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Unattainability of nodes

by

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Abstract

It is discussed how a probabilistic generalization of classical mechanics of a point-particle to its quantum version, as a straight forward application of the Copenhagen interpretation of quantum mechanics, intrinsically motivates a multi-particle concept. This corresponds to the unattainability of the nodes by the modeling stochastic process. That property is shown here to be fulfilled for physically interesting potentials. The language of nonstandard analysis is applied not only intending to make the involved microstructure more transparent but also using it as a handy tool.

Contents

1 Introduction

Perhaps, everybody knows about Einstein's "god does not throw the dice" point of view concerning the description of quantum mechanics, and the debate of its interpretation. This article contributes to this topic presenting quantum mechanics as an probabilistic extension of classical mechanics. It deals with a stochastic process that models the (stationary) Schrödinger equation with a potential (especially around a node) according to Born's, the so called Copenhagen interpretation of quantum mechanics.

At about a decade ago there was the last revival of that old physical problem, now in terms of stochastic processes. Perhaps the most popular (monographic) representative is Nelson's "Quantum fluctuations" [13]. In addition [7] should be mentioned. The idea of such a kind of approach goes back to the article [8] of Fényes, stated on p. 77 of Nelson's book. In spite of modeling with

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Markovian processes, as it is done in the latter work, there is another (stochastic) attempt going back to Schroedinger himself, cf. [15]. Modern versions of that can be found in [3], where the processes satisfy the Bernsteinian property instead, and in Nagasawa's monograph [12] from which the reader may get a lot of background information.

Common to both approaches it seems to be the careful implementation of (the correct) time direction into an inner and an outer process. This is tried to expose in section 2 which could be read as a second introduction to the matter. In section 3 an asymptotic property of the potential, which is just excluding multi-poles, is obtained to assure the unattainability of the nodes driven by the stochastic process finally. For the belonging potentials the main result is derived in section 4; i.e., for $\psi(x) \leq |\mathcal{O}(\|x\|)|$ the particle does not attain a node at $x = 0$. This requirement concerning the tunneling is already found by Albeverio et al. [1] (Theorem 4.1.) with help of Dirichlet forms – it is achieved here in a different (and simple) way (applying Anderson's random walk) by using nonstandard analysis[1].

With this method the process' starting distribution is not restricted to be the Copenhagen one (what is usually required in the stochastic approaches, cf. [5]), it allows the particle to be concentrated at a (starting-) point what seems to be much closer to an initiated classical description. This may also serve to contribute a physical interpretation which is suggested in [12] considering the exited quantum particle from a multi-particle point of view.

2 Extension of classical mechanics via stochastic processes

Taken respectfully this section develops what the above cited inventors have called stochastic mechanics and Euclidian quantum mechanics. But its content is presented a little bit different from the originals and that is why it is without respect referred to as an "extension" simply. The extension is made at the *kinematic* level, that means one starts from the velocity $v(t,x) \in \mathbb{R}^d$ at time $t \in \mathbb{R}_+$ and position $x \in \mathbb{R}^d$ of the considered mass[2]point. The infinitesimal[3] change $dx(t) = x(t+dt) - x(t)$ of the particle's position for an fixed infinitesimal time dt then writes as

$$dx(t) = v(t,x)dt. \tag{1}$$

The velocity field $v \in \mathrm{C}^1(\mathbb{R}_+, \mathbb{R}^d)$ of a classical mass-m-particle is determined by an *outer* force $F(t,x) \in \mathbb{R}^d$ via its time derivative by Newton's *dynamical* law $m\, dv(t,x) = F(t,x)\, dt$, with $F(t,x) = -\nabla V(t,x)$. V symbolizes the potential (*exterior* energy), and w.l.o.g. $m := 1$.

The reader may wonder about the initiated "splitting of terminology" or may simply ask after (the velocity field u of) an *inner* force (or *interior* potential U). As a comfort to the impatient one let's anticipate that the latter

[1]NSA is understood in Robinson's sense here, but the terminology is sometimes used in Nelson's way. So *(un)limited* is preferred to (in)finite, but *hyperfinite* is used instead of IST-finite.

[2]also called (quantum) *particle*

[3]At this stage nonstandard terminology is perhaps more virtuously applied (for mnemonic reasons) than seriously required; add the missing integral signs and everything reads in the standard manner.

corresponds to the Bohmian quantum potential, cf. [6]. The probabilistic extension is constructed by adding the *quantum fluctuation*, $\hbar\, u(t,x)\, dt + \sqrt{\hbar}\, dB_t$, to the right hand side of (1) and considering $x(\cdot)$ to be a stochastic process X (i.e. a family of measurable[4] maps in $\mathbb{R}^d$ indexed by elements of a certain time interval) where $\hbar < 1$ is the constant[5] related to a quantum's action, and B denotes the Brownian motion. The indicated duality between outer and inner seems to be the crucial point for the final success of modeling quantum mechanics from a classical point of view. This will therefore be implemented in the first definition. But before it is given the physical ideas behind the probabilistic extension are explained. See the end of §18 of [11] for an classical quantum mechanical approach. One will then recognize the content of the announced definition completely.

