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by

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Abstract. We derive necessary and sufficient optimality conditions for a relaxed (in terms of Young measures) variational problem governing steady states of ferromagnetic materials. Such conditions here stated in the form of a generalized Weierstrass maximum principle enable us to establish uniqueness of a solution in some specific situations and can also be used in efficient numerical algorithms solving the relaxed problems, for instance.

Key words. *Calculus of variations, convexification, ferromagnetism, micromagnetics, optimality conditions, relaxation, Young measures.*

AMS subject classifications: 49K20, 49K35, 49S05

1. Introduction

Steady-state configurations of mechanical systems are usually governed by an energy-minimization type principle. In past centuries, this led to a development of the variational calculus, which resulted in formulations of optimality conditions in terms of Euler-Lagrange equations or Weierstrass maximum principle. Sometimes, the involved energy is not convex in highest derivatives, which causes "physically" a development of a microstructure and "mathematically" a failure of existence of a solution. To describe the microstructure in detail and to overcome the failure of existence, the original problem is to be extended suitably. In some situations, it may happen that the extended (relaxed) problem has a convex structure with respect to some geometry not necessarily compatible with the "natural" geometry of the original nonconvex problem. Then one can formulate the optimality conditions. For the case of the scalar variational problems this results in one half of the Euler-Lagrange equation combined with the Weierstrass maximum principle, see [28, Section 5.3]. The identification of a linear structure that makes the relaxed problem convex and formulation of corresponding optimality conditions is a basis for construction of effective numerical algorithms for relaxed problems, cf. [7, 20, 28]. Let us still remark that other geometries applied to the relaxed problem may lead to other optimality conditions, cf. e.g. Chipot and Kinderlehrer [8], DeSimone [12] or Pedregal [26].

The goal of this paper is to adapt the above ideas to a steady-state micromagnetics. The variational problem, stated in Section 2, was already formulated in [3, 4, 5, 16] while its extension, stated here in Section 3, was formulated in [12, 24, 25, 27]. Our original results, i.e., the optimality conditions for the extended problem, are formulated in Sections 4 and 5 in terms of a Weierstrass-type maximum principle in the integral form (Propositions 1 and 3) and also pointwise (Propositions 2 and 4). Some consequences are mentioned in Section 6.

2. Steady-state model of micromagnetics

In the classical theory of rigid ferromagnetic bodies, based mainly on works of Landau and Lifshitz [22], a magnetization $m : \Omega \to \mathbb{R}^n$, describing the state of the body $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, depends on a position $x \in \Omega$ and has a given temperature-dependent magnitude

$$|m(x)| = \mathrm{const}(T) \quad \text{for almost all } x \in \Omega \ ,$$

with $m(x) = 0$ for $T \geq T_c$ the so-called Curie point. We will treat the case when the temperature is fixed below the Curie point and thus we shall assume that $|m| = 1$ almost everywhere in Ω. In the so-called no-exchange formulation, the energy of a large rigid ferromagnetic body $\Omega \subset \mathbb{R}^n$ consists of three parts and the variational principle governing steady-state configurations can be stated as follows (see e.g. Brown [3, 4, 5], Choksi and Kohn [9], James and Kinderlehrer [16], James and Müller [17], Kinderlehrer and Ma [18], Tartar [29], etc.):

$$\text{(1)} \quad \begin{cases} \text{minimize} & E(m,u) := \int_\Omega \Big[\varphi(m(x)) - H_{\mathrm{e}}(x)\cdot m(x)\Big]\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x \ , \\ \text{subject to} & |m| = 1 \quad \text{on } \Omega \ , \\ & \mathrm{div}(\nabla u - m\chi_\Omega) = 0 \ \text{ in } \mathbb{R}^n \ , \\ & m \in L^\infty(\Omega;\mathbb{R}^n), \quad u \in W^{1,2}(\mathbb{R}^n), \end{cases}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuous, $m : \Omega \to \mathbb{R}^n$ is the magnetization, $H_{\mathrm{e}} : \mathbb{R}^n \to \mathbb{R}^n$ is a given external magnetic field, and $u : \Omega \to \mathbb{R}$ is a potential of the induced magnetic field, and $\chi_\Omega : \mathbb{R}^n \to \{0, 1\}$ denotes the characteristic function of Ω. The first term in E is an anisotropy energy with a density φ which is supposed to be an even nonnegative function depending on material properties and exhibiting crystallographic symmetry. Two important cases are the uniaxial case, where φ attains its minimum along one axis, and the cubic case when it attains its minimum along three axes. The second term involving H_{e} is an interaction energy and the last term is a magnetostatic energy related with the magnetization field m through $\Delta u = \mathrm{div}(m\chi_\Omega)$. This equation stems from the Maxwell equations (omitting constants)

$$\text{(2)} \qquad \mathrm{div}\, B = 0 \ \ , \ \mathrm{curl}\, H = 0 \ ,$$

where B is magnetic induction and H intensity of the magnetic field. By the definition, $B = H + m\chi_\Omega$ and $H = -\nabla u$. Then $\Delta u = \mathrm{div}(m\chi_\Omega)$ follows immediately. Let us notice that the weak formulation of this equation reads

$$\text{(3)} \qquad \forall v \in W^{1,2}(\mathbb{R}^n) : \qquad \int_{\mathbb{R}^n} \Big[\nabla u(x) - m(x)\chi_\Omega(x)\Big]\,\nabla v(x)\,\mathrm{d}x = 0 \ \ .$$

