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Abstract

We introduce the notion of a special complex manifold: a complex manifold (M, J) with a flat torsionfree connection ∇ satisfying the condition $d^{\nabla} J = 0$. A special symplectic manifold is then defined as a special complex manifold together with a ∇-parallel symplectic form ω. The Hodge components ω^{11}, ω^{20}, ω^{02} are shown to be closed. If the form ω^{11} is nondegenerate, it defines a (pseudo) Kähler metric $g = \omega^{11} \circ J$ on M and if ω^{11} is ∇-parallel (e.g., if $\omega = \omega^{11}$) then $(M, J, \nabla, \omega^{11})$ is a special Kähler manifold in the sense of Freed. We give an extrinsic realisation of simply connected special complex, symplectic and Kähler manifolds as immersed complex submanifolds of $T^*\mathbb{C}^n$. Locally, any special complex manifold is realised as the image of a local holomorphic 1-form $\xi : \mathbb{C}^n \to T^*\mathbb{C}^n$. Such a realisation induces a canonical ∇-parallel symplectic structure on M and any special symplectic manifold is locally obtained this way. Special Kähler manifolds are realised by complex Lagrangian submanifolds and correspond to closed forms ξ. We include special complex manifolds (M, J, ∇) in a one-parameter family (M, J, ∇^{θ}), $\theta \in S^1$, and define projective versions of special complex, symplectic and Kähler manifolds in terms of an action of $\mathbb{C}^*$ on M which is transitive on this family. Finally, we discuss the natural geometric structures on the cotangent bundle of a special symplectic manifold, which are generalisations of the known hyper-Kähler structure on the cotangent bundle of a special Kähler manifold.

Key words: special geometry, special Kähler manifolds, hypercomplex manifolds, flat connections.

MSC: 53C30

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Introduction

Special Kähler manifolds have attracted a great deal of interest in both string theory and differential geometry since they first arose in the pioneering paper of de Wit and Van Proeyen [dW-VP] as the allowed target spaces for Maxwell supermultiplets coupled to 4-dimensional N=2 supergravity. These manifolds play a crucial role as admissible target spaces for scalar and vector couplings in both *rigid* supersymmetric theories and in supergravity theories, where the supersymmetry algebra is 'locally' realised. The special Kähler manifolds occurring in rigid and local supersymmetric theories correspond respectively to the affine and projective variants of special Kähler manifolds in the mathematical literature [C1, C2, F, H]. Special Kähler geometries, moreover, occur as natural geometric structures on certain moduli spaces. Projective special Kähler manifolds occur, for example, as moduli spaces of Calabi–Yau 3-folds (see e.g. [C1, C2]) and affine special Kähler manifolds occur as moduli spaces of complex Lagrangian submanifolds of hyper-Kähler manifolds [H]. Further, the base of any algebraic integrable system is also affine special Kähler [DW, F].

In this paper we introduce the notion of a *special complex manifold* as a complex manifold (M, J) with a flat torsionfree connection ∇ such that

$$d^{\nabla} J \quad = \quad 0\,. \tag{1}$$

We call it *special symplectic* if, in addition, a ∇-parallel symplectic form ω is specified. Further, if ω is J-invariant, or equivalently, of type $(1,1)$, it is precisely a *special Kähler* manifold in the sense of [F].

The main property of a special complex manifold is that any affine function f (i.e. a function satisfying $\nabla df = 0$) can be extended to a holomorphic function F such that $\mathrm{Re}\, F = f$. In particular, for a special symplectic manifold any local affine symplectic coordinate system $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ can be extended to a system of holomorphic functions $(z^1, \ldots, z^n, w_1, \ldots, w_n)$, which defines a local holomorphic immersion of M into $\mathbb{C}^{2n}$, such that the special symplectic structure is induced by certain canonical stuctures on $\mathbb{C}^{2n}$.

