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Asymptotically flat manifolds
and cone structure at infinity

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ASYMPTOTICAL FLATNESS AND CONE STRUCTURE AT INFINITY

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Abstract. We investigate asymptotically flat manifolds with cone structure at infinity. We show that any such manifold M has a finite number of ends, and we classify (except for the case dim $M = 4$, where it remains open if one of the theoretically possible cones can actually arise) for simply connected ends all possible cones at infinity. This result yields in particular a complete classification of asymptotically flat manifolds with nonnegative curvature: The universal covering of an asymptotically flat m-manifold with nonnegative sectional curvature is isometric to $\mathbb{R}^{m-2} \times S$, where S is an asymptotically flat surface.

0. Introduction

Let (M, g) be a complete noncompact Riemannian manifold. Choose a point $p \in M$ and set

$$A(M) = \limsup_{|px| \to \infty} \{|K_x| \cdot |px|^2\},$$

where $|K_x|$ denotes the maximal absolute value of the sectional curvatures at the point $x \in M$.

One easily checks that $A(M)$ does not depend on the choice of the reference point p, so that the quantity $A(M)$ yields a nice geometric invariant of M which is, in particular, invariant under rescalings of the metric.

Definition. *A noncompact complete Riemannian manifold* (M, g) *is called* asymptotically flat *if* $A(M) = 0$.

Note that the mere condition of being asymptotically flat places in general no restrictions whatsoever on the topology of a manifold. For instance, by a result of Abresch (see [Ab]) *any* noncompact surface carries a complete and asymptotically flat Riemannian metric.

Definition. *A noncompact complete Riemannian manifold* (M, g) *is said to have* cone structure at infinity *if there is a metric cone* C *with vertex* o *such that the pointed Gromov-Hausdorff limit of* $(M, \epsilon g, p)$ *exists for any sequence of numbers* $\epsilon > 0$ *converging to zero and such that this limit is isometric to* (C, o).

★This work was done while we enjoyed our stay at the MPI Leipzig and finished during a joint week at the IHES.

As a Gromov-Hausdorff limit of proper metric spaces, i.e., metric spaces such that any closed ball of finite radius is compact, the cone C which arises in the above definition is in particular proper and locally compact.

Note that large classes of Riemannian manifolds have cone structure at infinity. In fact, Kasue (see [K1]) showed that under certain lower curvature limitations (e.g., if for some $C < \infty$ and $\delta > 0$ it holds that $K_x \geq -C/|px|^{2+\delta}$) a noncompact (complete) Riemannian manifold always has this property. Thus in particular any noncompact Riemannian manifold with faster-than-quadratic curvature decay (i.e., any noncompact Riemannian manifold for which there exists some C and $\delta > 0$ such that $|K_x| \leq C/|px|^{2+\delta}$), and, especially, any noncompact Riemannian manifold with nonnegative curvature has cone structure at infinity.

Note also that on the other hand by Abresch's result one can easily construct asymptotically flat surfaces (S, g) such that the Gromov-Hausdorff limit of $(S, \epsilon_n g, p)$ indeed depends on the choice of the sequence $\epsilon_n \to 0$ and such that for some sequences this limit is not even a metric cone. In particular, by considering products $T^{m-2} \times (S, g)$ one thus obtains examples of asymptotically flat manifolds without cone structure at infinity in any dimension $m \geq 2$. (Actually any such example we know of looks on a big scale always two-dimensional; for more on this see section 3).

Theorem A. *Let M be an asymptotically flat m-manifold. Assume that M has cone structure at infinity. Then*

(i) There exists an open ball $B_R(p) \subset M$ such that $M \setminus B_R(p)$ is a disjoint union $\bigcup_i N_i$ of a finite number of ends, i.e., N_i is a connected topological manifold with closed boundary ∂N_i which is homeomorphic to $\partial N_i \times [0, \infty)$. For each N_i the limit $C_i = GH\text{-}\lim_{\epsilon \to 0} \epsilon N_i$ exists.

(ii) If the end N_i is simply connected, then N_i is homeomorphic to $S^{m-1} \times [0, \infty)$.
In this case moreover the following holds:
(a) if $m \neq 4$, then C_i is isometric to $\mathbb{R}^m$;
(b) if $m = 4$, then C_i is isometric to one of the following spaces: $\mathbb{R}^4$, $\mathbb{R}^3$, or $\mathbb{R} \times [0, \infty)$.

A finiteness of ends statement as in part (i) of Theorem A was proved by Abresch (see [Ab]) in a related setting.

