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$\mathcal{H}$-Matrix Approximation on Graded Meshes*

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Abstract

In a preceding paper [6], a class of matrices ($\mathcal{H}$-matrices) has been introduced which are data-sparse and allow an approximate matrix arithmetic of almost linear complexity. Several types of $\mathcal{H}$-matrices were analysed in [6, 7, 8, 9] which are able to approximate integral (nonlocal) operators in FEM and BEM applications in the case of quasi-uniform unstructured meshes.

In the present paper, the special class of $\mathcal{H}$-matrices on graded meshes is analysed. We investigate two types of separation criteria for the construction of the cluster tree which allow to optimise either the approximation power (cardinality–balancing strategy) or the complexity (distance–balancing strategy) of the $\mathcal{H}$–matrices under consideration. For both approaches, we prove the optimal complexity and approximation results in the case of composite meshes and tensor-product meshes with polynomial/exponential grading in $\mathbb{R}^d$, $d = 1, 2, 3$.

Keywords. fast algorithms, hierarchical matrices, data-sparse matrices, mesh refinement, BEM, FEM.

1 Introduction

Consider the h-version of the Galerkin FE method for approximation of the integral operator $A \in \mathcal{L}(W, W')$ defined in the Sobolev space $W = H^r(\Sigma)$. In the typical BEM applications, we deal with integral operators of the form

$$(Au)(x) = \int_\Sigma s(x,y)u(y)dy, \qquad x \in \Sigma,$$

with s being the fundamental solution (singularity function) associated with the *pde* under consideration or with s replaced by a suitable directional derivatives Ds of s. Here Σ is either a bounded $(d-1)$-dimensional manifold (surface) or a bounded domain in $\mathbb{R}^d$, $d = 2, 3$. In this paper, we confine ourselves to the case of an ansatz space $W_h := \mathrm{span}\{\varphi_i\}_{i\in I} \subset W$ of piecewise constant/linear basic functions with respect to the graded tensor-product meshes (and the associated triangulation if necessary) on the computational domain $\Sigma = (0,1)^{d_\Sigma}$. Therefore, we specify $H^r(\Sigma) = H^r_{00}(\Sigma)$ and $d_\Sigma = d - 1$ in the BEM applications, while $W = H^{2r}(\Sigma)$ and $d_\Sigma = d$ for the volume integral calculations. The extension of our approach to the case of closed surfaces is rather straightforward.

We assume that the singularity function s is asymptotically smooth[1], i.e.,

$$|\partial_x^\alpha \partial_y^\beta s(x,y)| \le c(|\alpha|, |\beta|)|x-y|^{-|\alpha|-|\beta|} g(x,y) \qquad \text{for all } |\alpha|, |\beta| \le m \tag{1.1}$$

and for all $x, y \in \mathbb{R}^d, x \neq y$, where α, β are multi-indices with $|\alpha| = \alpha_1 + \ldots + \alpha_d$. We consider two particular choices of the function $g \ge 0$ defined on $\Sigma \times \Sigma$. The first case $g(x,y) = s(x,y)$, $s(x,y) \ge 0$, is discussed in [7]. The second variant to be considered is $g(x,y) = |x-y|^{1-d-2r}$. Here $2r \in \mathbb{R}$ is the order of the integral operator $A : H^r(\Sigma) \to H^{-r}(\Sigma)$ in the BEM applications. Similar smoothness prerequisites are usually required in the wavelet or multi-resolution techniques, see also the related mosaic-skeleton approach in [12] as well as [3].

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[1]The estimate (1.1) with $g = s(x,y)$ is valid in many situations, e.g., in the case of the singularity function $\frac{1}{4\pi}|x-y|^{-1}$ for $d = 3$.

We analyse a data-sparse $\mathcal{H}$-matrix approximation for the integral operators A with asymptotically smooth kernels, see (1.1). The construction consists of three essential stages:

(a) the *admissible* block-partitioning P_2 of the tensor product index set $I \times I$, where – in our example – I is isomorphic to the set of supports of the FE basis functions from W_h;

(b) the construction of an approximate integral operator $A_{\mathcal{H}} \in \mathcal{L}(W, W')$ with the kernel $s_{\mathcal{H}}(\cdot,\cdot)$ defined on each block $X(\sigma) \times X(\tau) \in \Sigma \times \Sigma$, $\sigma \times \tau \in P_2$, by a *separable expansion* $s_{\tau,\sigma}(x,y) = \sum_{i \le k} a_i(x) c_i(y)$ of the order $k \ll n = \dim W_h$;

(c) the setup of approximating the Galerkin $\mathcal{H}$-matrix $\mathcal{A}_{\mathcal{H}} = \langle A_{\mathcal{H}} \varphi_i, \varphi_j \rangle_{i,j \in I}$ for the operator $A_{\mathcal{H}}$, where the near-field (respectively far-field) components are evaluated with the exact (respectively approximate) kernel.