The main example of a special complex manifold M is associated to a (local) holomorphic 1-form $\alpha = \sum F_i dz^i$ on $\mathbb{C}^n$ with invertible real matrix $\mathrm{Im}\, \frac{\partial F_i}{\partial z_j}$ as follows: The complex manifold $M = M_\alpha$ is the image of the section $\alpha : \mathbb{C}^n \to T^*\mathbb{C}^n = \mathbb{C}^{2n}$. The flat torsionfree connection ∇ on M is defined by the condition that the real part $\mathrm{Re}\, F$ of any complex affine function F on $\mathbb{C}^{2n}$ restricts to a ∇-affine function on M. Such a manifold M carries a natural ∇-parallel symplectic form ω and can therefore be considered as a special symplectic manifold as well. If, in addition, the 1-form α is closed, then M_α is a Lagrangian submanifold and ω is of type $(1,1)$. So M_α is then a special Kähler manifold. Conversely, we prove that any special complex, symplectic or Kähler manifold can be locally obtained by this construction. More generally, we show that any totally complex holomorphic immersion ϕ of a complex n-manifold M into $\mathbb{C}^{2n}$ induces on M the structure of a special symplectic manifold. Here, we call an immersion *totally complex* if the intersection $d\phi(T_pM) \cap \mathbb{R}^{2n} = 0$ for all $p \in M$. If in addition, the immersion ϕ is Lagrangian (i.e. $d\phi(T_pM)$ is a Lagrangian subspace of $T^*\mathbb{C}^n$), then M is a special Kähler manifold. Our main result is that any simply connected special complex, symplectic or Kähler manifold can be constructed in this fashion.

In section 2 we consider the projective version of special geometry. Our approach is based on the following observation: Any special complex manifold (M, J, ∇) can be included in a one-parameter family (M, J, ∇^θ) of special complex manifolds, with the connection ∇^θ defined by

$$\nabla^\theta X := e^{\theta J} \nabla (e^{-\theta J} X) , \tag{2}$$

where $e^{\theta J} X = (cos\theta) X + (sin\theta) JX$. A complex manifold (M, J) with a flat torsionfree connection ∇ is called a *conic complex manifold* if it admits a local holomorphic $\mathbb{C}^*$-action φ_λ with differential $d\varphi_\lambda X = re^{\theta J} X$ for all ∇-parallel (local) vector fields X, where $\lambda = re^{i\theta}$. This implies $\varphi_\lambda^* \nabla = \nabla^\theta$. We show that a conic complex manifold is automatically special.

Assume that a manifold $M_\alpha \subset T^*\mathbb{C}^n$, $\alpha = \sum F_i dz^i$, is a complex cone, i.e. it is invariant under complex scalings. This is the case when the coefficient functions F_i are homogeneous of degree one. The induced special geometry on M_α is then conic. Conversely, we prove that any conic (special) complex, symplectic or Kähler manifold can be locally realised as such a cone. In particular, any conic special Kähler manifold is locally described by the differential $\alpha = dF$ of a holomorphic homogeneous function F of degree two. In the simply connected case, we give a global description of conic special manifolds in terms of holomorphic immersions.

Then we define a projective special complex, symplectic or Kähler manifold as the orbit space $\overline{M}$ of a conic complex, symplectic or Kähler manifold M by the local $\mathbb{C}^*$-action, assuming that $\overline{M}$ is a (Hausdorff) manifold. From the realisation of simply connected conic manifolds as immersed submanifolds of $T^*\mathbb{C}^n$, we obtain an analogous realisation of projective special manifolds as immersed submanifolds of complex projective space $P(T^*\mathbb{C}^n)$. From this it follows that our definition of projective special Kähler manifolds is consistent with that given by Freed [F].

