coordinates and the remaining parameters appearing in the metric in such a way that the roots of $m(\mu)$ and $X(\mu)$ coincide. Thus, we use our residual freedom to replace t by a constant linear combination of t and φ to add the same arbitrary constant to $p(r)$ and $m(\mu)$. The roots of $m(\mu)$ will then be located at the endpoints of a symmetric interval $(-c, c)$. In order for the roots of $X(\mu)$ to also be located at c and $-c$, the coefficients g_1, f_2 and f_0 must necessarily satisfy the relations

$$g_1 \;=\; 0 \;\;,\quad f_2 \;>\; 0 \;\;,\quad f_0 \;>\; 0 \quad .$$

We can now set $f_2 = f_0 = 1$ by rescaling μ and r, so that the range of μ becomes the interval $(-1, 1)$. With these normalizations, the parameter a has now become an essential parameter in the metric. The metric functions p, m, W and X thus reduce to

$$\begin{aligned} p(r) \;&=\; r^2 + a^2 \;\;, & m(\mu) \;&=\; a(1-\mu^2) \;\;, \\ W(r) \;&=\; r^2 - 2Mr + a^2 + Q^2 + P^2 \;\;, & X(\mu) \;&=\; 1 - \mu^2 \quad . \end{aligned}$$

We see that by letting *cos* $\mu = \vartheta$, where $-\pi < \vartheta < \pi$, we recover the Kerr-Newman metric in Boyer-Lindquist coordinates (2.1).

We are now ready to state a theorem which generalizes Theorem 1.1 to the most general family of stationary axisymmetric metrics in which the Dirac equation is solvable by separation of variables. We will consider a normal form for the metrics (3.7) of Case A in which the stationary and axisymmetric character of the metric is made manifest through an appropriate choice of coordinates and by the imposition of suitable restrictions on the singularities of the metric functions. This normal form is easily established by a procedure similar to the one we described above for the Kerr-Newman metric.

Theorem 3.1 *Consider the stationary, axisymmetric metric and the vector potential given by*

$$\begin{aligned} ds^2 &= \frac{W(r)}{r^2+a^2\mu^2}\,(dt\,-a\,(1-\mu^2)\,d\varphi)^2 \,-\, \frac{r^2+a^2\mu^2}{W(r)}\,dr^2 \\ &\quad -\frac{X(\mu)}{r^2+a^2\mu^2}\,(a\,dt\,-\,(r^2+a^2)\,d\varphi)^2 \,-\, \frac{r^2+a^2\mu^2}{X(\mu)}\,d\mu^2\;, \end{aligned} \tag{3.15}$$

$$\mathcal{A} = \frac{1}{r^2+a^2\mu^2}\Big(H(r)\,(dt\,-a\,(1-\mu^2)\,d\varphi)+K(\mu)\,(a\,dt\,-\,(r^2+a^2)\,d\varphi)\Big)\;, \tag{3.16}$$

where $a>0$, $-\infty<t<\infty$, $0\le\varphi<2\pi$, $H(r)\in C^\infty(\mathbb{R})$, and $K(\mu)\in C^\infty([-1,1])$. Assume that the functions $X(\mu)\in C^\infty([-1,1])$ and $W(r)\in C^\infty(\mathbb{R})$ have simple zeros at $\mu=1,-1$ and $r=r_1,\ldots,r_N$, so that the range of the coordinates is $\mu\in(-1,1)$ and $r\in\mathbb{R}\setminus\{r_1,\ldots,r_N\}$. Suppose furthermore that the metric is asymptotically Minkowskian,

$$0\;<\;\lim_{r\to\pm\infty} r^{-2}\,W(r)\;<\;\infty\quad.$$

Then the Dirac equation $(G-m)\,\Psi=0$ has no normalizable time-periodic solutions.