Part (ii)(a) of Theorem A is new even in the special case of faster-than-quadratic curvature decay (recall that, as noted above, faster-than-quadratic curvature decay implies cone structure at infinity).

Theorem A generalizes here work of Greene, Petersen, and Zhu (see Theorem 1 in [GPZ], where the same conclusion as in (ii)(a) was proved under the additional assumptions of faster-than-quadratic curvature decay and nontriviality of the tangent bundle of ∂N_i.

When combined with the fact that volume growth of exactly Euclidean order implies flatness, Theorem A also yields the following result, which generalizes for manifolds of dimension $m \neq 4$ Theorem 2 in [GPZ] from faster-than-quadratic curvature decay to asymptotical flatness with cone structure at infinity:

Corollary. *Let M be an asymptotically flat m-manifold of dimension $m \neq 4$ which has cone structure at infinity. If M has nonnegative Ricci curvature and one simply connected end, then M is isometric to $\mathbb{R}^m$.*

Part (ii)(b) of Theorem A shows that four-dimensional manifolds play in Theorem A a peculiar role and indicates that in this dimension special phenomena can arise. Indeed, Unnebrink (see [U]) showed that there are examples of asymptotically flat 4-manifolds which have (cone structure at infinity and) a simply connected end N_1 such that, in the notation of Theorem A, $C_1 = \mathbb{R}^3$. It is not clear if there exists an asymptotically flat 4-manifold with cone structure at infinity with a simply connected end N_1 so that $C_1 = \mathbb{R} \times [0, \infty)$. (We actually conjecture that there is no such example; see also section 3.)

Note also that in dimension 2 all ends are homeomorphic to $S^1 \times [0, \infty)$, so that the ends of an asymptotically flat surface are never simply connected.

Combining Theorem A and a result from [GP] we obtain proofs of statements of Gromov (see [BGS], p.59) which till now have been treated in the literature (compare [D] and the references therein) as conjectures, and which completely classify the asymptotically flat manifolds with non-negative sectional curvature:

Theorem B. *Let M be an asymptotically flat m-manifold with nonnegative sectional curvature. Then the universal covering of M is isometric to $\mathbb{R}^{m-2} \times S$, where S is an asymptotically flat surface. If, in particular, M is simply connected and $m \geq 3$, then M is isometric to $\mathbb{R}^m$.*

Assuming faster-than-quadratic curvature decay and assuming that the unit normal bundle of the soul of M has nontrivial tangent bundle, the second part of Theorem B was proved by Drees ([D]).

As a direct consequence of Theorem B one also obtains an affirmative answer to a question of Hamilton (see [H], §19; this paper also contains some nice relations between asymptotical curvatures and the Ricci flow), which is equivalent to the following one:

Let M be a complete noncompact Riemannian manifold of dimension $m \geq 3$ with positive sectional curvature. Is it true that $A(M) > 0$?

That in odd dimensions the answer to this question is "yes" was already known and proved by Eschenburg, Schröder, and Strake ([ESS]).

Our results obviously also have a relation to the positive mass conjecture; in [GPZ] the reader will find explained the precise connections.

The main idea of the proof of Theorem A can be described as follows:

Inside an end N_i of an asymptotically flat manifold with cone structure at infinity we construct a continuous family of "spheres", which after rescaling have uniform curvature bounds and which Gromov-Hausdorff converge to the "unit sphere" in the cone C_i.

To this continuous family of spheres we now apply two results from [PRT]: The first says that any continuously collapsing family with bounded curvature contains an infinite "stable" subsequence. To this sequence then a corollary of the Limit of Covering Geometry Theorem from [PRT] applies.

(This corollary actually also holds without using a stability assumption, and the proof of Theorem A can be given without relying on [PRT], but instead on results from [PT], see section 1).

This in turn enables us to prove some inequalities for the ranks of certain homotopy groups. These imply that in fact collapse is not possible except for the case where the dimension of the manifold is equal to 4. Therefore in all nice cases C_i is nothing but $\mathbb{R}^m$.

There is a vast amount of literature on noncompact complete Riemannian manifolds whose sectional curvature at infinity is zero (and on many different specific ways in which the curvature is allowed to go to zero). For a detailed account of what is known and wanted to be known about such spaces, the reader is recommended to look at the survey article [Gre] and the paper [GPZ]. Here we just mention (besides the references already given) some papers in the field which are most closely related to the results of this note: [ES], [GW1], [GW2], [KS] and [LS].

The remaining parts of the paper are organized into a preliminaries, a proof, and a problem section which contains further remarks and several open questions.