Finally, we discuss the natural geometric structures on the cotangent bundle of a special symplectic manifold which are generalisations of the known hyper-Kähler structure on the cotangent bundle of a special Kähler manifold [CFG, C2, F, H]. We prove that the cotangent bundle $N = T^*M$ of a special symplectic manifold M carries two canonical complex structures : the standard complex structure J_1 induced by J and a complex structure J^ω, defined by ω and ∇. If the form ω^{11} is nondegenerate, then $N = T^*M$ carries also a natural almost hyper-Hermitian structure (J_1, J_2, g_N), i.e. a Riemannian metric g_N (which is an extension of the Kähler metric $g = \omega^{11} \circ J$) and two anticommuting g_N-orthogonal almost complex structures J_1, J_2. This almost hyper-Hermitian structure is integrable, i.e. J_1 and J_2 are integrable, if and only if ω^{11} is ∇-parallel. In this case (J_1, J_2, g_N) is a hyper-Kähler structure and we recover the known hyper-Kähler structure on the cotangent bundle of a special Kähler manifold. Similarly, if $\omega' = \omega^{20} + \omega^{02}$ is nondegenerate, then $N = T^*M$ carries a natural almost para-hypercomplex structure, that is a pair (J_1, J_2) of commuting almost complex structures. Here J_1 is the standard integrable complex structure and J_2 is integrable if and only if the form ω' is ∇-parallel.

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1 (Affine) special geometry

1.1 Special manifolds

Definition 1 *A* special complex manifold (M, J, ∇) *is a complex manifold* (M, J) *together with a flat torsionfree connection* ∇ *(on its real tangent bundle) such that*

$$d^\nabla J = 0 .$$

Here the complex structure J *is considered as a 1-form with values in* TM *and* d^∇ *denotes the covariant exterior derivative defined by* ∇.

A special symplectic manifold (M, J, ∇, ω) *is a special complex manifold* (M, J, ∇) *together with a* ∇*-parallel symplectic structure* ω.

A special Kähler manifold *is a special symplectic manifold* (M, J, ∇, ω) *for which* ω *is* J*-invariant, i.e. of type* $(1, 1)$. *The (pseudo-)Kähler metric* $g(\cdot, \cdot) := \omega(J\cdot, \cdot)$ *is called the* special Kähler metric *of the special Kähler manifold* (M, J, ∇, ω).

Remark 1: The evaluation of the TM-valued 2-form $d^\nabla J = alt(\nabla J)$ on two tangent vectors X and Y is given by:

$$d^\nabla J(X, Y) = (\nabla_X J)Y - (\nabla_Y J)X .$$

Remark 2: Since we do not assume the definiteness of the metric, it would be more accurate to speak of special *pseudo*-Kähler manifolds/metrics. However, as the signature of the metric is not relevant for our present discussion, we shall omit the prefix *pseudo*.

Given a linear connection ∇ on a manifold M and an invertible endomorphism field A on a manifold M, we denote by $\nabla^{(A)}$ the connection defined by

$$\nabla^{(A)} X = A\nabla(A^{-1}X)\,.$$

Given a flat connection ∇ on (the real tangent bundle of) a complex manifold (M, J), we define a one-parameter family of connections $\nabla^\theta = \nabla^{(e^{\theta J})}$ parametrized by the projective line $P^1 = \mathbb{R}/\pi\mathbb{Z}$, where $e^{\theta J} = (\cos\theta)\mathrm{Id} + (\sin\theta)J$. The connection ∇^θ is flat, since:

$$\nabla^\theta X = 0 \Longleftrightarrow \nabla(e^{-\theta J}X) = 0\,,$$

where X is a local vector field on M.

