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by

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ANTI-COMPLEXIFIED RICCI FLOW ON COMPACT SYMPLECTIC MANIFOLDS

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ABSTRACT. In this note we introduce a gradient flow equation for compatible metrics on symplectic manifolds with respect to the Blair-Ianus energy functional and prove the short time existence of the flow. We provide an example where the flow exists globally.

1. INTRODUCTION.

Let (M, ω) be a symplectic manifold of dimension $2n$. An almost complex structure J on M is called **compatible** with the symplectic structure ω if

$$g_J := \omega(\cdot, J\cdot) \text{ defines a Riemannian metric on } M.$$

Any Riemannian metric of the form g_J is called a compatible metric with the symplectic form ω.

Let $\mathcal{J}(M, \omega)$ (simply denoted by $\mathcal{J}(M)$) be the space of compatible almost complex structures. We identify $\mathcal{J}(M, \omega)$ with the space $Met(M, \omega)$ of all compatible metrics by

$$G : \mathcal{J}(M) \to Met(M, \omega), J \mapsto g_J(\cdot, \cdot). \tag{1.1}$$

A compatible almost complex structure is called **harmonic** if it is a critical point of the following functional

$$E(J) = \int_M |\nabla_{g_J} J|^2 dvol g_J \tag{1.2}$$

in $\mathcal{J}(M)$. Here ∇_{g_J} is the Levi-Civita connection with respect to the metric g_J. The functional E measures the extend of how close of a compatible almost structure being integrable. The functional $E(J)$ was proposed by James Eells to the first named author in 1995 to find a "best" almost complex structure on symplectic manifolds. However,

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later we realized that it was already introduced by Blair an Ianus in [BI].

Let Ric_{g_J} denotes the Ricci curvature of the metric g_J. The **anti-complexified Ricci curvature** is defined by

$$Ric^c = Ric_{g_J} - J^*(Ric_{g_J}),$$

where J^* denotes the induced action of J on the space of 2-symmetric forms S: $(J^*S)(X,Y) = S(JX,JY)$. It is easy to check (see also next section) that $Ric^c \in T_J Met(M,\omega)$.

Blair and Ianus [B-I] has shown that a compatible almost complex structure J is harmonic if and only if its anti-complexified Ricci vanishes, i.e.,

$$Ric^c = 0. \tag{1.3}$$

This fact is a consequence of the following first variational formula for E ([B-I], see also [Le]):

$$\frac{dE(J_t)}{dt}_{|t=0} = 2\int_M \langle -Ric^c_{g_J}, G_*(V)\rangle_{\tilde{g}_J}, \tag{1.4}$$

where $V := dJ_t/dt_{|t=0}$ is the direction of the variation and $G_*(V)$ is the associated variation in the space of $Met(M,\omega)$ and $\tilde{g}_J$ denotes the induced metric, which is the natural metric on the space $Met(M,\omega)$.

Since J can be seen as a section of twistor bundle over M with fiber $Sp(2n,\mathbf{R})$ (see [Le] or (2.1) below), (1.4) means that J is a harmonic section. However, unlike the usual harmonic sections (maps), (1.4) is a quasi-linear equation, for the harmonicity is defined with respect to J itself.

It is clear that if (M,ω) is Kähler then its Kähler complex structure is harmonic (that is actually our motivation to study the energy functional E). From equation (1.3) (we note that $Ric^c_{g_J}$ is the orthogonal projection of the Ricci curvature on the tangent space $TMet(M,\omega)$) and taking into account the fact that the volume element $dvol_{g_J} = \omega^n$ is fixed, we see that, if g_J is an Einstein metric, then the associated almost complex structure is also harmonic. When $n = 1$ the functional E is trivial. In [D-M] there are examples of harmonic almost complex structures which are not Kähler, for $n \geq 3$.

There are several works about the classification of the harmonic almost complex structures. We are interested in the existence of harmonic almost complex structures. In this paper, as a first step to study the existence of harmonic almost complex structures, we consider the (negative) gradient flow equation

$$\frac{d}{dt}g_J = Ric^c_{g_J}. \tag{1.5}$$

Equivalently,

$$\frac{d}{dt}J = \omega^{-1} \cdot Ric^c, \tag{1.6}$$

where $\omega^{-1} : TM \times TM \to End(TM)$ is defined, in local coordinates, $(\omega^{-1} \cdot T)^i_j = (\omega^{-1})^{ik} T_{kj}$.

