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Abstract

We discuss the geometric formulation of the classical Hamiltonian system of equations and the problem of existence of periodic solutions for such systems. After a brief discussion of the Poincaré-Birkhoff theorem we give the most general statement of the Arnold conjecture. In the remaining article we sketch the developments in symplectic topology motivated by the Arnold conjecture culminating in its proof.

1 Introduction

"Would the Sun rise tomorrow?" must have been one of the most important questions asked by the earliest observers of the universe. I have used this question as a title of my talks to mathematics clubs at colleges and high schools to describe the history of the problem of establishing the stability of the solar system. It is an important special problem in the area of the existence and stability of periodic solutions of dynamical systems and remains unsolved today. A solution in the case of the three-body problem in which one of the bodies is very small compared with the other two was obtained in the 1960s. This solution was used in designing the orbit of the moon shot that landed the first man on the moon [1]. While the question of the stability of the solar system may have been asked by ancient observers, its precise mathematical formualtion was obtained only in the 17th century. Kepler's announcement of his laws of planetary motion came as a great surprise to the scientific community of the time and understanding them and the theory behind them became the most important contemporary problem. Here is perhaps the most important example of physical results serving as a driving force for the development of new mathematical tools needed to understand and explain these results. Newton's development of differential and integral calculus and his theory of gravitational attraction and motion came as a direct consequence of the Kepler problem. The success of Newton's theories was astounding. His solution of the two-body problem provided a complete explanation of Kepler's laws. But the theory went far beyond that. For example, it was known to astronomers that Kepler's laws did not fully describe the orbit of Mercury; the farthest point from the Sun on its elliptic orbit shifted after each revolution. This was known as the precession of the perihelion. A complete solution for all the planetary orbits would require considering their motion under mutual attraction in addition to that of the Sun. This became known as the n-body problem. The system of equations for this problem looked quite intractable, and no-closed form solution of the system in terms of elementary functions is known for $n > 2$. However, in Newton's theory one could account for the observed motion of Mercury by considering the effects of other planets as a small perturbation of the elliptic orbit of Mercury under the gravitational attraction of the Sun. This accounted for a large part of the observed precession. How could one account for the part not predicted by the theory after taking into account all the known planets? Perhaps there were other as yet unobserved planets in the Solar system that could be affecting the motion of Mercury. Theoretical calculations confirmed the existence of two new planets. Once their perturbative effect was considered, almost all the shift could be accounted for. Careful measurement showed that the remaining shift was about 43 seconds of arc(1 degree = 3600 seconds of arc) per 100 years. No other planet was found to account for this small remaining shift and no reasonable explanation was found in Newton's theory. The residual shift was explained by a new theory of gravity propossed by Einstein in 1915. This theory, called the general theory of relativity, may be called a geometric theory of gravity. In Einstein's theory, the concept of absolute space and absolute time as distinct entities, which is fundamental in Newton's theory, is abandoned. The basic object is a four dimensional manifold with a Lorentz metric whose curvature represents the gravitational field.

Important theoretical advances in the study of dynamical systems were made in the 18th century. We consider the Lagrangian and the Hamiltonian approach in the next section and introduce the notion of a symplectic manifold. As we have seen in the above paragraph, the perturbative methods provided a very satisfactory first approximation to solutions of the n-body problem. But the series solutions used in the perturbative calculations could not be used to answer theoretical questions such as the stability of the solar system. Poincaré was the first to prove the general divergence of these series. We shall discuss his fundamental contributions in section 3. We begin section 4 with a discussion of various special solutions of the Arnold conjecture. We then briefly discuss Floer homology and its extension which paved the way for the general solution of the Arnold conjecture in the nondegenerate case by Liu and Tian [17] and independently by Fukaya and Ono [11].

2 Hamiltonian Systems

The three-body problem and, in particular, the problem of Lunar motion attracted the attention of several famous mathematicians; including Euler, Lagrange, Jacobi, and Hamilton. We now discuss two of the most important theoretical developments in which these mathematicians played a fundamental role. First of these is the introduction of variational methods in the study of dynamical systems. In particular, it was shown that every solution of Newton's equations of motion under gravity arises as a critical point in a variational problem. The equations of motion for a system with n degrees of freedom subject to a conservative force (i.e. a force derivable from a potential function V) can be written in the form

$$\frac{d^2q^i}{dt^2} = -\frac{\partial V}{\partial q^i}\ , \quad 1 \leq i \leq n\ . \tag{1}$$