Motivation 1 An stochastic model can preserve the classical picture of continuous trajectories by giving up their smoothness. It keeps the correspondence of the mass-particle to a point in coordinate space, at least ideally, by supposing this property to the solutions of the stochastic equations, even when those are described by a probability measure. But the concept of a finite local velocity is given up. With relativity in mind[6] this does not sound promising at a first look, but those infinities would only occur with measure zero (after a finite time), that means they are most unlikely.

Interpretation 2 For $\hbar \to 0$ the quantum fluctuation disappears, this shows the compatibility with classical mechanics.

Motivation 3 An unspoken prerequisite of classical mechanics is the possibility of permanent observation of the trajectory. Light does usually not disturb a flying football. Observing[7] the ball with wind instead wouldn't be fun for an intense player. But at the quantum level light is windy. Any observation requires interaction. This fact is taken account of, but it is stuck to the possibility of permanent observations. Therefore the quantum fluctuation term models a continual mechanical disturbance (by the Brownian motion) of the particle together with its interior inertia (by the drift).

Definition 4 The stochastic differential equation (SDE) in the Itô sense

$$dX_t = v(t, X_t)dt + \hbar\, u(t, X_t)dt + \sqrt{\hbar}\, dB_t \tag{2}$$

is called *quantum diffusion law* in the (exterior) potential V, if the exterior and interior drift, the so called *current velocity* (field) v and *osmotic velocity* (field) u respectively, have to transform like

$$v(t,x) \mapsto v^*(t^*, x) := -v(t,x), \quad u(t,x) \mapsto u^*(t^*, x) := u(t,x), \tag{3}$$

under a time reflection[8] $t \mapsto t^* := -t$, to leave the kinematic law (2) unchanged for the stared (time-reflected) magnitudes. Its continuous solutions would then, according to Motivation 1, simply deserve the name *quantum diffusions.*

[4] where the underlying measure space would have to be determined
[5] on the physical scale it holds $\hbar \ll 1$ for the so called Planck constant.
[6] only; ...extension to relativity will not be a topic here
[7] in spite the fact that the author has no clue how to realize this
[8] Its nontrivial outcome belongs to the (new) freedom that a stochastic model offers.

To illustrate the definition Itô's formula is applied giving the corresponding Fokker-Planck equation

$$\partial_t \rho = -\nabla \cdot \rho \,(v + \hbar\, u) + \frac{\hbar}{2} \triangle \rho, \tag{4}$$

cf. [9][9], p. 30, or [10], p. 324. But this is done only formally. The drift $\hbar u$ of (2) is singular if $\rho = 0$, this will be shown in the next section. So the SDE's coefficients do not satisfy the necessary regularity conditions for an classical approach, e.g. (global) Lipschitz conditions are required in the cited literature, on p. 92 and p. 319 respectively. However one can avoid arguing "formally" by considering several copies of the diffusion (2) in disconnected regions $\{x \,|\, \rho \geq c\}$ where $0 < c \in \mathbb{R}$ is a constant chosen arbitrary small.

At the end of the paper it will have been shown that for certain potentials V (cf. next section) one can add the "rest", i.e. the x with $\rho(x) = 0$, the so called *nodes*, to the domain of definition without changing the process' separation behavior.

Applying the proposed rules, performing the time reflection and dropping the star again, this leads to $-\partial_t \rho = \nabla \cdot \rho \,(v - \hbar\, u) + \frac{\hbar}{2} \triangle \rho$, Now adding this one to (4), $2\hbar \nabla \cdot \rho \, u = \hbar \, \triangle \rho$, or subtracting it from (4) respectively, one finally gets the next two statements.

Corollary 5 The osmotic velocity u of the quantum diffusion (2) is determined by $\frac{\nabla \rho}{2\rho}$ up to a time-dependent constant.

Interpretation 6 Subtraction reproduces $0 = \partial_t \rho + \nabla \cdot \rho\, v$, the continuity equation of the probability density with the outer current $\rho\, v$. That means ρ is driven by the outer force only and can therefore be regarded as a mass distribution, i.e. $\int x \rho \, dx$ are the classical trajectories.

In a stochastic framework mean values correspond to classical magnitudes. This is the expected way how classical concepts reappear.

Interpretation 7 It holds a stochastic Newton law $\frac{d^2}{dt^2} E[X_t] = -\nabla V(X_t)$ (for the conditional expectation $E[\cdot] := E(\cdot \,|\, \mathcal{P}_t^X)$ w.r.t. the associated filtration $\mathcal{P}_t^X = \sigma \langle (X_s)_{s \leq t} \rangle$. But there are several according to different definitions of the stochastic acceleration, e.g. Nelson's $(DD^* + D^*D)X/2$. There D and D^* respectively denote the *forward* and the *backward stochastic derivatives* for which $\frac{d}{dt} E[X_t] = E[DX_t] = E[D^* X_t]$.

