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Nontrivial extreme networks.
Singularities of Lagrange functions
and criterion of extremality

by

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Introduction

In the present work we investigate branching extremals (extreme networks) for one-dimensional variational functionals of Lagrange type. We mean the extremality in the following broad sense: admissible deformations can split vertices and thus change the topology of initial networks. To distinguish the networks whose vertices are not allowed to be splitted, and the networks whose vertices can be splitted, we call the former ones by **parametric networks** (i.e., the networks whose parameterizations are fixed), and the latter ones by **networks–traces**, or simply **traces**.

The authors are mostly interested in the following two questions:

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- what Lagrangians' properties lead to the existence of nontrivial branching extremals,
- what is the local structure of the branching extremals.

Notice that, generally speaking, an additional freedom appearing due to the possibility to split the vertices can essentially reduce the class of extreme networks–traces with respect to the class of extreme parametric networks and, moreover, it can be a reason for nonexistence of extremals. It turns out that the existence of nontrivial extreme networks is related closely with the presence of singularities of Lagrangian, see Theorem 3.1 and Corollaries 3.1–3.4. Triviality of extremals is understood here in the following sense: an extreme network is said to be **trivially extreme**, if (1) each of its edges is extreme with respect to arbitrary deformations (in particular, with respect to deformations moving boundary vertices of the network), and (2) every pointwise curve whose image coincides with an arbitrary vertex of the network is also extreme with respect to arbitrary deformations. In particular, if we cut a trivially extreme network over an arbitrary set of its vertices, then as a result we get the union of networks each of which is extreme too. The authors proved that if a Lagrangian is smooth, then the corresponding functional does not have a nontrivial extreme network, see Corollary 3.1.

The authors introduced a class of so-called quasiregular Lagrangians, i.e., the Lagrangians having in some sense the simplest singularities whose presence is necessary for the appearance of nontrivial branching extremals. For such Lagrangians the authors obtained a criterion of networks' extremality, see Theorem 3.2. The main difficulty here is the necessity to control all possible splittings of the network's vertices (the number of such splittings is infinite, generally speaking).

Besides that, a criterion of a parametric networks extremality (Theorem 2.1), and also a criterion of a trace extremality under the assumption of quasiregular Lagrangian's smoothness (Theorem 3.1) are obtained.

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1 Preliminaries

We consider graphs from a topological point of view. A **topological graph** G is a topological space obtained from a finite number of segments $\{I_\alpha\}$ by means of some gluing over their ending points. Let $\pi\colon \sqcup_\alpha I_\alpha \to G$ be the corresponding canonical projection. The images of the interiority of the segments I_α under the mapping π are called the **edges** of the graph G. The π-images of the ending points of the segments I_α are called the **vertices**. If for each α a specific segment $[a_\alpha, b_\alpha] = I_\alpha$ of the real axis is fixed, then such graph is called **framed**. Notice that since the ends of an arbitrary segment $[a_\alpha, b_\alpha]$ are ordered in the natural way, namely, $a_\alpha < b_\alpha$, then the orientation of each edge of a framed graph is fixed and we can speak about the **beginning** and **ending** vertices of the edges.

A homeomorphism $\varphi\colon G_1 \to G_2$ of framed graphs is called an **equivalence** if it takes vertices onto vertices and for each edge $[a,b]$ of the graph G_1 the mapping $\varphi|_{[a,b]}$ is the identical mapping of the segment $[a,b]$ onto itself. Framed graphs G_1 and G_2 are said to be **equivalent** if their exists an equivalence $\varphi\colon G_1 \to G_2$.

Assume that some subset B of vertices of a graph G is fixed. Such graph G is said to be a **graph with the boundary** $\partial G = B$. Vertices from ∂G are called **boundary** or **fixed** and all remaining vertices are called **interior** or **movable**. An edge of the graph incident to a boundary vertex is also called **boundary** and an edge not incident to boundary vertices is called **interior**.

Definition. Let G be an arbitrary connected framed graph and ∂G be some its boundary. A **parametric (framed) network of the topology** G on a manifold W is a continuous mapping Γ from G into W. The graph G in that case is called the **parameterizing graph of the parametric network** Γ, or its **topology**.

All the terminology of Graph Theory and Topology can be naturally applied to the case of parametric networks. For example, the restrictions of a mapping Γ onto vertices, edges, a boundary, a connected subgraph of the parameterizing graph, a local graph, etc., are called **vertices**, **edges**, a **boundary**, a **subnetwork**, etc., of the parametric network Γ.

Remark. Above we represented each framed graph as a collection of segments factorized over an equivalence gluing some ending points of these segments. In the same way, a parametric network can be represented as a collection of continuous curves in a manifold some of whose ending points are identified.

A parametric network $\Gamma\colon G \to W$ is said to be **smooth** (**regular, piecewise-smooth, piecewise-regular**) if the restriction of the mapping Γ onto the closure of each edge of the graph G is such a curve. Notice that the notion of a smooth parametric network is a natural generalization of the notion of a smooth curve.

Let $\Gamma\colon G \to X$ be an arbitrary parametric network and $I = [a,b]$ be a segment.

Definition. A continuous mapping $\Psi\colon G \times I \to W$ such that $\Psi(g,a) = \Gamma(g)$ for all $g \in G$ is called a **deformation of the parametric network** Γ. If the initial parametric network Γ is smooth (regular, piecewise-smooth, piecewise-regular), then we will assume that each parametric network $\Psi(\cdot,t) = \Gamma_t$ is also such a network and that for each edge e of the graph G the restriction of the mapping Ψ onto $\bar{e} \times I$ is smooth (here by $\bar{e}$ we denote the closure of e). The family of the velocity vectors of the curves $\Gamma_t(g)$ at the initial moment $t = 0$ over all points $g \in G$ is called the **field of deformation** Γ_t.