Consider the important step (a) in more details. Let I be the index set of unknowns (e.g., the FE-nodal points). For each $i \in I$, the support of the corresponding basis function φ_i is denoted by $X(i) := \mathrm{supp}(\varphi_i)$. The *cluster tree* $T(I)$ is characterised by the following properties:

(i) all vertices of $T(I)$ are subsets of I,

(ii) $T \in T(I)$ is the root;

(iii) if $\tau \in T(I)$ contains more than one element, the set $S(\tau)$ of sons of τ consists of at least 2 disjoint subsets satisfying $\tau = \bigcup_{\sigma \in S(\tau)} \sigma$;

(iv) the leaves of the tree are $\{i\}$ for all $i \in I$.

For $\tau \in T(I)$ we extend the definition of $X(\cdot)$ by $X(\tau) = \bigcup_{i \in \tau} X(i)$.

In the standard quasiuniform FE application, the cluster tree $T(I)$ is obtained by a recursive division of I into subsets of almost equal size having a diameter as small as possible. In the quasiuniform case, the term "almost equal size" can be understood in a geometrical sense (i.e., $\mathrm{diam}(X(\tau')) \approx \mathrm{diam}(X(\tau''))$ as well as with respect to the cardinality $\#\tau' \approx \#\tau''$. An appropriate construction of $T(I)$ will fulfil both criteria. However, in the *non-quasiuniform* case, these two properties cannot be satisfied in parallel. The remedy is that the first property can be substituted by $\mathrm{diam}(X(\tau')) \approx dist(X(\tau'), X(\tau''))$ due to admissibility condition (1.3) below.

The matrix entries belong to the index set $I \times I$. In a canonical way (cf. [7]), a block-cluster tree $T(I \times I)$ can be constructed from $T(I)$, where all vertices $b \in T(I \times I)$ are of the form $b = \tau \times \sigma$ for $\tau, \sigma \in T(I)$. Given a matrix $M \in \mathbb{R}^{I \times I}$, the block-matrix corresponding to $b \in T(I \times I)$ is denoted by $M^b = (m_{ij})_{(i,j) \in b}$. A *block partitioning* $P_2 \subset T(I \times I)$ is a set of disjoint blocks $b \in T(I \times I)$, whose union equals $I \times I$. A block partitioning P_2 determines the $\mathcal{H}$-matrix format. We use the following explicit definition of $\mathcal{H}$-matrices.

Definition 1.1 *Let a block partitioning P_2 of $I \times I$ and $k \ll n$ be given. The set of real $\mathcal{H}$-matrices induced by P_2 and k is*

$$\mathcal{M}_{\mathcal{H},k}(I \times I, P_2) := \{M \in \mathbb{R}^{I \times I} : \forall\, b \in P_2, \text{ there holds } \mathrm{rank}(M^b) \le k\}. \tag{1.2}$$

The admissibility conditions are used to incorporate the singularity location of the kernel function $s(x,y)$, $(x,y) \in \Sigma \times \Sigma$, in order to balance the size of matrix-blocks b and their distance from the singularity points, see [8] for more details. For the BEM applications, we assume that the following admissibility condition

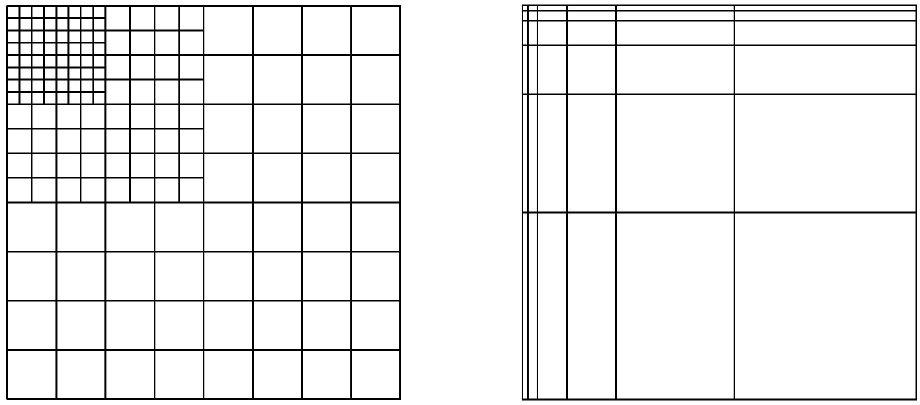
$$\min\{\mathrm{diam}(\sigma), \mathrm{diam}(\tau)\} \le 2\eta \ \mathrm{dist}(\sigma, \tau) \tag{1.3}$$

holds for all $\sigma \times \tau \in P_2$, where $\eta \le 1$ is a fixed parameter. We estimate the approximation error $||A - A_{\mathcal{H}}||_{W \to W'}$ and the global perturbation of the solution arising at the stage (**b**) above as well as the computational complexity of certain $\mathcal{H}$-formats in the following cases:

(i) J-level composite meshes characterised by $h_0 = 2^{-p}$ and $h_{min} = 2^{-J} h_0$ (see Fig. 1a);

(ii) polynomially graded tensor-product meshes $\{\omega_i\}^d$, where $\omega_i \sim \left(\frac{i}{N}\right)^\beta$, $\beta \ge 1$ (see Fig. 1b), with $N^d = \dim W_h$;

(iii) exponentially graded tensor-product geometric meshes $\{\omega_i\}^d$, with $\omega_i \sim \sigma^{N-i}$, $\sigma < 1$.



a) $p = 3,\ J = 2$ b) $\sigma = 0.5;\ N = 5$

Figure 1: Composite and tensor–product graded meshes

Note that in the cases (i) and (ii) the proper h-version of the Galerkin BEM is directly applied in the setup phase **(c)**. A possible extension of this scheme to the case of hp-version of FEs is based on the corresponding construction for the exponentially graded meshes, see item (iii) above. However, in this paper we shall not discuss approximations with higher order finite elements.