See the literature for the general definitions. Here, i.e. for (2), they are identical to $DX_t = v(t, X_t) + \hbar u(t, X_t)$ and $D^* X_t = -(-v(t, X_t) + \hbar u(t, X_t))$, the backward and forward drift respectively.

There is of course a connection to conventional understood quantum mechanics but which finally (cf. Problem 27) tells that the latter itself is (still) needed for modeling (at the current state of art).

Interpretation 8 From the time dependencies in the definition (2) and by only supposing a starting distribution one realizes a quantum diffusion to be a

[9]Here the equation is also called Kolmogorov's forward. Caution, it is a lot of notation around.

Markov[10] process. According to the Copenhagen interpretation, i.e. setting the quantum particle's probability density $\rho := |\psi|^2$, that property corresponds to what physicists refer to as *reduction of the wavefunction* – the way they interpret the wavefunction ψ in Schrödinger's time-dependent equation.

Remark 9 Even for the time-independent Schrödinger equation the stochastic process is running, reconstructing the stationary distribution locally up to a multiple of the density $|\psi|^2$.

At the end of this introduction it should be mentioned how Heisenberg's uncertainty relation, the distinguished quantum theoretical necessity reappears.

Interpretation 10 Assuming ρ to vanish at the boundary (cf. next section) integration by parts yields $E[X_t \otimes \frac{D-D^*}{2} X_t] = \int x \otimes \hbar u(t,x)\, dx = \frac{\hbar}{2} \int x \otimes \nabla\rho(t,x)\, dx = -\frac{\hbar}{2} I$.

3 The "unattainability of nodes" property

This section will have a closer look at the numerical implications of the extension. Therefore the superposing classical motion is not taken into account any more, i.e. the corresponding velocity v is set to zero so that one only *deals with the stationary quantum diffusion* $dX_t = \hbar u(X_t)\, dt + \sqrt{\hbar}\, dB_t$.

Looking at the drift term $u(x) = \frac{\nabla\rho(x)}{2\rho(x)}$ of the stationary process (cf. Corollary 5) one realizes singularities for all x with $\rho(x) = 0$. Those x are called nodes referring to the property which, by the Copenhagen interpretation, has to hold for the wave function as well, i.e. $\psi(x) = 0$. In this section a physically motivated estimation for the drift term at a node is given. The considered node is put into the origin from now on, this will hold w.l.o.g.

According to the assumptions above the quantum particle is modeled by the time-independent Schrödinger equation $-(\hbar^2/2)\triangle\psi + V(x)\,\psi = E\psi$, with real valued ψ. Excluding ρ to be nontrivial ψ has to be a bounded state, i.e., if V is set to zero at the boundary then for the eigenenergy it has to be $E < 0$. Now some properties for ψ, extracted from §18 in [11], are listed – a global one due to the integrability (normalization) of ψ,

(i) ψ vanishes at the boundary – and some local ones

(ii) $\psi \in \mathcal{C}^1$ except of those x where $V(x) = \pm\infty$,

(iii) ψ is finite if V is so, and $\psi(x) = 0$ for $V(x) = \pm\infty$.

The local properties are certainly not be the weakest assumptions for a "function", or something generalized, satisfying a (Schrödinger) differential equation. However it is intended to estimate the wave function around a node to see what kind of restrictions on the drift $u = \frac{\nabla\rho}{2\rho} = \frac{2|\psi|\,\nabla|\psi|}{2|\psi|^2} = \frac{\nabla|\psi|}{|\psi|}$, at the node, are implied by Schrödinger's equation. Now the quantum diffusion, which is to model, looks like

$$dX_t = \hbar\, \frac{\nabla|\psi|}{|\psi|}\, dt + \sqrt{\hbar}\, dB_t. \tag{5}$$

[10]The second mentioned approach in the Introduction starts with a weaker time dependence identifying the model process' probability density by the product of a forward and backward diffusion process' density. But this finally leads back to a Markovian property.

Proposition 11 *If the potential V behaves like $-|\mathcal{O}(\|x\|^s)| \le V(x)$ at the node, i.e. $x \to 0$, for an $s > -2$ then it does not hold $\psi(x) = |\mathcal{O}(\|x\|^\alpha)|$ for $0 < \alpha < 1$.*

Proof. Putting the "wrong" ansatz into the Schrödinger equation, this leads to $-|\mathcal{O}(\|x\|^{\alpha+s})| \le \frac{\hbar^2}{2}\alpha(\alpha-1)\|x\|^{\alpha-2} + E\|x\|^\alpha$, i.e. $|\mathcal{O}(\|x\|^{2+s})| \ge \frac{\hbar^2}{2}\alpha(1-\alpha)$ due to the negative sign of E. But according to the choice of α the right hand side is positive and therefore the inequality is false (for small $x > 0$). $\square$

Interpretation 12 The restriction to $s > -2$ includes all non-singular potentials, as well as the (singular) Coulomb potential, but it excludes those of multi-poles. The consequences seem to be physically expected, cf. [11], §18 for further motivation.