In particular, putting $v := u$, we have

$$\text{(4)} \qquad \int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x = \int_\Omega m(x)\cdot\nabla u(x)\,\mathrm{d}x \ ,$$

which gives $\|\nabla u\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \leq \|m\|_{L^2(\Omega;\mathbb{R}^n)}$ by the Hölder inequality. It follows from the Lax-Milgram lemma that (3) has for any $m \in L^2(\Omega;\mathbb{R}^n)$ the unique solution $u \in W^{1,2}(\mathbb{R}^n)$

and that the mapping $m \mapsto \nabla u$ is linear and weakly continuous. Hence the magnetostatic energy $m \mapsto \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2 \mathrm{d}x$ is sequentially weakly lower semicontinuous.

As the set of admissible magnetizations $\{m \in L^\infty(\Omega;\mathbb{R}^n);\ |m| = 1\}$ is not convex, we cannot rely on direct methods (see e.g. [11]) in proving the existence of a solution to (1); cf. [16] for failure of existence of a solution in a uniaxial case. More precisely, if the weak limit of some minimizing sequence of m's in (1) lives for almost all $x \in \Omega$ in the unit sphere then this is the strong limit; cf. [21, p. 99]. Therefore, a so-called fine structure (or, in the "limit" we will speak about a microstructure) in m will typically develop, and we have to look for a notion of generalized solutions and to formulate a so-called relaxed problem. Let us emphasize that the fine structure in m is actually observed in real ferromagnetic materials, see [15].

3. Relaxation in terms of Young measures

We need to describe suitably an oscillating character of sequences $\{(m^k, u^k)\}_{k\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ minimizing sequence (1). It is well known (see [2, 10, 30]) that we can extract a subsequence (denoted, for simplicity, by the same indices) and find $u \in W^{1,2}(\mathbb{R}^n)$ and a family of probability measures $\nu \equiv \{\nu_x\}_{x\in\Omega}$ such that $\mathrm{supp}(\nu_x) \subset S^{n-1} := \{s \in \mathbb{R}^n;\ |s| = 1\}$ which is weakly measurable in the sense that $v \bullet \nu$ is Lebesgue measurable for any $v \in C(S^{n-1})$, and

$$\text{(5)} \qquad \underset{k\to\infty}{\text{w-lim}}\ u^k = u \qquad \& \qquad \underset{k\to\infty}{\text{w*-lim}}\ v \circ m^k = v \bullet \nu$$

for any continuous function $v : \mathbb{R}^n \to \mathbb{R}$, where the limits refer respectively to the weak topology in $W^{1,2}(\mathbb{R}^n)$ and the weak* topology in $L^\infty(\Omega)$, and $[v \bullet \nu](x) := \int_{S^{n-1}} v(s)\nu_x(\mathrm{d}s)$ for almost all $x \in \Omega$. Let us denote the set of all $\nu \equiv \{\nu_x\}_{x\in\Omega}$ with the above listed properties by $\mathcal{Y}(\Omega; S^{n-1})$; such ν's are called Young measures. Conversely, for any $\nu \in \mathcal{Y}(\Omega; S^{n-1})$ there is a sequence of measurable functions $m^k : \Omega \to S^{n-1}$ such that the later convergence in (5) is fulfilled.

The relaxation of the problem (1) was done by DeSimone [12], Pedregal [24, 25], Rogers [27] etc. The continuously extended problem obtained by this way looks as follows:

$$\text{(6)} \qquad \left\{ \begin{array}{ll} \text{minimize} & \bar{E}(\nu, u) := \displaystyle\int_\Omega \int_{S^{n-1}} \Big(\varphi(s) - H_{\mathrm{e}}(x)\cdot s\Big)\, \nu_x(\mathrm{d}s)\, \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\, \mathrm{d}x, \\ \text{subject to} & \displaystyle\int_{\mathbb{R}^n} \Big[\nabla u(x) - \int_{S^{n-1}} \chi_\Omega(x)\, s\, \nu_x(\mathrm{d}s)\Big] \nabla v(x)\, \mathrm{d}x = 0 \quad \forall v \in W^{1,2}(\mathbb{R}^n)\ , \\ & \nu \in \mathcal{Y}(\Omega; S^{n-1}), \quad u \in W^{1,2}(\mathbb{R}^n)\ . \end{array} \right.$$

The probability measure ν_x describes in a proper (we may say "mesoscopic") way the microstructure of the "limit" magnetization at a point x.

The extended problem (6) is a correct relaxation for the original problem (1). Indeed, by [12, 25], the infimum of $\bar{E}$ is attained and it is equal to the infimum of E. Moreover, having (ν, u) a solution to (6), then there is a sequence $(m^k, u^k) \in L^\infty(\Omega;\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ satisfying $\Delta u^k = \mathrm{div}(m^k\chi_\Omega)$ in the weak sense, $|m^k| = 1$ a.e., minimizing E, and attaining (ν, u) in the sense (5). Conversely, every sequence $(m^k, u^k) \in L^\infty(\Omega;\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$

satisfying $\Delta u^k = \mathrm{div}(m^k\chi_\Omega)$ weakly, $|m^k| = 1$, and minimizing E, contains a subsequence attaining some $(\nu, u) \in \mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)$ in the sense (5), and every (ν, u) obtained by such way solves the relaxed problem (6).