Lemma 1 *Let ∇ be a flat connection with torsion T on a complex manifold (M, J). Then*

$$\nabla^\theta = \nabla + A^\theta\,, \quad \textit{where} \quad A^\theta = e^{\theta J}\nabla(e^{-\theta J}) = -(\sin\theta)e^{\theta J}\nabla J\,.$$

The torsion T^θ of the connection ∇^θ is given by:

$$T^\theta = T + alt(A^\theta) = T - (\sin\theta)e^{\theta J}d^\nabla J\,. \tag{3}$$

Proposition 1 *Let ∇ be a flat torsionfree connection on a complex manifold (M, J). Then the triple (M, J, ∇) defines a special complex manifold if and only if one of the following equivalent conditions holds:*

a) $d^\nabla J = 0$.

b) The flat connection ∇^θ is torsionfree for some $\theta \not\equiv 0 \pmod{\pi\mathbb{Z}}$.

b') The flat connection ∇^θ is torsionfree for all θ.

c) There exists $\theta \not\equiv 0 \pmod{\pi\mathbb{Z}}$ such that $[e^{\theta J}X, e^{\theta J}Y] = 0$ for all ∇-parallel local vector fields X and Y on M.

c') $[e^{\theta J}X, e^{\theta J}Y] = 0$ *for all θ and all ∇-parallel local vector fields X and Y on M.*

d) There exists $\theta \not\equiv 0 \pmod{\pi\mathbb{Z}}$ such that $d(\xi \circ e^{-\theta J}) = 0$ for all ∇-parallel local 1-forms ξ on M.

d') $d(\xi \circ e^{-\theta J}) = 0$ *for all θ and all ∇-parallel local 1-forms ξ on M.*

Proof: a) is the property defining special complex manifolds. Since ∇ is torsionfree, the torsion T^θ of ∇^θ is related to $d^\nabla J$ in virtue of (3) by:

$$T^\theta = -(\sin\theta)e^{\theta J}d^\nabla J\,.$$

If $\theta \not\equiv 0 \pmod{\pi\mathbb{Z}}$ the endomorphism $(\sin\theta)e^{\theta J}$ is invertible. This implies the equivalence of a), b) and b'). Let X and Y be ∇-parallel local vector fields. Then $e^{\theta J}X$ and $e^{\theta J}Y$ are ∇^θ-parallel, by the definition of ∇^θ, and hence

$$T^\theta(e^{\theta J}X, e^{\theta J}Y) = -[e^{\theta J}X, e^{\theta J}Y]\,.$$

This yields b) $\Leftrightarrow$ c) and b') $\Leftrightarrow$ c'). For a ∇-parallel local 1-form ξ and X, Y as above we compute:

$$\begin{aligned}
&d(\xi\circ e^{-\theta J})(e^{\theta J}X, e^{\theta J}Y)\\
&= -\xi(e^{-\theta J}[e^{\theta J}X, e^{\theta J}Y]) + e^{\theta J}X\xi(Y) - e^{\theta J}X\xi(X)\\
&= -\xi(e^{-\theta J}[e^{\theta J}X, e^{\theta J}Y])
\end{aligned}$$

since the functions $\xi(X)$ and $\xi(Y)$ are constant. This proves the equivalences c) $\Leftrightarrow$ d) and c') $\Leftrightarrow$ d'), completing the proof of the proposition. $\Box$

Given a complex manifold (M, J) with a flat connection ∇, we say that the connection

$$\nabla^{\frac{\pi}{2}} = \nabla^{(e^{\frac{\pi}{2}J})} = \nabla^{(J)} = \nabla - J\nabla J$$

is its *conjugate connection.*

Corollary 1 *Let (M, J) be a complex manifold with a flat torsionfree connection ∇. Then the following are equivalent:*

a) (M, J, ∇) is a special complex manifold.

b) The conjugate flat connection $\nabla^{(J)}$ is torsionfree.

c) $[JX, JY] = 0$ for all ∇-parallel local vector fields X and Y on M.

d) $d(\xi\circ J) = 0$ for all ∇-parallel local 1-forms ξ on M.

Corollary 2 *If (M, J, ∇) is a special complex manifold then (M, J, ∇^θ) is a special complex manifold for any θ. If (M, J, ∇, ω) is a special Kähler manifold then $(M, J, \nabla^\theta, \omega)$ is a special Kähler manifold for any θ.*

The next proposition shows that any special complex manifold also has a canonical torsionfree *complex* connection, which in general is not flat.

