Anti-complexified Ricci flow on compact symplectic manifolds

by

Hông-Vân Lê and Guofang Wang

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ANTI-COMPLEXIFIED RICCI FLOW ON COMPACT SYMPLECTIC MANIFOLDS

HÔNG-VÁN LÊ AND GUOFANG WANG

MAX-PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES
INSELSTRASSE 22-26
D-04103 LEIPZIG

Abstract. In this note we introduce a gradient flow equation for compatible metrics on symplectic manifolds with respect to the Blair-Ianus energy functional and prove the short time existence of the flow. We provide an example where the flow exists globally.

1. Introduction.

Let \((M,\omega)\) be a symplectic manifold of dimension \(2n\). An almost complex structure \(J\) on \(M\) is called compatible with the symplectic structure \(\omega\) if

\[ g_J := \omega(\cdot, J\cdot) \]

defines a Riemannian metric on \(M\).

Any Riemannian metric of the form \(g_J\) is called a compatible metric with the symplectic form \(\omega\).

Let \(\mathcal{J}(M,\omega)\) (simply denoted by \(\mathcal{J}(M)\)) be the space of compatible almost complex structures. We identify \(\mathcal{J}(M,\omega)\) with the space \(\text{Met}(M,\omega)\) of all compatible metrics by

\[ G : \mathcal{J}(M) \rightarrow \text{Met}(M,\omega), J \mapsto g_J(\cdot,\cdot). \]

A compatible almost complex structure is called harmonic if it is a critical point of the following functional

\[ E(J) = \int_M |\nabla g_J|^2 dvol g_J \]

in \(\mathcal{J}(M)\). Here \(\nabla g_J\) is the Levi-Civita connection with respect to the metric \(g_J\). The functional \(E(J)\) measures the extend of how close of a compatible almost structure being integrable. The functional \(E(J)\) was proposed by James Eells to the first named author in 1995 to find a “best” almost complex structure on symplectic manifolds. However,
later we realized that it was already introduced by Blair an Ianus in [BII].

Let $Ric_{g_J}$ denotes the Ricci curvature of the metric $g_J$. The **anti-complexified Ricci curvature** is defined by

$$Ric^c = Ric_{g_J} - J'(Ric_{g_J}),$$

where $J'$ denotes the induced action of $J$ on the space of 2-symmetric forms $S$: $(J'S)(X,Y) = S(JX,JY)$. It is easy to check (see also next section) that $Ric^c \in T_J Met(M,\omega)$.

Blair and Ianus [B-I] has shown that a compatible almost complex structure $J$ is harmonic if and only if its anti-complexified Ricci vanishes, i.e.,

$$(1.3) \quad Ric^c = 0.$$  

This fact is a consequence of the following first variational formula for $E$ ([B-I], see also [Le]):

$$(1.4) \quad \frac{dE(J_t)}{dt}_{|t=0} = 2 \int_M \langle -Ric^c_{g_J}, G_s(V) \rangle_{\tilde{g}_J},$$

where $V := dJ_t/dt_{|t=0}$ is the direction of the variation and $G_s(V)$ is the associated variation in the space of $Met(M,\omega)$ and $\tilde{g}_J$ denotes the induced metric, which is the natural metric on the space $Met(M,\omega)$.

Since $J$ can be seen as a section of twistor bundle over $M$ with fiber $Sp(2n,\mathbb{R})$ (see [Le] or (2.1) below), (1.4) means that $J$ is a harmonic section. However, unlike the usual harmonic sections (maps), (1.4) is a quasi-linear equation, for the harmonicity is defined with respect to $J$ itself.

It is clear that if $(M,\omega)$ is Kähler then its Kähler complex structure is harmonic (that is actually our motivation to study the energy functional $E$). From equation (1.3) (we note that $Ric^c_{g_J}$ is the orthogonal projection of the Ricci curvature on the tangent space $TMet(M,\omega)$) and taking into account the fact that the volume element $dvol_{g_J} = \omega^n$ is fixed, we see that, if $g_J$ is an Einstein metric, then the associated almost complex structure is also harmonic. When $n = 1$ the functional $E$ is trivial. In [D-M] there are examples of harmonic almost complex structures which are not Kähler, for $n \geq 3$.