One can also think about a "coarser" relaxation in terms of the original "macroscopic" magnetization m. We denote by $\delta_{S^{n-1}}$ the indicator function of the unit sphere, i.e.

$$\delta_{S^{n-1}}(s) = \begin{cases} 0 & \text{if } |s| = 1 \\ +\infty & \text{otherwise} . \end{cases}$$

Furthermore, by v^{**} we denote the second Fenchel conjugate (the convex envelope of v), i.e. $v^{**} = \sup\{w \text{ convex};\ w \le v\}$. This can be used to pose the following relaxed problem:

$$(7)\quad \begin{cases} \text{minimize } \tilde{E}(m,u) = \int_\Omega [\varphi + \delta_{S^{n-1}}]^{**}(m(x)) - H_{\mathrm{e}}(x)\cdot m(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x, \\ \text{subject to} \quad \mathrm{div}(\nabla u - m\chi_\Omega) = 0 \ \text{ in } \mathbb{R}^n , \\ \qquad\qquad m \in L^\infty(\Omega;\mathbb{R}^n) , \quad u \in W^{1,2}(\mathbb{R}^n) . \end{cases}$$

Note that $[\varphi + \delta_{S^{n-1}}]^{**}$ equals $+\infty$ outside the unit ball in $\mathbb{R}^n$ so that any minimizer (m, u) of (7) must satisfy $|m(x)| \le 1$ for a.a. $x \in \Omega$. DeSimone [12] showed that $\tilde{E}$ always attains its minimum on the considered admissible set, and this minimum is equal to the infimum of (1). For any $s \in \mathbb{R}^n$, one has

$$(8)\qquad [\varphi + \delta_{S^{n-1}}]^{**}(s) = \inf_{\substack{\mu \text{ probability measure on } S^{n-1} \\ \int_{S^{n-1}} \sigma\mu(\mathrm{d}\sigma) = s}} \int_{S^{n-1}} \varphi(\sigma)\mu(\mathrm{d}\sigma) .$$

Note that, for $|s| > 1$, the set of μ's considered in (8) is empty so that the infimum in (8) is $+\infty$. It is clear that if (ν, u) minimizes $\bar{E}$ then (m, u) with

$$m(x) = \int_{S^{n-1}} s\nu_x(\mathrm{d}s)$$

minimizes $\tilde{E}$. Said differently, a unique minimizer of $\bar{E}$ implies a unique minimizer of $\tilde{E}$. The opposite implication does not hold because, fixing some $m \in L^\infty(\Omega;\mathbb{R}^n)$ with values in the unit ball centered at the origin, we might still have many (even continuum of) minimizers of $\bar{E}$ with the first moment m, cf. Example 3 below. Clearly, the only term responsible for uniqueness/nonuniqueness is φ. Recently, Carstensen and Prohl [6] showed that if φ corresponds to uniaxial ferromagnets, i.e. φ is nonnegative and equals zero at two points $\pm s \in S^{n-1}$, and has a given representation, then $\tilde{E}$ has a unique minimizer. The Euler-Weierstrass-type optimality conditions for the corresponding $\bar{E}$ will enable us to prove that under some properties of φ there also exists a unique Young measure-valued minimizer, see Examples 1–2 below. See also DeSimone [12] for earlier results on uniqueness of Young measure solutions in the uniaxial case.

4. Optimality conditions in terms of ν and u

It is usual to identify a given Young measure $\nu \in \mathcal{Y}(\Omega; S^{n-1})$ with the linear functional in $L^1(\Omega; C(S^{n-1}))^*$ defined by

$$\langle \nu, h\rangle = \int_\Omega \int_{S^{n-1}} h(x,s)\nu_x(\mathrm{d}s)\mathrm{d}x. \tag{9}$$

Thus $\mathcal{Y}(\Omega; S^{n-1})$ can be considered as a convex weakly* compact subset of $L^1(\Omega; C(S^{n-1}))^*$, see [28, Corollary 3.1.7]. Furthermore, let us define $\Pi : L^1(\Omega; C(S^{n-1}))^* \times W^{1,2}(\mathbb{R}^n) \to W^{-1,2}(\mathbb{R}^n) \cong W^{1,2}(\mathbb{R}^n)$ by the formula

$$\langle v, \Pi(\nu,u)\rangle = -\langle \nu, \chi_\Omega \nabla v \otimes \mathrm{id}\rangle \ + \int_\Omega \nabla v(x) \cdot \nabla u(x)\,\mathrm{d}x \tag{10}$$

for $v \in W^{1,2}(\mathbb{R}^n)$, where naturally $[\nabla v \otimes \mathrm{id}](x,s) := \nabla v(x) \cdot s$. Let us note that $\Pi(\nu,u) = 0$ just means that u solves

$$\int_{\mathbb{R}^n} \left[\nabla u(x) - \int_{S^{n-1}} \chi_\Omega(x)\, s\, \nu_x(\mathrm{d}s)\right] \nabla v(x)\,\mathrm{d}x = 0 \qquad \forall v \in W^{1,2}(\mathbb{R}^n)\ . \tag{11}$$

