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Geometric diffeomorphism finiteness in low dimensions and homotopy group finiteness

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GEOMETRIC DIFFEOMORPHISM FINITENESS IN LOW DIMENSIONS AND HOMOTOPY GROUP FINITENESS

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Abstract. The main results of this note consist in the following two geometric finiteness theorems for diffeomorphism types and homotopy groups of closed simply connected manifolds:
1. For any given numbers C and D the class of closed smooth simply connected manifolds of dimension $m < 7$ which admit Riemannian metrics with sectional curvature bounded in absolute value by $|K| \leq C$ and diameter bounded from above by D contains at most finitely many diffeomorphism types. In each dimension $m \geq 7$ there exist counterexamples to the preceding statement.
2. For any given numbers C and D and any dimension m there exist for each natural number $k \geq 2$ up to isomorphism always at most finitely many groups which can occur as the k-th homotopy group of a closed smooth simply connected m-manifold which admits a metric with sectional curvature $|K| \leq C$ and diameter $\leq D$.

Introduction

In dimensions less than 7 the presence of a metric with given curvature and diameter bounds suffices, as the first result of this note shows, to restrict the diffeomorphism type of a simply connected closed smooth manifold always to finitely many possibilities:

1.1 Theorem (a) *For any given numbers C and D there is at most a finite number of diffeomorphism types of simply connected closed smooth m-manifolds, $m < 7$, which admit Riemannian metrics with sectional curvature $|K| \leq C$ and diameter $\leq D$.*
(b) *In each dimension $m \geq 7$ there exist counterexamples to the preceding statement.*

Note that Theorem 1.1 does not require a positive lower bound on volume or injectivity radius.

1.2 Remark Theorem 1.1 and the Bonnet-Myers theorem imply that given $m < 7$ and any $\delta > 0$, there is at most a finite number of diffeomorphism types of simply connected closed smooth m-dimensional manifolds M which admit Riemannian metrics with Ricci curvature $Ricc \geq \delta > 0$ and sectional curvature $K \leq 1$. This explains in particular why 7 is the first dimension where infinite sequences of closed simply connected manifolds of mutually distinct diffeomorphism type and uniformly positively pinched sectional curvature (compare [E], [AW]) can appear.

1.3 Remark Theorem 1.1 implies as well that in dimensions less than 7 there cannot exist infinite sequences of mutually nondiffeomorphic closed simply connected nonnegatively curved manifolds which satisfy uniform upper curvature and diameter bounds. When combined with recent work of K. Grove and W. Ziller, this fact yields a distinction between the class of nonnegatively curved manifolds and the classes of nonnegatively curved manifolds which are subject to some upper curvature

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bound which cannot be obtained by any of the previously known finiteness results (compare however Theorem 1.11 below).
Namely, Grove and Ziller constructed metrics of nonnegative curvature on an infinite sequence of pairwise nondiffeomorphic six-manifolds M_n which are total spaces of S^2 bundles over S^4 and whose third homotopy group $\pi_3(M_n)$ is isomorphic to the cyclic group $\mathbb{Z}_n$ of order n (see [GZ]). If now, without loss of generality, the diameters of all these manifolds are scaled to, say, one, Theorem 1.1 implies that for any given number C, at most finitely many M_n can satisfy the upper curvature bound $K \leq C$.

Theorem 1.1 can of course be used as well to distinguish almost nonnegatively curved manifolds from manifolds of almost nonnegative curvature and upper curvature bound. For instance, it follows from a construction of Fukaya and Yamaguchi (see [FY]) that there exist almost nonnegatively curved metrics on infinite sequences of S^2 bundles over $\mathbb{CP}^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. But Theorem 1.1 implies that only finitely many of these manifolds can be subject to any given upper curvature bound.

1.4 Remark In dimension 4 there exist closed simply connected topological manifolds of fixed homeomorphism type which admit infinitely many nondiffeomorphic smooth structures (e.g., the Dolgachev surfaces $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}^2}$, where $9 \leq k \in \mathbb{N}$; see [FM]). Theorem 1.1 shows here that geometric quantities as basic as curvature and diameter bounds suffice to distinguish and control Donaldson and Seiberg-Witten invariants.

1.5 Remark In dimension 4 Theorem 1.1 actually holds for all closed smooth manifolds with non-zero Euler characteristic, and therefore in particular for all closed smooth four-manifolds with finite fundamental groups (see below).

1.6 Remark Since the total space of a principal torus bundle over a Riemannian manifold always carries a metric with similar absolute curvature and upper diameter bounds as the base manifold, Theorem 1.1 has the purely topological corollary that for $k+l < 7$ there are over a given smooth closed l-manifold only finitely many principal T^k bundles with simply connected and non-diffeomorphic total spaces.