There are several works about the classification of the harmonic almost complex structures. We are interested in the existence of harmonic almost complex structures. In this paper, as a first step to study the existence of harmonic almost complex structures, we consider the (negative) gradient flow equation

$$(1.5) \quad \frac{d}{dt}g_J = Ric^c_{g_J}.$$
Equivalently,
\[
\frac{d}{dt} J = \omega^{-1} \cdot \text{Ric}^\mathbb{C},
\]
where \( \omega^{-1} : TM \times TM \to \text{End}(TM) \) is defined, in local coordinates,
\[
(\omega^{-1} \cdot T)^i_j = (\omega^{-1})^{ik} T_{kj}.
\]

Our anti-complexified Ricci flow is motivated by the Hamilton’s Ricci flow, which is inspired by the heat flow of harmonic maps introduced by Eells-Sampson [ES]. In this paper, we shall show the short time existence of (1.5) (see Theorem 3.17). We first like to use a more geometric way to prove the local existence of (1.5). Since (1.5) is not a parabolic equation, we want to change it to an equivalent parabolic equation by using an automorphism group as in [D]. (Here the automorphism group is the symplectomorphism group.) As in the Ricci flow case, the complexified Ricci operator \( \text{Ric}^\mathbb{C} \) is not elliptic. The degeneracy of the anti-complex Ricci operator \( \text{Ric}^\mathbb{C} \) is related to the fact that it is invariant under the action of the symplectomorphism group. Using this invariance we obtain a Bianchi type identity
\[
\frac{d}{dt} J \delta \text{Ric}^\mathbb{C} = 0.
\]
for the complexified Ricci operator \( \text{Ric}^\mathbb{C} \). However we observe that unlike the Ricci operator, the degeneracy of \( \text{Ric}^\mathbb{C} \) is not completely determined by the symplectomorphism group. The reason is simple: the symbol of the operator \( \text{Ric}^\mathbb{C} \) \( \sigma_\xi D\text{Ric}^\mathbb{C} \) has null of dimension two generated by \( \xi \otimes J\xi + J\xi \otimes \xi \) and \( \xi \otimes \xi - J\xi \otimes J\xi \), while the (Hamiltonian) symplectomorphism group only generates a one dimension line (which is generated by \( \xi \otimes J\xi + J\xi \otimes \xi \) in the symbol level. Furthermore, one even can show that the symbol \( \sigma_\xi D\text{Ric}^\mathbb{C} \) is complex (with respect to the Kähler structure on the space of compatible almost complex structures, see below). So it is not enough to use the symplectomorphism group to deal with the degeneracy of the operator \( \text{Ric}^\mathbb{C} \). This leads us to consider complexifying the Hamiltonian symplectomorphism group of the symplectomorphism group. However, in general, there is no such a complexified (Hamiltonian) symplectomorphism group, see [Do2]. Hence we consider to use the complexification of the action of the Hamiltonian symplectomorphism group on the space \( \mathcal{J}(M, \omega) \). Such a way of complexification of the action was first suggested by Donaldson in [Do]. It is a Hamiltonian flow on the infinite dimensional Kähler manifold \( \mathcal{J}(M, \omega) \). It is difficult to show its global existence, but we can show the local existence, see also [Do2]. Unfortunately, this flow may not preserve the complexified Ricci operator. At least till now we are unable to show it. So we have to use a more analytic way to deal with (1.5) which was introduced in [H1]. The Bianchi type identity (1.7) gives us a “half” of the integrability condition introduced in [H1], see section 2 below. The discussion above leads us to find another
“half” of the integrability condition. Altogether guarantees that we can apply the Nash-Moser inverse function theorem ([H1] and [H2]) to our problem (1.5) to obtain the short time existence.

We will consider the global existence of (1.4) for certain symplectic manifolds in forthcoming papers.

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2. Space of compatible almost structures

In this section we describe the space $\mathcal{J}(M,\omega)$ of compatible almost complex structures.

(2.1). The space $\mathcal{J}(M)$ of compatible almost complex structures can be identified with the space of the sections $\Gamma(P)$, where $P$ is a (twistor) bundle over $M$ whose fiber is $Sp(2n,\mathbb{R})/U(n)$. Thus a compatible almost complex structure $J$ is a section of $P$. The tangent space $T_J\mathcal{J}(M)$ is therefore a space of sections of the vector bundle $V(J)$ with the adjoint $U_n$ action. (Every $J$ gives rise to a $U_n(J)$-principal bundle, and $V(J)$ is the orthogonal complement of the subalgebra $u_n(J)$ in $sp(2n,\mathbb{R})$.) Since $Sp(2n,\mathbb{R})/U(n)$ is a Hermitian symmetric space, the space $\mathcal{J}(M)$ inherits the natural Kähler structure. We denote by $I$ the complex structure on $\mathcal{J}(M)$, and by $\Omega$ the symplectic structure on $\mathcal{J}(M)$ (see also [Do]). Of course we can consider $L_q^p$ or $C^k$ compatible almost complex structures and provide $\mathcal{J}(M)$ with certain Banach (or Hilbert) structures. But in this note we are concerned only with smooth compatible almost structures and the smooth structure on $\mathcal{J}(M,\omega)$.