We introduce an equivalence ρ on the class of all parametric networks on W as follows. We say that a parametric network Γ_1 **can be projected onto** Γ_2 if there exists a projection $\pi\colon G_1 \to G_2$ such that $\Gamma_2 \circ \pi = \Gamma_1$. Here the projection $\pi\colon G_1 \to G_2$ is a canonical projection of the space G_1 to the quotient space $G_2 = G_1/H$, where H is a subgraph in G_1. The projection π induces the mapping $\pi\colon \Gamma_1 \to \Gamma_2$ of one network onto another one which is also called a **projection**. Two parametric networks $\Gamma_i\colon G_i \to X$ are said to be **ρ-adjacent** if one of them can be projected onto another. Notice that the relation of ρ-adjacency is reflective and symmetric, but not transitive. We extend this relation upto an equivalence relation as follows. Two parametric networks Γ and Γ' are said to be **ρ-equivalent** if there exists a finite sequence $\{\Gamma = \Gamma_1, \Gamma_2, \dots, \Gamma_n = \Gamma'\}$ of parametric networks such that each two neighboring networks Γ_i and Γ_{i+1} are ρ-adjacent. The classes of ρ-equivalence are called (framed) **networks–traces**, or simply (framed) **networks**. If a parametric network Γ is contained in a trace Υ, then we will write $\Upsilon = [\Gamma]$.

A **canonical representative of a trace** Υ is a parametric network $\Gamma \in \Upsilon$ such that each parametric network Γ' from Υ can be projected onto Γ. One can prove that every trace possesses exactly one (up to equivalence) canonical representative, see [4].

A (local) **deformation of a trace** Υ is an one-parametric family $\Upsilon_t = [\Gamma_t]$, $t \in [t_1, t_2]$, of networks, where Γ_t is a deformation of some parametric network $\Gamma = \Gamma_{t_1}$ from Υ such that for $t > t_1$ each parametric network Γ_t is the canonical representative of the network Υ_t.

2 Extreme Parametric Networks Local Structure

Let W be a smooth manifold, $\pi : TW \to W$ be the tangent bundle, $L : TW \to \mathbb{R}$ be a continuous Lagrangian. Let Ω be the space of all piecewise smooth curves on W and $\psi_L\colon \Omega \to \mathbb{R}$ be the classical variational functional corresponding to the Lagrangian L, i.e.,

$$\psi_L(\gamma) = \int_a^b L\big(\gamma(t), \dot\gamma(t)\big)\, dt,$$

Speaking about a mapping of one network $\Gamma_1\colon G_1 \to W$ to another one $\Gamma_2\colon G_2 \to W$, we mean a mapping of the corresponding sets $\Big\{\big(g, \Gamma_i(g)\big)\Big\}$.

For simplicity we restrict ourselves only by the case of autonomous Lagrangians, however, all the results can be obtained in non autonomous case too.

where $\gamma\colon [a,b]\to W$ is a curve from Ω.

Let G be a framed graph, ∂G be some its boundary, $\beta\colon \partial G\to W$ be an arbitrary mapping, and $\Gamma\colon G\to W$ be an arbitrary piecewise smooth parametric network with the boundary $\partial\Gamma=\beta$. On the set of such networks the following functional Ψ_L is defined: the value of Ψ_L on a network Γ is equal to the sum of the values of the functional ψ_L on the edges of the network Γ.

Let (x,ξ) be an arbitrary point from TW. We assume that the Lagrangian L is twice continuously differentiable at the point (x,ξ). Denote by $p(x,\xi) = L_\xi(x,\xi)$ the following point from the cotangent space T^*_xW. If x^i are some coordinates on W in some neighbourhood of the point x and ξ^i are the corresponding coordinates in the tangent space T_xW, then the ith component of the covector $p(x,\xi)$ has the form

$$p(x,\xi)_i = \frac{\partial L}{\partial \xi^i}(x,\xi).$$

The covector $p(x,\xi)$ is called a (generalized) **impulse at the point** (x,ξ). Further, we define another covector $[L](x,\xi)$ as follows. We consider an arbitrary smooth curve $x(t)$ such that $\dot x(0)=\xi$. We put:

$$[L](x,\xi)_i = \frac{d}{dt}\Big|_{t=0}\Big\{p_i\big(x(t),\dot x(t)\big)\Big\} - \frac{\partial L}{\partial x^i}\big(x(0),\dot x(0)\big).$$

It is well known that this definition does not depend on the choice of the curve $x(t)$, see [1]. The covector $[L]$ is called the **Lagrangian derivative** of the function L (at the point (x,ξ)).

A curve $\gamma(t)$ is called **quasiregular**, if it is either regular, or pointwise (i.e., a mapping into a point). A parametric network Γ is called **quasiregular**, if all its edges are quasiregular curves. An edge of the graph G parameterizing a regular (a pointwise) edge of the network Γ is called **regular** (respectively, **pointwise**).

A Lagrangian L is called **quasiregular**, if

(1) the function L is smooth on all TW except, may be, the zero section $W_0\subset TW$;

(2) the restriction of the function L onto the zero section $W_0\subset TW$ equals zero;

(3) for each vector $\zeta\in T_{(P,\xi)}(TW)$ there exists a derivative $\zeta(L)$ of the function L with respect to the direction of the vector ζ;

(4) for an arbitrary smooth deformation γ_ε, $\varepsilon\in[0,1]$, of a pointwise curve γ_0, such that all the curves γ_ε, $\varepsilon>0$, are regular the function $\frac{\partial}{\partial\varepsilon}L(\gamma_\varepsilon(t),\dot\gamma_\varepsilon(t))$

Here and below we understand a derivative of a function with respect to a direction as a limit for $\varepsilon\to 0+$. In particular, derivatives of a function with respect to opposite directions need not be equal, generally speaking.

is continuous on (t,ε). That condition is called the **concordance condition**.

As an example of quasiregular nonsmooth Lagrangian one can consider the Lagrangian corresponding to the length functional on a Riemannian manifold. More general example can be obtained by means of a norm given on each tangent space of a manifold and smoothly depending on points and nonzero tangent vectors. Notice that strict convexity of the norm is not assumed.