We would like to thank Patrick Ghanaat for pointing out to us a simplified proof of the sublemma in section 2 as well as Luis Guijarro for useful comments.

1. Preliminaries

In this section we review some results about manifolds which collapse with bounded curvature and diameter. More on this can be found in the references given in [PRT] and [PT].

Definition. *A sequence of metric spaces M_i is called stable if there is a topological space M and a sequence of metrics d_i on M such that (M, d_i) is isometric to M_i and such that the metrics d_i converge as functions on $M \times M$ to a continuous pseudometric.*

Proposition (Continous Collapse implies Stability) ([PRT]). *Suppose that a simply connected manifold M admits a continuous one-parameter family of metrics $(g_t)_{0<t\leq 1}$ with $\lambda \leq K_{g_t} \leq \Lambda$ such that, as $t \to 0$, the family of metric spaces $M_t = (M, g_t)$ Hausdorff converges to a compact metric space X of lower dimension. Then the family M_t contains a stable subsequence.*

The version of the Limit of Covering Geometry Theorem from [PRT] we need in this paper (it is straightforward to check that the proof given in [PRT] also proves the result below) can be stated as follows:

Theorem (Limit of Covering Geometry Theorem ([PRT]). *Let M_n be a stable sequence of Riemannian m-manifolds with curvature bounds $|K(M_n)| \leq 1$ such that for $n \to \infty$ the sequence of metric spaces M_n Hausdorff converges to a compact metric space X of lower dimension. Consider any sequence of points $p_n \in M_n$ and balls $B_n = B_{\pi/2} \subset T_{p_n}$ which are equipped with the pull back metrics of the exponential maps $\exp_{p_n} : T_{p_n} \to M_n$. Assume that for any such converging subsequence $B_n \to B$, the limit B has curvature ≥ 0 in the sense of Alexandrov.*

Then for any converging subsequence $B_n \to B$, the limit B has the same dimension as the manifolds M_n, and in a neighbourhood of its center, the metric on B coincides with that of a metric product $\mathbb{R} \times N$, where N is a manifold with two-sided bounded curvature $0 \leq K(N) \leq 1$ in the sense of Alexandrov.

Our proof of Theorem A will in fact only use the following corollary of this theorem. At first sight this corollary looks almost obvious, but it doesn't seem easy to adopt any of the known proofs of injectivity radius estimates to this case.

Corollary. *Let M_n be a (stable) sequence of closed simply connected Riemannian manifolds of dimension $m \geq 2$ with curvature $|K(M_n)| \leq C$ and uniformly bounded diameters. Consider any sequence of points $p_n \in M_n$ and balls $B_n = B_{\pi/2\sqrt{C}} \subset T_{p_n}$ which are equipped with the pull back metrics of the exponential maps $\exp_{p_n} : T_{p_n} \to M_n$. Assume that for any such converging subsequence $B_n \to B$, the limit B has at all interior points curvature $= 1$. Then the manifolds M_n converge to a standard sphere.*

The stability condition is actually not necessary for the above result to hold. This can be seen from the following independent proof of the Corollary, which does not use stability at all. The proof itself is very short, but since it uses the notion of Grothendieck-Lipschitz convergence and Riemannian megafolds from [PT], we decided to also incorporate the above [PRT] approach, which might be easier to understand.

Proof of the Corollary without stability assumption.

The only nontrivial part is to establish a lower positive bound for the injectivity radii of all manifolds M_n.

Since because of the Gauss-Bonnet theorem the case $m = 2$ is trivial, we may assume that $m \geq 3$.

If the manifolds M_n would collapse, then, after passing to a subsequence if necessary, one may assume that the manifolds M_n Grothendieck-Lipschitz converge to a Riemannian megafold $\mathfrak{M}$ which is not a manifold.

The assumptions of the Corollary imply that the limit M has constant curvature $= 1$, so that $\mathfrak{M} = (S^m : G)$, where G is a commutative group of isometries of S^m. However, by ([PT], Theorem A.7) we have that $0 \neq H^2_{dR}(\mathfrak{M}) = H^2_{dR}(S^m)$, which is a contradiction. $\square$

2. Proofs

Proof of part (i) of Theorem A.

The first statement of the theorem will follow from the fact that the distance function to p, dist_p, for sufficiently large values does not have any critical points.

By assumption, for any sequence of numbers $\epsilon > 0$ converging to zero, the pointed Gromov-Hausdorff limit of $(M, \epsilon g, p)$ exists and is isometric to a locally compact metric cone C with vertex o. The cone C obviously has curvature ≥ 0 (in the sense of Alexandrov) everywhere except the origin $o \in C$.