Our anti-complexified Ricci flow is motivated by the Hamilton's Ricci flow, which is inspired by the heat flow of harmonic maps introduced by Eells-Sampson [ES]. In this paper, we shall show the short time existence of (1.5) (see Theorem 3.17). We first like to use a more geometric way to prove the local existence of (1.5). Since (1.5) is not a parabolic equation, we want to change it to an equivalent parabolic equation by using an automorphism group as in [D]. (Here the automorphism group is the symplectomorphism group.) As in the Ricci flow case, the complexified Ricci operator Ric^c is not elliptic. The degeneracy of the anti-complex Ricci operator Ric^c is related to the fact that it is invariant under the action of the symplectomorphism group. Using this invariance we obtain a Bianchi type identity

$$d^* J \delta Ric^c = 0. \tag{1.7}$$

for the complexified Ricci operator Ric^c. However we observe that unlike the Ricci operator, the degeneracy of Ric^c is not completely determined by the symplectomorphism group. The reason is simple: the symbol of the operator Ric^c $\sigma_\xi DRic^c$ has null of dimension *two* generated by $\xi \otimes J\xi + J\xi \otimes \xi$ and $\xi \otimes \xi - J\xi \otimes J\xi$, while the (Hamiltonian) symplectomorphism group only generates a *one* dimension line (which is generated by $\xi \otimes J\xi + J\xi \otimes \xi$) in the symbol level. Furthermore, one even can show that the symbol $\sigma_\xi DRic^c$ is complex (with respect to the Kähler structure on the space of compatible almost complex structures, see below). So it is not enough to use the symplectomorphism group to deal with the degeneracy of the operator Ric^c. This leads us to consider complexifying the Hamiltonian symplectomorphism group of the symplectomorphism group. However, in general, there is no such a complexified (Hamiltonian) symplectomorphism group, see [Do2]. Hence we consider to use the complexification of the action of the Hamiltonian symplectomorphism group on the space $\mathcal{J}(M, \omega)$. Such a way of complexification of the action was first suggested by Donaldson in [Do]. It is a Hamiltonian flow on the infinite dimensional Kähler manifold $\mathcal{J}(M, \omega)$. It is difficult to show its global existence, but we can show the local existence, see also [Do2]. Unfortunately, this flow may not preserve the complexified Ricci operator. At least till now we are unable to show it. So we have to use a more analytic way to deal with (1.5) which was introduced in [H1]. The Bianchi type identity (1.7) gives us a "half" of the **integrability condition** introduced in [H1], see section 2 below. The discussion above leads us to find another "half" of the integrability condition. Altogether guarantees that we can apply the Nash-Moser inverse function theorem ([H1] and [H2]) to our problem (1.5) to obtain the short time existence.

We will consider the global existence of (1.4) for certain symplectic manifolds in forthcoming papers.

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2. Space of compatible almost structures

In this section we describe the space $\mathcal{J}(M, \omega)$ of compatible almost complex structures.

(2.1). The space $\mathcal{J}(M)$ of compatible almost complex structures can be identified with the space of the sections $\Gamma(P)$, where P is a (twistor) bundle over M whose fiber is $Sp(2n, \mathbf{R})/U(n)$. Thus a compatible almost complex structure J is a section of P. The tangent space $T_J\mathcal{J}(M)$ is therefore a space of sections of the vector bundle $V(J)$ with the adjoint U_n action. (Every J gives rise to a $U_n(J)$-principal bundle, and $V(J)$ is the orthogonal complement of the subalgebra $u_n(J)$ in $sp(2n, \mathbf{R})$.) Since $Sp(2n, \mathbf{R})/U(n)$ is a Hermitian symmetric space, the space $\mathcal{J}(M)$ inherits the natural Kähler structure. We denote by $\mathcal{I}$ the complex structure on $\mathcal{J}(M)$, and by Ω the symplectic structure on $\mathcal{J}(M)$ (see also [Do].) Of course we can consider L^p_q or C^k compatible almost complex structures and provide $\mathcal{J}(M)$ with certain Banach (or Hilbert) structures. But in this note we are concerned only with smooth compatible almost structures and the smooth structure on $\mathcal{J}(M, \omega)$.

We also present here the second description of the tangent space $T\mathcal{J}(M)$ which is more convenient for calculations.