1.7 Remark In the relevant dimensions Theorem 1.1 complements and improves other geometric finiteness results. It shows that for dimensions $m < 7$ in Cheeger's Finiteness Theorem (see [C] and [Pt]) the geometric assumption of a lower positive bound on volume can be replaced by the topological condition of simply-connectedness. Theorem 1.1 also shows that the π_2-Finiteness Theorem from [PT] holds for dimensions less than seven without the requirement that the second homotopy group be finite.

The previously known finiteness theorems in Riemannian geometry which require at most bounds on volume, curvature, and diameter, can be stated as follows:

For closed smooth manifolds M of a given fixed dimension m, the conditions

- $vol(M) \geq v > 0$, $|K(M)| \leq C$ and $\operatorname{diam}(M) \leq D$ imply

  finiteness of diffeomorphism types
  (Cheeger ([C]) 1970 and Peters ([Pt]) 1984);

- $vol(M) \geq v > 0$, $\int_M |R|^{m/2} \leq C$, $|\mathrm{Ric}_M| \leq C'$, $\operatorname{diam}(M) \leq D$ imply

  finiteness of diffeomorphism types
  (Anderson-Cheeger ([AC]) 1991);

- $vol(M) \geq v > 0$, $K(M) \geq C$ $\operatorname{diam}(M) \leq D$ imply finiteness of homotopy types

(Grove-Petersen ([GP]) 1988);

finiteness of homeomorphism types

(Perelman ([Pr]) 1991 (preprint) and, if $m \neq 3$, Grove-Petersen-Wu ([GPW]) 1990);

finiteness of diffeomorphism types, if $m \neq 3, 4$

(Grove-Petersen-Wu ([GPW]) 1990);

- $|K(M)| \leq C$, $\operatorname{diam}(M) \leq D$, $\pi_1(M) = 0$ and finite $\pi_2(M)$ imply

  finiteness of diffeomorphism types

  (Petrunin-T. ([PT]) 1999);

- $K(M) \geq C$ and $\operatorname{diam}(M) \leq D$ imply

  for any coefficient field F a uniform bound for the total Betti number $dim\, H_*(M; F)$

  (Gromov ([G]) 1981).

1.8 Remark Combining Gromov's Betti number theorem and Freedman's classification of simply connected topological four-manifolds (see [FQ]) one obtains (compare [T2]) that Theorem 1.1(a) holds in dimension 4 under the conditions $K \geq C$ and $\operatorname{diam} \leq D$, provided that one replaces diffeomorphism by homeomorphism types. This result may be compared to the situation in the finiteness theorem of Grove-Petersen-Wu and gives rise to the following

1.9 Question In which dimensions can the upper curvature bound in Theorem 1.1(a) be discarded?

In the first dimension of interest, namely, dimension 3, the answer might well depend on the still unresolved Poincaré Conjecture.

Due to the special topological phenomena which one encounters in dimension 4 (compare Remark 1.4) and the lack of general smooth classification results for four-manifolds, here an answer to Question 1.9 seems presently to be out of reach.

In dimension 5 the answer to the above question should be positive. This is supported by the fact that (see [T2]) here the conclusion of Theorem 1.1(a) holds under the conditions $K(M) \geq C$, $\operatorname{diam}(M) \leq D$, and $|\mathrm{Tor}\ H_2(M; \mathbb{Z})| \leq E$.

As the Grove-Ziller examples (compare Remark 1.3) show, in dimension 6 the upper curvature bound is indispensable for Theorem 1.1 to be true.

It is also of interest to note that results of LeBrun (see [L]) imply that in dimension 4 the sectional curvature bound in Theorem 1.1 can not be replaced by an absolute bound on Ricci curvature.

In order to obtain, in particular, a better understanding of the existence and properties of infinite sequences of mutually nondiffeomorphic uniformly positively pinched closed simply connected 7-manifolds (see [E], [AW]) it will be of interest to find topological conditions which will yield geometric finiteness results for seven-manifolds.

1.10 Conjecture For given finite constants B, C and D there are at most finitely many diffeomorphism types of closed simply connected 7-dimensional manifolds M whose fourth cohomology group is bounded in order by $|H^4(M; \mathbb{Z})| \leq B$ and which admit Riemannian metrics with sectional curvature $|K| \leq C$ and diameter $\leq D$.

The proof of Theorem 1.1(a) uses in dimensions 5 and 6 the classification results of Barden ([B]) and Žubr ([Z]). Its geometric ingredients consists of collapsing arguments and the following homotopy group finiteness theorem which, in contrast to Theorem 1.1(a), is valid in any dimension:

1.11 Theorem *Given $m \in \mathbb{N}$, C and D, for each natural number $k \geq 2$ there exists a finite set $\Pi_k = \Pi_k(m, C, D)$ of isomorphism classes of finitely generated Abelian groups such that if M is a closed smooth simply connected m-dimensional manifold which admits a Riemannian metric with sectional curvature $|K(M)| \leq C$ and diameter diam$(M) \leq D$, then $\pi_k(M) \in \Pi_k$.*

1.12 Remark Theorem 1.11 extends from closed simply connected manifolds to closed manifolds with finite fundamental group and improves Theorem 0.3 in [R] as well as Theorem 4.2 in [FR2a] from finiteness of isomorphism classes of rational homotopy groups $\pi_k(M) \otimes \mathbb{Q}$ to finiteness of possibilities for the homotopy groups $\pi_k(M)$ themselves.