Let us assume that there exists a sequence of points x_n such that $|px_n| \to \infty$ as $n \to \infty$, and such that each x_n is a critical point for dist_p.

Consider the sequence of rescaled manifolds $(M, g/|px_n|, p)$. By the assumption of the theorem, this sequence converges to (C, o), and the points $x_n \in (M, g/|px_n|)$ (after passing to a subsequence) converge to a point $x \in C$ which has distance 1 to the origin o.

Since C is a cone, we can consider $y := 2x \in C$. Choose a sequence of points $y_n \in (M, g/|px_n|)$ which converge to y, and consider minimal geodesics $x_n y_n$ from x_n to y_n. Since x_n is a critical point of dist_p, there is for each n a minimal geodesic px_n which makes an angle less than $\pi/2$ with the minimal geodesic $x_n y_n$. Therefore Toponogov's comparison theorem implies that $\lim_{n\to\infty} |py_n|/|px_n| \le \sqrt{2}$. But obviously $\lim_{n\to\infty} |py_n|/|px_n| = |oy|/|ox| = 2$, a contradiction.

Thus for some $R > 0$ the function dist_p does not have any critical points outside the open ball $B_R(p)$. In particular, as follows from Morse theory for distance functions, see ([Grov], Cor. 1.9.), M has finite topological type, i.e., M is homeomorphic to the interior of a compact manifold with boundary (which in our case is simply the closed ball $\bar{B}_R(p)$).

This also implies that the manifold M has only finitely many ends.

Note that the cone C_i is nothing but the closure of the connected component of $C \backslash o$ that corresponds to N_i, in particular for each N_i the limit $C_i = \text{GH-}\lim_{\epsilon\to 0} \epsilon N_i$ exists.

Thus part (i) of Theorem A is proved. $\square$

Proof of part (ii) of Theorem A.

The fact that N is homeomorphic to $S^{m-1} \times [0, \infty)$ will follow directly from the proofs of (ii)(a) and (ii)(b). Therefore we only need to prove these two statements.

Note that if $\dim C_i = m = \dim M$, then parts (ii)(a) and (ii)(b) of the theorem are trivially true: Indeed, if so we have that the curvature of C_i is zero everywhere except the origin. We can assume that $m \neq 2$ (otherwise all ends would be homeomorphic to $S^1 \times [0, \infty)$, and therefore in particular they would not be simply connected). It follows that $C_i = \mathbb{R}^m/F$, where F is a finite group of rotations which acts freely on $\mathbb{R}^m \backslash 0$. Since $C_i \setminus B_1(o)$ is homeomorphic to N_i, it follows that $F = \pi_1(\partial N_i)$. Since by assumption ∂N_i is simply connected, F must be trivial, and thus for $\dim C_i = m = \dim M$ our claims are proved, since the above also implies that in this case N_i is homeomorphic to $S^{m-1} \times [0, \infty)$.

From now on we will assume that $\dim C_i < m$.

We can view C_i as a cone over its space of directions, $C_i = C(\Sigma_i)$, where Σ_i is an Alexandrov space of curvature ≥ 1 or $\dim \Sigma_i = 1$. Σ_i can be viewed as a "unit sphere" in C_i.

We will first construct a continuous family of hypersurfaces $S_{2-1,\epsilon}$ in $(M, \epsilon g)$ which collapse to Σ_i such that the sectional curvatures of $S_{2-1,\epsilon}$ stay uniformly bounded. The following construction is very close to one used by Kasue in [K2]. We will therefore only explain it here; all of its details can be found in ([K2, §2]).

For each rescaling $(M, \epsilon g, p)$, consider the sphere of radius 2, $S_{2,\epsilon}(p) \subset (M, \epsilon g)$. Its principal curvatures for outcoming normal directions lie in the range $[-C(\epsilon), \infty]$, where $C(\epsilon) \to 1/4$ as $\epsilon \to 0$.

Next consider, for an inward direction (to p) the equidistant hypersurface $S_{2-1,\epsilon}$ at distance 1 to $S_{2,\epsilon}(p)$. Then $S_{2-1,\epsilon}$ has uniformly bounded principal curvatures which in fact lie in the range $[-C'(\epsilon), C'(\epsilon)]$, where $C'(\epsilon) \to 1$ as $\epsilon \to 0$.

Therefore, since M is asymptotically flat, $S_{2-1,\epsilon}$ has uniformly bounded sectional curvature as $\epsilon \to 0$. For sufficiently small ϵ it follows that $S_{2-1,\epsilon}$ (equipped with the induced intrinsic metric) is a continuous family which, as $\epsilon \to 0$, collapses to Σ_i.