The motion on the interval $t_0 \leq t \leq t_1$ is a function $q : [t_0,\ t_1] \to \mathbf{R}^n$ whose components $q^i(t)$ solve the system of equations (1). The solution is a path in $\mathbf{R}^n$ beginning at $q(t_0) = q_0$ and ending at $q(t_1) = q_1$. The velocity $\dot{q}(t)$ is a vector in the tangent space $T_{q(t)}\mathbf{R}^n$ at time t. For this reason the tangent bundle $T\mathbf{R}^n$ of $\mathbf{R}^n$ is called the velocity phase space of the dynamical system. Now define the Lagrangian function $L(q, \dot{q})$ by

$$L(q, \dot{q}) := \frac{1}{2}||\dot{q}||^2 + V(q), \quad L : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}, \tag{2}$$

where we have identified $T\mathbf{R}^n$ with $\mathbf{R}^n \times \mathbf{R}^n$ and $||\dot{q}||$ is the standard Euclidean norm of the velocity. Then it can be shown that every solution of equations (1) is a critical point in a variational problem for the Lagrangian action $\mathcal{A}_L$ defined by

$$\delta(\mathcal{A}_L) = 0, \text{ where } \mathcal{A}_L := \int_{t_0}^{t_1} L(q, \dot{q}) \; dt \tag{3}$$

The variation is taken over all C^1 functions $q : [t_0, \; t_1] \to \mathbf{R}^n$ that satisfy the boundary conditions $q(t_0) = q_0$ and $q(t_1) = q_1$. In fact, it can be shown that the solutions correspond to the minima of the action. The Lagrangian formulation can be generalized by considering q^i as local coordinates on a manifold M. In classical mechanics M is called the configuration space of the system and TM is called the velocity phase space. The Lagrangian function can also be generalized to allow explicit dependence on time. Thus it is now a function

$$L : \mathbf{R} \times TM \to \mathbf{R}, \text{ or locally } L = L(t, q, \dot{q}). \tag{4}$$

The critical points of the corresponding action satisfy the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i} \; , \quad 1 \le i \le n \; . \tag{5}$$

We note that the Lagrangian formalism has been extended from systems with finitely many degrees of freedom to those with infinitely many degrees of freedom and constitutes a basic tool in obtaining the field equations in physical theories. For example, Einstein's field equations of gravitation and the Yang-Mills equations can be obtained as the Euler-Lagrange equations for appropriate choices of Lagrangians [18].

It is easy to see that the system of n second order differential equations (1) for the variables q^i is equivalent to the following system of $2n$ first order differential equations for the variables q^i, p_i

$$\frac{dq^i}{dt} = p_i \; , \frac{dp_i}{dt} = -\frac{\partial V}{\partial q^i} \; , \quad 1 \le i \le n \; . \tag{6}$$

If we define the Hamiltonian function $H(q, \; p)$ by

$$H(q, p) := \frac{1}{2}||p||^2 - V(q) \; , \quad H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}, \tag{7}$$

then equations (6) for the variables $q^i,\ p_i$ can be written in the form

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{8}$$

The system of equations (8) for the variables $q^i,\ p_i$ is called the Hamiltonian system of equations. The variables p_i are called the momenta conjugate to the coordinates q^i . The Hamiltonian function in the system of equations (8) for the variables $q^i,\ p_i$ has a simple physical interpretation as the total energy of the system. Using the system of equations (8) it is easy to check that

$$\frac{dH}{dt} = \sum_{i=1}^{n} \frac{dq^i}{dt}\frac{\partial H}{\partial q^i} + \frac{dp_i}{dt}\frac{\partial H}{\partial p_i} = 0.$$

This means that H is constant on the trajectories (solutions) of the system. This statement is the law of conservation of energy in classical mechanics.

As we shall see later, the domain of the Hamiltonian must be identified with the cotangent bundle $T^*\mathbf{R}^n$ of $\mathbf{R}^n$, and not with the tangent bundle as in the Lagrangian case. In general, if the manifold M is the configuration space of the system, T^*M is called the momentum phase space or simply the phase space. The Hamiltonian function can also be generalized to allow explicit dependence on time. Thus it is now a function

$$H : \mathbf{R} \times T^*M \to \mathbf{R} \text{ or locally } H = H(t,q,p) = H_t(q,p) \tag{9}$$

It is possible that H, or the family H_t, may be defined for t in some subset of $\mathbf{R}$.

In the system of equations that we are considering it is easy to establish the equivalence of the Lagrangian and Hamiltonian formalisms. To see this we observe that the p_i and H are given in terms of the Lagrangian by

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \text{ and } H = p_i\dot{q}^i - L\ . \tag{10}$$

The first set of equations (10) can be solved for $\dot{q}^i$ as a function of $(q^i,\ p_i)$ and the result used in the definition of H to obtain a function of $(q^i,\ p_i)$ that satisfies equations (8). The transformation of variables from $(q^i,\ \dot{q}^i)$ to $(q^i,\ p_i)$ is called the Legendre transformation. Conversely, starting with Hamilton's equations we can obtain in this case the Euler-Lagrange equations. However, the Legendre transformation need not exist for arbitrary L. The condition that guarantees the existence of the Legendre transformation is the Legendre condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0. \tag{11}$$

Now we can state the following theorem.