Let $j = d/dt_{t=0} J_t \in T_{J_0}\mathcal{J}(M)$. Denote by $s*$ the skew-star (adjoint) operator with respect to the symplectic form ω i.e.

$$\omega(Av, w) = \omega(v, A^{s*}w).$$

Then clearly an element $j \in End(TM)$ is in $T_{J_0}\mathcal{J}(M)$ iff the two following conditions holds

$$jJ_0 + J_0 j = 0 \tag{2.2.a}$$

$$j + j^{s*} = 0 \tag{2.2.b}$$

By a direct computation we get

2.3. Lemma. *The skew-star operator* $(s*)$ *can be defined via* J *and star* $(*)$ *operator as follows*

$$A^{s*} = -JA^*J$$

As a direct consequence we get

2.4. Corollary. *The conditions (2.2.a) and (2.2.b) are equivalent to the following*

$$jJ_0 + J_0j = 0 \tag{2.2.a}$$

$$j^* = j. \tag{2.2.b'}$$

Thus the orthogonal projection to the tangent space $T_J\mathcal{J}$ equals the composition of the symmetrization and the operator of taking anti-complex linear part.

2.5. Remark. Let $j' := J_0 \cdot j$. Then the conditions (2.2.a) and $(2.2.b')$ are equivalent to the following

$$j'J_0 + J_0j' = 0, \tag{2.2.a'}$$

$$(j')^* = j'. \tag{2.2.b''}$$

Thus we see: the multiplication with J_0 defines the natural $Symp(M)$-invariant complex structure $\mathcal{I}$ on $\mathcal{J}(M)$ which is arisen in 2.1 (see also [Do]).

Under the identification G of the space $\mathcal{J}(M)$ with the space $Met(M,\omega)$ of compatible Riemannian metrics on M, we can rewrite the conditions (2.2.a) and (2.2.b) as follows. The symmetric tensor

$$D = \frac{d}{dt} g_{J\,|t=0}$$

satisfies the following anti-complex-linear condition

$$J^*(D)(X,Y) = \omega(JX, jJY) = -\omega(JX, JjY) = -D(X,Y).$$

We can write this condition as

$$J^*(D) = -D. \tag{2.2$'$}$$

In term of matrix multiplication [1] we get from $(2.2')$

$$J^*DJ = -D \Longleftrightarrow DJ + JD = 0. \tag{2.6}$$

Thus the tangent spaces of $T_J\mathcal{J}$ and $T_{g_J}\mathcal{J}$ are canonically identified.

2.7. We denote by $Symp_{Ham}(M,\omega)$ the group of Hamiltonian symplectomorphisms of (M,ω). (The reason why we are interested in $Symp_{Ham}$ is that this group is a normal subgroup of the symplectomorphism group $Symp(M,\omega)$, and a) the quotient $Symp/Symp_{Ham}$ is of finite dimension, b) The group $Symp_{Ham}$ is easier to handle from the analytical point of view.) Clearly this group acts on the space

[1] as in [BI]

$\mathcal{J}(M,\omega)$. First we recall that any vector field v on M acts on the space of metrics by taking the Lie derivative (resp. the space of almost complex structures) as follows (see e.g. [Be, lemma 1.60])

$$\mathcal{L}_v(g) = 2\delta^* v^\flat, \tag{2.8}$$

where δ^* is the symmetrization of the covariant derivative, and $v^\flat$ denotes the dual 1-form (w.r.t. the metric g) of the vector field v. An explicit formula for $\mathcal{L}_v J$ is more complicated. We do not need it here. We refer to [Do, Lemma 10] for such an expression. But when v is symplectic vector fields, its expression is simple:

$$\mathcal{L}_v J = 2\omega^{-1} \cdot \delta^* v^\flat. \tag{2.9}$$

3. A Bianchi type identity and the integrability condition

In this section, we obtain a Bianchi type identity by using the invariance of the energy functional under the action of the symplectomorphism group. As mentioned in the introduction, this identity is not enough in order to show the short time existence of (1.5). We find the another part of the integrability condition using the Kähler structure $\mathcal{I}$ defined in the previous section.

Recall the definition of action of symplectomorphism on $\mathcal{J}$. Let $\phi \in Symp(M)$ and $J \in \mathcal{J}$. One define a new compatible almost complex structure by

$$\phi^* J(X) = \phi_*^{-1}(J(\phi_*(x))),$$

for any $X \in TM$.

3.1 Lemma. *The following formula*

$$g_{\phi^* J} = \phi^*(g_J)$$

holds if ϕ is a symplectomorphism.