We also present here the second description of the tangent space $T\mathcal{J}(M)$ which is more convenient for calculations.

Let $j = d/dt_{t=0} J_t \in T_{J_0}\mathcal{J}(M)$. Denote by $s*$ the skew-star (adjoint) operator with respect to the symplectic form $\omega$ i.e.

$$\omega(Av, w) = \omega(v, A^{*s}w).$$

Then clearly an element $j \in End(TM)$ is in $T_{J_0}\mathcal{J}(M)$ iff the two following conditions holds

\begin{align*}
(2.2.a) & \quad jJ_0 + J_0j = 0 \\
(2.2.b) & \quad j + j^{*s} = 0
\end{align*}

By a direct computation we get
2.3. Lemma. The skew-star operator \((s\ast)\) can be defined via \(J\) and star \((\ast)\) operator as follows
\[
A^{ss} = -JA^s J
\]
As a direct consequence we get

2.4. Corollary. The conditions (2.2.a) and (2.2.b) are equivalent to the following

\[
\begin{align*}
(2.2.a) & \quad jJ_0 + J_0j = 0 \\
(2.2.b') & \quad j^s = j.
\end{align*}
\]
Thus the orthogonal projection to the tangent space \(T_J J\) equals the composition of the symmetrization and the operator of taking anti-complex linear part.

2.5. Remark. Let \(j' := J_0 \cdot j\). Then the conditions (2.2.a) and (2.2.b') are equivalent to the following

\[
\begin{align*}
(2.2.a') & \quad j'J_0 + J_0j' = 0, \\
(2.2.b'') & \quad (j')^s = j'.
\end{align*}
\]
Thus we see: the multiplication with \(J_0\) defines the natural \(\text{Sym}(M)\)-invariant complex structure \(I\) on \(J(M)\) which is arisen in 2.1 (see also [Do]).

Under the identification \(G\) of the space \(J(M)\) with the space \(\text{Met}(M, \omega)\) of compatible Riemannian metrics on \(M\), we can rewrite the conditions (2.2.a) and (2.2.b) as follows. The symmetric tensor

\[
D = \frac{d}{dt} g_{j=0}
\]
satisfies the following anti-complex-linear condition
\[
J^s(D)(X, Y) = \omega(JX, jJY) = -\omega(JX, JjY) = -D(X, Y).
\]
We can write this condition as
\[
(2.2') \quad J^s(D) = -D.
\]
In term of matrix multiplication \(^1\) we get from (2.2')
\[
(2.6) \quad J^s DJ = -D \iff DJ + JD = 0.
\]
Thus the tangent spaces of \(T_J J\) and \(T_{g_0} J\) are canonically identified.

2.7. We denote by \(\text{Sym}_\text{Ham}(M, \omega)\) the group of Hamiltonian symplectomorphisms of \((M, \omega)\). (The reason why we are interested in \(\text{Sym}_\text{Ham}\) is that this group is a normal subgroup of the symplectomorphism group \(\text{Sym}(M, \omega)\), and a) the quotient \(\text{Sym}/\text{Sym}_\text{Ham}\) is of finite dimension, b) The group \(\text{Sym}_\text{Ham}\) is easier to handle from the analytical point of view.) Clearly this group acts on the space

\(^1\)as in [BI]
$\mathcal{J}(M, \omega)$. First we recall that any vector field $v$ on $M$ acts on the space of metrics by taking the Lie derivative (resp. the space of almost complex structures) as follows (see e.g. [Be, lemma 1.60])

\[(2.8) \quad \mathcal{L}_v(g) = 2\delta^* v',\]

where $\delta^*$ is the symmetrization of the covariant derivative, and $v'$ denotes the dual 1-form (w.r.t. the metric $g$) of the vector field $v$. An explicit formula for $\mathcal{L}_vJ$ is more complicated. We do not need it here. We refer to [Do, Lemma 10] for such an expression. But when $v$ is symplectic vector fields, its expression is simple:

\[(2.9) \quad \mathcal{L}_vJ = 2\omega^{-1} \cdot \delta^* v'.\]

### 3. A BIANCHI TYPE IDENTITY AND THE INTEGRABILITY CONDITION

In this section, we obtain a Bianchi type identity by using the invariance of the energy functional under the action of the symplectomorphism group. As mentioned in the introduction, this identity is not enough in order to show the short time existence of (1.5). We find the another part of the integrability condition using the Kähler structure $\mathcal{I}$ defined in the previous section.