Theorem 2.1 *Let $L = L(t, q, \dot{q})$ be a C^2 function satisfying the Legendre condition (11). Let p_i, H be defined by the set of equations (10). Then $q(t)$ is a solution of the Euler-Lagrange equations (5) if and only if q^i, p_i satisfy Hamiltons equations (8). Conversely, Hamiltons equations can be transformed into the Euler-Lagrange equations if H satisfies the condition*

$$det\left(\frac{\partial^2 H}{\partial p_i \partial p_j}\right) \neq 0. \tag{12}$$

In what follows we shall be mainly concerned with Hamiltonian systems. Hamiltons equations point to a special structure on the phase space called the symplectic structure. In general, a Hamiltonian function is defined on a manifold that carries such a symplectic structure. The concepts of classical mechanics such as Poisson bracket, integrals of motion, etc. can be expressed in terms of the symplectic structure. We now consider this in some detail.

Symplectic Manifolds

Let M be an m-dimensional manifold and let $\omega \in \Lambda^2(M)$, the space of differential forms of degree 2 on M. We say that ω is nondegenerate if, $\forall p \in M$,

$$\omega(p)(u, v) = 0, \forall v \in T_pM \Rightarrow u = 0. \tag{13}$$

If $\omega_{ij}(p)$ are the components of $\omega(p)$ in a local coordinate system at p, then the above condition (13) is equivalent to

$$det\ \omega_{ij}(p) \neq 0, \quad \forall p \in M. \tag{14}$$

Condition (14) and the skew symmetry of ω imply that the dimension m must be even, i.e. $m = 2n$. Then condition (13) is equivalent to the condition that $\omega^n := \omega \wedge \omega \wedge \ldots \wedge \omega$ is a volume form on M, i.e.

$$\omega^n(p) \neq 0, \quad \forall p \in M. \tag{15}$$

Recall that any 2-form α can be regarded as a bilinear map on T_pM, and hence induces a linear map

$$\alpha^\flat(p) : T_pM \to T_p^*M$$

defined by the equality

$$\alpha^\flat(p)(u)(v) = \alpha(p)(u, v),$$

where $u, v \in T_pM$. The nondegeneracy of $\omega \in \Lambda^2(M)$ defined above is equivalent to $\omega^\flat$ being an isomorphism. Its inverse is denoted by $\omega^\sharp$. Thus a nondegenerate 2-form sets up an isomorphism between vector fields and 1-forms.

Definition 2.1 *A* **symplectic structure** *on a manifold M is a 2-form ω that is nondegenerate and closed (i.e. $d\omega = 0$). A* **symplectic manifold** *is a pair (M, ω), where ω is a symplectic structure on the manifold M.*

Example 2.1 *Let Q be an n-dimensional manifold. Let $P = T^*Q$ be the cotangent space of Q; then P carries a natural symplectic structure ω defined as follows. Let θ be the 1-form on P defined by*

$$\theta(\alpha_p)(X) = \alpha_p(\pi_*(X)), \forall \alpha_p \in T^*Q, X \in T_{\alpha_p}P,$$

where π is the canonical projection of $P = T^*Q$ to Q. We define $\omega = -d\theta$. The form θ is called the **canonical** 1-**form** *and ω the* **canonical symplectic structure** *on T^*Q. By definition, ω is exact, and hence closed. Its nondegeneracy follows from a local expression for ω in a special coordinate system, called a* **canonical coordinate system** *defined as follows. Let $\{q^i\}$ be local coordinates at $p \in Q$. Then $\alpha_p \in P$ can be expressed as $\alpha_p = p_i dq^i$. We take $Q^i = q^i \circ \pi, P_i = p_i \circ \pi$ as the canonical coordinates of $\alpha_p \in P$. Using these coordinates, we can express the canonical 1-form θ as*

$$\theta = P_i dQ^i.$$

It is customary to denote the canonical coordinates on P by the same letters q^i, p_i and from now on we follow this usage. The canonical symplectic structure ω is given by

$$\omega = -d(p_i dq^i) = dq^i \wedge dp_i. \tag{16}$$

From this expression it follows that

$$\omega^n = dq^1 \wedge dp_1 \wedge \ldots \wedge dq^n \wedge dp_n \neq 0.$$

Further, the components of ω *in this coordinate system are given by the matrix*

$$(\omega_{ij}) = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

where I_n *(resp.* 0_n*) denotes the* $n \times n$ *unit (resp. zero) matrix.*

When $Q = \mathbf{R}^n$*,* P *can be identified with* $\mathbf{R}^{2n}$*, and in this case equation (16) is valid globally and defines the standard symplectic structure on* $\mathbf{R}^{2n}$*.*

The above example is of fundamental importance in the theory of symplectic manifolds in view of the following theorem, which asserts that, at least locally, every symplectic manifold looks like the standard symplectic $\mathbf{R}^{2n}$.