The properties defining quasiregular Lagrangians appeared under an attempt to understand what characteristics of Riemannian length functional lead to existence of nontrivial extreme networks. The first property implies that regular edges of an extreme network satisfy the standard Euler–Lagrange equations. The second property is necessary to define correctly the functional on the space of the networks–traces (to do that, we need the vanishing of the functional on pointwise edges). The third and the fourth properties seem to be necessary for using the standard technique of Calculus such as differentiation of integrals depending on a parameter.

The following Assertion calculates for the functional ψ_L with a quasiregular Lagrangian its derivative with respect to a direction.

Assertion 2.1 *Let L be a quasiregular Lagrangian and ψ_L be the corresponding classical variational functional. Let $\gamma(t)$, $t\in[a,b]$, be an arbitrary quasiregular curve and γ_ε, $\varepsilon\in[0,1]$, be a smooth deformation of the curve γ such that for $\varepsilon>0$ all the curves γ_ε are regular. By $\eta(t)$ we denote the field of the deformation γ_ε, i.e., $\eta(t)=\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\gamma_\varepsilon(t)$. If the curve $\gamma(t)$ is regular, then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\psi_L\bigl(\gamma_\varepsilon\bigr)=p(\gamma,\dot\gamma)(\eta)\Big|_{t=a}^{t=b}+\int_a^b[L](\gamma,\dot\gamma)(\eta)\,dt.$$

Otherwise, i.e., if the curve γ is pointwise, then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\psi_L\bigl(\gamma_\varepsilon\bigr)=\int_a^b\zeta(L)(\gamma,0)\,dt,$$

where $\zeta=(\eta,\dot\eta)$ is the corresponding vector field along the curve $t\mapsto\bigl(\gamma(t),0\bigr)$ in TW. If we assume additionally that the Lagrangian L is smooth in a neighbourhood of the point $(x,0)$, where $x=\gamma(t)$, then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\psi_L\bigl(\gamma_\varepsilon\bigr)=p(x,0)(\eta)\Big|_{t=a}^{t=b}+\int_a^b[L](x,0)(\eta)\,dt.$$

Proof. The first statement is well known, see for example [1].

Let us prove the second statement. We consider the function

$$f(t,\varepsilon)=L\bigl(\gamma_\varepsilon(t),\dot\gamma_\varepsilon(t)\bigr),$$

where $t \in [a,b]$ and $\varepsilon \in [0,1]$. The partial derivative $f_\varepsilon(t,\varepsilon)$ of the function $f(t,\varepsilon)$ with respect to ε is equal to $\zeta(L)\big(\gamma(t),\dot\gamma(t)\big)$. Since the Lagrangian L is quasiregular, we conclude that the function $f_\varepsilon(t,\varepsilon)$ is continuous. Therefore,

$$\frac{d}{d\varepsilon}\int_a^b L\big(\gamma_\varepsilon(t),\dot\gamma_\varepsilon(t)\big)\,dt = \frac{d}{d\varepsilon}\int_a^b f(t,\varepsilon)\,dt = \int_a^b f_\varepsilon(t,\varepsilon)\,dt$$
$$= \int_a^b \zeta(L)\big(\gamma(t),\dot\gamma(t)\big)\,dt,$$

q.e.d.

A quasiregular (parametric) network Γ is said to be **extreme** for a functional Ψ_L with a quasiregular Lagrangian, if for an arbitrary deformation Γ_ε, $\varepsilon \in [0,\varepsilon_0]$, of the network $\Gamma = \Gamma_0$ we have:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Psi_L(\Gamma_\varepsilon) \ge 0.$$

Remark. To avoid cumbersome statements, we define extremality as nonnegativity of the derivations with respect to directions. The networks satisfying the opposite inequality also have been considered as extreme ones. However we do not restrict the class of extreme networks because the latter networks can be obtained from the former one by means of changing the Lagrangian's sign.

Let $\Gamma\colon G \to W$ be an arbitrary quasiregular parametric network. The maximal connected subgraphs of the graph G all of whose edges are pointwise are called **pointwise components of the graph** G. **Reduced components of the graph** G are either its pointwise components, or its vertices not incident to pointwise edges. The corresponding elements of parametric networks are called in the same way.

For each reduced component $H \subset \Gamma$ we denote by E_H the set of all the edges of the network H, by I_H the set of edges from $\Gamma \setminus H$ adjacent with H and such that their ending vertices lie in H, and by O_H the set of edges from $\Gamma \setminus H$ adjacent with H and such that their beginning vertices lie in H. Notice that the sets I_H and O_H can intersect each other. Besides that, the union of all the sets I_H and O_H over all reduced components H of the network Γ coincides with the set of all regular edges of the network Γ. At last, if γ is an edges of the network Γ, then by $[a_\gamma, b_\gamma]$ we denote the segment parameterizing that edge. By ∂H we denote the set of vertices from H which belong to the boundary of the network Γ. By $\bar\partial H$ we denote the set of the vertices from H which are incident with the regular edges of the network Γ. Further, for every vertex $x \in \bar\partial H$ we denote by $I_H(x)$ and $O_H(x)$ the subsets of I_H and O_H, respectively, consisting of all the edges incident to x. We put $N_H(x) = I_H(x) \cup O_H(x)$.

It is convenient to introduce the following notation. We put $p_x(\gamma)$ to be equal to $\pm p(\gamma,\dot\gamma)$ at a vertex x of an edge γ of the network Γ, where the sign

$+$ corresponds to the case when x is the ending vertex of γ and the sign $-$ corresponds to the opposite case.

Assertion 2.1 leads easily to the following result.