Recall the definition of action of symplectomorphism on $\mathcal{J}$. Let $\phi \in \text{Symp}(M)$ and $J \in \mathcal{J}$. One define a new compatible almost complex structure by

\[\phi^* J(X) = \phi^{-1}_x (J(\phi_x(x))),\]

for any $X \in TM$.

**3.1 Lemma.** The following formula

\[g_{\phi^* J} = \phi^* (g_J)\]

holds if $\phi$ is a symplectomorphism.

**Proof.** For completeness, we give the proof of the Lemma. For any symplectomorphism $\phi$, we have, for any $p \in M$ and $X, Y \in T_pM$,

\[\omega_p(\phi^* J(X), Y) = \omega_p((\phi^{-1})_* J(\phi_* X), Y) = \omega_{\phi_* p}(J(\phi_* X, \phi_* Y) = \phi^* g_J(X, Y),\]

The second equality holds, since $\phi$ is a symplectomorphism. \qed

**3.2 Proposition.** The functional $E$ is invariant under the action of $\text{Symp}(M, \omega)$. 

Proof. From Lemma 3.1, we have
\[ E(\phi^* J) = \int_M \|
abla \phi^*(g_J) \phi^* J\|^2_{\phi^*(g_J)} vol_{\phi^*(g_J)} \]
\[ = \int_M \phi^*(\nabla g_J J)^2_{\phi^*(g_J)} vol_{\phi^*(g_J)} \]
\[ = \int_M \|
abla g_J J\|^2_{g_J} vol_{g_J} = E(J). \]
\[ \square \]

3.3 Corollary. The group \( \text{Symp}(M, \omega) \) preserves \( \text{Ric}^c \), i.e., for any symplectomorphism \( \phi \),
\[ (3.3) \quad \phi^* (\text{Ric}^c) = \text{Ric}^{\phi^* (J)}. \]
\[ \square \]

Equality (3.3) implies that the harmonicity of an almost complex structure is preserved under the action of symplectomorphisms. Hence the operator \( \text{Ric}^c \) is not elliptic.

Let \( S^2(TM) \) denote the space of symmetric 2-tensors. Recall that for any \( J \in \mathcal{J} \), the tangent space of \( \text{Met}(M, \omega) \) at \( g_J, T_{g_J} \text{Met}(M, \omega) \) is the space of sections of \( S^2(TM) \) satisfying (2.2'). Let \( \Omega^2(M) \) be the space of 1-forms and \( \Omega^0_0(M) \) the space of functions with average zero. Define an operator \( \delta^* : \Omega^1(M) \to S^2(TM) \) by
\[ \delta^* = \mathcal{L}_{\alpha} g_J, \]
where \( \alpha^\# \) is the dual vector of \( \alpha \) with respect to the metric \( g_J \) and \( \mathcal{L}_{\alpha^\#} \) is the Lie derivative. Giving \( S^2(TM) \) and \( \Omega^1(M) \) the induced metrics of \( g_J \), we can define the adjoint operator of \( \delta^* \), \( \delta : S^2(TM) \to \Omega \) which can be given, in local coordinates,
\[ (\delta h)_k = g^{ij} h_{jk,i}. \]

3.4 Proposition. We have the following Bianchi type identity
\[ (3.4) \quad d^a J \delta \text{Ric}^c = 0. \]

Proof. For any function \( f \) on \( M \), one can define a so-called Hamiltonian vector field \( X_f \) by
\[ \omega(X_f, Y) = -df(Y), \]
for any \( Y \in TM \). Let \( \phi_t \) be a family of symplectomorphisms generated by \( X_f \), i.e.,
\[ \frac{d}{dt} \phi_t = X_f(\phi_t). \]
Since symplectomorphisms preserve the energy functional \( E \), we have
\[ (3.5) \quad 0 = \frac{d}{dt} E(\phi_t * (J)) = \int_M \langle \text{Ric}^c(g_J), L_{X_f} g \rangle vol(M). \]
From (3.5), we have

\[ 0 = \int_M f d^* \delta\hat{R} \hat{c} \text{vol}(M), \]

for all function \( f \), which yields (3.4).