Theorem 2.2 *(Darboux) Let* ω *be a nondegenerate 2-form on a* $2n$ *dimensional manifold* M*. Then* ω *is symplectic if and only if each* $p \in M$ *has a local coordinate neighborhood* U *with coordinates* $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ *such that*

$$\omega_{|U} = dq^i \wedge dp_i.$$

One consequence of Darboux's theorem is that there are no local invariants of symplectic manifolds. This is in stark contrast to the situation in Riemannian manifolds and is the simplest expression of the symplectic rigidity.

Example 2.1 is also associated with the geometrical formulation of classical Hamiltonian mechanics, where Q is the configuration space of the mechanical system and P is the corresponding phase space. We now explain this formulation.

If (M, ω) is a symplectic manifold, then the charts guaranteed by Darboux's theorem are called **symplectic charts** and the corresponding coordinates (q^i, p_i) are called **canonical coordinates**. If $M = T^*Q$ and ω is the canonical symplectic structure on it, then, in the physical literature, the q^i's are called **canonical coordinates** and the p_i's the corresponding **conjugate momenta**. This terminology arises from the formulation of classical mechanics on T^*Q. We now indicate briefly the connection between the classical Hamilton's equations and symplectic manifolds.

Let (M, ω) be a symplectic manifold. A vector field $X \in \mathcal{X}(M)$ is called **Hamiltonian** (resp. **locally Hamiltonian** or **symplectic**) if $\omega^\flat(X)$ is exact (resp. closed). The set of all Hamiltonian (resp. locally Hamiltonian) vector fields is denoted by $\mathcal{HX}(M)$ (resp. $\mathcal{LHX}(M)$). If $X \in \mathcal{HX}(M)$, then there exists an $H \in \mathcal{F}(M)$ such that

$$\omega^\flat(X) = dH. \tag{17}$$

The function H is called a **Hamiltonian** corresponding to X. If M is connected, then any two Hamiltonians corresponding to X differ by a constant. Conversely, given any $H \in \mathcal{F}(M)$, $\omega^\sharp(dH)$ defines the corresponding Hamiltonian vector field which is denoted by X_H. The integral curves of X_H are said to represent the evolution of the classical mechanical system specified by the Hamiltonian H. In a local coordinate system, these integral curves appear as solutions of the system of differential equations

$$\frac{dx^i}{dt} = X_H, \quad 1 \leq i \leq 2n. \tag{18}$$

In particular, in a local canonical coordinate system, these differential equations can be expressed as

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \tag{19}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{20}$$

This is the form of the classical **Hamilton's equations**, which were obtained for the special Hamiltonian in (8). Let $f, g \in \mathcal{F}(M)$. The **Poisson bracket** of f and g, denoted by $\{f, g\}$, is the function

$$\{f, g\} := \omega(X_f, X_g).$$

If X is a Hamiltonian vector field with flow F_t, then Hamilton's equations can be expressed in the form

$$\frac{d}{dt}(f \circ F_t) = \{f \circ F_t, H\}. \tag{21}$$

We now introduce the symmetry group of a Hamiltonian system.

Definition 2.2 *Let* (M, ω) *be a symplectic manifold. A diffeomorphism of* M *that preserves the symplectic structure is called a* **symplectomorphism** *of* (M, ω)*. Thus the set* $Symp(M, \omega)$ *of all symplectomorphisms is given by*

$$Symp(M, \omega) := \{\phi \in Diff(M) \mid \omega = \phi^*(\omega)\} \ . \tag{22}$$

It is easy to check that the set $Symp(M, \omega)$ *is a subgroup of the group* $Diff(M)$ *of diffeomorphisms of* M*. Now let* H_t , $t \in [0, 1]$ *be a smooth time-dependent family of Hamiltonians. A symplectomorphism* $\phi \in Symp(M, \omega)$ *is called a* **Hamiltonian symplectomorphism** *if there exists a smooth time-dependent family of symplectomorphisms* ϕ_t , $t \in [0, 1]$*, satisfying the condition*

$$\frac{d}{dt}(\phi_t) = X_{H_t} \circ \phi_t \ , \quad \phi_0 = id_M. \tag{23}$$

The family ϕ_t , $t \in [0, 1]$*, satisfying the above condition is called a* **Hamiltonian isotopy***. It can be shown that the set* $Ham(M, \omega)$ *of all Hamiltonian symplectomorphisms is a normal subgroup of* $Symp(M, \omega)$*.*

Arnold's conjecture about periodic solutions of time dependent Hamiltonian systems provided the major impetus for many important developments in symplectic geometry, topology, and their applications in the second half of this century (see, for example, [21]). In the next section we return to the developments in the first half of this century that paved the way for later work.