To have the node property satisfied, i.e. $\psi(0) = 0$, it has to be $\alpha > 0$ for the asymptotic estimate. The proposition tells that $\psi(x) = |\mathcal{O}(\|x\|^\alpha)|$ is only possible for $\alpha \ge 1$. This restricts the behavior of $u(x) = \alpha \frac{x}{\|x\|}\|x\|^{\alpha-1}/\|x\|^\alpha$.

Corollary 13 Thus the SDE estimating the quantum diffusion at a node (in the origin) is $dX_t = \alpha\,\hbar \frac{X_t}{\|X_t\|^2}\,dt + \sqrt{\hbar}\,dB_t$ for $\alpha \ge 1$.

Motivation 14 Taking spherical coordinates one can *go over to one dimension* only (to the radial coordinate of course). The resulting stochastic process satisfying

$$dX_t = \alpha\hbar\frac{dt}{X_t} + \sqrt{\hbar}\,dB_t \quad \text{for } \alpha \ge 1 \tag{6}$$

is referred to as *stationary quantum diffusion at* (the right hand side of) *a node*. Here it is possible to make the additional generalization compared to the arguments in the "standard" literature by choosing the initial distribution ϵ_{x_0} to be a Dirac one, where $0 < x_0 \in \mathbb{R}$. This further implies an ambiguity of the starting distribution which would be allowed to choose. Usually (cf. introduction), according to the modeled object, it is discussed being built by the Copenhagen density $|\psi|^2$ only.

But, as in the previous section, it is not clear yet whether such a stochastic process really exists, and in the further exposition neither the probability distributions of (5) or (6) are explicitly calculated nor continuity-arguments are given, only estimates will be achieved (based on the distribution of Brownian motion). But the following arguments provide a way to the positive answer.

Remark 15 Showing the process' unattainability of the node (done in the next section) generally constitutes one part towards the proof of the existence of stochastic process. The other part of an SDE's existence proof would deal with the exclusion of an explosion, cf. [5], p. 426.

The latter would only be relevant if the quantum diffusion did not be completely enclosed by nodes. But this case is reducible to the first one just by considering the boundary being a node, cf. (i). In case the boundary is at infinity then the drift points away from infinity. Dropping the corresponding term out of (6) this directly leads to a Brownian motion, i.e. to an estimate against the possibility of explosions.

However the existence of the stochastic process (5) will have been proven at the end of the next section, and there is no better one to fit the probability distribution of a solution of the Schrödinger equation. But, as it will be concluded, it depends on the initial conditions of the process whether it reproduces the Copenhagen distribution globally. Looking at (5) one realizes the drift being independent of the normalization of ψ, i.e. of ρ.

4 The behavior at a node

With the help of NSA one can easily falsify the possibility of the particle's tunneling through a node. Only to show its departure away from an infinitesimal region of the node requires a more sensible discussion. The final answer is given in the theorem of the article, at the end of this section.

For all what follows it is enough to assume countable saturation for the nonstandard extension. One may start by choosing an unlimited $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ and taking the infinitesimal $\delta t =: 1/\nu$, to be the mesh of a hyperfinite lattice[11] $\mathrm{T} := \{n\,\delta t \,|\, 1/\delta t \geq n \in {}^*\mathbb{N}\} \subset {}^*[0,1]$ (having not more than one real number of $[0,1]$ sitting inside a mesh) which is also called *hyperfinite time line* on $[0,1]$.

The stationary process at the node (6) can be written in nonstandard fashion[12], by letting $n \in \mathrm{T}/\delta t$ and

$$\chi_{(n+1)\delta t} = \chi_{n\delta t} + \alpha\hbar\frac{\delta t}{\chi_{n\delta t}} + \sqrt{\hbar}\,(\beta_{(n+1)\delta t} - \beta_{n\delta t}) \quad \text{with } 0 < \chi_0 \in {}^*\mathbb{R}, \tag{7}$$

as a hyperfinite random walk in ${}^*\mathbb{R}$. Here Anderson's walk $\beta_{(n+1)\delta t} = \beta_{n\delta t} \pm \sqrt{\delta t}$, each case with probability $\mathrm{p} = 1/2$, and $\beta_0 = 0$ is applied. As [4] just tells, using the representation $B_t := {}^\circ\beta_{[t\nu]\delta t}$, $(\beta_\tau)_{\tau\in\mathrm{T}}$ serves as a nonstandard version of a Brownian motion $(B_t)_{t\in[0,1]}$ being based on Loeb's measurable space $(\{-1,+1\}^\nu, L({}^*\mathcal{P}(\{-1,+1\}^\nu)))$. Loeb's "$\pm 1$ coin tossing" corresponds to Anderson's successive $\pm\sqrt{\delta t}$ addition, or more technical

$$\beta_{n\delta t}(\omega) = \sum_{i=1}^{n} \sqrt{\delta t}\,\omega_i \;\text{ for any } \omega \in \{-1,+1\}^\nu =: \Omega, \text{ and}$$