3.6. Now we compute the symbol of the operator \( \hat{R} \hat{c} \). First, the linearization operator of the Ricci operator in the direction \( h \in S^2(TM) \) is

\[ D\hat{R} \hat{c}(g)h = \frac{1}{2} \Delta_L h - \delta^*(\delta G(h)), \]

where \( \Delta_L \) is the Lichnerowicz Laplacian defined by \( \Delta_L h = D^*Dh + \hat{R} \hat{c} \circ h + \hat{R} \hat{c} - 2\hat{R}_g \). (See [Be].) Locally the derivative of the Ricci operator can be expressed as follows:

\[ DR \hat{c}(g_{\alpha\beta})h_{\alpha\beta} = g^{\gamma\delta}\left\{ \frac{\partial^2 h_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} - \frac{\partial^2 h_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 h_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 h_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} \right\} + \cdots, \]

where the dots denote lower order terms. The symbol of the linear differential operator \( DR \hat{c}(g_{\alpha\beta}) \), which maps \( S^2(R^{2n}) \) to itself, in the direction \( \xi \) is

\[ \sigma DR \hat{c}(g_{\alpha\beta})(\xi)h_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}[h_{\alpha\gamma}\xi_{\beta}\xi_\delta + h_{\beta\gamma}\xi_\alpha\xi_\gamma - h_{\alpha\beta}\xi_\gamma\xi_\delta - h_{\gamma\delta}\xi_\alpha\xi_\beta]. \]

(See [D] and [H1].)

Let \( S^2_2(TM) \) denote the subspace of symmetric tensors which satisfy (2.2'). In fact, \( S^2_2(TM) = T_J Met(M, \omega) \). Let \( c(J) = Id - J^2 \) such that \( \frac{1}{2}c(J) \) is a projection operator from \( S^2(TM) \) to \( S^2_2(TM) \). The linearization operator \( DR \hat{c} \) of \( \hat{R} \hat{c} \) is

\[ DR \hat{c}(g_{\beta\gamma})(h_{\alpha\beta}) = c(J) DR \hat{c}(g_{\beta\gamma})(h_{\alpha\beta}) + \cdots = \]

\[ = c(J)c_{\alpha\beta}g^{\delta\gamma}\left\{ \frac{\partial^2 h_{\alpha\beta}}{\partial x^\gamma \partial x^\delta} - \frac{\partial^2 h_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 h_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 h_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} \right\} + \cdots, \]

where \( h_{\alpha\beta} \in T_J Met(M, \omega) \). The symbol of the linear differential operator \( DR \hat{c} \), is

\[ (3.7) \]

\[ \sigma_\xi DR \hat{c}(g_{\beta\gamma})(h_{\alpha\beta}) = \frac{1}{2}(\delta^p_{\alpha}\delta^q_{\beta} - J^p_{\alpha} J^q_{\beta})g^{\gamma\delta}[h_{\gamma\xi_\alpha}\xi_\delta + h_{\xi_\alpha}\xi_\gamma - h_{\gamma\xi_\delta}\xi_\beta]. \]

Clearly \( h_{\alpha\beta} = (\delta^p_{\alpha}\delta^q_{\beta} - J^p_{\alpha} J^q_{\beta})(\xi_\alpha\xi_\beta + \xi_\beta\xi_\alpha) \) and \( h_{\alpha\beta} = J^p_{\alpha}\xi_\alpha + J^q_{\alpha}\xi_\beta \) are the zero eigenvalues of \( \sigma DR \hat{c}(\xi)(g_{\beta\gamma})(h_{\alpha\beta}) \). (See also the proof of Proposition 3.12.)

Globally, the linearization operator of \( \hat{R} \hat{c} \) is

\[ DR \hat{c}(g_{\beta})(h) = c(J)\left\{ \frac{1}{2} \Delta_L h - \delta^*(\delta G(h)) \right\} + Dc(J) R h, \]

for \( h \in S^2_2(TM) \). Here we use the notation given in [H1].
A crucial point is that the (highest) symbol $\sigma_\xi DRic^\varepsilon$ is complex with respect to the complex structure $I$, namely for any $h \in S^2(TM)$,
\begin{equation}
\sigma_\xi DRic^\varepsilon(Ih) = I\sigma_\xi DRic^\varepsilon(h).
\end{equation}

3.9. For a given metric $g$ on $M$ let us denote by $O_g$ be the subspace of $\Omega^1(M)$ defined by $O_g = \text{Im}\{d : \Omega^0(M) \to \Omega^1(M)\}$ and
\begin{equation}
p : \Omega^1(M) \to O_g \subset \Omega^1(M)
\end{equation}
the orthogonal projection. It is to see that $p$ is a zero order pseudodifferential operator. In fact,
\begin{equation*}
p = d\Delta^{-1}d^*,
\end{equation*}
where $\Delta^{-1}$ is the inverse operator of the Laplacian $\Delta = d^*d : \Omega^0(M) \to \Omega^0(M)$. The operator $\Delta^{-1} : \Omega^0(M) \to \Omega^0(M)$ is well-defined. Define $L_1 : S^2(TM) \to \Omega^1(M)$ by $L_1 = pJ\delta$. By (3.4) we have
\begin{equation}
L_1 Ric^\varepsilon = 0.
\end{equation}
Let $L_2 : S^2(TM) \to \Omega^1(M)$ be defined by $L_2 = L_1 I$. Though $L_2 Ric^\varepsilon$ may not vanish, we will show that the operator $L_2 Ric^\varepsilon$, at most, has degree 1 w.r.t. $g_J$. Now we define $L_{g_J} : S^2(TM) \to \Omega^1(M)$ by $L = (L_1, L_2)$.