Key Lemma. *Take any sequence of points $p_{\varepsilon_n} \in S_{2-1,\varepsilon_n}$, $\varepsilon_n \to 0$. Consider the balls $B_n = B_1(0) \subset T_{p_{\varepsilon_n}}(S_{2-1,\varepsilon_n})$, equipped with the pull back metrics.*

Then as $n \to \infty$, the B_n Lipschitz converge to the ball of radius 1 in $S^{l-1} \times \mathbb{R}^{m-l}$, for some fixed l depending on M.

Moreover $\Sigma_i = \widetilde{S}^{l-1}/A$, where A is an Abelian group of isometries of $\widetilde{S}^{l-1}$ (here by $\widetilde{S}^{l-1}$ we understand the standard $l-1$-sphere if $l \geq 3$, $\mathbb{R}$ if $l = 2$, and a point if $l = 1$).

The proof of the Key Lemma will be given below. Let us now continue with the proof of part (ii) of Theorem A:

Obviously all $S_{2-1,\epsilon}$ are homeomorphic to ∂N_i and therefore simply connected. Now applying the Corollary in section 1 we see that $l < m$.

Using that ∂N_i is simply connected we can moreover show that the group A in the Key Lemma is connected: Let A^o be the identity component of A. Then $\widetilde{\Sigma}_i = S^{l-1}/A^o$ is a branched covering of Σ_i, and it is easy to see that one can find a covering $\widetilde{\partial N_i} \to \partial N_i$ which is a lifting of $\widetilde{\Sigma}_i \to \Sigma_i$. But since ∂N_i is simply connected we have that $\widetilde{\Sigma}_i = \Sigma_i$. Therefore $A^o = A$, i.e., A is connected.

Now if $l = 1$, then Σ_i is a point, so that ∂N_i must be an infranil manifold. But any infranil manifold has infinite fundamental group, which contradicts the fact that ∂N_i is simply connected.

If $l = 2$ it follows that Σ_i is homeomorphic to a point or $\mathbb{R}$, since A is connected. The first case cannot occur by the above reasoning, and the second contradicts that C is locally compact.

Therefore the only serious case to deal with is the case $l \geq 3$.

From the above we have that in this case Σ_i is isometric to $S^{l-1}/T^{k'}$.

Since for $\varepsilon \to 0$ the hypersurfaces $S_{2-1,\varepsilon}$ collapse to Σ_i and since $S_{2-1,\varepsilon}$ is homeomorphic to ∂N_i, we know that Σ_i is homeomorphic to $\partial N_i/T^k$ and that this homeomorphism can be chosen to preserve the natural stratifications of these spaces.

Let us now do some topological calculations:

Let O_{T^k} be a regular orbit of the T^k action on ∂N_i. Consider the relative homotopy sequence of pairs

$$\pi_2(\partial N_i, O_{T^k}) \to \pi_1(T^k) = \mathbb{Z}^k \to \pi_1(\partial N_i) = 0.$$

Therefore $\mathrm{rk}_{\mathbb{Q}}\, \pi_2(N_i, O_{T^k}) \geq k$.

Next consider the corresponding homotopy sequence for S^{l-1}:

$$0 = \pi_2(T^{k'}) \to \pi_2(S^{l-1}) \to \pi_2(S^{l-1}, O_{T^{k'}}) \to \pi_1(T^{k'}) = \mathbb{Z}^{k'} \to \pi_1(S^{l-1})$$

Therefore $\mathrm{rk}_{\mathbb{Q}}\, \pi_2(S^{l-1}, O_{T^{k'}}) = k' +$ $\mathrm{rk}_{\mathbb{Q}}\, \pi_2(S^{l-1})$.

On the other hand one has that $k = \dim \partial N_i -$ $\dim \Sigma_i$ and $k' = l-1-$ $\dim \Sigma_i$.

Let $\Sigma_i^{\#}$ denote Σ_i with the singular sets removed. Consider now the following three cases:

1. Σ_i has no boundary. Then obviously $\mathrm{rk}_{\mathbb{Q}}\, \pi_2(S^{l-1}, O_{T^{k'}}) = \mathrm{rk}_{\mathbb{Q}}\, \pi_2(\Sigma_i^{\#}) = \mathrm{rk}_{\mathbb{Q}}\, \pi_2(\partial N_i, O_{T^k})$.