with probability $p = 1/2$ for an atomic event $\{\beta_{n\delta t}(\omega) - \beta_{(n-1)\delta t}(\omega) = \sqrt{\delta t}\,\omega_n\}$ (a single coin toss), $\mathrm{P}\{\cdot\} := \mathrm{p}^\nu\,\mathrm{card}\{\cdot\} = \mathrm{card}\{\cdot\}/2^\nu$ defines the *hyperfinite counting measure* which extends to Loeb's measure $L(\mathrm{P})$. The latter defines the measure of the stochastic process, and the finite dimensional distributions of Anderson's Brownian motion B can be expressed by P, e.g. (the 1-dim ones) for any $r \in \mathbb{R}$ and $t \in [0,1]$

$$L(\mathrm{P})\,\underbrace{\{\omega\in\Omega \,|\, B_t(\omega) \leq r\,\}}_{\in L({}^*\mathcal{P}(\{-1,+1\}^\nu))} = \lim_{n\to\infty} {}^\circ\mathrm{P}\,\underbrace{\{\omega\in\Omega \,|\, \beta_{[t\nu]\delta t}(\omega) \leq r + 1/n\,\}}_{\in {}^*\mathcal{P}(\{-1,+1\}^\nu)}. \tag{8}$$

It is not the aim to find an expression for the finite dimensional distributions of X this way. But being able to estimate χ away from the node ensures (cf.

[11] By the way, this models (a lot of) real analysis, and especially stochastic processes, in an "infinitesimally close" or "nearly by" fashion as the reader, unfamiliar with this terms, could become convinced from studying [2] or [14].

[12] The reader may excuse the individualistic usage of notation, caution, it only looks similar to [2] or [4].

Remark 15) the existence of the stationary quantum diffusion (6) belonging to χ, which according to (8) and (7) is the standard process X.

Remark 16 Modeling the start of the hyperfinite walk χ by $\chi_0 = x_0$, and $0 < x_0 \in \mathbb{R}$, which translates (and therefore justifies Motivation 14) to the considered initial distribution ϵ_{x_0} of X, one gets

$$X_t = {}^\circ\chi_{[t\nu]\delta t} \ \ L(\mathrm{P}) \text{ a.s. for any } t \in (0,1]. \tag{9}$$

Later on the following density representation of β, based on Bernoulli's combinatorial scheme, will have been used.

Proposition 17 $\mathrm{P}\{\beta_{n\delta t} = m\sqrt{\delta t}\} = \binom{n}{m}\frac{1}{2^n}$ *for all hyperfinite* $m \leq n \in T/\delta t$ *if* m *and* n *are both even or odd.*

The *Proof* does not contain any nonstandard arguments except of the remark that "hyper" comes right from transfer (of the standard Bernoulli scheme). So the number of paths can be described by a hyperfinite "isosceles" (implying the even/odd condition) Pascalian triangle, i.e., supposing the walk to be at the m^{th} position coefficient at the n^{th} time step fixes the cardinality (number of possible ways), up to this time, to be $\binom{n}{m}$. Including (i.e. multiplying) the cardinality $2^{\nu-n}$ of the remaining time steps, up to ν, finally leads to the proposed formula for the hyperfinite counting measure. $\square$

Back at the node-crossing problem one might firstly try to argue the Brownian motion's worst case using the (deterministic) process

$$x_{(n+1)\delta t} := x_{n\delta t} + \alpha\hbar\frac{\delta t}{x_{n\delta t}} - \sqrt{\hbar}\sqrt{\delta t} \quad \text{with} \quad x_{0\delta t} := \chi_0 > 0. \tag{10}$$

This is an estimate from the left, $x_{n\delta t} \leq \chi_{n\delta t}$ p a.s. Thus, for each (infinitesimal small)

$$x_{n\delta t} < \ \alpha\sqrt{\hbar}\sqrt{\delta t} =: x_\infty \ \approx 0 \tag{11}$$

the particle's motion turns away from zero indeed, $x_{(n+1)\delta t} > x_{n\delta t}$, i.e., $\chi_{n\delta t}$ keeps positive p a.s. But simply to throw away the probability will not lead to a satisfying result. The particle would move towards a fixed point being still infinitesimal close to zero.

Lemma 18 *The sequence (10) has the only fixed point* x_∞*, defined in (11), which especially is attracting. (For* $\alpha > 1$ *an iteration moves the particle, being in a certain region, as indicated below.)*

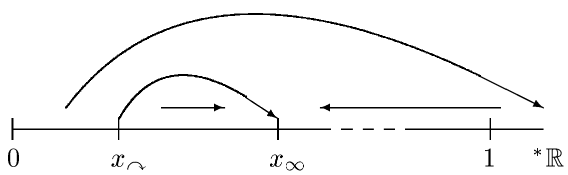

Figure 1: The iteration map $x_{n\delta t} \mapsto x_{(n+1)\delta t}$.