Theorem 2.1 (On Local Structure) *Let L be a quasiregular Lagrangian. A quasiregular network $\Gamma\colon G\to W$ with a boundary β is an extreme network for the functional Ψ_L if and only if each regular edge $\gamma\colon[a_\gamma,b_\gamma]\to W$ of the network Γ is extreme for the functional ψ_L, i.e., $[L](\gamma,\dot\gamma)=0$, and for each reduced component $H\colon G_H\to W$ of the network Γ and for each smooth network $\eta\colon G_H\to TW$, where $\pi\circ\eta=\Gamma|_{G_H}$ and $\eta(\partial G_H)\subset W_0$, the following expression is nonnegative:*

$$\sum_{\gamma\in I_H} p(\gamma,\dot\gamma)(\eta_\gamma)\big|_{t=b_\gamma}-\sum_{\gamma\in O_H} p(\gamma,\dot\gamma)(\eta_\gamma)\big|_{t=a_\gamma}+\sum_{\gamma\in E_H}\int_{a_\gamma}^{b_\gamma}\zeta_\gamma(L)\big(\gamma,\dot\gamma\big)\,dt$$
$$=\sum_{x\in\bar\partial H}\Bigg[\sum_{\gamma\in N_H(x)} p_x(\gamma)\Bigg]\big(\eta(x)\big)+\sum_{\gamma\in E_H}\int_{a_\gamma}^{b_\gamma}\zeta_\gamma(L)\big(\gamma,\dot\gamma\big)\,dt,$$

where η_γ is the edge $\eta|_{[a_\gamma,b_\gamma]}$ of the network η and $\zeta_\gamma=(\eta_\gamma,\dot\eta_\gamma)$ is vector field along the curve $(\gamma,\dot\gamma)$ in TW.

We define the **lift** of a parametric network $\Gamma\colon G\to W$ up to the tangent bundle as the set $\Big\{\big(\gamma(t),\dot\gamma(t)\big)\Big\}$ of curves over all the edges γ of the network Γ.

Corollary 2.1 *Under assumptions of Theorem 2.1, let the Lagrangian L be smooth in a neighbourhood of the lift of the network Γ (for example, it takes place if the network Γ is regular). Then the quasiregular parametric network $\Gamma\colon G\to W$ with boundary β is extreme for the functional Ψ_L if and only if each edge $\gamma\colon[a_\gamma,b_\gamma]\to W$ of the network Γ is extreme for the functional ψ_L, i.e., $[L](\gamma,\dot\gamma)=0$, and for each movable vertex x of Γ the following equality holds:*

$$\sum_\gamma p_x(\gamma)=0,$$

where the summation is taken over all the edges γ incident to the vertex x.

3 Local Structure of Extreme Networks

In this section we consider extreme networks–traces and investigate their local structure. It turns out that the nontriviality of the local structure is a consequence of nonsmoothness of the Lagrangian L generating the classical variational functional.

Let L be a quasiregular Lagrangian. Then the functional Ψ_L defined on parametric quasiregular networks generates a functional on the set of the corresponding networks–traces. The latter functional will be denoted in the same way. Notice that the canonical representatives of such networks are regular parametric networks. We start with the case of smooth Lagrangians L.

3.1 Smooth Lagrangians

Let a Lagrangian L be smooth on entire TW and suppose that L equals zero on the zero section W_0 of TW. Then L is quasiregular, in particular. The following result holds.

Theorem 3.1 *A network Υ is extreme for a functional Ψ_L with smooth Lagrangian L which vanishes on the zero section $W_0 \subset TW$, if and only if the following properties hold.*

(1) *Each edge γ of the network Υ is extreme for the functional ψ_L.*

(2) *For any vertex x of the network Υ and for any edge γ incident to x the equalities $p(x, 0) = 0$ and $p_x(\gamma) = 0$ hold.*

Proof. By definition, the network Υ is extreme if and only if every its representative Γ is extreme for the corresponding functional defined on parametric networks. In particular, the canonical representative Γ of the network Υ possesses the conditions of Theorem 2.1, thus, its edges are extreme for ψ_L. That proves the necessity of the first condition.

Similarly, Assertion 2.1 implies that each pointwise edge of an arbitrary representative Γ of the network Υ is extreme for the functional ψ_L, i.e., it satisfies Euler–Lagrange equations.

Further, let γ be an arbitrary edge of the network Υ incident to the vertex x.

Consider a representative Γ of the network Υ such that

- the reduced component of the vertex x of the network Γ consists of one edge γ';
- the edge γ' is incident to a vertex x' of degree 2 with respect to Γ and such that x' is incident to the edge γ.

The constructed parametric network Γ is extreme, thus, Corollary 2.1 applied to the vertex x' implies $p_{x'}(\gamma) \pm p(x', 0) = 0$, where the sign depends on the orientation of the edge γ'. Since this orientation can be chosen in an arbitrary way, and since $p_{x'}(\gamma) = p_x(\gamma)$, we have

$$p_x(\gamma) = p(x', 0) = -p(x', 0) = p(x, 0) = 0,$$

q.e.d.

The converse statement follows immediately from Corollary 2.1, because all the impulses vanish by the assumption. Theorem is proved.

Corollary 3.1 *Let Υ be extreme for a functional Ψ_L with smooth Lagrangian L which vanishes on the zero section $W_0 \subset TW$. Then each edge of the network Υ is extreme for the functional ψ_L with respect to arbitrary deformations and each pointwise curve whose image coincides with an arbitrary vertex of the network Υ is also extreme for the functional ψ_L with respect to arbitrary deformations. In other words, each extreme network for the functional Ψ_L is trivially extreme.*

Corollary 3.2 *Let Υ be extreme for a functional Ψ_L with smooth Lagrangian L which vanishes on the zero section $W_0 \subset TW$. Let x be a vertex of the network Υ and $\gamma_1, \dots, \gamma_k$ be the edges of Υ incident to x. By ξ_i we denote the velocity vector of the edge γ_i at the vertex x. Then each point (x, ξ_i) and also the point $(x, 0)$ is critical for the restriction of the Lagrangian L onto T_xW.*

Corollary 3.3 *Let L be a smooth Lagrangian vanishing on the zero section $W_0 \subset TW$. If the restriction of the Lagrangian L onto T_xW has no critical points outside $0 \in T_xW$ for any $x \in W$, then each extreme network for the functional Ψ_L is pointwise.*

Example. Let $L(x, \xi) = \langle \xi, \xi \rangle$ be the Lagrangian corresponding to the energy functional on Riemannian manifold W. Corollary 3.3 implies that all the extreme traces for this functional are pointwise.