3.12 Proposition. (1) The operators $L_{g_J}$ and $L_{g_J} Ric^\varepsilon$ have degree 1.
(2) All the eigenvalues of $\sigma_\xi DRic^\varepsilon$ in $\text{Null}\sigma_\xi L$ are positive.

Proof. (1). Since the projection $p$ is a pseudodifferential operator of degree 0, we get that $L_{g_J}$ is a pseudodifferential operator of degree 1. We have to show that $L_2 Ric^\varepsilon$ has degree 1. We know that $L_2 Ric^\varepsilon$ is a third order operator. Now we first show that the third order symbol of $D(L_2 Ric^\varepsilon)$ is zero. It is easy to check that
\begin{equation*}
\sigma_\xi^3 D(L_2 Ric^\varepsilon) = I\sigma_\xi^3 D(L_1 Ric^\varepsilon).
\end{equation*}
Hence (3.11) implies that $\sigma_\xi^2 L_2 Ric^\varepsilon$ vanishes. Now we compute the second order symbol of $D(L_2 Ric^\varepsilon)$. We first have the following relation
\begin{equation}
\sigma_\xi^2 D(L_2 Ric^\varepsilon) = \sigma_\xi^0 (L_1 I) \sigma_\xi^2 DRic^\varepsilon + \sigma_\xi^1 (L_1 I) \sigma_\xi^1 DRic^\varepsilon.
\end{equation}
For a point $x \in M$ we use normal coordinates around $x$ to compute that at the given point $x$ we have $\sigma_\xi^1 DRic^\varepsilon(x) = 0$. On the other hand, from (3.8) and (3.11), we get
\begin{equation*}
\sigma_\xi^0 (L_1 I) \sigma_\xi^2 DRic^\varepsilon = \sigma_\xi^0 (L_1 I) \sigma_\xi^2 DRic^\varepsilon I = 0.
\end{equation*}
Altogether yields that $\sigma_\xi^2 L_2 Ric^\varepsilon(x) = 0$. Hence $L_2 Ric^\varepsilon$ is a first order operator, so is $L$.

(2). For convenience, we use the similar calculation presented in [H1] to show (2). We choose coordinates at a point such that $\partial / \partial x_1, \partial / \partial x_2, \cdots, \partial / \partial x_n$,
$J_{\frac{\partial}{\partial x_i}}, J_{\frac{\partial}{\partial x_{n+k}}}, \ldots, J_{\frac{\partial}{\partial x_n}}$ is an orthonormal basis. Sometimes, we denote $k = n + k$ for $k = 1, 2, \ldots, n$. Assume $\xi_1 = 1$ and $\xi_i = 0$ for $i \neq 1$. Let $T_{jk}$ be a symmetric tensor and $h_{jk} = T_{jk} - J^p J_{kq} T_{pq}$ its anti-complex part with respect to the almost complex structure $J$. Then the symbol of $DRic'$ acts on the tensor $h_{jk}$ is

$$[\sigma_{\xi} DRic' h]_{jk} = 2h_{jk}, \quad \text{if } j \neq 1 \text{ or } k \neq 1 \text{ or } 1,$$

$$[\sigma_{\xi} DRic' h]_{jk} = h_{jk}, \quad \text{if } k \neq 1 \text{ or } 1,$$

$$[\sigma_{\xi} DRic' h]_{11} = 0,$$

$$[\sigma_{\xi} DRic' h]_{11} = \sum_{j=2}^{n} (h_{jj} + h_{jj}).$$

Let $h$ be an element of Null space of $\sigma_{\xi} L(g)$. It is clear that $h$ is an element of null space of $\sigma_{\xi} \{d^s L(g)\} = \sigma_{\xi} d^s \circ \sigma_{\xi} L(g)$. Since $d^s p = d^s$, $d^s L = (d^s J\delta, d^s J\delta I)$. Hence $h$ satisfies

(3.15.a) \quad $[\sigma_{\xi} d^s J\delta h] = 0$

and

(3.15.b) \quad $[\sigma_{\xi} d^s J\delta I h] = 0$.