Also note that Π is (weak*×weak,weak)-continuous and surjective in the sense that

$$\forall f \in W^{-1,2}(\mathbb{R}^n)\ \ \exists u \in W^{1,2}(\mathbb{R}^n)\ \ \exists \nu \in \mathcal{Y}(\Omega; S^{n-1}) : \qquad \Pi(\nu,u) = f\ , \tag{12}$$

which follows immediately from the Lax-Milgram lemma. The relaxed problem (6) now takes the following abstract form:

$$\begin{cases} \text{minimize} & \bar{E}(\nu,u) \\ \text{subject to} & \Pi(\nu,u) = 0\ , \\ & (\nu,u) \in \mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n). \end{cases} \tag{13}$$

Note that $\bar{E}$ is convex, Π is linear, and $\mathcal{Y}(\Omega; S^{n-1})$ is convex, so that the problem (13) has a convex structure. As $\bar{E}$ is Gateâux differentiable and $0 \in \mathrm{int}(\Pi(\mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)))$ due to (12), it is known (see e.g. Aubin and Ekeland [1, p.175]) that the first-order optimality conditions looks as follows:

$$\begin{aligned} \bar{E}'(\nu,u) &\in -N_{\mathrm{Ker}\,\Pi \,\cap\, (\mathcal{Y}(\Omega;S^{n-1}) \times W^{1,2}(\mathbb{R}^n))}(\nu,u) \\ &= \mathrm{Range}\,\Pi^* - N_{\mathcal{Y}(\Omega;S^{n-1}) \times W^{1,2}(\mathbb{R}^n)}(\nu,u) = \mathrm{Range}\,\Pi^* - N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu) \times \{0\}, \end{aligned}$$

where $\bar{E}' = (\bar{E}'_\nu, \bar{E}'_u) : L^1(\Omega; C(S^{n-1}))^* \times W^{1,2}(\mathbb{R}^n) \to (L^1(\Omega; C(S^{n-1}))^{**} \times W^{-1,2}(\mathbb{R}^n)) \cong L^1(\Omega; C(S^{n-1}))^{**} \times W^{1,2}(\mathbb{R}^n))$ denotes the Gâteaux differential of E and $\Pi^* = (\Pi^*_\nu, \Pi^*_u) : W^{1,2}(\mathbb{R}^n) \to L^1(\Omega; C(S^{n-1}))^{**} \times W^{-1,2}(\mathbb{R}^n)$ is the adjoint operator to Π. Moreover, $N_{\mathcal{Y}(\Omega;S^{n-1}) \times W^{1,2}(\mathbb{R}^n)}(\nu,u)$ denotes the normal cone to the convex set $\mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)$ at the point (ν,u), and analogously $N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu)$ is the normal cone to $\mathcal{Y}(\Omega; S^{n-1})$ at ν, i.e.

$$N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu) := \left\{\xi \in L^1(\Omega; C(S^{n-1}))^{**};\ \ \forall \tilde{\nu} \in \mathcal{Y}(\Omega; S^{n-1}) :\ \ \langle \xi, \tilde{\nu} - \nu\rangle \le 0\right\}.$$

Therefore, we can deduce that, if $(\nu,u)\in L^1(\Omega;C(S^{n-1}))^*\times W^{1,2}(\mathbb{R}^n)$ solves (13), then there is a Lagrange multiplier $\lambda\in W^{1,2}(\mathbb{R}^n)\cong W^{-1,2}(\mathbb{R}^n)^*$ such that

$$\Pi_u^*\lambda-\bar{E}_u'(\nu,u)=0\ , \tag{14}$$

$$\Pi_\nu^*\lambda-\bar{E}_\nu'(\nu,u)\in N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu)\ , \tag{15}$$

see [28, Sec. 5.3]. As the problem (13) is convex, the conditions (14)–(15) are also sufficient in the sense that, if $(\nu,u)\in\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n)$ satisfies $\Pi(\nu,u)=0$ and (14)–(15) for some multiplier $\lambda\in W^{1,2}(\mathbb{R}^n)$, then (ν,u) solves (13).

The abstract conditions (14)–(15) turns for the concrete data $\bar{E}$ from (6), Π from (10) and $\mathcal{Y}(\Omega;S^{n-1})$ into the following integral maximum principle:

Proposition 1. *Let $H_{\rm e}\in L^2(\Omega;\mathbb{R}^n)$, $\varphi:\mathbb{R}^n\to\mathbb{R}$ be continuous, let $(\nu,u)\in L^1(\Omega;C(S^{n-1}))^*\times W^{1,2}(\mathbb{R}^n)$ solve (13) with the data from (6) and (10). Then*

$$\langle\nu,\mathcal{H}_u\rangle=\sup_{\substack{m\in L^\infty(\Omega;\mathbb{R}^n)\\ |m|=1\text{ a.e.}}}\int_\Omega\mathcal{H}_u(x,m(x))\,{\rm d}x\ , \tag{16}$$

where the Hamiltonian $\mathcal{H}_u:\Omega\times\mathbb{R}^n\times\mathbb{R}$ is defined by

$$\mathcal{H}_u(x,s):=-\nabla u(x)\cdot s+H_{\rm e}(x)\cdot s-\varphi(s)\ . \tag{17}$$

Conversely, if $(\nu,u)\in\mathcal{Y}(\Omega;S^{n-1})\times W^{1,2}(\mathbb{R}^n)$ satisfies $\Pi(\nu,u)=0$ and if the maximum principle (16) holds, then (ν,u) solves (13).