We develop two different strategies to construct the $\mathcal{H}$-matrices on graded meshes. The first approach is as closed as possible to the structure of the *block partitioning* P_2 of $I \times I$ on the uniform mesh with $h_i = h$. It is based on the cardinality-balanced cluster tree and yields almost linear complexity. The analysis of the arising geometrical partition of $\Sigma \times \Sigma$ leads to the optimal approximation result. The second concept is based on a separation criterion providing a cluster tree with well balanced geometrical decompositions of Σ on each level. The corresponding admissible block partitioning P_2 is built using a ternary tree $T(I)$ with $\#I = O(3^p)$. However, in the case $h_{i-1} \ll h_i$, the ternary tree approaches a binary one, see the figure in Section 3.

2 Cardinality-Balanced Partitions

In this section, we introduce the cardinality-balanced separation strategy for constructing the cluster tree. We show that the corresponding hierarchical matrices are dense enough in the case of graded meshes, i.e., they lead to the same asymptotically optimal approximations as the exact finite element / boundary element Galerkin schemes. Consider the tensor-product grid $\omega = \{\omega_i\}^d$, where the grid points ω_i, $i = 0, 1, ..., N$, are defined by a sequence of mesh parameters $\{h_i\}_{i \in I_1}$, $h_i > 0$, $I_1 := \{1, ..., N\}$,

$$\omega_0 = 0,\ \omega_N = 1,\ \omega_i = \omega_{i-1} + h_i, \qquad i \in I_1.$$

The associated tensor-product index set I is given by

$$I := \{\mathbf{i} = (i_1, ..., i_d) : 1 \le i_k \le N,\ k = 1, ..., d\}, \qquad N = 2^p,$$

where each multi-index $\mathbf{i} \in I$ corresponds to the box $\delta_\mathbf{i} := [\omega_{i_k}, \omega_{i_k - 1}]_{k=1}^{d_\Sigma} \subset [0,1]^{d_\Sigma}$. In the case of Sobolev spaces $W = H_{00}^r(\Sigma)$ with negative index $r < 0$, we use the ansatz space W_h of piecewise constant FEs on a rectangular mesh, while for $r \ge 0$ the linear elements on the associated triangulation are chosen. For each $t \in \mathbb{R}$ define the weight-function $\mu(x) \in L^2(\Sigma)$ by

$$\mu(x) := h_\mathbf{i}^t \quad \text{at } x \in \delta_\mathbf{i} \qquad \text{with } h_\mathbf{i} = \min_{1 \le k \le d}\{h_{i_k}\}.$$

Assumption 2.1 *(Inverse inequality) For $t \in [0,1]$, the following inequality holds:*

$$||\mu(x) \cdot v||_{0,\Sigma} \le c||v||_{-t,\Sigma} \qquad \textit{for all } v \in W_h. \tag{2.1}$$

The estimate (2.1) was discussed in [2] in an equivalent form for rather general nonuniform grids, see conditions (A1)-(A3) therein. These conditions remain valid for the grids under consideration, see (i)-(iii) in Introduction.

In the case of quasi-uniform meshes (i.e., there are constants $c_1,\ c_2 > 0$ such that $c_1 h \le h_i \le c_2 h,\ i \in I_1$ with $h = N^{-1}$), a class of hierarchical matrix formats $\mathcal{M}_{\mathcal{H},k}(I \times I, P_2)$ was shown to have almost linear complexity and optimal approximation in the BEM applications. Our goal here is an extension of this result to the situation with strongly refined grids. Set $d = 1$ for the moment. Let $T(I)$ be the binary cluster tree