Using the same form of $\xi$ and $g$ in local coordinates, one can readily to compute the symbols of $\sigma_{\xi} d^s J\delta$ and $\sigma_{\xi} d^s J\delta I$,

(3.16) \quad $[\sigma_{\xi} d^s J\delta h] = h_{11}$ and $[\sigma_{\xi} d^s J\delta I h] = h_{11}$.

By (3.15) and (3.16), an element in the null space of $\sigma_{\xi} \{d^s L(g)\}$ satisfies $h_{11} = h_{11} = 0$. By (3.14), we have shown that the eigenvalues of $\sigma_{\xi} DRic'$ in Null $\sigma_{\xi} L$ are positive. \hfill $\square$

Following [Ha1], we call the operator $L$ the integrability condition.

3.17 Theorem. For any smooth compatible almost complex structure $J_0$, there exists $T > 0$ such that (1.5) admits a unique solution $J(t)$ satisfying $J(0) = J_0$.

Proof. Using Proposition 3.12, we can follow the argument of Hamilton in [Ha1, sections 5 and 6] to show the Theorem. The only difference is that the projection operator $p = d\Delta^{-1} d^s$ in our integrability condition $L$ is a pseudo-differential operator. The crucial point to apply the Hamilton-Nash-Moser inverse function theorem for tame Frechet spaces developed in [Ha2, Part III, Theorem 1.1.1] is to check that all relevant operators are tame and the linearized equation has a unique solution. This can be done as in [Ha1] for our operators, since $p$ is a bounded operator.

For convenience, we sketch the proof of Theorem 3.17 adapted from the proof of Hamilton.

Step 1. Reduction to the Hamilton-Nash-Moser inverse function theorem. We denote by $Met(M \times [0, 1], \omega)$ the space of smooth sections
of the induced bundle of compatible metrics over $M \times [0, 1]$. Let us consider the operator

$$\mathcal{E} : \text{Met}(M \times [0, 1], \omega) \to \text{Met}(M \times [0, 1], \omega) \times \text{Met}(M, \omega),$$

$$\mathcal{E}(g) = \left( \frac{dg}{dt} - \text{Ric}^\varepsilon(g), g|\{t = 0\} \right).$$

In order to apply the Hamilton-Nash-Moser inverse function theorem we have to show that the linearized equation

$$D\mathcal{E}(g)\dot{g} = \frac{\partial \dot{g}}{\partial t} - D\text{Ric}^\varepsilon(g)\dot{g} = \ddot{\tilde{h}},$$

has a unique solution for the initial value problem $\dot{\tilde{g}} = \tilde{g}_0$ at $t = 0$ and verify that the solution $\ddot{\tilde{g}}$ is a smooth tame function of $\tilde{h}$ and $\tilde{g}_0$.

**Step 2.** The integrability condition reduces the linearized equation to a system of two PDEs. We denote by $Q(g)$ the composition $L(g) \circ \text{Ric}^\varepsilon(g)$. Let us consider new two differential operators $M(g)$ and $P(g)$ such that

$$M(g)\ddot{\tilde{g}} = DL(g)\{\tilde{g}, \frac{\partial \tilde{g}}{\partial t}\} - DL(g)\{\text{Ric}^\varepsilon(g), \tilde{g}\} + DQ(g)\ddot{\tilde{g}},$$

$$P(g) = D\text{Ric}^\varepsilon(g) + L^*(g)L(g).$$

Operator $M$ appears in the evolution equation for $\ddot{\tilde{e}} = L(g)\ddot{\tilde{g}}$. Namely we have

$$\frac{\partial \ddot{\tilde{e}}}{\partial t} - M(g)\ddot{\tilde{g}} = \ddot{k},$$

where $\ddot{k} = L(g)\ddot{\tilde{h}}$. The operator $P$ is obtained from $D\text{Ric}^\varepsilon$ by “killing” the kernel of its symbol and hence is a a parabolic pseudo differential operator. Now as in [Hal] the uniqueness of the solution of the linearized equation $D\mathcal{E}(g)$ is equivalent to the uniqueness of the (smooth) solution of the system

$$\begin{cases}
\frac{\partial \tilde{g}}{\partial t} - P(g)\tilde{g} + L^*(g)\tilde{e} = \ddot{\tilde{h}}, \\
\frac{\partial \tilde{e}}{\partial t} - M(g)\tilde{g} = \ddot{\tilde{k}},
\end{cases}$$

(3.18)

for unknown sections $\tilde{g}, \tilde{e}$ and for given $\ddot{\tilde{h}}, \ddot{\tilde{k}}$, and given $g$ with initial data $\ddot{\tilde{g}} = \tilde{g}_0$ and $\ddot{\tilde{e}} = \tilde{e}_0 = L(g_0)(\tilde{g}_0)$ at $t = 0$.