3 Poincaré's Last Geometric Theorem

Inspite of the great success of perturbative methods in the study of the solar system, the problem of obtaining periodic solutions in the n-body problem seemed to be insurmountable even in the case $n = 3$. In 1877, more than 100 years after Lagrange's discovery of special periodic solutions, the American mathematician and astronomer G. W. Hill found new periodic solutions of the three body problem. Hill's work was greatly appreciated by Poincaré and is considered a milestone not only in the three body problem but in the study of dynamical systems in general. Poincaré's thesis of 1879 and his early papers already contain several new ideas on the qualitative study of dynamical systems. The most important among these is to regard a solution of a system of differential equations as a curve in the configuration space and then to study it by using global geometric and topological methods. He firmly believed that many questions regarding properties of dynamical systems and, in particular, those of the solar system should really be thought of as questions of qualitative geometry, and that the answers to these questions would only come when one can construct qualitatively the trajectories of the system. Using these ideas Poincaré was able to generalize Hill's work to prove the existence of an uncountable family of periodic solutions in a special three body problem. A general study of this problem was the core of his memoir that won the gold medal in the prize competition sponsored by king Oscar. It was this event that spread Poincaré's fame as a great mathematician in the mathematical community and in the public at large. This memoir and its subsequent revision formed the basis for his celebrated three volume work "Les Méthodes Nouvelles de la Mécanique Céleste". The work contains his discovery of asymptotic solutions of Hamiltonian systems and the results on the existence of periodic solutions for such systems. In particular, Poincaré's theory shows the existence of an infinite number of asymptotic and periodic solutions in the n-body problem. Weierstrass, one of the judges of the prize competition, described these discoveries as epoch-making. However, Poincaré was fully aware that he was nowhere near a complete solution of the n-body problem.

After nearly ten years Poincaré returned to the problem of periodic orbits in a paper on the existence and stability of closed geodesics on a convex surface that he presented at the St. Louis congress in 1904. Using methods of variational calculus he was able to show the existence of at least one stable closed geodesic and to conclude that, in general, their number should be odd. In fact, he strongly believed that the minimum number of closed geodesics should be three. Birkhoff proved this result in the late 1920s with certain restrictions. A complete proof was given a bit later by Lyusternik and Schnirelman. There is a large body of work on the higher dimensional generalization of this problem with many interesting applications. In 1909 K. F. Sundman, an astronomer at the Helsinki Observatory, was a major contribution to the 3-body problem. However, Sundman's solution is in terms of series which converge too slowly to be of practical computational use. Furthermore, the results could not be used to obtain any qualitative information about the trajectories such as their periodicity or stability and work on these questions continues to this day. An excellent account of the 3-body problem with special emphasis on Poincaré's work may be found in [3].

In his last work Poincaré developed and used methods of algebraic topology to formulate a certain statement which if true, would imply the existence of an infinite number of periodic solutions in the restricted three body problem. This statement has come to be known as "Poincaré's last geometric theorem". Even though he did not prove this theorem, he felt that it was important to bring it to the attention of the mathematical community and it was published in 1912. In the following year, shortly after Poincaré's death this result was proved by Birkhoff [4] and is now often called the Poincaré-Birkhoff theorem. We now state this theorem.

Theorem 3.1 *Let A be the annular region bounded by circles with radii a and b. Thus*

$$A = \{(x, y) \in \mathbf{R}^2 \mid a^2 \leq x^2 + y^2 \leq b^2\}.$$

Any area-preserving homeomorphism of A that leaves the boundary circles invariant but twists them in opposite directions must have at least two fixed points.

Poincaré had shown that the existence of one fixed point would imply the existence of a second one, but the complete proof eluded him. Birkhoff's elegant proof reflected his great interest in and understanding of the topological methods introduced by Poincaré in the study of dynamical systems. He pursued this approach in his later work and founded the modern subject of dynamical systems and separated it from astronomy.

In 1925 Birkhoff [5] extended his proof to apply to ring shaped regions with arbitrary boundary curves. This result can be used to prove the existence of periodic orbits in dynamical systems with two degrees of freedom. Birkhoff's work raised the following natural question "What is the appropriate generalization of the Poincaré-Birkhoff theorem to higher dimensions?" Birkhoff knew that the right generalization was not obtained by considering volume-preserving diffeomorphisms. We have here a situation that occurs frequently in mathematics. In low dimensions several different structures may be equivalent and the generalization to higher dimensions is not clear. A suitable generalization often depends on a reformulation of the problem under consideration. Thus from equation (15) we see that in two dimensions any symplectic structure is a multiple of the volume (here area) form. So an area preserving diffeomorphism is also a symplectomorphism. But an arbitrary symplectomorphism of a compact symplectic manifold need not have any fixed points, hence some restrictions must be imposed on allowable symplectomorphism. A reasonable conjecture was formulated nearly 40 years after Birkhoff's proof and is the celebrated "Arnold Conjecture". In the next section we discuss various attempts at proving the Arnold conjecture in some special cases and comment on its recent complete proof.