Example. Let $L(x, \xi) = T(\xi) - U(x)$ be the Lagrangian corresponding to the functional describing the motion of a mass point in the conservative forces field with a potential $U(x)$ on a Riemannian manifold W. Corollary 3.3 implies that all the extreme traces for this functional are pointwise.

Corollary 3.4 *Let L be a smooth Lagrangian vanishing on the zero section $W_0 \subset TW$. Suppose that the Cauchy problem for the Euler–Lagrange equations corresponding to the Lagrangian L possesses the uniqueness property. If the Lagrangian L restricted onto T_xW has at most two critical points for any $x \in W$, then each extreme network for the functional Ψ_L is a curve, possibly pointwise* (*the canonical representative has vertices of degrees* 1 *and* 2 *only*).

3.2 Quasiregular Lagrangians

Let us return to the case of quasiregular Lagrangians of general type.

Assertion 3.1 *Let L be a quasiregular Lagrangian and Υ be an extreme network for the functional Ψ_L. Then $\zeta(L)(x, 0) \geq 0$ for any vertex $x \in \Upsilon$ and any vector $\zeta \in T_{(x,0)}TW$.*

Proof. Suppose otherwise, namely, assume that for some vertex x and some $\zeta_0 \in T_{(x,0)}TW$ the inequality $\zeta_0(L)(x, 0) < 0$ holds. Consider a representative Γ of the network Υ for which the preimage of the vertex x consists of one edge $\gamma\colon [0,\delta] \to x$ incident to a vertex of degree 1 such that the latter vertex corresponds to the end point δ of the parameterizing segment $[0,\delta]$.

Let us fix local coordinates x^i in the manifold W in a neighbourhood of x. This gives us local coordinates (x^i, ξ^j) in the bundle TW. Suppose that the vector ζ_0 is of the form $\zeta = \alpha^i \partial_{x^i} + \beta^i \partial_{\xi^i}$ in this coordinates. Put $\alpha = (\alpha^1, \dots, \alpha^n)$ and $\beta = (\beta^1, \dots, \beta^n)$, where $n = \dim W$. We construct a deformation of the parametric network Γ fixed on all edges of Γ except γ. On the edge γ define the deformation as follows: $\gamma_\varepsilon(t) = x + \varepsilon(\alpha + t\beta)$. Then the field $\eta(t)$ of this deformation has the form $\eta(t) = \alpha + t\beta$ and the corresponding field $\zeta(t)$ equals $\bigl(\eta(t), \dot\eta(t)\bigr) = (\alpha + t\beta, \beta)$. In particular, $\zeta(0) = \zeta_0$. Therefore, by choosing δ to be sufficiently small, the concordance condition implies

$$\int_0^\delta \zeta(L)\, dt < 0.$$

The latter contradicts to extremality of Γ, see Theorem 2.1. Assertion 3.1 is proved.

Assertion 3.2 *Let L be a quasiregular Lagrangian and Υ be an extreme network for the functional Ψ_L. Then at each movable vertex x of the network Υ the sum of impulses $p_x(\gamma)$ over all edges γ from Υ incident to x equals zero.*

Proof. This is a direct consequence from Corollary 2.1 because the canonical representative is a regular extreme network.

By $V(TW)$ we denote the subbundle of $T(TW) \to TW$ whose fibers consist of the vectors tangent to the corresponding fibers of the tangent bundle $TW \to W$. By $H(W_0)$ we denote the distribution of the spaces tangent to the zero section $W_0 \subset TW$. At each point x of W the tangent space $T_{(x,0)}TW$ can be decomposed into the direct sum of the spaces $V_{(x,0)}TW$ and $H_{(x,0)}W_0$. The spaces $H_{(x,0)}W_0$ are called **horizontal**.

Notice that each vector $\zeta \in T_{(x,0)}TW$ can be uniquely decomposed into the sum of its **vertical** $\zeta_v \in V_{(x,0)}TW$ and **horizontal** $\zeta_h \in H_{(x,0)}W_0$ components.

We say that a quasiregular Lagrangian L **is differentiable along the base** W if for each $x \in W$ and each vector $\zeta \in T_{(x,0)}TW$ the following equality holds:

$$\zeta(L) = \zeta_v(L) + \zeta_h(L),$$

and the function $\zeta_h(L)$ is linear on $\zeta_h \in H_{(x,0)}W_0$.

Assertion 3.3 *Let L be a quasiregular Lagrangian and Υ be an extreme network for the functional Ψ_L. Suppose that the Lagrangian L is differentiable along the base. Then for each vertex x of the network Υ the restriction of the function $\zeta(L)$ onto the horizontal space $H_{(x,0)}W_0$ equals zero.*

Proof. Assertion 3.1 implies $\zeta(L)(x,0) \geq 0$ for any $\zeta \in T_{(x,0)}TW$, in particular, for all ζ from the horizontal space $H_{(x,0)}W_0$. Since the Lagrangian L is differentiable along the base, the restriction of the function $\zeta(L)(x,0)$ onto the space $H_{(x,0)}W_0$ is a linear function, thus, this function equals zero. The proof is complete.

A quasiregular Lagrangian L is called **proper** if for each point $x \in W$ the function $f(\zeta) = \zeta(L)(x,0)$ defined on $T_{(x,0)}TW$ is smooth outside $\zeta = 0$.

A vertex x of a trace Υ (the corresponding vertex of the canonical representative for Υ) is called **free** if the impulses $p_x(\gamma)$ of the edges γ incident to this vertex vanish.

Recall that there exists a canonical isomorphism μ between the tangent spaces T_xW to a manifold W and the corresponding vertical spaces $V_{(x,0)}TW$. This isomorphism can be defined as follows. Let ξ be an arbitrary vector from T_xW. Consider a smooth curve $\gamma(t) = (x, t\xi)$ in TW outgoing from the point $\gamma(0) = (x,0)$. The velocity vector $\dot\gamma(0)$ belongs, by definition, to $V_{(x,0)}TW$. The isomorphism μ takes the vector ξ into the vector $\dot\gamma(0)$.