This theorem can be thought of as an analogue in the axisymmetric context of the generalization presented in [13, Remark 4.3] of the non-existence theorem for normalizable time-periodic solutions of the Dirac equation in the Reissner-Nordström background. We can likewise argue here that the proof of Theorem 3.1 is similar to that of Theorem 1.1. Indeed, we can see from (3.10) that the separated equations obtained for the metric (3.15) are similar in structure to those obtained in the Kerr-Newman case. Next, we remark that in view of the assumptions made in Theorem 3.1 to ensure the regularity of the metric at the symmetry axis, it follows that the maximal analytic extension of the metric (3.15) will have a conformal diagram similar in structure to the one obtained for the non-extreme Kerr-Newman metric (see Figure 1). We thus conclude that matching conditions for the spinor fields across the horizons take a form identical to the equations (2.33)–(2.36) which were obtained for the Kerr-Newman case. The key observation is then that the assumption that we made about the zeros of $W(r)$ will guarantee that the estimate required to prove the analogue of Lemma 2.1 is valid in this more general context. Finally, the regularity of the eigenfunctions of the angular equation is established by the same procedure as in the Kerr-Newman case, by showing that the angular operator can be viewed as an essentially self-adjoint elliptic operator on the 2-sphere S^2 with C^∞ coefficients.

We conclude by remarking that non-existence theorems similar to Theorems 1.1 and 3.1 likewise hold true in Cases B_+, B_-, and C^{00}.

A Regularity of the Angular Part

In this appendix, we will show that all solutions of the angular equation (2.21) are regular. More precisely, we will see that these functions are of class C^∞ in the open interval $0<\vartheta<\pi$ and uniformly bounded on the closed interval $0\le\vartheta\le\pi$. The method is to reduce the problem to an elliptic eigenvalue equation on the 2-sphere, where standard elliptic regularity theory can be applied.

Consider on S^2 the PDE

$$\left(i\sigma^1\left(\frac{\partial}{\partial\vartheta}+\frac{\cot\vartheta}{2}\right)\,+\,i\sigma^2\,\frac{1}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}\,+\,a\omega\,\sin\vartheta\,\sigma^2\,-\,am\,\cos\vartheta\,\sigma^3\right)\alpha\;=\;\lambda\,\alpha \tag{A.1}$$

for a two-component, complex function α (σ^j are again the Pauli matrices (2.6)). With the ansatz

$$\alpha \;=\; e^{-ik\varphi} \begin{pmatrix} iY_- \\ Y_+ \end{pmatrix} \quad ,$$

the eigenvalue equation (A.1) simplifies to (2.21). Thus it suffices to show regularity for the solutions of (A.1).

Unfortunately, the coefficients in (A.1) have singularities (which are not just removable by a coordinate transformation on S^2). However, after performing the transformation

$$\alpha \;\to\; \tilde{\alpha} = U\alpha \quad \text{with} \quad U(\vartheta, \varphi) \;=\; \exp\!\left(-i\,\frac{\varphi}{2}\,\sigma^3\right)\, \exp\!\left(-i\,\frac{\vartheta}{2}\,\sigma^2\right) \quad ,$$

which is uniformly bounded on S^2 and smooth away from the poles, we obtain for $\tilde{\alpha}$ the equation $A\tilde{\alpha} = \lambda\tilde{\alpha}$ with the smooth operator

$$A \;=\; i\left(\vec{\sigma}\vec{\nabla} - \sigma^r \frac{\partial}{\partial r}\right) \;+\; a\omega\, \sin\vartheta\, \sigma\varphi \;-\; am\, \cos\vartheta\, \sigma^r \quad ,$$

where σ^r and σ^φ denote the Pauli matrices in polar coordinates,

$$\begin{aligned} \sigma^r &= \sin\vartheta\, \cos\varphi\, \sigma^1 \;+\; \sin\vartheta\, \sin\varphi\, \sigma^2 \;+\; \cos\vartheta\, \sigma^3 \\ \sigma^\varphi &= -\sin\varphi\, \sigma^1 \;+\; \cos\varphi\, \sigma^2 \quad . \end{aligned}$$

The operator A is essentially self-adjoint on $C^\infty(S^2)^2 \subset L^2(S^2)^2$. Thus its square A^2 is a positive, essentially self-adjoint operator with smooth coefficients on a compact domain. Standard elliptic theory yields that A^2 has a purely discrete spectrum with finite-dimensional eigenspaces and smooth eigenfunctions. Since the eigenvectors of A are obtained by diagonalizing A on the finite-dimensional eigenspaces of A^2, they are also smooth.

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