Proof. For completeness, we give the proof of the Lemma. For any symplectomorphism ϕ, we have, for any $p \in M$ and $X, Y \in T_pM$,

$$\begin{aligned} \omega_p(\phi^* J(X), Y) &= \omega_p((\phi^{-1})_* J(\phi_* X), Y) \\ &= \omega_{\phi(p)}(J\phi_* X, \phi_* Y) \\ &= \phi^* g_J(X, Y), \end{aligned}$$

The second equality holds, since ϕ is a symplectomorphism. □

3.2 Proposition. *The functional E is invariant under the action of $Symp(M,\omega)$.*

Proof. From Lemma 3.1, we have

$$\begin{aligned} E(\phi^* J) &= \int_M |\nabla_{\phi^*(g_J)} \phi^* J|^2_{\phi^*(g_J)} vol_{\phi^*(g_J)} \\ &= \int_M |\phi^*(\nabla_{g_J} J)|^2_{\phi^*(g_J)} vol_{\phi^*(g_J)} \\ &= \int_M |\nabla_{g_J} J|^2_{g_J} vol_{g_J} = E(J). \end{aligned}$$

□

3.3 Corollary. *the group $Symp(M, \omega)$ preserves Ric^c, i.e., for any symplectomorphism ϕ,*

$$\phi^*(Ric^c) = Ric^{\phi^*(J)}. \tag{3.3}$$

□

Equality (3.3) implies that the harmonicity of an almost complex structure is preserved under the action of symplectomorphisms. Hence the operator Ric^c is not elliptic.

Let $S^2(TM)$ denote the space of symmetric 2-tensors. Recall that for any $J \in \mathcal{J}$, the tangent space of $Met(M, \omega)$ at g_J, $T_{g_J} Met(M, \omega)$ is the space of sections of $S^2(TM)$ satisfying (2.2′). Let $\Omega^1(M)$ be the space of 1-forms and $\Omega^0_0(M)$ the space of functions with average zero. Define an operator $\delta^* : \Omega^1(M) \to S^2(TM)$ by

$$\delta\alpha = \mathcal{L}_{\alpha^\#} g_J,$$

where $\alpha^\#$ is the dual vector of α with respect to the metric g_J and $\mathcal{L}_{\alpha^\#}$ is the Lie derivative. Giving $S^2(TM)$ and $\Omega^1(M)$ the induced metrics of g_J, we can define the adjoint operator of δ^*, $\delta : S^2(TM) \to \Omega$ which can be given, in local coordinates,

$$(\delta h)_k = g^{ij} h_{jk,i}.$$

3.4 Proposition. *We have the following Bianchi type identity*

$$d^* J \delta Ric^c = 0. \tag{3.4}$$

Proof. For any function f on M, one can define a so-called Hamiltonian vector field X_f by

$$\omega(X_f, Y) = -df(Y),$$

for any $Y \in TM$. Let ϕ_t be a family of symplectomorphisms generated by X_f, i.e.,

$$\frac{d}{dt}\phi_t = X_f(\phi_t).$$

Since symplectomorphisms preserve the energy functional E, we have

$$0 = \frac{d}{dt} E(\phi_t * (J)) = \int_M \langle Ric^c(g_J), L_{X_f} g \rangle vol(M). \tag{3.5}$$

From (3.5), we have

$$0 = \int_M f d^* J \delta Ric^c vol(M),$$

for all function f, which yields (3.4). □

3.6. Now we compute the symbol of the operator Ric^c. First, the linearization operator of the Ricci operator in the direction $h \in S^2(TM)$ is

$$DRic(g)h = \frac{1}{2}\Delta_L h - \delta^*(\delta G(h)),$$

where Δ_L is the Lichnerowicz Laplacian defined by $\Delta_L h = D^*Dh + Ric_g \circ h + h \circ Ric_g - 2\overset{\circ}{R}_g h$. (See [Be].) Locally the derivative of the Ricci operator can be expressed as follows:

$$DRic(g_{\alpha\beta})h_{\alpha\beta} = g^{\gamma\delta}\{\frac{\partial^2 h_{\alpha\beta}}{\partial x^\gamma \partial x^d} - \frac{\partial^2 h_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 h_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 h_{\gamma\delta}}{\partial x^\alpha \partial x^\beta}\} + \cdots ,$$

where the dots denote lower order terms. The symbol of the the linear differential operator $DRic(g_{\alpha\beta})$, which maps $S^2(R^{2n})$ to itself, in the direction ξ_i is

$$\sigma DRic(g_{\alpha\beta})(\xi_\epsilon)h_{\alpha\beta} = \frac{1}{2}g^{\gamma\delta}[h_{\alpha\gamma}\xi_\beta\xi_\delta + h_{\beta\delta}\xi_\alpha\xi_\gamma - h_{\alpha\beta}\xi_\gamma\xi_\delta - h_{\gamma\delta}\xi_\alpha\xi_\beta].$$

(See [D] and [H1].)