Assertion 3.4 *Let Υ be an extreme network–trace for a functional Ψ_L, where L is a proper quasiregular Lagrangian differentiable along the base. Let x be a nonfree movable vertex of the network Υ. Then the restriction of the function $f(\zeta) = \zeta(L)(x,0)$ onto $V_{(x,0)}TW \setminus \{0\}$ is positive. Also, $\zeta(L)(x,0) = \zeta_v(L)(x,0)$.*

Proof. Assertion 3.3 implies that $\zeta(L)(x,0) = \zeta_v(L)(x,0)$. Besides that, Assertion 3.1 implies that $\zeta(L)(x,0) \geq 0$. Thus, to complete the proof it suffices to verify that $\zeta(L)(x,0)$ can not be equal to zero for vertical nonzero vectors $\zeta \in V_{(x,0)}TW$.

Suppose otherwise, i.e., assume that at a nonfree vertex x of the network Υ the equality $\zeta_0(L)(x,0) = 0$ holds for some vertical vector $\zeta_0 \neq 0$. Since the vertex x is not free, there exists an edge γ incident to x such that its impulse $p_x(\gamma)$ does not vanish. Consider a representative Γ of the network Υ for which the preimage of the vertex x consists of one edge $\gamma' \colon [-a,a] \to x$. Also, we assume that one of the vertices incident to γ', say x', is incident to the edge γ and the other vertex, say x'', is incident to all the other edges $\gamma_1, \dots, \gamma_k$ incident to x in Υ.

Put $\mu^{-1}(\zeta_0) = \nu_0$. We construct a deformation of the network Γ' remaining fixed all the vertices of the network Γ' except the ones of the edge γ'.

Suppose first that $p_x(\gamma)(\nu_0) \neq 0$. If $p_x(\gamma)(\nu_0) > 0$, then we choose the edge γ' in such a way that the point $-a$ from the parameterizing segment $[-a,a]$ corresponds to x'. If $p_x(\gamma)(\nu_0) < 0$, then we assume that $-a$ corresponds to the vertex x''. To be definite, we assume that the second possibility takes place.

Assume that in a neighbourhood of the point $(x,0)$ of the tangent bundle TW local coordinates (x^i, ξ^j) generated by some coordinates x^i in W are given. Define the deformation on the edge γ' as follows: $\gamma'_\varepsilon(t) = \varepsilon\nu_0(t+a)/(2a)$, where

$\varepsilon \in [0, \varepsilon_0]$. Then the field of the deformation $\eta(t)$ of the edge γ' has the following form: $\eta(t) = \nu_0(t+a)/(2a)$, and its lift $\zeta(t)$ equals

$$\big(\eta(t), \dot\eta(t)\big) = \Big(\frac{t+a}{2a}\nu_0, \frac{1}{2a}\nu_0\Big).$$

The vertical component of the field $\zeta(t)$ equals $\dot\eta(t)$ and has the form $\frac{1}{2a}\nu_0$, thus, $\zeta_v(L)(x,0) = 0$.

On the other hand, $\eta(-a) = 0$ and $\eta(a) = \nu_0$, therefore, the first variation of the functional Ψ_L with respect to the deformation of the network Γ in consideration equals

$$p_x(\gamma)(\nu_0) + \int_{-a}^{a} \zeta_v(L)(x,0)\,dt = p_x(\gamma)(\nu_0) < 0,$$

a contradiction. It remains to note that the total impulse p of the edges incident to the vertex x'' of the network Γ is opposite to the impulse $p_x(\gamma)$. Thus, if $p_x(\gamma)(\nu_0)$ is positive, then $p(\nu_0)$ is negative and the similar reasons can be applied.

Now, let $p_x(\gamma)(\nu_0) = 0$. Since $p_x(\gamma) \neq 0$, then there exists a direction $\theta \in T_xW$ such that $p_x(\gamma)(\theta) < 0$. Consider the function $A(\delta) = (\nu_0 + \delta\theta)(L)(x,0)$. By definition of proper Lagrangian the function A is smooth for sufficiently small δ. Moreover, the point $\delta = 0$ is a minimum point for the function A. Really, $A(0) = 0$ by the choice of ν_0. By Assertion 3.1 we have $A(\delta) \geq 0$. Thus,

$$A(\delta) = o(\delta) \quad \delta \to 0,$$

because 0 is an interior point of the domain for the parameter δ.

Consider a deformation of the network Γ constructed in the same way as in the first case but with the direction ν_0 replaced with the direction $\nu_0 + \delta\theta$. Write down the formula of the first variation:

$$p_x(\gamma)(\nu_0 + \delta\theta) + \frac{1}{2a}\int_{-a}^{a} A(\delta)\,dt = \delta\big(p_x(\gamma)(\theta) + o(1)\big) \quad \text{for } \delta \to 0.$$

Thus, for sufficiently small $\delta > 0$ the first variation is negative, that contradicts to the extremality of the network Γ. The proof is completed.

Corollary 3.5 *Let Υ be an extreme network–trace of a functional Ψ_L, where L is a proper quasiregular Lagrangian differentiable along the base. Let x be a nonfree movable vertex of the network Υ. Then for some number $c(x) > 0$ depending on the point x only and for any $\eta \in T_xW$ we have*

$$\lim_{\varepsilon \to 0+} \frac{L(x, \varepsilon\eta)}{\varepsilon} \geq c(x)\|\eta\|.$$

In particular, the restriction of the function L onto an arbitrary linear subspace of $V_{(x,0)}TW$ is not differentiable at the origin.

Let us prove one more property of extreme traces.

Assertion 3.5 *Let Υ be an extreme network–trace for a functional Ψ_L, where L is a proper quasiregular Lagrangian differentiable along the base. Then at each vertex x of the network Υ for any edges $\gamma_1, \dots, \gamma_k$ incident to x and for any vector $\eta \in T_xW$ the following inequality holds:*

$$\sum_{i=1}^{k} p_x(\gamma_i)(\eta) + \mu(\eta)(L)(x,0) \geq 0.$$

Proof. Partition the edges of the network Υ incident to the vertex x into two classes by putting into the first one the edges $\gamma_1, \dots, \gamma_k$ and by putting into the second one all the remaining edges. Consider a representative Γ of the network Υ for which the preimage of the vertex x consists of one edge $\gamma\colon [-a,a] \to x$. Also, assume that one of the vertices of γ, say x', is incident to the edges $\gamma_1, \dots, \gamma_k$ and the other one, say x'', is incident to the remaining edges incident to the vertex x in Υ. Without loss of generality, we assume that the vertex x' corresponds to the ending point a of the parameterizing segment $[-a,a]$.