**Step 3.** The existence and uniqueness of the solution of (3.18). The existence and uniqueness is obtained by the iteration method and by using uniform estimate for the second equation in (3.18) delayed in time $\delta$. Roughly speaking, we can assume without loss of generality that all $\ddot{\tilde{e}}, \ddot{\tilde{g}}, \ddot{\tilde{k}}, \ddot{\tilde{h}}$ vanish in the negative time $t \leq 0$. Then we solve the second delayed equation on the time interval $(0, \delta)$, then use it to solve the first parabolic equation on the time interval $(0, \delta)$, and advance this procedure further. To get the true solution we need to consider the limit solution when $\delta$ goes to zero and get a uniform estimate on the solutions. This will be done in the last step.
Step 4. The tameness of the solution \((\hat{g}, \hat{e})\) of the system \((3.18)\). We recall that [Ha 2] a continuous nonlinear map \(P\) of a Frechet space \(F_1\) to a Frechet space \(F_2\) is **tame** if it satisfies a tame estimate in a neighborhood \(U\) of each point in \(F_1\)

\[
||P(g)||_n \leq C(1 + ||g||_{n+r})
\]

for all \(f \in U\) and all \(n \geq b = \text{const}(U)\) and \(C\) is a constant which may depend on \(n\). Here the norm \(||g||_n\) is defined as follows

\[
||g||_n^2 = \sum_{2j \leq n} \int_0^T |(\partial/\partial t)^j g|_{n-2j} dt,
\]

with \(||g||_n\) measure \(L_2\) norm of \(g\) and its derivates up to degree \(n\). Now it is easy to verify that all the tame estimates (Lemma 6.10 in [Ha 1]) are also valid here (for the delayed equation in Step 3) also valid in our case of pseudo-differential operator \(L\).

\[\square\]

4. Example

In this section, we discuss the anti-complexified Ricci flow on the Thurston manifold and show that the flow exists globally and converges to a degenerate metric. It is easy to see that the energy of the evolving metrics tends to zero.

Let \(G = H^3 \times S^1\) be the product of the Heisenberg group and \(S^1\). I.e, \(G\) is a closed Lie subgroup of \(GL(4, \mathbb{C})\) defined by

\[
\begin{pmatrix}
1 & x & z & 0 \\
0 & 1 & y & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & t
\end{pmatrix}
\]

with \(x, y, z, t \in \mathbb{R}^1\). Let \(\Gamma\) be a subgroup of \(G\) consisting of all matrices of \(G\) which entries are integers. The quotient \(M = G/\Gamma\) is the Thurston manifold. Differential forms \(dx, dy, dz - xdy, dt\) are invariant under the left translation by any element in \(G\). A left invariant metric on \(G\) is given in [A] by

\[
ds^2 = dx^2 + dy^2 + (dz - xdy)^2 + dt^2.
\]

Clearly, from this metric we can get an metric on \(M\), which is also denoted by the same form. Set

\[
\omega = dx \wedge dt + dy \wedge (dz - xdy).
\]

It is a symplectic form on \(G\) and invariant under the translation. Hence, we can see it as a symplectic form on \(M\). The metric \(ds^2\) is a compatible metric with \(\omega\), since there is an almost complex structure

\[
Jdx = dt, \quad Jdt = -dx, \quad Jdy = dz - xdy, \quad J(dz - xdy) = -dy
\]
such that $ds^2 = \omega(\cdot, J\cdot)$. Clearly, following metric

$$g = A^{-2}dx^2 + B^{-2}dy^2 + B^2(dz - xdy)^2 + A^2dt^2,$$

for any $A, B > 0$ are compatible metrics. The corresponding almost complex structures are

$$Jdx = A^{-2}dt, \quad Jdt = -A^2dx,$$
$$Jdy = B^{-2}(dz - xdy), \quad J(dz - xdy) = -B^2dy.$$

The anti-complexified Ricci flow starting from the Abbena metric ($A = B = 1$) is

$$\begin{align*}
\frac{d}{dt}A^{-2} &= \frac{1}{2}A^4B^4, \\
\frac{d}{dt}B^{-2} &= A^2B^6.
\end{align*} \tag{4.1}$$

For this example, it is easy to solve (4.1). Its solution is

$$A^2 = \sqrt{2}(5\sqrt{2}t + 1)^{-\frac{1}{2}},$$
$$B^2 = (5\sqrt{2}t + 1)^{-\frac{1}{2}}.$$

So, the anti-complexified Ricci flow (4.1) exists globally. However, at infinity, it becomes degenerate.

References