Let $S^2_J(TM)$ denote the subspace of symmetric tensors which satisfy (2.2′). In fact, $S^2_J(TM) = T_{\mathcal{J}}Met(M,\omega)$. Let $c(J) = Id - J^*$ such that $\frac{1}{2}c(J)$ is a projection operator from $S^2(TM)$ to $S^2_J(TM)$. The linearization operator $DRic^c$ of Ric^c is

$$\begin{aligned} DRic^c(g^\alpha_\beta)(h_{\alpha\beta}) &= c(J)DRic(g^\alpha_\beta)(h_{\alpha\beta}) + \cdots \\ &= c(J)_{\alpha\epsilon}g^{\gamma\delta}\{\tfrac{\partial^2 h_{\epsilon\beta}}{\partial x^\gamma \partial x^d} - \tfrac{\partial^2 h_{\epsilon\delta}}{\partial x^\gamma \partial x^\beta} - \tfrac{\partial^2 h_{\gamma\beta}}{\partial x^\epsilon \partial x^\delta} + \tfrac{\partial^2 h_{\gamma\delta}}{\partial x^\epsilon \partial x^\beta}\} + \cdots \\ &= (\delta^p_\alpha\delta^q_\beta - J^p_\alpha J^q_\beta)g^{\gamma\delta}\{\tfrac{\partial^2 h_{pq}}{\partial x^\gamma \partial x^d} - \tfrac{\partial^2 h_{p\delta}}{\partial x^\gamma \partial x^q} - \tfrac{\partial^2 h_{\gamma q}}{\partial x^p \partial x^\delta} + \tfrac{\partial^2 h_{\gamma\delta}}{\partial x^p \partial x^q}\} \\ &\quad + \cdots , \end{aligned}$$

where $h_{\alpha\beta} \in T_J Met(M,\omega)$. The symbol of the linear differential operator $DRic^c$, is

(3.7)

$$\sigma_\xi DRic^c(g^\alpha_\beta)(h_{\alpha\beta}) = \frac{1}{2}(\delta^p_\alpha\delta^q_\beta - J^p_\alpha J^q_\beta)g^{\gamma\delta}[h_{p\gamma}\xi_q\xi_\delta + h_{q\delta}\xi_p\xi_\gamma - h_{pq}\xi_\gamma\xi_\delta - h_{\gamma\delta}\xi_p\xi_q].$$

Clearly $h_{\alpha\beta} = (\delta^p_\alpha\delta^q_b - J^p_\alpha J^p_\beta)(\xi_p\xi_q + \xi_q\xi_p)$ and $h_{\alpha\beta} = J^\gamma_\beta\xi_\alpha\xi_\gamma + J^\gamma_a\xi_r\xi_\beta$ are the zero eigenvalues of $\sigma DRic^c(\xi)(g^\alpha_\beta)(h_{\alpha\beta})$. (See also the proof of Proposition 3.12.)

Globally, the linearization operator of Ric^c is

$$DRic^c(g_J)(h) = c(J)\{\frac{1}{2}\Delta_L h - \delta^*(\delta G(h))\} + Dc(J)Ric^c(g_J), h,$$

for $h \in S^2_J(TM)$. Here we use the notation given in [H1].

A crucial point is that the (highest) symbol $\sigma_\xi DRic^c$ is complex with respect to the complex structure $\mathcal{I}$, namely for any $h \in S^2_J(TM)$,

$$\sigma_\xi DRic^c(\mathcal{I}h) = \mathcal{I}\sigma_\xi DRic^c(h). \tag{3.8}$$

3.9. For a given metric g on M let us denote by O_g be the subspace of $\Omega^1(M)$ defined by $O_g = Im\{d : \Omega^0(M) \to \Omega^1(M)\}$ and

$$p : \Omega^1(M) \to O_g \subset \Omega^1(M) \tag{3.10}$$

the orthogonal projection. It is to see that p is a zero order pseudo-differential operator. In fact,

$$p = d\Delta^{-1}d^*,$$

where Δ^{-1} is the inverse operator of the Laplacian $\Delta = d^*d : \Omega^0_0(M) \to \Omega^0_0(M)$. The operator $\Delta^{-1} : \Omega^0_0(M) \to \Omega^0_0(M)$ is well-defined. Define $L_1 : S^2_J(TM) \to \Omega^1(M)$ by $L_1 = pJ\delta$. By (3.4) we have

$$L_1 Ric^c = 0. \tag{3.11}$$

Let $L_2 : S^2_J(TM) \to \Omega^1(M)$ be defined by $L_2 = L_1\mathcal{I}$. Though L_2Ric^c may not vanish, we will show that the operator L_2Ric^c, at most, has degree 1 w.r.t. g_J. Now we define $L_{g_J} : S^2_J(TM) \to \Omega^1(M) \oplus \Omega^1(M)$ by $L = (L_1, L_2)$.