Proof. (10) immediately yields the single fixed point. From the quotient

$$q := \frac{x_{(n+1)\delta t} - x_\infty}{x_{n\delta t} - x_\infty} = 1 - \frac{\sqrt{\hbar}\sqrt{\delta t}}{x_{n\delta t}} =: 1 - \frac{x_\frown}{x_{n\delta t}} \tag{12}$$

one reads off that it is attracting for every $x_0 > x_\frown$. Now all possible cases for x_0 are considered. If $x_{n\delta t} > x_\infty$, this includes $x_0 > x_\infty$, one has $0 < 1-(1/\alpha) < q < 1$, i.e., $x_\infty < x_{(n+1)\delta t} < x_{n\delta t}$. (That means, once being on the right hand side of the fixed point the particle stays there and moves towards the fixed point.) For $x_0 < \sqrt{\hbar}\sqrt{\delta t} = x_\frown < x_\infty$, the first iteration gives, $x_{\delta t} > x_\infty$, the prerequisite for the previous case (puts the particle to the right hand side of the fixed point). And finally, for $x_\frown < x_0 < x_\infty$, it holds again $0 < q < 1-(1/\alpha)$, here it means, $x_{n\delta t} < x_{(n+1)\delta t} < x_\infty$. (The particle stays infinitesimal closed to zero all the time.) □

And it comes even worse, looking at the arrow pointing to the left in Figure 1. As the next proposition states, a particle having a limited distance from the fixed point (on the right hand side of it), and riding on x needs only an infinitesimal time to reach an infinitesimal neighborhood of the node.

Proposition 19 *Let $x_\infty < x_0 \in \mathbb{R}$. After an infinitesimal time, $\approx \frac{x_0}{\sqrt{\hbar}}\sqrt{\delta t}$, the process x is infinitesimal close to zero.*

Proof. By induction on all $1 \le n \le x_0/(\sqrt{\hbar}\sqrt{\delta t})$ in ${}^*\mathbb{N}$ one gets

$$x_{n\delta t} \le x_0 - n\sqrt{\hbar}\sqrt{\delta t} + \alpha\hbar\,\delta t \sum_{m=0}^{n-1} \frac{1}{x_0 - m\sqrt{\hbar}\sqrt{\delta t}},$$

repeatedly using the estimate $x_{n\delta t} \ge x_{(n-1)\delta t} - \sqrt{\hbar}\sqrt{\delta t}$, which is obtained from (10). The sum is certainly understood as a hyperfinite one.

Let $\eta := x_0/(\sqrt{\hbar}\sqrt{\delta t}) - [x_0/(\sqrt{\hbar}\sqrt{\delta t})] \in {}^*[0,1)$. After $N := [x_0/(\sqrt{\hbar}\sqrt{\delta t})]$ time steps δt, i.e. after the infinitesimal time $N\delta t = x_0\sqrt{\delta t}/\sqrt{\hbar} - \eta\delta t \approx x_0\sqrt{\delta t}/\sqrt{\hbar}$, the above inequality reduces to

$$x_{N\delta t} \le \eta\,\delta t + \alpha\sqrt{\hbar}\,\sqrt{\delta t}\sum_{m=0}^{N-1} \frac{1}{\eta + N - m}.$$

The function inside the sum is increasing, m taken to be in ${}^*\mathbb{R}$, and therefore it can be dominated by the Riemann integral in ${}^*\mathbb{R}$.

$$\sum_{m=0}^{N-1} \frac{1}{\eta + N - m} \le \int_0^{N-1} \frac{dm}{\eta + N - m} \le \ln\frac{N+\eta}{1+\eta} < \ln(N+1),$$

i.e.,

$$x_{N\delta t} < \eta\,\delta t + \alpha\sqrt{\hbar}\,\sqrt{\delta t}\,\ln\frac{x_0 + \sqrt{\hbar}\sqrt{\delta t}}{\sqrt{\hbar}\sqrt{\delta t}}.$$

Using $0 \lessapprox x_\infty < x_{N\delta t}$ from the previous lemma and applying the property of $\sqrt{\delta t}$ to be infinitesimal (e.g. via l'Hospital's rule) result in

$$0 < x_{N\delta t} \lessapprox \left(\alpha\sqrt{\hbar}\,\ln\frac{x_0}{\hbar}\right)\sqrt{\delta t}\,\ln\frac{1}{\sqrt{\delta t}} \approx 0,$$

and the proof is finished. $\square$

It was shown so far, the particle being somewhere out of a node, will never cross nor reach it. This also holds for $\alpha = 1$. In the remaining case $\alpha < 1$, i.e. if the step size (lattice) is bigger than the fixed point being away from zero, the walk just runs over the node. But anyway without using probabilistic arguments one would still have to assume the particle being sucked straight (within an infinitesimal time) into a fixed point being in an infinitesimal neighborhood of the node. But this is in fact not the case.

As the proof of the following lemma will show, another estimation λ from the left of the hyperfinite walk χ, being different from (10), can actually be done (by dropping drift terms out of (7) carefully); for $n \in \mathrm{T}/\delta t$

$$\lambda_{n\delta t} := \sqrt{\hbar}\, r(\beta_{n\delta t}) + (\alpha - 1)\sqrt{\hbar}\sqrt{\delta t} \tag{13}$$

where $r(\beta) := \beta^+ - \beta^-$ denotes *the reflected Anderson walk at zero*, with $\beta^\pm := H(\pm\beta)\sqrt{\delta t}$, and $H(x) := \left\{ \begin{array}{ll} 1\,, & x \geq 0 \\ 0\,, & x < 0 \end{array} \right.$ defining a step function.