4 The Arnold Conjecture

As with other famous conjectures, the Arnold conjecture is easy to state and understand. Coming from one of the great mathematicians of this century, it attracted immediate attention. Attempts at proving it have led to many important developments in symplectic geometry and topology. We now state it in its most general form.

Theorem 4.1 *(The Arnold Conjecture)*
Let (M, ω) be a compact symplectic manifold. Then a symplectomorphism generated by a time-dependent Hamiltonian vector field has at least as many fixed points as the minimal number of critical points of a function on the manifold. If all the fixed points are nondegenerate then their number is at least the number of critical points of a Morse function on the manifold.

Originally Arnold formulated the conjecture for the standard 2-torus. The general form stated above was given later [2]. Even in two dimensions the proof did not come quickly. In an unpublished paper, Eliashberg proved it for Riemann surfaces in 1979. His methods are specific to the two dimensional case. In 1982, Conley and Zehnder [6] used new ideas to prove the conjecture for standard Tori T^{2n}, $n \geq 1$. Their result was extended to some other quotients of $\mathbf{R}^n$ (which include, in particular, Riemann surfaces) by Floer [8] and Sikorav [27].

A fundamental change in the mehods of proof came through Floer's proof [9, 10] of the Arnold conjecture for a class of manifolds called monotone symplectic manifolds. The main new ingredient in Floer's proof was the introduction of a new homology theory for symplectic manifolds, now well known as the **Symplectic Floer Homology**. Floer was motivated by the construction of the Morse cohomology given in Witten's celebrated paper [28].

Classical Morse theory on a finite dimensional compact differentiable manifold M relates the behavior of critical points of a suitable function on M with topological information about M. The relation is generally stated as an equality of certain polynomials as follows. Recall first that a smooth function $f : M \to \mathbf{R}$ is called a **Morse function** if its critical points are isolated and nondegenerate. If $x \in M$ is a critical point (i.e. $df(x) = 0$), then the Taylor expansion of f around x yields the Hessian of f at x defined by

$$\{\frac{\partial^2 f}{\partial x^i \partial x^j}(x)\}.$$

The nondegeneracy of the critical point x is equivalent to the nondegeneracy of the quadratic form determined by the Hessian. The dimension of the negative eigenspace of this form is called the **Morse index**, or simply the index, of f at x and is denoted by $\mu_f(x)$, or simply $\mu(x)$ when f is understood. It can be verified that these definitions are independent of the choice of the local coordinates. Let m_k be the number of critical points with index k. Then the **Morse series** of f is the formal power series

$$\sum_k m_k t^k.$$

Recall that the Poincaré series of M is given by $\sum_k b_k t^k$, where $b_k \equiv b_k(M)$ is the k-th Betti number of M. The relation between the two series is given by

$$\sum_k m_k t^k = \sum_k b_k t^k + (1+t) \sum_k q_k t^k, \tag{24}$$

where q_k are non-negative integers. Comparing the coefficients of the powers of t in this relation leads to the well-known **Morse inequalities**

$$\sum_{k=0}^{i} m_{i-k}(-1)^k \geq \sum_{k=0}^{i} b_{i-k}(-1)^k \ , \quad 0 \leq i \leq n-1, \tag{25}$$

and to the expression for the Euler characteristic χ of M in terms of the Morse indices of the Morse function f.

$$\chi := \sum_{k=0}^{n} b_k(-1)^k \ = \sum_{k=0}^{n} m_k(-1)^k. \tag{26}$$

In his fundamental paper [28], Witten used a suitable supersymmetric quantum mechanical Hamiltonian and its ground states (identified with the critical points of f) to construct his Morse complex. He showed how the standard Morse theory can be modified by considering the gradient flow of the Morse function f between pairs of critical points of f. One may think of this as a sort of relative Morse theory. He was motivated by the phenomenon of the quantum mechanical tunneling. In a classical system the transition from one ground state to another is forbidden, but in a quantum mechanical system it is possible to have tunneling paths between two ground states. In gauge theory (for an introductory account see, for example, Marathe and Martucci [18]) the role of such tunneling paths is played by instantons. Indeed, Witten uses the prescient words "instanton analysis" to describe the tunneling effects obtained by considering the gradient flow of the Morse function f between two ground states (critical points). The relation to Morse theory arises in the following way. A Morse function f on M defines a one-parameter family of operators

$$d_t = e^{-ft} d e^{ft}, \quad \delta_t = e^{ft} \delta e^{-ft}, \quad t \in \mathbf{R} \tag{27}$$