3.12 Proposition. *(1) The operators L_{g_J} and $L_{g_J}Ric^c$ have degree 1.*
(2) All the eigenvalues of $\sigma_\xi DRic^c$ in $Null\sigma_\xi L$ are positive.

Proof. (1). Since the projection p is a pseudodifferential operator of degree 0, we get that L_{g_J} is a pseudodifferential operator of degree 1. We have to show that L_2Ric^c has degree 1. We know that L_2Ric^c is a third order operator. Now we first show that the third order symbol of $D(L_2Ric^c)$ is zero. It is easy to check that

$$\sigma^3_\xi D(L_2Ric^c) = \mathcal{I}\sigma^3_\xi D(L_1Ric^c).$$

Hence (3.11) implies that $\sigma^3_\xi L_2Ric^c$ vanishes. Now we compute the second order symbol of $D(L_2Ric^c)$. We first have the following relation

$$\sigma^2_\xi D(L_2Ric^c) = \sigma^0_\xi(L_1\mathcal{I})\sigma^2_\xi DRic^c + \sigma^1_\xi(L_1\mathcal{I})\sigma^1_\xi DRic^c. \tag{3.13}$$

For a point $x \in M$ we use normal coordinates around x to compute that at the given point x we have $\sigma^1_\xi DRic^c(x) = 0$. On the other hand, from (3.8) and (3.11), we get

$$\sigma^0_\xi(L_1\mathcal{I})\sigma^2_\xi DRic^c = \sigma^0_\xi(L_1)\sigma^2_\xi DRic^c\mathcal{I} = 0.$$

Altogether yields that $\sigma^2_\xi L_2Ric^c(x) = 0$. Hence L_2Ric^c is a first order operator, so is L.

(2). For convenience, we use the similar calculation presented in [H1] to show (2). We choose coordinates at a point such that $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}$,

$J\frac{\partial}{\partial x^1}, J\frac{\partial}{\partial x^2}, \cdots, J\frac{\partial}{\partial x^n}$ is a orthonormal basis. Sometimes, we denote $\bar{k} = n + k$ for $k = 1, 2, \cdots, n$. Assume $\xi_1 = 1$ and $\xi_i = 0$ for $i \neq 1$. Let T_{jk} be a symmetric tensor and $h_{jk} = T_{jk} - J_j^p J_{kq} T_{pq}$ its anti-complex part with respect to the almost complex structure J. Then the symbol of $DRic^c$ acts on the tensor h_{jk} is

(3.14)

$$\begin{array}{lll} [\sigma_\xi DRic^c h]_{jk} & = 2h_{jk}, & \text{if } j \neq 1 \text{ or } \bar{1}, k \neq 1 \text{ or } \bar{1}, \\ [\sigma_\xi DRic^c h]_{1k} & = h_{1k}, & \text{if } k \neq 1, \text{or } \bar{1}, \\ [\sigma_\xi DRic^c h]_{1\bar{1}} & = 0, & \\ [\sigma_\xi DRic^c h]_{11} & = \sum_{j=2}^n (h_{jj} + h_{\bar{j}\bar{j}}). & \end{array}$$

Let h be an element of Null space of $\sigma_\xi L(g)$. It is clear that h is an element of null space of $\sigma_\xi\{d^* L(g)\} = \sigma_\xi d^* \circ \sigma_\xi L(g)$. Since $d^* p = d^*$, $d^* L = (d^* J\delta, d^* J\delta\mathcal{I})$. Hence h satisfies

$$[\sigma_\xi d^* J\delta h] = 0 \qquad (3.15.a)$$

and

$$[\sigma_\xi d^* J\delta\mathcal{I} h] = 0. \qquad (3.15.b)$$

Using the same form of ξ and g in local coordinates, one can readily to compute the symbols of $\sigma_\xi d^* J\delta$ and $\sigma_\xi d^* J\delta\mathcal{I}$,

$$[\sigma_\xi d^* J\delta h] = h_{1\bar{1}} \text{ and } [\sigma_\xi d^* J\delta\mathcal{I} h] = h_{11}. \qquad (3.16)$$

By (3.15) and (3.16), an element in the null space of $\sigma_\xi\{d^* L(g)\}$ satisfies $h_{11} = h_{1\bar{1}} = 0$. By (3.14), we have shown that the eigenvalues of $\sigma_\xi DRic^c$ in $Null \sigma_\xi L$ are positive. $\square$

Following [Ha1], we call the operator L the **integrability condition**.