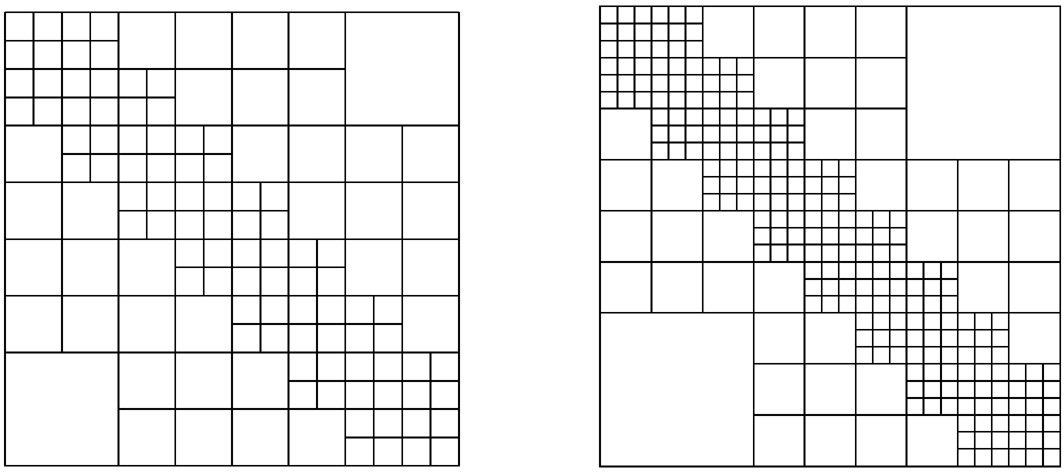


Figure 2: Block-partitionings using (a) binary and (b) ternary trees

of the uniform depth p, where $\#I = 2^p$, which has the root $I_1^0 = I$ on level $\ell = 0$. It is built by successive splitting of each vertex into two parts of equal cardinality. $T(I)$ contains the subsets I_j^ℓ, $0 \le \ell \le p$, $1 \le j \le 2^\ell$, on each level ℓ such that at level p, we reach the one-element sets (leaves) $I_1^p = \{1\}, ..., I_n^p = \{n\}$. The sons $I_{j_1}^{\ell+1}$, $I_{j_2}^{\ell+1}$ of I_j^ℓ are defined by the separation criterion $\#I_{j_1}^{\ell+1} = \#I_{j_2}^{\ell+1}$, see Fig. 3a. The cluster tree

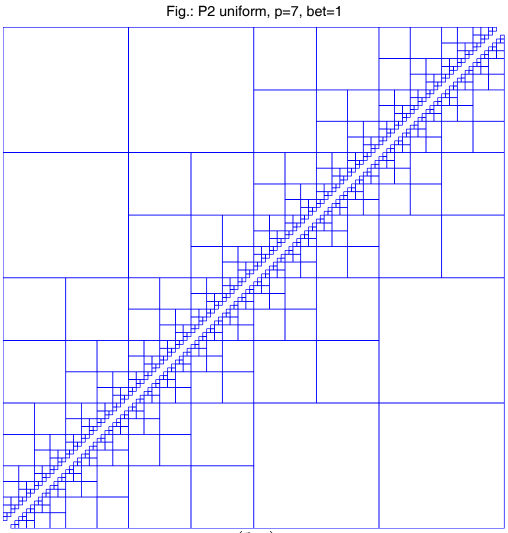
(a) $\#I_1^1 = \#I_2^1$ (b) I_k^1, $k = 1, 2, 3$, satisfy (3.1)

Figure 3: (a) Cardinality- and (b) distance-balanced separation criteria.

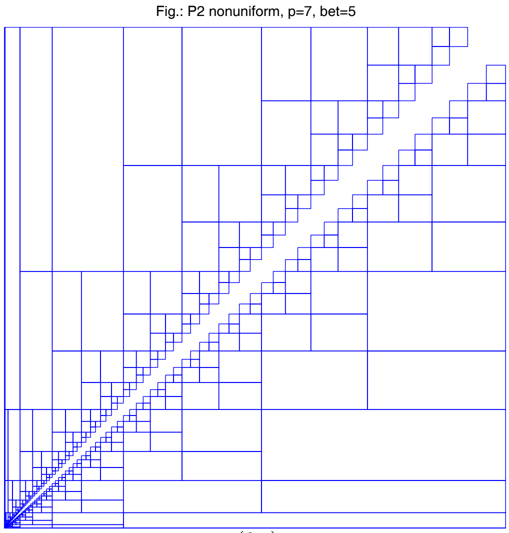
$T_2 := T(I \times I)$ has the following set of vertices, $\mathbf{I}_{ij}^\ell := I_i^\ell \times I_j^\ell$ for $0 \le \ell \le p,\ 1 \le i, j \le 2^\ell$. The set $S_2(t)$ of sons for $t = \mathbf{I}_{ij}^\ell \in T_2$ is given by $S_2(t) := \{\tau \times \sigma : \tau \in S_1(I_i^\ell),\ \sigma \in S_1(I_j^\ell)\}$, where $S_1(f)$ denotes the set of sons belonging the parent cluster $f \in T(I)$. Finally, we obtain the explicit block partitioning $P_2 := \cup_{\ell=2}^p P_2^\ell$, where $P_2^2 = \{\mathbf{I}_{14}^2\} \cup \{\mathbf{I}_{41}^2\}$ and

$$P_2^\ell = \{\mathbf{I}_{ij}^\ell \in T_2 :\ |i - j| \ge 1 \text{ and } \mathbf{I}_{ij}^\ell \cap P_2^{\ell'} = \emptyset,\ \ell' < \ell\} \qquad \text{for } \ell = 3, ..., p.$$

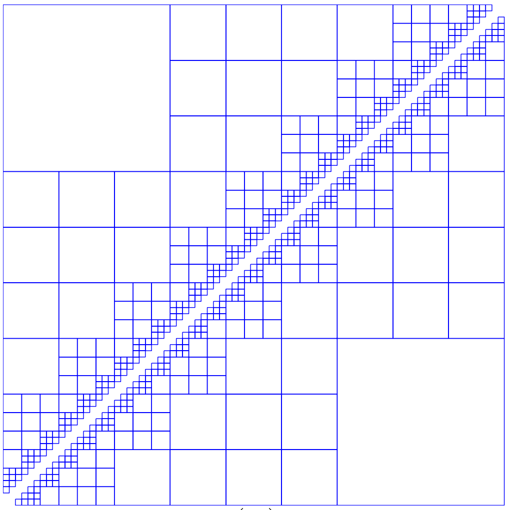
The construction for $d = 2, 3$ is completely similar.