It is easy to verify that $d_t^2 = \delta_t^2 = 0$. Witten defines C_p, the set of p-chains of his complex, to be the free group generated by the critical points of f of Morse index p. He then argues that the operator d_t defined in (27) defines in the limit as $t \to \infty$ a coboundary operator

$$d_\infty : C_p \to C_{p+1}$$

and that the cohomology of this complex, called the Morse cohomology, is isomorphic to the deRham cohomology of M. The parameter t interpolates between the deRham cohomology and the Morse cohomology as t goes from 0 to $+\infty$. The idea of instanton tunnelling and the corresponding Witten complex was extended by Floer to do Morse theory on the infinite dimensional moduli space of gauge potentials on a homology 3-sphere Y and to define a new (co)homology theory. This cohomology is called Floer's instanton cohomology. It is different from the deRham cohomology of Y and leads to new topological invariants of Y. Instanton (co)homology and some of its generalizations are discussed in [19]. Floer also used these ideas to define a "symplectic (co)homology" associated to a symplectic manifold and it is this (co)homology that enters into his proof of the Arnold conjecture. A detailed study of the homological concepts of finite dimensional Morse theory in analogy with Floer homology may be found in M. Schwarz [25]. While many basic concepts of "Morse homology" can be found in the classical investigations of Milnor, Smale, and Thom, its presentation as an axiomatic homology theory in the sense of Eilenberg and Steeenrod [7] is given for the first time in [25]. One consequence of this axiomatic approach is the uniqueness result for "Morse homology" and its natural equivalence with other axiomatic homology theories defined on a suitable category of topological spaces. Witten's isomorphism is then a corollary of this result.

An important new tool for the study of symplectic manifolds is provided by the pseudo-holomorphic curves introduced by Gromov in [12]. They play an essential role in Floer's definition of symplectic cohomology. We now review Gromov's definition of pseudo-holomorphic curves. Recall first that a **complex structure** on a real vector space V is a linear transformation $J : V \to V$ such that $J^2 = -I_V$, where I_V is the identity transformation of V. A complex structure on the tangent bundle TM is called an **almost complex structure** on M. Thus for each $x \in M$, J_x is a complex structure on T_xM. If (M, ω) is a symplectic manifold and J is an almost complex structure on M, then we say that J is compatible with ω if

$$\omega(X, Y) := \omega(JX, JY), \quad \forall X, Y \in TM \tag{28}$$

In that case the bilinear form g_J defined by

$$g_J(X, Y) := \omega(X, JY), \quad \forall X, Y \in TM \tag{29}$$

is a J-invariant Riemannian metric on M i.e.

$$g_J(X, Y) := g_J(JX, JY), \quad \forall X, Y \in TM \tag{30}$$

Given a symplectic manifold (M, ω) there always exists an almost complex structure J compatible with ω. The set of all such compatible J forms a contractible space. An almost complex structure J can be used to define local complex coordinates on some neighborhood of each point. When two such neighborhoods intersect the transition functions between the two complex coordinate systems is smooth but, in general, is not holomorphic (i.e. complex analytic). If there is a covering of M by neighborhoods such that the transition functions on every intersection are holomorphic then J is said to be **integrable**, The manifold M is then a complex manifold. A symplectic manifold (M, ω) with an integrable almost complex structure J compatible with ω is called a **Kähler** manifold. We note that every almost complex structure on a Riemann surface is integrable, and hence a Riemann surface is a Kähler manifold. We are now in a position to define a pseudo-holomorphic curve in M.

Definition 4.1 *Let (M, ω) be a symplectic manifold with an almost complex structure J compatible with ω. Let Σ be a Riemann surface with com-*

plex structure J_Σ. *A map* $f : \Sigma \to M$ *is called pseudo-holomorphic or J-holomorphic if*

$$J \circ df \;=\; df \circ J_\Sigma. \tag{31}$$

In particular, if f is an embedding, then the image $f(\Sigma)$ is a complex curve in M.

Floer combined the ideas of Morse theory with the variational methods of Conley and Zehnder [6] and the pseudo-holomorphic curves introduced by Gromov [12] to define his "symplectic (co)homology" associated with a closed monotone symplectic manifold and used it to prove the Arnold conjecture for this class of manifolds. We briefly describe the main features of his construction. If H is a time-dependent Hamiltonina function on a symplectic manifold (M, ω) then in local coordinates, Hamilton's equations of motion are given by