3.17 Thereom. *For any smooth compatible almost complex structure J_0, there exists $T > 0$ such that (1.5) admits a unique solution $J(t)$ satisfying $J(0) = J_0$.*

Proof. Using Proposition 3.12, we can follow the argument of Hamilton in [Ha1, sections 5 and 6] to show the Theorem. The only difference is that the projection operator $p = d\Delta^{-1} d^*$ in our integrability condition L is a pseudo-differential operator. The crucial point to apply the Hamilton-Nash-Moser inverse function theorem for tamed Frechet spaces developed in [Ha2, Part III, Theorem 1.1.1] is to check that all relevant operators are **tame** and the linearized equation has a unique solution. This can be done as in [Ha1] for our operators, since p is a bounded operator.

For convenience, we sketch the proof of Theorem 3.17 adapted from the proof of Hamilton.

Step 1. *Reduction to the Hamilton-Nash-Moser inverse function theorem.* We denote by $Met(M \times [0, 1], \omega)$ the space of smooth sections of the induced bundle of compatible metrics over $M \times [0,1]$. Let us consider the operator

$$\mathcal{E} : Met(M \times [0,1], \omega) \to Met(M \times [0,1], \omega) \times Met(M, \omega),$$

$$\mathcal{E}(g) = (\frac{dg}{dt} - Ric^c(g), g|\{t = 0\}).$$

In order to apply the Hamilton-Nash-Moser inverse function theorem we have to show that the linearized equation

$$D\mathcal{E}(g)\tilde{g} = \frac{\partial \tilde{g}}{\partial t} - DRic^c(g)\tilde{g} = \tilde{h},$$

has a unique solution for the initial value problem $\tilde{g} = \tilde{g}_0$ at $t = 0$ and verify that the solution $\tilde{g}$ is a smooth tame function of $\tilde{h}$ and $\tilde{g}_0$.

Step 2. *The integrability condition reduces the linearized equation to a system of two PDEs.* We denote by $Q(g)$ the composition $L(g) \circ Ric^c(g)$. Let us consider new two differential operators $M(g)$ and $P(g)$ such that

$$M(g)\tilde{g} = DL(g)\{\tilde{g}, \frac{\partial g}{\partial t}\} - DL(g)\{Ric^c(g), \tilde{g}\} + DQ(g)\tilde{g},$$

$$P(g) = DRic^c(g) + L^*(g)L(g).$$

Operator M appears in the evolution equation for $\tilde{e} = L(g)\tilde{g}$. Namely we have

$$\frac{\partial \tilde{e}}{\partial \tilde{t}} - M(g)\tilde{g} = \tilde{k},$$

where $\tilde{k} = L(g)\tilde{h}$. The operator P is obtained from $DRic^c$ by "killing" the kernel of its symbol and hence is a a *parabolic pseudo differential operator*. Now as in [Ha1] the uniqueness of the solution of the linearized equation $D\mathcal{E}(g)$ is equivalent to the uniqueness of the (smooth) solution of the system

$$(3.18) \begin{cases} \frac{\partial \tilde{g}}{\partial t} - P(g)\tilde{g} + L^*(g)\tilde{e} = \tilde{h}, \\ \frac{\partial \tilde{e}}{\partial t} - M(g)\tilde{g} = \tilde{k}, \end{cases}$$

for unknown sections $\tilde{g}, \tilde{e}$ and for given $\tilde{h}, \tilde{k}$, and given g with initial data $\tilde{g} = \tilde{g}_0$ and $\tilde{e} = \tilde{e}_0 = L(g_0)(\tilde{g}_0)$ at $t = 0$.

Step 3. *The existence and uniqueness of the solution of (3.18).* The existence and uniqueness is obtained by the iteration method and by using uniform estimate for the second equation in (3.18) delayed in time δ. Roughly speaking, we can assume without loss of generality that all $\tilde{e}, \tilde{g}, \tilde{k}, \tilde{h}$ vanish in the negative time $t \le 0$. Then we solve the second delayed equation on the time interval $(0, \delta)$, then use it to solve the first parabolic equation on the time interval $(0, \delta)$, and advance this procedure further. To get the true solution we need to consider the limit solution when δ goes to zero and get a uniform estimate on the solutions. This will be done in the last step.

Step 4. *The tameness of the solution* $(\tilde{g}, \tilde{e})$ *of the system* (3.18). We recall that [Ha 2] a continuous nonlinear map P of a Frechet space F_1 to a Frechet space F_2 is **tame** if it satisfies a tame estimate in a neighborhood U of each point in F_1

$$||P(g)||_n \le C(1 + ||g||_{n+r})$$

for all $f \in U$ and all $n \ge b = const(U)$ and C is a constant which may depend on n. Here the norm $||g||_n$ is defined as follows

$$||g||_n^2 = \sum_{2j \le n} \int_0^T |(\partial/\partial t)^j g|_{n-2j} dt,$$

with $|g|_n$ measure L_2 norm of g and its derivates up to degree n. Now it is easy to verify that all the tame estimates (Lemma 6.10 in [Ha 1]) are also valid here (for the delayed equation in Step 3) also valid in our case of pseudo-differential operator L.