Proof. Let us evaluate the differential of $\bar{E}$. As for $\bar{E}_u'(\nu,u)\in\mathcal{L}(W^{1,2}(\mathbb{R}^n),\mathbb{R})=W^{-1,2}(\mathbb{R}^n)$, we have

$$\langle E_u'(\nu,u),v\rangle=\int_\Omega\nabla u\cdot\nabla v\,{\rm d}x, \tag{18}$$

while for $\bar{E}_\nu'(\nu,u)\in L^1(\Omega;C(S^{n-1}))^{**}=\mathcal{L}(L^1(\Omega;C(S^{n-1}))^*,\mathbb{R})$ we have

$$[E_\nu'(\nu,u)](\tilde\nu)=\langle\tilde\nu,1\otimes\varphi-H_{\rm e}\otimes{\rm id}\rangle \tag{19}$$

where naturally $[1\otimes\varphi](x,s)=\varphi(s)$ and $[H_{\rm e}\otimes{\rm id}](x,s)=H_{\rm e}(x)\cdot s$.

The equation (14) now gives

$$\int_\Omega\nabla v\cdot\nabla\lambda\,{\rm d}x=\langle\lambda,\Pi(0,v)\rangle=\langle\Pi_u^*\lambda,v\rangle=\Big\langle\bar{E}_u'(\nu,u),v\Big\rangle=\int_\Omega\nabla u\cdot\nabla v\,{\rm d}x \tag{20}$$

for any $v\in W^{1,2}(\mathbb{R}^n)$, from which we get simply $\lambda=u+$ constant. As λ should live in $W^{1,2}(\mathbb{R}^n)$, this constant must vanish so that we eventually $\lambda=u$.

The inclusion (15) results in the inequality

$$\begin{aligned}0\geq\Big\langle\Pi_\nu^*\lambda-\bar{E}_\nu'(\nu,u),\tilde\nu-\nu\Big\rangle &= \langle\lambda,\Pi(\tilde\nu-\nu,0)\rangle\ -\ \langle\tilde\nu-\nu,1\otimes\varphi-H_{\rm e}\otimes{\rm id}\rangle\\ &= \langle\tilde\nu-\nu,-\chi_\Omega\nabla\lambda\otimes{\rm id}-1\otimes\varphi+H_{\rm e}\otimes{\rm id}\rangle\end{aligned}$$

for all $\tilde{\nu} \in \mathcal{Y}(\Omega; S^{n-1})$. This gives $\langle \tilde{\nu} - \nu, \mathcal{H}_\lambda \rangle \le 0$ with the Hamiltonian $\mathcal{H}_\lambda = \mathcal{H}_u$ given by (17). By (20), $\mathcal{H}_\lambda = \mathcal{H}_u$. In other words, we got

$$\langle \nu, \mathcal{H}_u \rangle = \max_{\tilde{\nu} \in \mathcal{Y}(\Omega; S^{n-1})} \langle \tilde{\nu}, \mathcal{H}_u \rangle \,. \tag{21}$$

As E'_ν as well as Π^*_ν take their values in $L^1(\Omega; C(S^{n-1}))$ rather than in $L^1(\Omega; C(S^{n-1}))^{**}$, we can take into considerations only the intersection of the normal cone $N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu) \subset L^1(\Omega; C(S^{n-1}))^{**}$ with $L^1(\Omega; C(S^{n-1}))$ as was already done in [28]. Hence,

$$\begin{aligned} N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu) \cap L^1(\Omega; C(S^{n-1})) &= \Big\{ h \in L^1(\Omega; C(S^{n-1}));\ \forall \tilde{\nu} \in \mathcal{Y}(\Omega; S^{n-1}) :\ \langle \tilde{\nu}, h \rangle \le \langle \nu, h \rangle \Big\} \\ &= \Big\{\ h \in L^1(\Omega; C(S^{n-1}));\ \langle \nu, h \rangle = \sup_{\substack{m \in L^\infty(\Omega; \mathbb{R}^n) \\ |m|=1 \text{ a.e.}}} \int_\Omega h(x, m(x))\, \mathrm{d}x\ \Big\}, \end{aligned}$$

which eventually gives us (16).

As the problem (13) is convex, the maximum principle (16) is also sufficient in the above specified sense.

□

Thanks to the special form of the set of admissible magnetizations in (1) admitting arbitrary oscillations of m, the integral maximum principle (16) can be localized into the following pointwise maximum principle, which gives a very explicit restriction on possible steady-state microstructures.

Proposition 2. *Let $(\nu, u) \in \mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)$ solve the relaxed problem (13). Then*

$$\int_{S^{n-1}} \mathcal{H}_u(x, s) \nu_x(\mathrm{d}s) = \max_{s \in S^{n-1}} \mathcal{H}_u(x, s) \quad \text{for a.a. } x \in \Omega \tag{22}$$

with the Hamiltonian again from (17). In other words,

$$\mathrm{supp}(\nu_x) \subset \mathrm{Argmax}\mathcal{H}_u(x, \cdot)\ , \tag{23}$$

where $\mathrm{Argmax}\mathcal{H}_u(x, \cdot) := \{ s \in S^{n-1};\ \mathcal{H}_u(x, s) = \max \mathcal{H}_u(x, S^{n-1}) \}$. *Conversely, if $(\nu, u) \in \mathcal{Y}(\Omega; S^{n-1}) \times W^{1,2}(\mathbb{R}^n)$ satisfies $\Pi(\nu, u) = 0$ and (23) holds for a.a. $x \in \Omega$, then (ν, u) solves the relaxed problem (13).*

Proof. We will show that (16) and (22) are equivalent to each other. Due to [13, Th 1.2. Ch. VIII], there exists $\tilde{m} : \Omega \to S^{n-1}$, measurable such that $\mathcal{H}_u(x, \tilde{m}(x)) = \max_{s \in S^{n-1}} \mathcal{H}_u(x, s)$ for almost all $x \in \Omega$.