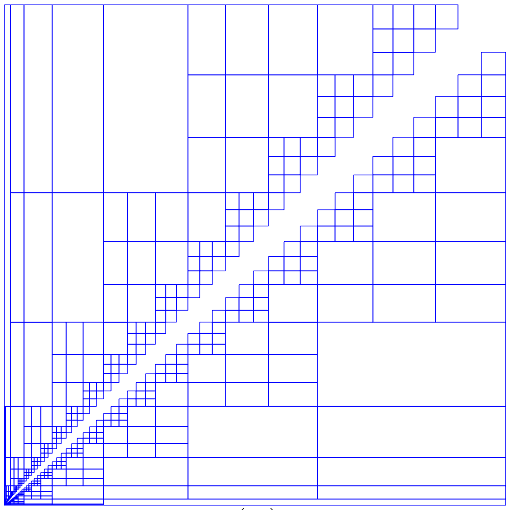
$(b1)$



$(b2)$



$(t1)$



$(t2)$

Figure (b1) depicts the P_2 partitioning of the product index set for $n = 2^7$ based on the binary tree with the cardinality-balanced separation criterion. Figure (b2) presents the associated geometrical decomposition of the product domain $[0,1]^2$ for the grading parameter $\beta = 5$. Figures (t1) and (t2) depict the corresponding partitionings for the same separation strategy in the case of the ternary tree with $n = 3^4$ and with the grading parameter $\beta = 4$. In both figures (b1) and (t1), the left bottom corner corresponds to the index $(1,1)$. Note that the unshaded diagonal stripes in figures (b1) and (t1) correspond to 1×1 blocks of the near-field zone in $I \times I$.

We make a technical assumption which is connected with a certain kind of monotonicity of the refinement.

Assumption 2.2 *Let μ and t be as in (2.1). For each $\tau \times \sigma \in P_2^\ell$, $\ell = 1, ..., p$, there holds*

$$\int\limits_{X(\tau)\times X(\sigma)} \mu^{-2}(x)\mu^{-2}(y)dxdy \leq c\, dist(\tau,\sigma)^{2(d_\Sigma+2t)} \qquad \textit{for } t = -\tfrac{1}{2} \textit{ and } t = 0. \tag{2.2}$$

Assumption 2.2 may be varified for the mesh-refinements considered. There are many opportunities to build separable expansions of the form

$$s_{\tau,\sigma}(x,y) = \sum_{j=1}^{k} a_j(x)c_j(y), \qquad (x,y) \in X(\tau) \times X(\sigma) \tag{2.3}$$

for each cluster $\tau \times \sigma \in P_2$, where $k = O(\log^{d-1} N)$ is the order of expansion. Let x, y vary in the respective sets $X(\tau)$ and $X(\sigma)$ corresponding to the admissible clusters $\tau, \sigma \in T(I)$ and assume without loss of generality that $\mathrm{diam}(X(\sigma)) \le \mathrm{diam}(X(\tau))$. The optimal centre of expansion is the Chebyshev centre[2] y_* of $X(\sigma)$, since then $\|y - y_*\| \le \frac{1}{2}\,\mathrm{diam}(X(\sigma))$ for all $y \in X(\sigma)$. We recall the familiar approximation result (see, e.g., [7] for the proof) based on the Taylor expansion with respect to y.

Lemma 2.3 *Assume that (1.1) is valid and that the admissibility condition (1.3) holds with η satisfying $c(0,1)\eta < 1$. Then for $m \ge 1$, the remainder of the Taylor expansion satisfies the estimate*

$$|s(x,y) - \sum_{|\nu|=0}^{m-1} \frac{1}{\nu!}(y_* - y)^\nu \frac{\partial^\nu s(x,y_*)}{\partial y^\nu}| \le \frac{c(0,m)}{m!}\eta^m \max_{y\in X(\sigma)} |g(x,y)|, \tag{2.4}$$

for $x \in X(\tau)$, $y \in X(\sigma)$.