Lemma 20 *Let* $\chi_0 > x_\infty$*, and* $\alpha > 1$*. Then* $\lambda_\tau \leq \chi_\tau$ P a.s. *for* $\tau \in \mathrm{T}$.

Proof. Start χ at $\chi_0 = x_\infty$, and drop the drift term $\alpha\hbar\delta t/\chi_{n\delta t} \geq 0$ out of (7) whenever χ is not at x_∞ but also there if the (following) Anderson step points to the right. This leads to a modified χ, denoted by $\check{\chi}$. By this construction one gets $\check{\chi}_\tau \leq \chi_\tau$ for $\tau \in \mathrm{T}$; and it remains to proof that the modified process really is built by (13), i.e., $\check{\chi}_\tau = \lambda_\tau$ P a.s.

Taking into account that at x_∞ an Anderson step pointing to the left cancels with the drift (This actually is the definition of the fixed point x_∞, cf. Lemma 11.) the modified process reads as

$$\check{\chi}_{(n+1)\delta t} = \check{\chi}_{n\delta t} + \sqrt{\hbar}\sqrt{\delta t} \cdot \left\{ \begin{array}{rl} 1 & \\ 0 & \text{if } \check{\chi}_{n\delta t} = x_\infty \\ -1 & \text{otherwise} \end{array} \right. \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \quad ,$$

with the same probability p=1/2 for both cases. The corresponding walk is illustrated below.

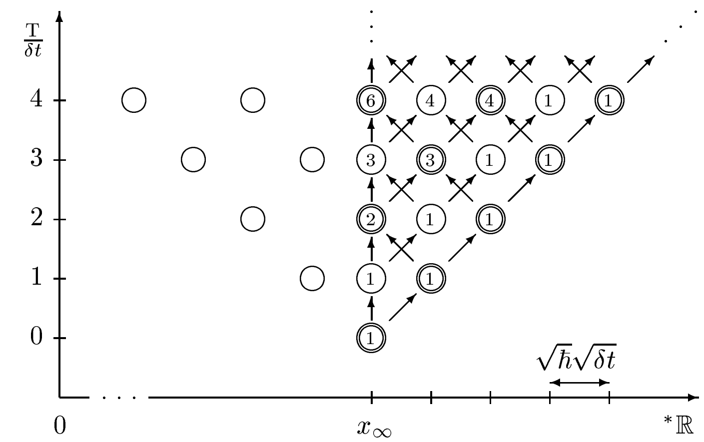

Figure 2: The modified Pascalian triangle.

Each arrow in the figure represents one atomic choice (made with p=1/2), and therefore the number of possible ways (printed into the circles) is obtained by recursive addition of the previous number of arrows (just as in the Pascalian triangle).

Now remembering the Bernoulli scheme of Anderson's walk one recognizes the claim $\check{\chi} = \lambda$. Mirroring the unlabeled circles from the isosceles Pascalian triangle (of Proposition 17) at x_∞, this corresponds to $-\beta^-_{n\delta t} + x_\infty$, and moving them $\sqrt{\hbar}\sqrt{\delta t}$ to the left; i.e. producing $\beta^+_{n\delta t} - \beta^-_{n\delta t} + x_\infty - \sqrt{\hbar}\sqrt{\delta t}$, and one obtains the modified Pascalian triangle corresponding to λ. $\square$

Remark 21 The reflection operator $r(\cdot)$ at zero of course posses the much simpler representation by the absolute value $|\cdot|$ which mnemonically already denotes its standard analogue. Using this notation one has to keep in mind that the modeling node in (6) is set to zero by hand.

The lemma straightforwardly translates into standard language.

Theorem 22 *Let $\alpha \geq 1$. Then $\sqrt{\hbar}|B_t| \leq X_t$ $L(P)$ a.s. for all $t \in [0,1]$.*

Proof. The lemma does not exclude $1 < \alpha \approx 1$. Therefore one can, using the overspill principle, include $\alpha = 1$ to the prerequisites here. Certainly $x_\infty < x_0$ for all $0 < x_0 \in \mathbb{R}$ (cf. Remark 16), i.e., the lemma applies. Together with the above remark and the identifications (9) and (13) the observation that ${}^\circ\lambda_{n\delta t} = \sqrt{\hbar}\,{}^\circ r(\beta_{n\delta t})$ leads to the stated result. For $t \in [0,1]$ it holds

$$\sqrt{\hbar}|B_t| = \sqrt{\hbar}\,{}^\circ r\big(\beta_{[t\nu]\delta t}\big) = {}^\circ\lambda_{[t\nu]\delta t} \leq {}^\circ\chi_{[t\nu]\delta t} = X_t \qquad L(\mathrm{P}) \text{ a.s.}$$

$\square$

This means one can estimate the quantum stochastic process X away from the node by (the probability distribution of) a Brownian motion reflected at this node. Thus one has the local existence of the stationary process at the node.