1.13 Remark Theorem 1.11 provides also new means to rule out the existence of metrics with uniformly bounded curvatures and diameters on given sequences of Riemannian manifolds and to distinguish nonnegatively or almost nonnegatively curved manifolds from manifolds with nonnegative or almost nonnegative curvature and upper curvature bound. For instance, by taking products of spheres of appropriate dimension with the six-dimensional Grove-Ziller examples (see Remark 1.3 above), one obtains in each dimension $m \geq 8$ infinite sequences of closed smooth simply connected and nonnegatively curved m-manifolds whose third homotopy groups are pairwise distinct. Theorem 1.11 implies that these manifolds do not admit any metrics with uniform upper curvature and diameter bounds, while this conclusion does not follow from Theorem 1.1 nor from any previously known finiteness result.

These conclusions may be compared to the fact that the Aloff-Wallach and Eschenburg examples (see [AW], [E]) show that there exist infinite sequences of closed simply connected manifolds which satisfy uniform curvature and diameter bounds and whose homotopy types are pairwise distinct, but which possess the property that for each $k \in \mathbb{N}$ their respective k-th homotopy groups are all isomorphic.

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Added in proof. A first version of this work, containing both the proofs of Theorems 1.1 and 1.11, was published on the web as arXiv e-print and Max Planck Institute Leipzig preprint in August 1999 (compare [T1]). Proofs of Theorem 1.1(a) and Theorem 1.11 have since then (see [FR1] and [FR2b]) also been given by F. Fang and X. Rong.

The Proofs of Theorems 1.1 and 1.11

Proof of Theorem 1.1(a) In dimensions one and two the statement is true by trivial reasons. Since a simply connected closed three-manifold is a homotopy three-sphere, in dimension three the π_2-Finiteness Theorem of [PT] applies. Thus one is left with the cases where the dimension m is equal to 4, 5, or 6.

Let us first note that the conclusion of Theorem 1.1(a) holds actually for all closed smooth manifolds of a given arbitrary dimension m whose Euler characteristic is non-trivial. This can be seen as follows: Suppose that there exists a sequence $(M_n)_{n \in \mathbb{N}}$ of pairwise non-diffeomorphic closed smooth Riemannian m-manifolds with uniformly bounded curvatures and diameters and non-zero

Euler characteristic. Cheeger's Finiteness Theorem implies that this sequence must collapse, i.e., it must hold that $vol(M_n) \to 0$ as $n \to \infty$. In particular, by [CG] for n sufficiently large all M_n admit a pure F-structure of positive rank and thus have vanishing Euler characteristic, which is a contradiction.

Since the Euler characteristic of a closed 4-manifold with finite fundamental group is at least 2, Theorem 1.1(a) therefore holds in particular for closed smooth 4-manifolds with arbitrarily large, but finite order of the fundamental group.

In the five-dimensional case Barden's classification result (see [B]) says that simply connected 5-manifolds M are classified up to diffeomorphism by the second homology group $H_2(M;\mathbb{Z})$ and an invariant $i(M)$ which is obtained as follows: Regarding the second Stiefel-Whitney class of M as a homomorphism $w : H_2(M) \to \mathbb{Z}_2$, one may arrange w to be non-zero on at most one element of a certain "basis" of H_2. This element has order 2^i for some i, and this i is the invariant $i(M)$.

Since there are for a given finitely generated group H always only finitely many homomorphisms to a fixed finite group, it follows in particular that there are always only finitely many distinct diffeomorphism types of closed smooth simply connected 5-manifolds with a given second homology group. This observation, combined with Theorem 1.11 (and the Hurewicz theorem) proves Theorem 1.1(a) for $m = 5$.

To prove Theorem 1.1(a) in dimension six, let us here first note that Žubr's diffeomorphism classification of closed oriented simply connected six-manifolds (see [Z]) implies that a given class of closed smooth simply connected manifolds of dimension 6 will contain at most finitely many diffeomorphism types if it satisfies the following conditions:

For the members M of this class there are, up to isomorphism, only finitely many possibilities for the second homology group $H_2(M;\mathbb{Z})$, the third Betti number $b_3(M;\mathbb{Z})$, the cup form μ_M (a symmetric trilinear form $H^2(M;\mathbb{Z}) \otimes H^2(M;\mathbb{Z}) \otimes H^2(M;\mathbb{Z}) \to H^6(M;\mathbb{Z}) \cong \mathbb{Z}$ given by the cup product evaluated on the orientation class, which determines the multiplicative structure of the cohomology ring), the first Pontryagin class $p_1(M)$ (which here is integral, i.e., $p_1(M) \in H^4(M;\mathbb{Z})$).