First, suppose that (16) is fulfilled. Therefore,

$$\begin{aligned} \langle \nu, \mathcal{H}_u \rangle &= \int_\Omega \int_{S^{n-1}} \mathcal{H}_u(x, s) \nu_x(\mathrm{d}s)\, \mathrm{d}x = \sup_{\substack{m \in L^\infty(\Omega; \mathbb{R}^n) \\ |m|=1 \text{ a.e.}}} \int_\Omega \mathcal{H}_u(x, m(x))\, \mathrm{d}x \\ &\ge \int_\Omega \mathcal{H}_u(x, \tilde{m}(x))\, \mathrm{d}x = \int_\Omega \max_{s \in S^{n-1}} \mathcal{H}_u(x, s)\, \mathrm{d}x\ . \end{aligned}$$

In other words,

$$\int_\Omega \Big(\int_{S^{n-1}} \mathcal{H}_u(x,s)\nu_x(\mathrm{d}s) - \max_{s\in S^{n-1}} \mathcal{H}_u(x,s)\Big) \,\mathrm{d}x \ge 0 \;.$$

At the same time, $\int_{S^{n-1}} \mathcal{H}_u(x,s)\nu_x(\mathrm{d}s) \le \max_{s\in S^{n-1}} \mathcal{H}_u(x,s)$ for almost all $x \in \Omega$, which shows that (22) holds.

Let now (22) be satisfied. Integrating it over Ω one gets

$$\langle \nu, \mathcal{H}_u\rangle = \int_\Omega \max_{s\in S^{n-1}} \mathcal{H}_u(x,s)\,\mathrm{d}x \ge \sup_{\substack{m\in L^\infty(\Omega;\mathbb{R}^n)\\ |m|=1 \text{ a.e.}}} \int_\Omega \mathcal{H}_u(x,m(x))\,\mathrm{d}x \ge \int_\Omega \mathcal{H}_u(x,\tilde m(x))\,\mathrm{d}x \;.$$

On the other hand, $\int_\Omega \max_{s\in S^{n-1}} \mathcal{H}_u(x,s)\,\mathrm{d}x = \int_\Omega \mathcal{H}_u(x,\tilde m(x))\,\mathrm{d}x$ and thus (16) holds. $\square$

5. Optimality conditions in terms of ν and m

One can also alternatively consider optimality conditions when the energy functional is supposed to depend on the "mesoscopic" Young-measure magnetization ν and the "macroscopic" magnetization m. Interestingly, it turns out that such optimality conditions are the same as those derived in the previous section. In order to show this we will define a new functional $e : L^1(\Omega; C(S^{n-1}))^* \times L^2(\Omega;\mathbb{R}^n) \to \mathbb{R}$ by

$$e(\nu,m) = \langle \nu, 1\otimes\varphi\rangle - \int_\Omega H_\mathrm{e}(x)\cdot m(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\,\mathrm{d}x \;,$$

with ∇u determined via $\Delta u = \mathrm{div}(m\chi_\Omega)$. Eventually, we define $\pi : L^1(\Omega; C(S^{n-1}))^* \times L^2(\Omega;\mathbb{R}^n)^* \to L^2(\Omega;\mathbb{R}^n)$ by

$$\pi(\nu,m) = \mathrm{id}\bullet\nu - m$$

and we see that $\pi\left(\mathcal{Y}(\Omega;S^{n-1}) \times L^2(\Omega;\mathbb{R}^n)\right) = L^2(\Omega;\mathbb{R}^n)$. Thus we are concerned with

$$(24)\qquad \begin{cases} \text{minimize} & e(\nu,m) \\ \text{subject to} & \pi(\nu,m) = 0 \;, \\ & (\nu,m) \in \mathcal{Y}(\Omega;S^{n-1}) \times L^2(\Omega;\mathbb{R}^n) \;. \end{cases}$$

Note that π is continuous and linear and e is convex. We will also show that e is Gateâux differentiable. The first order optimality conditions read in this case that, if $(\nu,m) \in \mathcal{Y}(\Omega;S^{n-1}) \times L^2(\Omega;\mathbb{R}^n)$ solve (24) then there is a Lagrange multiplier $\ell \in L^2(\Omega;\mathbb{R}^n)$ such that

$$(25)\qquad \pi^*_m \ell - e'_m(\nu,m) = 0 \;,$$

$$(26)\qquad \pi^*_\nu \ell - e'_\nu(\nu,m) \in N_{\mathcal{Y}(\Omega;S^{n-1})}(\nu) \;,$$

where $e' = (e'_\nu, e'_m) : L^1(\Omega; C(S^{n-1}))^* \times L^2(\Omega;\mathbb{R}^n) \to L^1(\Omega; C(S^{n-1}))^{**} \times L^2(\Omega;\mathbb{R}^n)$ and $\pi^* = (\pi^*_\nu, \pi^*_m) : L^2(\Omega;\mathbb{R}^n) \to L^1(\Omega; C(S^{n-1}))^{**} \times L^2(\Omega;\mathbb{R}^n)$.