Let $A_{\mathcal{H}}$ be the integral operator with s replaced by the Taylor expansion $s_{\sigma,\tau}$ for $(x,y) \in X(\tau) \times X(\sigma)$ provided that $\tau \times \sigma \in P_2$ is an admissible block and no leaf. Construct the Galerkin system matrix from $A_{\mathcal{H}}$ instead of A. The perturbation of the matrix induced by $A_{\mathcal{H}} - A$ yields a perturbed discrete solution of the original variational equation

$$\langle (\lambda I + A)u, v\rangle = \langle f, v\rangle \qquad \forall v \in W := H^r(\Sigma),\ r \le 1,$$

where $\lambda \in R$ is a given parameter. The effect of this perturbation in the panel clustering methods is studied in several papers (cf. [10]). Here, we give the consistency error estimate for the $\mathcal{H}$-matrix approximation. Define the integral operator $\widehat{A}$ with the kernel

$$\widehat{s}(x,y) := \begin{cases} \rho(\sigma,\tau) \max\limits_{y\in\sigma} |g(x,y)| & \text{for } (x,y) \in X(\tau)\times X(\sigma),\ \ \#\tau, \#\sigma \ge m^{d_\Sigma}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.5}$$

where $\rho(\sigma,\tau) = (\frac{diam(\sigma)}{dist(\sigma,\tau)})^m$. For the given ansatz space $W_h \subset W$ of piecewise constant/linear FEs, consider the perturbed Galerkin equation for $u_{\mathcal{H}} \in W_h$,

$$\langle (\lambda I + A_{\mathcal{H}})u_{\mathcal{H}}, v\rangle = \langle f, v\rangle \qquad \forall v \in W_h.$$

Theorem 2.4 *Assume that (1.1) is valid and set $\eta = \frac{\sqrt{d}}{2}$. Suppose that the operator $\lambda I + A \in \mathcal{L}(W, W')$ is W-elliptic. Then there holds*

$$||u - u_{\mathcal{H}}||_W \lesssim \inf_{v_h\in V_h} ||u - v_h||_W + \frac{c(0,m)}{m!}\ \eta^m\ ||\widehat{A}||_{W_h\to W_h'}||u||_W. \tag{2.6}$$

Under Assumptions 2.1 and 2.2 the norm of $\widehat{A}$ is estimated by

$$||\widehat{A}||_{W_h\to W_h'} \lesssim \begin{cases} ||A|| & \text{if } g = s(x,y) \text{ and } s(x,y)\ge 0 \\ c\,N^{d_\Sigma/2} & \text{if } g = |x-y|^{1-d-2r}, \end{cases} \tag{2.7}$$

for the range of Sobolev index $-1 \le 2r \le 1 + d_\Sigma - d$.

Proof. The continuity and strong ellipticity of A imply

$$||u - u_{\mathcal{H}}||_W \lesssim \inf_{v\in W_h} ||u - v||_W + \sup_{u,v\in W_h} \frac{|\langle (A - A_{\mathcal{H}})u, v\rangle|}{||u||_W ||v||_W} ||u_{\mathcal{H}}||_W$$

(cf. first Strang lemma). On the other hand, under the assumption (1.1), Lemma 2.3 yields

$$\begin{aligned} |\langle (A - A_{\mathcal{H}})u, v\rangle| &\lesssim \frac{c(0,m)}{m!}\eta^m \sum_{\tau\times\sigma\in P_2} \int_{X(\tau)\times X(\sigma)} |\widehat{s}(x,y)u(y)\,v(x)|dxdy \\ &\lesssim \frac{c(0,m)}{m!}\eta^m\ ||\widehat{A}||_{W_h\to W_h'}||u||_W||v||_W \qquad \forall u, v \in W_h. \end{aligned} \tag{2.8}$$

[2] Given a set X, the centre of the minimal sphere containing X is called the *Chebyshev centre*.

Now, assuming that $\frac{c(0,m)}{m!}\eta^m \,||\widehat{A}||_{W_h\to W_h'}$ is sufficiently small, the estimate (2.7) and $\eta < 1$ imply the strong ellipticity of the discrete Galerkin operator yielding the stability $||u_{\mathcal{H}}||_W \le c||u||_W$. This implies (2.6). In the case $g = s(x,y)$, the first assertion in (2.7) follows from $\rho(\sigma,\tau) \le 1$ and from the bound $||\,|u|\,||_W \le ||u||_W$ for all $u \in W_h$. Consider the case $g = |x-y|^{1-d-2r}$. If $r \ge 0$, the standard L^2-norm estimate combined with the imbedding $H^r(\Sigma) \subset L^2(\Sigma)$ implies

$$|\langle \widehat{A}u, v\rangle| \le \left(\int_\Sigma \widehat{s}(x,y)^2 dxdy\right)^{1/2} ||u||_{r,\Sigma}||v||_{r,\Sigma}.$$

Setting $\varepsilon = 1 - d - 2r$, we then proceed

$$\begin{aligned}
||\widehat{A}||^2_{W_h\to W_h'} &\lesssim \int_\Sigma \widehat{s}(x,y)^2 dxdy \le \sum_{\sigma\times\tau\in P_2}\int_{X(\sigma)\times X(\tau)} \rho^2(\sigma,\tau)(dist(\sigma,\tau))^{2\varepsilon}\, dxdy \\
&\le \sum_{\ell=2}^{p}\sum_{\sigma\times\tau\in P_2^\ell}(dist(\sigma,\tau))^{2\varepsilon}|X(\sigma)||X(\tau)| \le \sum_{\ell=2}^{p}\sum_{\sigma\times\tau\in P_2^\ell}(dist(\sigma,\tau))^{2(1+d_\Sigma-d-2r)} \\
&\le \sum_{\ell=2}^{p}\sum_{\sigma\times\tau\in P_2^\ell} 1 \le \sum_{\ell=2}^{p} 2^{\ell d_\Sigma} \le c\,N^{d_\Sigma},
\end{aligned}$$

where the first estimate in the last line is based on the property of the admissible partitioning: $\#P_2^\ell = O(2^{d_\Sigma \ell})$.