$$\frac{dx^i}{dt} = \; X_{H_t}, \quad 1 \leq i \leq 2n. \tag{32}$$

Let ϕ_t denote the flow on M generated by the solutions of the system of Hamilton's equations. Let $\mathcal{P}(H)$ be the set of periodic solutions of equations (32) of period 1. Then the set of fixed points of the time one flow ϕ_1 is in one to one correspondence with the set $\mathcal{P}(H)$. For a generic H the graph of ϕ_1 is transversal to the diagonal $M \times M$. It follows that in this case the set $\mathcal{P}(H)$ is finite. We shall only consider this nondegenerate case. The Lefschetz fixed point theorem of algebraic topology then gives the Euler charactersistic $\chi(M)$ (the alternating sum of the Betti numbers) as a lower bound for the cardinality of the set $\mathcal{P}(H)$ whereas the Arnold conjecture gives the sum of the Betti numbers as a lower bound. This is yet another illustration of the symplectic rigidity. To find the periodic orbits Floer now uses a variational formulation of the problem as follows. Let $\mathcal{L}$ be the space of contractible loops on M and $\hat{\mathcal{L}}$ its universal covering. An element of $\hat{\mathcal{L}}$ can be represented by a pair $[\alpha, f]$, where $f : D \to M$ is a smooth map of the standard disc D with boundary values given by α. The symplectic action functional $\mathcal{A}_H$ is defined by

$$\mathcal{A}_H([\alpha, f]) := \int_D f^*(\omega) + \int_\alpha H_t(\alpha(t)), \quad \forall [\alpha, f] \in \hat{\mathcal{L}}. \tag{33}$$

Then the pair $[\alpha, f]$ is a critical point of the symplectic action functional $\mathcal{A}_H$ if and only if α is a periodic solution of equations (32) of period 1. With each critical point, and hence with each periodic orbit α, there is associated an integer $\mu(\alpha)$ called the Conley-Zehnder index [13, 24]. Let J be an almost complex structure compatible with ω and g_J the corresponding J-invariant metric. This metric induces an L^2-metric on $\hat{\mathcal{L}}$. Let c be a gradient flow line of the action $\mathcal{A}_H$ connecting periodic orbits α and β. These flow lines satisfy several conditions which ensure that for a generic choice of (J, H) the moduli space $\mathcal{M}(J, H, \alpha, \beta)$ of unparametrized flow lines quotiented out by the action of $\mathbf{R}$ is a smooth manifold of dimension $\mu(\beta) - \mu(\alpha) - 1$. Floer then shows that this moduli space has a natural compactification provided that M satisfies a certain condition (monotonicity). This compactified moduli space is denoted by $\mathcal{M}(\alpha, \beta)$ and has the expected dimension $\mu(\beta) - \mu(\alpha) - 1$. The rest of the construction is similar to that of Witten's Morse cohomology. The cochain C_k is a vector space generated by the critical points with the Conley-Zehnder index k. If $\mu(\alpha) = k$ and $\mu(\beta) = k + 1$ then the dimension of $\mathcal{M}(\alpha, \beta)$ is zero, and hence it is a collection of finitely many signed points. The algebraic sum of the signs measures the distinct connecting orbits with orientation and is denoted by $n(\alpha, \beta)$. The coboundary operator δ is defined as follows.

$$\delta(\alpha) := \sum n(\alpha, \beta)\beta \in C_{k+1}$$

where the sum is over all β such that $\mu(\beta) = k + 1$. It can be shown that $\delta^2 = 0$. The cohomology of this complex is the symplectic Floer cohomology (SFH for short). A succint introduction to SFH may be found in [26]. The total dimension of this complex is the cardinality of period one orbits. The final step in Floer's proof is to show that SFH is isomorphic to the standard deRham cohomology of M with total dimension equal to the sum of the Betti numbers. Thus Arnold's conjecture is a consequence of the SFH. Floer's ideas were extended by Hofer and Zehnder [14] and Ono [22] to prove the Arnold conjecture for other classes of manifolds. These manifolds include the family of Calabi-Yau manifolds which play fundamental role in string theory.

In view of these results it became clear that an extension of Floer cohomology to arbitrary symplectic manifolds would prove the Arnold conjecture. However, the Floer construction did not work for arbitrary symplectic manifolds for the following reason. The natural compactification of the moduli space $\mathcal{M}(J, H, \alpha, \beta)$ may, in general, add boundary components with dimension higher than that of the moduli space $\mathcal{M}(J, H, \alpha, \beta)$ itself. Thus Floer's construction can not be carried out in this setting. A similar problem arises in other applications of J-holomorphic curves, notably, in the study of new invariants of symplectic manifolds called the Gromov-Witten invariants and in the definition of quantum cohomology [20, 23]. Liu and Tian and their collaborators had successfully dealt with these problems by introducing the notion of virtual moduli cycles. They have now used a modification of this idea in the symplectic situation to define Floer cohomology for arbitrary symplectic manifolds. The proof of the general Arnold conjecture in the nondegenerate case follows from this [17]. In their proof of the general Arnold conjecture in the nondegenerate case Fukaya and Ono [11] use the idea of stable maps introduced by Kontsevich and Manin in [16, 15] and the Kuranishi structure on the moduli space of stable maps. Recall that Kuranishi's well known method was used by him to study the deformation theory of complex structures and has been extended to study moduli spaces of instantons in gauge theory. The Kuranishi structure of Fukaya and Ono is an extension of these ideas to the case of the moduli space of stable maps in symplectic manifolds.

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