Now suppose that for some C and D there exists an infinite sequence of pairwise non-diffeomorphic closed smooth simply connected Riemannian 6-manifolds M_n with curvature $|K| \leq C$ and diameter $\leq D$. By what has been said above, this sequence must collapse, and one may suppose that all manifolds M_n carry a pure F-structure of positive rank and have vanishing Euler characteristic.

Since M_n is simply connected, by [CG] (compare [PT]) the F-structure on each M_n is given by an effective smooth torus action without fixed points. Since all orbits of this action have positive dimension, it is easy to see that each M_n thus also admits a fixed-point free *circle* action.

Now ([H], Lemma 3.2) implies that each M_n has vanishing trilinear cup form μ and that the first Pontryagin class of M_n is torsion, i.e., $p_1(M_n) \in$ Tor $H^4(M_n;\mathbb{Z})$. By Poincaré duality, Tor $H^4(M_n;\mathbb{Z}) \cong$ Tor $H_2(M_n;\mathbb{Z})$. Also note that $\chi(M_n) = 0$ implies that $b_3(M_n) = 2\,b_2(M_n) + 2$.

But by Theorem 1.11 there are only finitely many possibilities for the second homology group of all M_n, and combining this with the facts above it follows that the collapsing sequence in question contains at most finitely non-diffeomorphic manifolds, which yields the desired contradiction. $\square$

Proof of Theorem 1.1(b) Uniformly positively pinched sequences of mutually non-diffeomorphic Aloff-Wallach or Eschenburg spaces (see [AW], [E]) and their products with spheres of appropriate dimension show immediately that Theorem 1.1(a) does not hold in dimension 7, nor in any dimension $m \geq 9$. Counterexamples to the validity of Theorem 1.1(a) in dimension 8 can be obtained in the following manner:

Starting with the connected sum $M := S^3 \times \mathbb{CP}^2 \# S^3 \times \mathbb{CP}^2$, using the Gysin sequence one sees that for any pair of relatively prime integers $p, q \in \mathbb{Z}$ there is a circle bundle $S^1 \to E_{p,q} \to M$ over M whose total space is simply connected and whose fourth cohomology is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$. In particular infinitely many non-diffeomorphic total spaces arise. It only remains to observe that given any Riemannian metric on M, one can easily construct on each total space $E_{p,q}$ a metric with similar absolute curvature and upper diameter bounds as the metric on M. $\square$

Proof of Theorem 1.11 Theorem 1.11 can be proven most easily by appealing to the following geometric classification theorem for closed simply connected closed manifolds from [PT]:

Theorem ([PT]) *For given $m \in \mathbb{N}$, C and D, there exists a finite number of closed simply connected smooth manifolds E_l with finite second homotopy groups such that any simply connected closed m-dimensional manifold M which admits a Riemannian metric with sectional curvature $|K| \leq C$ and diameter $\leq D$ is diffeomorphic to a factor space $M = E_l/T^{k_l}$, where $0 \leq k_l = \dim E_l - m = b_2(M;\mathbb{Z})$ and T^{k_l} acts freely on E_l.*

Now fix numbers m, C, and D, and let M be a closed smooth simply connected of dimension m which admits a Riemannian metric with sectional curvature $|K| \leq C$ and diameter $\leq D$. By the above theorem there exists a closed smooth simply connected manifold E with finite second homotopy group such that E is diffeomorphic to the total space of a principal T^k bundle over M. (In the terminology of [PT], the manifold E is the so-called *universal torus bundle* of M.)

Since tori are aspherical, the homotopy exact sequence

$$\cdots \to \pi_i(T^k) \to \pi_i(E) \to \pi_i(M) \to \cdots \to 0 = \pi_2(T^k) \to \pi_2(E) \to \pi_2(M) \to \pi_1(T^k) \cong \mathbb{Z}^k \to 0$$

of the principal bundle $T^k \to E \to M$ then shows that for $3 \leq i \in \mathbb{N}$ the homotopy group $\pi_i(M)$ is isomorphic to $\pi_i(E)$, and that $\pi_2(M)$ is isomorphic to $\pi_2(E) \oplus \mathbb{Z}^k$.

Since by the above geometric classification theorem for given numbers m, C, and D there are only finitely many non-diffeomorphic manifolds E, in noting that the homotopy groups of a closed simply connected manifold are finitely generated one sees that the proof of Theorem 1.11 is complete. $\square$

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