In the case $r < 0$, we first obtain the bound in the weighted L^2-norm and then apply the inverse inequality on graded meshes, see (2.2). It is enough to consider the value $r = -\frac{1}{2}$ only. For such a choice there holds

$$|\langle \widehat{A}u, v\rangle| \le (\int_\Sigma \mu^{-2}(x)\mu^{-2}(y)\widehat{s}(x,y)^2 dxdy)^{1/2}||u||_{r,\Sigma}||v||_{r,\Sigma};$$

$$\begin{aligned}
||\widehat{A}||^2_{W_h\to W_h'} &\lesssim \int_\Sigma \mu^{-2}(x)\mu^{-2}(y)\widehat{s}(x,y)^2 dxdy \\
&\le \sum_{\ell=2}^{p}\sum_{\sigma\times\tau\in P_2^\ell}(\mathrm{dist}(\tau,\sigma))^{2(1-d-2r)} \int\limits_{X(\sigma)\times X(\tau)} \mu^{-2}(x)\mu^{-2}(y)dxdy \\
&\le \sum_{\ell=2}^{p}\sum_{\sigma\times\tau\in P_2^\ell}(\mathrm{dist}(\tau,\sigma))^{2(1+d_\Sigma-d)} \le c\,N^{d_\Sigma}.
\end{aligned}$$

This completes our proof. ∎

By the construction, our P_2 partitioning generates the same block-structure of $\mathcal{H}$-matrices as the corresponding one for the case of uniform meshes. Then, following to [8], we obtain the almost linear complexity bound.

Proposition 2.5 *Let $d \in \{1, 2, 3\}$, $A \in \mathcal{M}_{\mathcal{H},k}(I\times I, P_2)$, and $\eta = \frac{\sqrt{d}}{2}$. Then the storage and matrix-vector multiplication expenses are bounded by*

$$\mathcal{N}_{st} \le (2^d - 1)(\sqrt{d}\eta^{-1} + 1)^d\, pkN, \qquad \mathcal{N}_{MV} \le \mathcal{N}_{st}, \tag{2.9}$$

where the cost unit of $\mathcal{N}_{MV}$ is one addition and one multiplication. Both estimates are asymptotically sharp.

The local Rk-approximations in the Galerkin method may be computed as follows. The block entry $\mathcal{A}_{\mathcal{H}}^{\tau\times\sigma}$ of the Galerkin matrix $\mathcal{A}_{\mathcal{H}} := \{\langle A_{\mathcal{H}}\varphi_i, \varphi_j\rangle\}_{i,j=1}^N$ associated with each cluster $\tau\times\sigma$ on level ℓ may be presented as a rank- k matrix $\mathcal{A}_{\mathcal{H}}^{\tau\times\sigma} = \sum\limits_{|\nu|=0}^{m-1} a_\nu * b_\nu^T$, where $k := \binom{d_\Sigma+m-1}{m-1} = O((m-1)^{d_\Sigma})$ is the number of terms. In turn,

$$a_\nu = \left\{\int_{X(\tau)} (y-y_*)^\nu \varphi_i(y)dy\right\}_{i=1}^{N_\tau}, \qquad b_\nu = \left\{\int_{X(\sigma)} \frac{\partial^\nu s(x,y_*)}{\partial y^\nu}\varphi_j(x)dx\right\}_{j=1}^{N_\sigma},$$

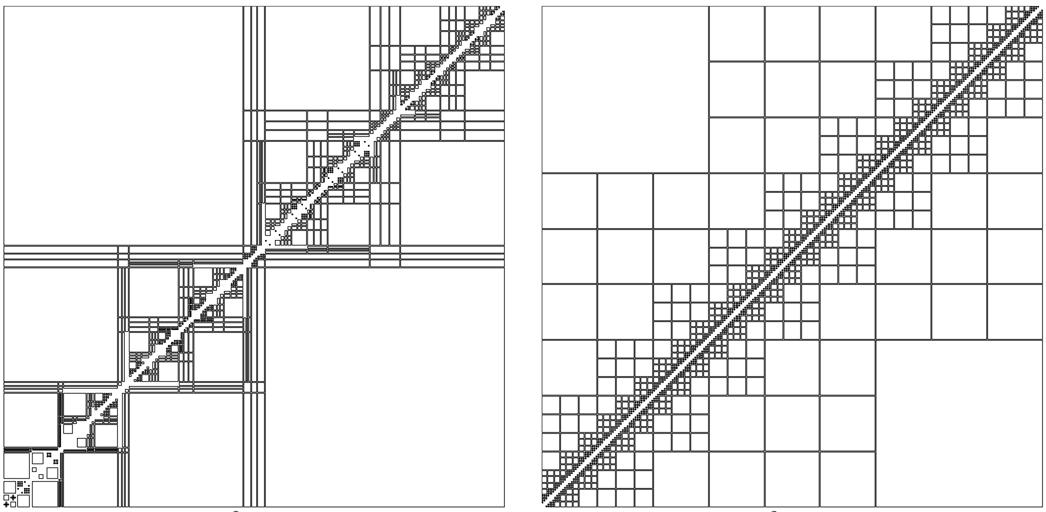
where $N_\tau = \#\tau = O(2^{d_\Sigma(p-\ell)})$ (resp. $N_\sigma = \#\sigma = O(2^{d_\Sigma(p-\ell)})$) is the cardinality of τ (resp. σ).