Let us fix some local coordinates x^i in W in a neighbourhood of the point x. Without loss of generality, we suppose that $x = 0$ in these coordinates.

Consider a deformation Γ_ε of the network Γ such that all the vertices of the network Γ except x' remain fixed and the vertex x' moves uniformly with the velocity ν: $x'(\varepsilon) = \varepsilon\nu$. Also, the edge γ is deformed linearly: $t \mapsto \varepsilon\nu\frac{t+a}{2a}$ for each $t \in [-a,a]$. Then the field of the deformation $\eta(t)$ has the following form along the edge γ: $\eta(t) = \nu\frac{t+a}{2a}$, thus, $\dot\eta(t) = \frac{1}{2\alpha}\nu$. Let us put $\zeta(t) = \big(\eta(t), \dot\eta(t)\big)$. Write down the condition that the first variation for such deformation is nonnegative using the fact that $\zeta(L) = \zeta_v(L) = \dot\eta(L)$ by Assertion 3.3. We have

$$\Big(\sum_{i=1}^{k} p_x(\gamma_i)\Big)(\nu) + \frac{1}{2a}\int_{-a}^{a} \nu(L)(x,0)\,dt = \Big(\sum_{i=1}^{k} p_x(\gamma_i)\Big)(\nu) + \nu(L)(x,0) \geq 0.$$

The proof is completed.

The following theorem gives a criterion of extremality for a network–trace with respect to a functional defined by a proper quasiregular Lagrangian differentiable along the base.

Theorem 3.2 *A network–trace Υ is extreme for a functional Ψ_L defined by a proper quasiregular Lagrangian L differentiable along the base if and only if the following conditions hold.*

(1) *Each edge γ of the network Υ is extreme for the functional ψ_L.*

(2) *At each vertex x of the network Υ for any edges $\gamma_1, \dots, \gamma_k$ incident to x and for any vector $\eta \in T_xW$ the following condition holds*:

$$\sum_{i=1}^{k} p_x(\gamma_i)(\eta) + \mu(\eta)(L)(x,0) \ge 0.$$

(3) *At each movable vertex of the network Υ the sum of impulses of the edges incident to this vertex equals zero.*

(4) *At each fixed vertex x of the network Υ for any $\zeta \in V_{(x,0)}W$ the inequality $\zeta(L)(x,0) \ge 0$ holds. Note that if we write down the ζ in the form $\mu(\eta)$, $\eta \in T_xW$, then the condition of this item can be rewritten as follows*: $\mu(\eta)(L)(x,0) \ge 0$.

Proof. Let Υ be an extreme network–trace for Ψ_L. Then the canonical representative of the network Υ is a regular network. Corollary 2.1 implies that all its edges are extreme for the functional ψ_L. The fact that all the remained properties of the network Υ take place was proved in Assertions 3.1–3.5.

Now, let us prove the sufficiency. Suppose that the network Υ satisfies Conditions (1)–(4) of Theorem. Let $\Gamma\colon G \to W$ be an arbitrary representative of the network Υ and $[\Gamma_\varepsilon]$ be some deformation of the trace Υ. We need to show that the first variation of the functional Ψ_L for such deformation is nonnegative. Since the first variation of the functional Ψ_L can be decomposed into the sum of expressions corresponding to the reduced components of the parametric network Γ, it suffices to verify the nonnegativity of the first variation under the assumption that the deformation Γ_ε preserves all reduced components of the network Γ except one of them. We denote the latter component by H. Also, by x we denote the vertex in Υ corresponding to H.

Let $\bar\partial H = \{x_1, \dots, x_k\}$ be the set of vertices of the network H incident to regular edges from Γ. By E^i we denote the set of nondegenerate edges of the network Γ incident to the vertex x_i. By $p_x(E^i)$ we denote the total impulse of the edges from E^i at the vertex x.

We need the following Lemma.

Lemma 3.1 *If x is a movable vertex of the network Υ, then for any $\eta \in T_xW$ the inequality $\mu(\nu)(L)(x,0) \ge 0$ holds.*

Proof. Let us use Condition (2) in the case when the set $\{\gamma_1, \dots, \gamma_k\}$ of edges coincides with the set of all edges from Υ incident to x. For an arbitrary vector η we have:

$$\sum_{i=1}^{\deg x} p_x(\gamma_i)(\eta) + \mu(\eta)(L)(x,0) \ge 0.$$

Since x is a movable vertex, then the sum in the left hand side of this inequality equals zero by Condition (3), that implies the result sought for. Lemma is proved.

Notice that by Condition (4) and by Lemma 3.1 it suffices to check the case when the network H is a tree.

Let γ be an arbitrary edge of the network H. By $[a_\gamma, b_\gamma]$ we denote the segment parameterizing this edge. Let x_a and x_b denote the vertices of the edge γ corresponding a_γ and b_γ. If we cut the tree H by the edge γ, then the network H is decomposed into two components H_1 and H_2, where H_1 denotes that component which contains the vertex x_a. Partition the set of nondegenerate edges of the network Γ incident to H into two classes E_1^γ and E_2^γ by putting into the class E_i^γ the edges adjacent with the component H_i.

Fix some local coordinates in W near the point x. Then the isomorphism μ is given by the identity matrix in the corresponding coordinates in $T(TW)$. That is why we shall omit μ in what follows.