Proposition 3. *Let $H_{\rm e} \in L^2(\Omega; \mathbb{R}^n)$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ continuous, let $(\nu, m) \in \mathcal{Y}(\Omega; S^{n-1}) \times L^2(\Omega; \mathbb{R}^n)$ solve (24) and let u solve (3). Then*

$$\langle \nu, h_\ell \rangle = \sup_{\substack{\tilde{m} \in L^2(\Omega;\mathbb{R}^n) \\ |\tilde{m}|=1 \text{ a.e.}}} \int_\Omega h_\ell(x, \tilde{m}(x))\,\mathrm{d}x \ , \tag{27}$$

where the Hamiltonian in now defined as

$$h_\ell := \ell \otimes \mathrm{id} - \varphi \tag{28}$$

with $\ell = H_{\rm e} - \nabla u$.

Proof. First we prove that $e'(\nu, m) = (\varphi, -H_{\rm e} + \nabla u)$. The first component of e', namely e'_ν, is obvious because $e(\cdot, m)$ is affine. As to the second component, we denote by w the solution to (3) with arbitrary $v \in L^2(\Omega; \mathbb{R}^n)$ instead of m. Then we have

$$\begin{aligned}
[e'_m(\nu, m)](v) &= -H_{\rm e} \cdot v + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} |\nabla u(x) + t \nabla w(x)|^2 \,\mathrm{d}x\,|_{t=0} \\
&= -H_{\rm e} \cdot v + \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla w(x)\,\mathrm{d}x \\
&= -H_{\rm e} \cdot v + \int_\Omega \nabla u(x) \cdot v(x)\,\mathrm{d}x = (-H_{\rm e} + \nabla u) \cdot v \ ,
\end{aligned}$$

where we used, beside (3) with v instead of m, also the linearity of the mapping $m \mapsto \nabla u$.
Furthermore, for any $\ell \in L^2(\Omega; \mathbb{R}^n)$ it holds $\pi^* \ell = (\ell \otimes \mathrm{id}, -\ell)$ because

$$\langle \pi^* \ell, (\nu, m) \rangle = \langle \ell, \pi(\nu, m) \rangle = \left\langle \ell, \mathrm{id} \bullet \nu \right\rangle - \langle \ell, m \rangle = \langle \nu, \ell \otimes \mathrm{id} \rangle - \langle \ell, m \rangle \ .$$

The relations (25) and (26) now turn respectively into

$$\ell = H_{\rm e} - \nabla u \ , \tag{29}$$

$$-\varphi + \ell \otimes \mathrm{id} \in N_{\mathcal{Y}(\Omega; S^{n-1})}(\nu) \ . \tag{30}$$

Again, since e'_ν as well as π^*_ν take their values in $L^1(\Omega; C(S^{n-1}))$ rather than in $L^1(\Omega; C(S^{n-1}))^{**}$, we obtain the claimed maximum principle. □

Proposition 4. *Under the assumptions of the previous proposition*

$$[h_\ell \bullet \nu](x) = \max_{s \in S^{n-1}} h_\ell(x, s) \quad \textit{for a.a. } x \in \Omega \ . \tag{31}$$

The proof of the above point-wise version is analogous as that one of Proposition 2.
We point out that $h_\ell = \mathcal{H}_u$ provided $\ell = H_{\rm e} - \nabla u$ so that, in fact, Propositions 1 and 2 are equivalent with Propositions 3 and 4, respectively.

6. Some consequences

The following proposition gives a sufficient condition under which the relaxed problem (13) has a unique minimizer. This condition is indeed satisfied in some physically relevant situations, see Examples 1 and 2 below, while in other situations admitting many minimizers is not satisfied, see Example 3.

Proposition 5. *Let (7) possess a unique minimizer and let, for any* $r \in \mathbb{R}^n$*, the function* $S^{n-1} \to \mathbb{R}$: $s \mapsto r\cdot s - \varphi(s)$ *attain its maximum at a finite number* $\kappa(r) \le n+1$ *of points* σ_r^l *such that* $\dim(\mathrm{sp}\{\sigma_r^l\}_{l=1}^{\kappa(r)}) = \kappa(r) - 1$*. Then (13) has a unique solution.*

Proof. The proof paraphrases that one of [28, Corollary 5.3.4]. Let $(m,u) \in L^2(\Omega;\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$ be a unique minimizer of (7) and let (ν^1,u^1) and (ν^2,u^2) be two different solutions to (13). Let us denote $m^1 = \mathrm{id}\bullet\nu^1$ and $m^2 = \mathrm{id}\bullet\nu^2$. As (m^1,u^1) and (m^2,u^2) must solve (7), we get