Remark 23 Following the method here α can not be chosen smaller than 1 as one realizes by looking at (13). But the time interval, the process is considered on, can be extended to any compact one simply by enlarging T.

5 Conclusions

Applying well known properties of a Brownian motion (which also hold for the reflected one), dealing with the local and with the recurrence behavior (respectively), the following (two) additional conclusions from the theorem of last section can be formulated.

Corollary 24 The quantum particle (supposed to be) modeled by a quantum diffusion described in section 3, i.e. by (5) finally, almost surely does not stay in an infinitesimal[13] neighborhood of a node longer than an infinitesimal time. And the quantum particle almost surely cannot enter an infinitesimal region of the node more than finitely many times within a finite time.

[13]The author apologizes for his lazy (nonstandard) tongue here.

Even if the "a.s." refers to the probability space of a Brownian motion and the language is a bit nonstandard here, nothing so far did stop a rigorous nonlocal definition of quantum diffusion which, as proven above, would not tunnel through nor being captured in the nodes. Due to Proposition 11 this holds for all potentials $V(x) \geq \mathcal{O}(\|x\|^{-2})$ for $x \to 0$. But it should clearly be stated again that nothing here has been proven about continuity, i.e. about the (commonly used) attribute "diffusion". Achieving this by using NSA could be a topic of subsequent work.

Additional to Nelson's discussion in [13] (and a bit opposite to his "heuristic" arguments on p. 82) the impossibility of "communication through the nodes" would also allow other physical interpretations. This is illustrated with the simplest model that produces discrete eigenstates.

Example 25 Let $V(t,x) = 0$ be a constant, time-independent potential in a one dimensional compact space, e.g. in $[0,1] \subset \mathbb{R}$, and being equal to $+\infty$ elsewhere in $\mathbb{R}$. Thus $X_t \in [0,1] \subset \mathbb{R}$ (the process' configuration space) for $t \in \mathbb{R}_+$. The Schrödinger equation gives the following set of wave functions (certainly not time dependent) $\psi(t,x) = \sqrt{2}\sin(\pi n x)$, $n \in \mathbb{N}$, for the energy eigenvalues $E_n = \pi^2\hbar^2 n^2/2$. The probability density resulting by the Copenhagen interpretation is $\rho(t,x) = 2\sin^2(\pi n x)$, with nodes at 0 and i/n for $i \in \{1,\dots,n\}$.

Caution[14], this is not the *stationary probability density* $\tilde{\rho}$ of the belonging quantum diffusion (5), i.e. $dX_t = \hbar\, dt/\sin(\pi n X_t) + \sqrt{\hbar} B_t$, if the initial probability distribution $P_{X_0} = P\{X_0 \in \cdot\}$ does not yield the same value $p_i = P_{X_0}(G_i) = \int_{G_i} \rho(0,x)\,dx = 1/n$ for all disconnected regions $G_i = ((i-1)/n, i/n)$ separated by nodes. Following [13], p. 81, one obtains $\tilde{\rho} = \rho \sum_{i \leq n} P_{X_0}(G_i) 1_{G_i}$. Despite of Nelson's conclusions one could model the situation of different p_i's, as well as the "proper" one, with n independent processes on configuration spaces G_i (and possibly) with (an arbitrary) starting distributions $P_{X_0}(\,\cdot \cap G_i)$ of n particles.

Interpretation 26 If one takes the stochastic picture serious the Copenhagen interpretation only holds in separated regions, e.g., once the particle has entered a region enclosed by nodes (especially for excited states) it stays there. This way a maximal number (some may have degenerated initial distributions, i.e. $P_{X_0}(G_i) = 0$) of n quantum particles, each having the eigenenergy E_n, would fit into the configuration space $[0,1]$. And this should hold for all $n \in \mathbb{N}$ of the example.

Further one may suggest the single quantum particle to "consists" of some sort of sub-particles which could differ in number depending on the particle's energy E_n (but being $\leq n$). In [12], on p. 161, a similar idea is formulated. There "photons" are suggested to play the rôle of sub-particles. Providing a single quantum particle with the energy E_n then in addition to the ground state particle, which may somehow be regarded as a sub-particle as well, $n-1$ "extra" photons are to consider "being" inside the configuration space $[0,1]$. One might understand the action of a surrounding photon (on the quantum particle) as a quantum fluctuation, a disturbance resulting in a Brownian motion which is tempered (may be as a collective effect) by the osmotic drift.

Certainly second quantization (or QFT) clears the picture. But it seems remarkable that the classical quantum model here already demands for it.

[14]remembering Remark 9 and the end of section 3

Another aspect should be emphasized at the end.

Problem 27 The stochastic process here, build by the quantum fluctuation, is only fitting the quantum mechanical description via Schrödinger's equation. Looking for an intrinsic stochastic model, reproducing the discrete eigenvalues etc., might be a challenging exercise for further research. This perhaps is already initiated by a different approach, called quantum chaos. However a stochastic theory should straightforwardly lead to numerical simulations.

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