2. $\partial\Sigma_i$ has one component. Then $\mathrm{rk}_{\mathbb{Q}}\pi_2(S^{l-1}, O_{T^{k'}}) = \mathrm{rk}_{\mathbb{Q}}\pi_2(\Sigma_i^{\#}) + 1 = \mathrm{rk}_{\mathbb{Q}}\pi_2(\partial N_i, O_{T^k})$.

3. $\partial\Sigma_i$ has more than one component.

In both case 1 and case 2 it follows that $k' \geq k -$ $\mathrm{rk}_{\mathbb{Q}}\pi_2(S^{l-1})$, and hence $m \leq l + \mathrm{rk}_{\mathbb{Q}}\pi_2(S^{l-1})$.

However, this contradicts the fact that $l < m$, except if $m = 4, l = 3$. In this particular case it follows that $\Sigma_i = S^2/A^{rot}$. Therefore, since Σ_i has not more than one boundary component, we have that A^{rot} is trivial and $\Sigma_i = S^2$. Thus $C_i = C(\Sigma_i) = \mathbb{R}^3$ (and that this indeed can happen was shown in [U]).

Case 3 can only occur if Σ_i is homeomorphic to $[0,1]$. Then, since N_i is simply connected, it must hold that $k, k' \leq 2$. Since the T^k action on ∂N_i has empty fixed point set, we have that $k = 2$, and since $l < m$, we have that $k' = 1$. Therefore $m = 4$, $l = 3$ and Σ_i is isometric to $S^2/S^1 = [0, \pi]$, so that $C_i = C(\Sigma_i) = \mathbb{R} \times [0, \infty)$.

The proof of Theorem A is complete. $\square$

Proof of Theorem B.

Let M be an asymptotically flat m-manifold with nonnegative sectional curvature. Then M has cone structure at infinity, and by [GP] the soul S of M is flat. This forces the universal cover $\tilde{M}$ of M to split isometrically as a Euclidean part, coming from the soul S, and a nonnegatively curved complete manifold F homeomorphic to $\mathbb{R}^k$.

Now F is also asymptotically flat and has one end $S^{k-1} \times [0, \infty)$. Therefore by Theorem A, if $k \neq 2, 4$, then the cone at infinity of F is isometric to $\mathbb{R}^k$. Since by Toponogov's Comparison Theorem any line in the cone corresponds to a line in F, it follows from the Toponogov Splitting Theorem that F itself is isometric to $\mathbb{R}^k$.

Thus to finish the proof we must only exclude the case $k = 4$. By Theorem A, if $k = 4$ we have that $C = \text{GH-lim}_{\epsilon \to 0}\, \epsilon F$ is isometric to one of the following: $\mathbb{R}^4$, $\mathbb{R}^3$ or $\mathbb{R} \times [0, \infty)$. In all of these cases we have that C contains a line, and therefore F splits isometrically as $\mathbb{R} \times F'$. But since F is asymptotically flat it follows that F' is flat, and therefore F is isometric to $\mathbb{R}^4$. $\square$

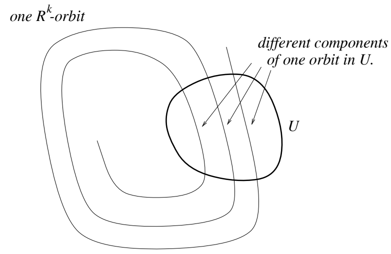
Proof of the Key Lemma.

Consider a ν-neighbourhood $U \supset \Sigma_i \subset C_i$. From the results of [CFG] (see section 1 of [PRT], where also further references can be found) we have an N-structure $\pi : E_\epsilon \to U$, where E_ϵ is a subset of $(M, \epsilon g)$ containing the hypersurface $S_{2-1,\epsilon}$. Since E_ϵ is homotopically equivalent to ∂N_i, it follows that E_ϵ is simply connected. Therefore the N-structure is given by an almost isometric smooth T^k-action without fixed points (see again section 1 in [PRT]).

Now take a point $x \in \Sigma_i \subset C_i$ (so $|ox| = 1$) and consider a spherical neighbourhood of $U_x \ni x$. Consider the preimage $V_\epsilon = \pi^{-1}(U_x) \subset E_\epsilon$ and let $\widetilde{V}_\epsilon$ be its universal Riemannian covering. Then the T^k-action induces an almost isometric $\mathbb{R}^k \times F$ action on $\widetilde{V}_\epsilon$, where F is a finite Abelian group.

From [CFG] one has a uniform bound for the injectivity radius of $\widetilde{V}_\epsilon$, so that, as $\epsilon \to 0$, $\widetilde{V}_\epsilon$ converges to a flat manifold $\widetilde{V}_0$ with boundary and isometric $\mathbb{R}^k \times F$ action (for the convergence claim see the first part of Lemma 2.1.4 in [PRT]).