$\square$

4. Example

In this section, we discuss the anti-complexified Ricci flow on the Thurston manifold and show that the flow exists globally and converges to a degenerate metric. It is easy to see that the energy of the evolving metrics tends to zero.

Let $G = H^3 \times \mathbb{S}^1$ be the product of the Heisenberg group and $\mathbb{S}^1$. I.e, G is a closed Lie subgroup of $GL(4, \mathbb{C})$ defined by

$$\begin{pmatrix} 1 & x & z & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}$$

with $x, y, z, t \in \mathbf{R}^1$. Let Γ be a subgroup of G consisting of all matrices of G which entries are integers. The quotient $M = G/\Gamma$ is the Thurston manifold. Differential forms $dx, dy, dz - xdy, dt$ are invariant under the left translation by any element in G. A left invariant metric on G is given in [A] by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2 + dt^2.$$

Clearly, from this metric we can get an metric on M, which is also denoted by the same form. Set

$$\omega = dx \wedge dt + dy \wedge (dz - xdy).$$

It is a symplectic form on G and invariant under the translation. Hence, we can see it as a symplectic form on M. The metric ds^2 is a compatible metric with ω, since there is an almost complex structure

$$Jdx = dt, \quad Jdt = -dx, \quad Jdy = dz - xdy, \quad J(dz - xdy) = -dy$$

such that $ds^2 = \omega(\cdot, J\cdot)$. Clearly, following metric

$$g = A^{-2}dx^2 + B^{-2}dy^2 + B^2(dz - xdy)^2 + A^2dt^2, \text{ for any} A, B > 0$$

are compatible metrics. The corresponding almost complex structures are

$$\begin{array}{ll} Jdx = A^{-2}dt, & Jdt = -A^2dx, \\ Jdy = B^{-2}(dz - xdy), & J(dz - xdy) = -B^2dy. \end{array}$$

The anti-complexified Ricci flow starting from the Abbena metric ($A = B = 1$) is

$$\left\{ \begin{array}{lcl} \frac{d}{dt}A^{-2} & = & \frac{1}{2}A^4B^4 \\ \frac{d}{dt}B^{-2} & = & A^2B^6. \end{array} \right. \tag{4.1}$$

For this example, it is easy to solve (4.1). Its solution is

$$\begin{array}{lcl} A^2 & = & \sqrt{2}(5\sqrt{2}t + 1)^{-\frac{1}{5}} \\ B^2 & = & (5\sqrt{2}t + 1)^{-\frac{1}{5}} \end{array}$$

So, the anti-complexified Ricci flow (4.1) exists globally. However, at infinity, it becomes degenerate.

References

[A] Abbena, E., *An example of an almost Kähler manifold which is not Kählerian,* Boll.-Un.-Mat.-Ital.-A, **3** (1984), 383–392.

[Be] A. Besse, *Einstein Manifolds*, Springer-Verlag 1987.

[B-I] D. E. Blair and S. Ianus, *Critical associated metrics on symplectic manifolds*, Cont. Math. **51** (1986), 23-29.

[D-M] J. Davidov and O. Muskarov, *Twistor spaces with Hermitian Ricci Tensor*, Proc. AMS, 109, **4** (1990), 1115-1120.

[Do] S. K. Donaldson, *Remark on Gauge theory, complex geometry and 4-manifold topology,* The Fields Medal Volume, Ed. M. F. Atiyah and D. Iagolnitzer, World Scientific, 1997.

[Do2] S. K. Donaldson, Private communication.

[DT] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. **18** (1983) 157-162; ibid., *improved version*, to appear in Selected Papers on the Ricci Flow, ed. H.-D. Cao, B. Chow, S.-C. Chu, and S.-T. Yau, International Press.

[ES] J. Eells and Sampson, *Harmonic mappings of Riemannian manifolds, Amer. J. Math.* ***86*** *(1964), 109-160*

[Ham] R. S. Hamilton *Three-manifolds with positive Ricci curvature*, JDG, **17** (1982), 255-306.

[H2] R. S. Hamilton *The inverse function theorem of Nash and Moser*, Bull. AMS (new series), **7** (1982) 65-222.

[Le] H. V. Le, *Harmonic almost complex structures on compact symplectic manifolds*, preprint 1998.

[Ta] M. E. Taylor, *Partial differential equations III,* Springer Verlag, 1996.