$$m^1 = m = m^2 \qquad \text{and} \qquad u^1 = u = u^2. \tag{32}$$

Then the Hamiltonian is determined uniquely, i.e. $\mathcal{H}_{u^1} = \mathcal{H}_{u^2}$. By (23), and the assumption, the probability measure ν_x^i must be supported at a finite number $k(x) = \kappa(H_\mathrm{e}(x) - \nabla u(x))$ of points $s^l(x) = \sigma^l_{H_\mathrm{e}(x)-\nabla u(x)}$, i.e., $\nu_x^i = \sum_{l=1}^{k(x)} a_l^i(x)\delta_{s^l(x)}$ with $a_l^i \ge 0$, $\sum_{l=1}^{k(x)} a_l^i = 1$ a.e. in Ω. By (32), we have

$$\sum_{l=1}^{k(x)} (a_l^1(x) - a_l^2(x))s^l(x) = m^1(x) - m^2(x) = 0\ .$$

The full-rank condition yields $a^1 = a^2$ a.e. in Ω. □

Proposition 6. *Let the assumptions of Proposition 5 be fulfilled. Then the original problem (1) possesses a solution (being equal just to* (u,m)*, the assumed unique solution to (7)) if and only if, for a.a.* $x \in \Omega$*, it holds*

$$m(x) \in \{\sigma_r^l\}_{l=1}^{\kappa(r)} \qquad \textit{with} \quad r = H_\mathrm{e}(x) - \nabla u(x)\ . \tag{33}$$

Proof. The unique solution (ν,u) to (13) is 1-atomic at a given $x \in \Omega$, i.e. of the form $\nu_x = \delta_{m(x)}$, if and only if (33) holds. If (33) holds a.e., (m,u) then solves (1). Conversely, if (m,u) solves (1), then (ν,u) with $\nu_x = \delta_{m(x)}$ solves (13). As this solution is unique, failure of (33) for x from a positive Lebesgue measure set implies failure of existence of any solution to (1). □

Example 1. (*Uniaxial magnets I.*) Let us take $n=2$, $\varphi : \mathcal{S}^1 \to \mathbb{R}$

$$\varphi(s) = s_1^2 + \min\{(s_2-1)^2, (s_2+1)^2\},$$

such potential is related with the so-called uniaxial ferromagnet. For this type of φ there is a unique solution to (7); cf. DeSimone [12] or also Carstensen and Prohl [6]. As $n=2$, we have $S^1 = \{(\cos\theta, \sin\theta);\ \theta \in [0, 2\pi]\}$ and, for $\phi(\theta) := \varphi(\cos\theta, \sin\theta)$, it is easy to see that $\phi(\theta) = 2 - 2|\sin\theta|$. For $r = (r_1, r_2) \in \mathbb{R}^2$, let us further denote $f(\theta, r) = \phi(\theta) - r_1\cos\theta - r_2\sin\theta$. Differentiating f with respect to θ we have

$$\frac{\partial f}{\partial\theta}(\theta, r) = \begin{cases} r_1\sin\theta - (r_2+2)\cos\theta & \text{if } \theta \in]0,\pi[\\ r_1\sin\theta - (r_2-2)\cos\theta & \text{if } \theta \in]\pi, 2\pi[\ . \end{cases}$$

We see that $f(\cdot, r)$ has at most four local extrema at $\theta = 0$, $\theta = \pi$, $\theta = \theta_1 \in [0,\pi]$ and $\theta = \theta_2 \in [\pi, 2\pi]$, where

$$\theta_1 := \begin{cases} \arctan\frac{r_2+2}{r_1} & \text{if } r_1 \neq 0, \\ \frac{\pi}{2} & \text{if } r_1 = 0, \end{cases} \qquad \text{and} \qquad \theta_2 := \begin{cases} \arctan\frac{r_2-2}{r_1} & \text{if } r_1 \neq 0, \\ \frac{3\pi}{2} & \text{if } r_1 = 0. \end{cases}$$

On the other hand, there is no $r \in \mathbb{R}^2$ for which $f(0,r) = f(\pi,r) = f(\theta_1,r) = f(\theta_2,r)$. This shows together with Proposition 3 that $\kappa(r) \leq 3$, $r \in \mathbb{R}^2$, and that for φ as above (13) has a unique solution. In truth, one can even show that $\kappa(r) \leq 2$, so that (13) has a solution (ν, u) with $\nu_x = \lambda(x)\delta_{s_1(x)} + (1-\lambda(x))\delta_{s_2(x)}$ for almost all $x \in \Omega$.

Example 2. (*Uniaxial magnets II.*) By similar arguments one obtains uniqueness also for $n = 2$, $\varphi : \mathcal{S}^1 \to \mathbb{R}$,

$$\varphi(s) = \varphi(s_1, s_2) = s_1^2$$

as already observed in [12].

Example 3. (*Cubic magnets.*) Let us take $n = 3$, $\varphi : \mathcal{S}^2 \to \mathbb{R}$,

$$\varphi(s) = \varphi(s_1, s_2, s_3) = s_1^2 s_2^2 + s_1^2 s_3^2 + s_2^2 s_3^2\ .$$

Then one can see that for $H_{\rm e} = 0$ there are many solutions to (13), for example, $\nu_x = \frac{1}{2}\delta_{(0,1)} + \frac{1}{2}\delta_{(0,-1)}$, $x \in \Omega$, $u = 0$ or $\nu_x = \frac{1}{2}\delta_{(1,0)} + \frac{1}{2}\delta_{(-1,0)}$, $x \in \Omega$, $u = 0$. Note that the assumptions of Proposition 5 are indeed not satisfied because $\kappa(0) = 6 > n + 1 = 4$.

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