By $\eta_\gamma(t)$, $t \in [a_\gamma, b_\gamma]$, we denote the field of the deformation Γ_ε along the edge γ. Let $p_x(E_i^\gamma)$ be the total impulse of the edges from E_i^γ at the vertex x. Then Condition (2) implies that for any $t \in [a_\gamma, b_\gamma]$ the following inequalities hold:

$$p_x(E_i^\gamma)\big(\dot\eta_\gamma(t)\big) + \dot\eta_\gamma(t)\big(L\big)(x,0) \geq 0, \qquad i = 1, 2.$$

If we integrate these inequalities over t from a_γ to b_γ, we obtain

$$p_x(E_i^\gamma)\big(\eta_\gamma(b_\gamma)\big) - p_x(E_i^\gamma)\big(\eta_\gamma(a_\gamma)\big) + \int_{a_\gamma}^{b_\gamma} \dot\eta_\gamma(t)\big(L\big)(x,0)\,dt \geq 0, \qquad i = 1, 2. \tag{1}$$

Suppose that the vertex x of the network Υ is movable. Then Condition (3) implies $p_x(E_1^\gamma) + p_x(E_2^\gamma) = 0$, thus

$$\begin{aligned} p_x(E_2^\gamma)\big(\eta_\gamma(b_\gamma)\big) - p_x(E_2^\gamma)\big(\eta_\gamma(a_\gamma)\big) + \int_{a_\gamma}^{b_\gamma} \dot\eta_\gamma(t)\big(L\big)(x,0)\,dt \\ = p_x(E_2^\gamma)\big(\eta_\gamma(b_\gamma)\big) + p_x(E_1^\gamma)\big(\eta_\gamma(a_\gamma)\big) + \int_{a_\gamma}^{b_\gamma} \dot\eta_\gamma(t)\big(L\big)(x,0)\,dt \geq 0. \end{aligned} \tag{2}$$

Let y be one of the vertices incident to γ. If $y = x_a \in H_1$, then we put $p_y^\gamma = p_x(E_1^\gamma)$ and $\overline{p}_y^\gamma = p_x(E_2^\gamma)$; otherwise, we put $p_y^\gamma = p_x(E_2^\gamma)$ and $\overline{p}_y^\gamma = p_x(E_1^\gamma)$.

Write down the inequality (2) for each edge γ from H and sum these inequalities. The obtained sum can be decomposed into two parts: the sum $\sum_2$ of the integrals over all edges from H (this sum coincides with the integral part of the formula of the first variation for the deformation Γ_ε) and the sum $\sum_1$ of impulses. We show that this sum of impulses coincides with the nonintegral part of the formula of the first variation for the deformation Γ_ε, and this will completes the proof in the case of movable vertex x. To do that, we gather the terms of the sum $\sum_1$ taking together the impulses applied to the value of the deformation field η at the same vertex y of the network H. Denote the obtained sum by $\sum_y$.

Let $y \in \bar{\partial} H$, i.e., $y = x_i$ for an appropriate i and let d be the degree of the vertex y in H. By $f_1, \dots, f_d$ we denote the edges from H incident to y. Then the sum $\sum_y$ is of the form $\sum_j p_y^{f_j}$, or

$$\sum_y = -\sum_j \overline{p}_y^{f_j}.$$

It remains to note that the right hand side is as follows:

$$-\sum_j \overline{p}_y^{f_j} = -\sum_{j \neq i} p_x(E^j) = p_x(E^i).$$

The last equality follows from Condition (3). Thus, $\sum_{x_i} = p_x(E^i)$.

Now, let $y \notin \bar{\partial} H$. Then $\sum_y = \sum_\gamma p_y^\gamma$, where the sum is taken over all edges γ from H incident to y. Since the vertex x is movable, then $p_y^\gamma + \overline{p}_y^\gamma = 0$ by assumption. Thus, $\sum_y = -\sum_\gamma \overline{p}_y^\gamma$. It is easy to see that the last sum equals the sum of impulses at the vertex x over all edges from Υ incident to x and, therefore, it is equal to zero. Thus, the case of movable vertex x is completely analized.

Now, let x be a fixed vertex of the network Υ. In this case among the vertices from H there are boundary ones. Recall that at any boundary vertex x the sum of impulses of the edges from Υ incident to x is not supposed to be equal to zero. We denote this sum by p. Let y be a fixed vertex from H. In the same way as above, by η we denote the field of the deformation Γ_ε and let η_y be the value of the field η at the vertex y. Since y is fixed, then $\eta_y = 0$.

Write down the formula of the first variation for the deformation Γ_ε and add to it the zero term $-p(\eta_y)$. To complete the proof, it suffices to show that the obtained expression is nonnegative. By p_y we denote the sum of impulses of nondegenerate edges of the network Γ incident to y at the vertex y (if such edges do not exist, we put $p_y = 0$).

We repeat the above reasoning for the case of the movable vertex x replacing $\bar{\partial} H$ with $\bar{\partial} H \cup \{y\}$ and the sum p_y of impulses at y with $p_y - p$. Notice that the sum impulses redefined in such a way equals zero at the vertices of the redefined set $\bar{\partial} H$. Besides that, inequalities (1) remain valid because the value of the deformation field equals zero at the vertex y. Thus, we are under the same assumptions as in the case of movable vertex x. The proof is completed.

Example. Let Ψ_L be the length functional on a Riemannian manifold W. Evidently, we are under assumptions of Theorem 3.2. Therefore, we get a description of the local structure of extreme traces obtained in [2], see also [3].

Corollary 3.6 *A network–trace Υ with a boundary $\partial\Upsilon$ in a Riemannian manifold W is an extreme for the length functional, if and only if the following conditions hold.*

(1) *The edges of* Υ *are geodesics.*

(2) *The angle between each two adjacent edges is more or equal than* $120°$.

(3) *Each vertex of degree* 1 *belongs to* $\partial\Upsilon$.

(4) *If a vertex of degree* 2 *does not belong to* $\partial\Upsilon$, *then the angle between the edges incident to the vertex equals* $180°$.

Notice that the first and the second conditions of Corollary 3.6 follow from the first and the second conditions of Theorem 3.2. The third and the fourth conditions of Corollary 3.6 follow from the third condition of Theorem 3.2. The fourth condition of Theorem 3.2 holds for the length functional automatically.

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