Since the interior of $\widetilde{V}_0$ is flat, there exists a map $\widetilde{V}_0 \to \mathbb{R}^m$ which is for all interior points a local isometry. Therefore the $\mathbb{R}^k \times F$ action on $\widetilde{V}_0$ can be extended to an action of whole $\mathbb{R}^m$, and the local factors $U \subset \mathbb{R}^m/\mathbb{R}^k$ are isometric to local branched coverings of subsets of C_i. (Here by local factors we understand factorizing U by the connected components of the $\mathbb{R}^k$-orbits in U, as is illustrated in the following picture).



Now the above group $\mathbb{R}^k$ can be regarded as an Abelian group of isometries of Euclidean space $\mathbb{R}^m$. We will show that in our case $\mathbb{R}^k$ actually splits into a direct sum of translations and rotations.

To this means first note the following:

Sublemma. *Let a connected Abelian group H act on Euclidean space $\mathbb{R}^m$ by isometries. Then one can represent $\mathbb{R}^m$ as an orthonormal sum $V \oplus W$ such that H is contained in a direct sum of translations and rotations,*

$$H < A^{tr} \oplus A^{rot},$$

so that the following holds: The group $A^{tr} = V$ is the group consisting of all parallel translations along V, and $A^{rot} \subset O(W)$ is an Abelian subgroup of rotations of W.

Proof of the Sublemma.

By [Al] one orbit of H is an affine subspace V (in fact, such an orbit corresponds to the origin o of C_i). Choose the origin of affine space $\mathbb{R}^m$ so that it is contained in this subspace. Each element $\alpha \in H$ can be viewed as $(r_\alpha, \phi_\alpha) \in V \times O(m)$, such that $\alpha(x) = r_\alpha + \phi_\alpha(x)$ for any $x \in \mathbb{R}^m$.

Then V can be viewed as the set of all pure translations of H, $A^{tr} = V = \{r : (r, \phi) \in H$ for some $\phi \in O(m)\}$. Let $A^{rot} := \{\phi : (r, \phi) \in H$ for some $r\}$ be the group of pure rotations of H. If each $\phi \in A^{rot}$ acts trivially on V, then obviously $H < A^{tr} \oplus A^{rot}$, which is exactly what we want.

Therefore we only have to prove that for any $\phi \in A^{rot}$ and any $v \in V$ we have that $\phi(v) = v$.

Take any $(r, \phi) \in H$ and $v \in V$. For all $n \in \mathbb{N}$ there exists $\phi_n \in O(m)$ such that $(nv, \phi_n) \in H$. Since H is Abelian, it follows that $(r, \phi)(nv, \phi_n) = (nv, \phi_n)(r, \phi)$ and therefore $nv + \phi_n r = r + \phi nv$. Dividing by n and letting $n \to \infty$ thus implies $\phi v = v$. $\square$

Thus our group $\mathbb{R}^k$ is contained in a direct sum $A^{tr} \oplus \tilde{A}^{rot}$, where $\tilde{A}^{rot}$ is universal covering Lie group of A^{rot}. Now note that since the local factors by $\mathbb{R}^k$ have a cone structure, $\mathbb{R}^k$ moreover itself splits as $\mathbb{R}^k = A^{tr} \oplus \tilde{A}^{rot}$:

Indeed, since the local factors $U/\mathbb{R}^k$ admit a cone structure, in radial directions their sectional curvatures must be zero. But this is impossible unless $\mathbb{R}^k$ is itself a direct product $\mathbb{R}^k = \tilde{H} = A^{tr} \oplus \tilde{A}^{rot}$:

To prove this, we only have to show that (in the notation of the Sublemma) it holds that $A^{rot} \subset H$. Assume that this is wrong. Then we can find a ray $c : [0, \infty) \to \mathbb{R}^m$ which is orthogonal to V, and there will be an element $\alpha \in a^{rot}$ in the Lie algebra of A^{rot} which is not contained in the Lie algebra of H, so that α defines a linear Jacobi field on the ray c which can assumed to be non-zero.

Consider now the projection $\bar{c}$ of c along some local factor. Then $\bar{c}$ is a piece of a ray in the cone C and the projection $\bar{J}$ of the field J is also a Jacobi field. But since C is a cone, any Jacoby field along $\bar{c}$ must be linear. On the other hand it is straightforward to show that $|\bar{J}(t)|$ is a strictly concave function, and this is a contradiction.