3 On distance-balanced partitions

In this section, we study the algorithm based on the concept of balanced geometrical partitionings. The main topic to be discussed here is an argument for the linear complexity of the resulting $\mathcal{H}$-matrices. For ease of presentation consider the one dimensional case. In contrast to the *cardinality-balanced partitioning*, here we use a ternary cluster tree $T_1 = T(I)$ with $N = 3^p$. Starting with the root $\{I\}$ on level $\ell = 0$, we introduce the triple of sons I_1^1, I_2^1 and I_3^1 on level $\ell = 1$ (see Fig. 3b), satisfying the *separation criteria*

$$(a)\ \#I_1^1 = \#I_3^1; \qquad (b)\ \operatorname{diam} I_1^1 = \operatorname{diam} I_2^1, \tag{3.1}$$

where $i < i < k$ for all $i \in I_1^1$, $j \in I_2^1$, $k \in I_3^1$. To analyse the complexity of the admissible P_2 partitioning, we first construct the nonary tree $T_2 = T(I \times I) := \{\mathbf{I}_{ij}^\ell\}_{\ell=0}^p$, where $\{\mathbf{I}_{ij}^\ell\} = I_i^\ell \times I_j^\ell$, $i, j \leq 3^\ell$, and then build P_2 with respect to the admissibility condition (1.3). Note that the intermediate cluster I_2^1 plays here an artificial role because in the case of strong refinement we have $\#I_2^1 \ll \#I_1^1$. Therefore, the corresponding branch of the block-cluster tree will be the shorter the stronger the refinement is. It it easy to see that for monotonous refinement the conditions (3.1a,b) imply that the clusters $\sigma = I_1^1$ and $\tau = I_3^1$ satisfy the standard admissibility requirement (1.3) (similar on each level ℓ). The latter immediately yields the local approximation property, see Lemma 2.3, for each admissible block $\sigma \times \tau \in P_2$. Then the global error estimate similar to Theorem 2.4 holds true. In this point, the number of blocks from P_2^ℓ is bounded from above and below by the corresponding one for the nonary and quad trees, respectively, constructed on the uniform grids. On the other hand, the ratio of the latter two values is uniformly bounded with respect to the problem size. In the case of quasi-uniform



$\beta = 8$ $\qquad\qquad$ $\beta = 1$

meshes, the complexity result is completely similar to those from Theorem 2.4 above, see [8] for the case of binary tree.

Lemma 3.1 *Let $d \in \{1, 2, 3\}$, $A \in \mathcal{M}_{\mathcal{H},k}(I \times I, P_2)$, $N = 3^{d\Sigma p}$ and $\eta = \frac{\sqrt{d}}{2}$. Then there holds*

$$\mathcal{N}_{st} \leq (3^d - 1)(\sqrt{d}\eta^{-1} + 1)^d\, kN \log_3 N; \qquad \mathcal{N}_{MV} \leq \mathcal{N}_{st} \tag{3.2}$$

for the storage and matrix-vector multiplication (the cost unit of $\mathcal{N}_{MV}$ is one addition and one multiplication). Both estimates are asymptotically sharp.

In the general case, the complexity analysis is based on the observation that due to assumption $h_i \leq h_{i+1}$, we obtain larger admissible blocks in P_2 on each level compared with those arising on the basis of the balanced nonary tree T_2 for uniform grids. Thus, we have $\mathcal{N}_{st,g} \leq c\mathcal{N}_{st}$ and $\mathcal{N}_{MV,g} \leq c\mathcal{N}_{MV}$, where the abbreviation "g" denotes graded meshes. Moreover, if $h_i \ll h_{i+1}$ the corresponding nonary tree approaches the *quad-tree* and the induced $P_{2,g}$ partitioning becomes very close to the simplest one based on the binary tree for $n = 2^p$ (see the figure in this section drawn for the case of $1D$ polynomially graded mesh with $n = 512$, $\beta = 1$ and $\beta = 8$).

Remark 3.2 *The analysis of $\mathcal{H}$-matrix approximations on the composite grids of Fig. 1a is a particular case of the arguments above. In the case of piecewise linear elements, a standard modification of the FE space by using slave nodes is required.*

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