Therefore the local factors W/A^{rot} are isometric to local branched coverings of C_i (everywhere except the origin). Thus $C_i \setminus o$ is isometric to a factor of its universal covering, $(\widetilde{W \setminus 0})/A$, by an Abelian Lie group A. Restricting this last isometry to the unit spheres of both cones it follows that $\Sigma_i = \tilde{S}^{l-1}/A$, and the second part of the Key Lemma is proved.

Let $\rho : \widetilde{V}_\epsilon \to V_\epsilon$ be the covering map and $\widetilde{S}_{2-1,\epsilon} = \rho^{-1}(S_{2-1,\epsilon})$. It converges to the preimage of Σ_i under the map $V_0 \to V_0/A = U_x \subset C_i$, so that it locally coincides with the cylinder $V \times S^{l-1}$, where S^{l-1} is the unit sphere in W. Therefore, since $x \in \Sigma_i$ can be chosen to be arbitrary, the covering geometry of $S_{2-1,\epsilon}$ converges to the one of $V \times S^{l-1}$, and this finishes the proof of the first part of the Key Lemma. $\square$

3. Remarks and Open Questions

Question 1. Let M be an asymptotically flat manifold, and let the sequence $(M, \epsilon_n g, p)$ converge to (G, o) as $\epsilon_n \to 0$. Assume that $\dim G \geq 3$ and that $G \backslash o$ has only one connected component.

Is it true that G is a metric cone with origin o?

A positive answer to this question could possibly lead to a general classification of asymptotically flat manifold of higher dimension. To obtain such a classification is particularly interesting because of the fact that Gromov (see [Grom], p.96 and also [LS]) showed that any (smooth paracompact) noncompact manifold M admits a complete Riemannian metric whose asymptotic curvature satisfies $A(M) \leq C$, where C depends only on the dimension of M.

Question 2. Does there exist in each dimension m a positive constant $C(m)$ such that any noncompact complete Riemannian m-manifold M with $A(M) \leq C(m)$ is asymptotically flat?

Note that thhe answer (positive or negative) to the following question would give a complete classification of the cone structures at infinity of simply connected ends of asymptotically flat manifolds:

Question 3. Can the cone $\mathbb{R} \times [0, \infty)$ be a cone at infinity of a simply connected end of an asymptotically flat 4-manifold?

It seems not possible to obtain such an example by a direct generalization of Unnebrink's example. Namely, one can exchange the Berger spheres $S^3_{f(t),h(t)}$ (in the notation of [U]) in Unnebrink's example by $S^3_{a(t),f(t),h(t)}$, where the number $a(t)$ describes along which one-dimensional subgroup of the T^2-action on the standard S^3 we shrink the distance (so $S^3_{a,f,h}$ is a Berger sphere if $a = \pm 1$). But direct calculation then shows that there is no triple of functions a, f, h which would give an asymptotically flat 4-manifold with $\mathbb{R} \times [0, \infty)$ as a cone at infinity.

However, on the other hand, if one would take as a a constant which is close to 1, then as a result one obtains an end N whose asymptotic curvature $A(N)$ is arbitrarily small and which has $\mathbb{R} \times [0, \infty)$ as cone at infinity.

Remark. The same arguments as the one which we used in the proof of Theorem A actually also make it possible to characterize the cones at infinity of complete noncompact manifolds whose asymptotic curvature is *small*:

Namely, if for some given sequence of simply connected m-dimensional ends N_n with $A(N_n) \to 0$ as $n \to \infty$ their cones at infinity C_n Gromov-Hausdorff converge to some metric space C (which then must be a cone), then for sufficently large n it holds that N_n is homeomorphic to $S^{m-1} \times [0, \infty)$ and moreover the following is true:

(a) if $m \neq 4$, then C is isometric to $\mathbb{R}^m$;

(b) if $m = 4$, then C is isometric to one of the following spaces: $\mathbb{R}^4$, $\mathbb{R}^3$, or $\mathbb{R} \times [0, \infty)$.

The above modification of the Unnebrink construction for constant a shows that for manifolds with small asymptotic curvature all cones which are mentioned in part (b) actually do arise.

As a last point we would like to mention that the methods we used in this paper do not distinguish between spaces which are asymtotically flat and sequences of spaces whose asymptotic curvature goes to zero.

Therefore, no matter how special our question whether $\mathbb{R} \times [0, \infty)$ can be a cone at infinity of a simply connected end of an asymptotically flat 4-manifold might at first sight look like, any negative answer to it will require more sensitive collapsing techniques.

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