Proposition 2 *Let (M, J, ∇) be a special complex manifold. Then $D := \frac{1}{2}(\nabla + \nabla^{(J)})$ defines a torsionfree connection such that $DJ = 0$.*

Proof: As a convex combination of torsionfree connections, D is a torsionfree connection. For any vector field X on M we compute:

$$D_X J = \nabla_X J - \frac{1}{2}[J\nabla_X J, J] = \nabla_X J - \nabla_X J = 0\,. \quad \square$$

Proposition 3 *Let (M, J, ∇, ω) be a special Kähler manifold with special Kähler metric g and Levi-Civita connection ∇^g. Then the following hold:*

(i) $\nabla^g = D = \frac{1}{2}(\nabla + \nabla^{(J)})$.

(ii) The conjugate connection $\nabla^{(J)}$ is g-dual to ∇, i.e.

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^{(J)} Z)$$

for all vector fields X, Y and Z on M.

(iii) The tensor ∇g is completely symmetric.

Proof: (i) is an immediate consequence of Proposition 2 since $g = \omega(\cdot, J\cdot)$. (ii) follows from a direct computation, which only uses the fact that ω is ∇-parallel and J-invariant:

$$\begin{aligned} Xg(Y, Z) &= X\omega(Y, JZ) = \omega(\nabla_X Y, JZ) + \omega(Y, \nabla_X JZ) \\ &= g(\nabla_X Y, Z) + \omega(JY, J\nabla_X JZ) = g(\nabla_X Y, Z) + g(Y, \nabla_X^{(J)} Z)\,. \end{aligned}$$

Finally, to prove (iii) it is sufficient to check that ∇g is symmetric in the first two arguments:

$$\begin{aligned} &(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) \\ &= Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) - Yg(X, Z) + g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \\ &\overset{(ii)}{=} -g(\nabla_X Y, Z) + g(\nabla_X^{(J)} Y, Z) + g(\nabla_Y X, Z) - g(\nabla_Y^{(J)} X, Z) \\ &= g(-[X, Y] + [X, Y], Z) = 0\,. \quad \square \end{aligned}$$

Proposition 4 *Let (M, J, ∇, ω) be a special symplectic manifold and $\omega = \omega^{11} + \omega^{20} + \omega^{02}$ the Hodge decomposition of the symplectic form. Then each of the components ω^{11}, ω^{20}, ω^{02} are closed.*

Proof: It is sufficient to check that the (1,1)-component $\omega^{11} = \frac{1}{2}(\omega + \omega(J\cdot, J\cdot))$ is closed. Since ∇ has no torsion, the exterior derivative is given by $d = alt \circ \nabla$. We compute:

$$2d\omega^{11} = d(\omega + \omega(J\cdot, J\cdot)) = alt \circ \nabla\omega(J\cdot, J\cdot)\,.$$

Since $\nabla\omega = 0$ for any $X_1, X_2, X_3 \in T_pM$ we obtain:

$$\begin{aligned}
&2d\omega^{11}(X_1, X_2, X_3)\\
=\ & \frac{1}{3}(\omega((\nabla_{X_1}J)X_2, JX_3) + \omega(JX_2, (\nabla_{X_1}J)X_3) + cycl.)\\
=\ & \frac{1}{3}(\omega((\nabla_{X_1}J)X_2, JX_3) + \omega(JX_2, (\nabla_{X_3}J)X_1) + cycl.)\,,\ \text{using } d^{\nabla}J{=}0\,,\\
=\ & \frac{1}{3}(\omega((\nabla_{X_1}J)X_2, JX_3) - \omega((\nabla_{X_3}J)X_1, JX_2) + cycl.) = 0\,. \qquad \Box
\end{aligned}$$

Proposition 5 *Let (M, J, ∇, ω) be a special symplectic manifold and assume that ω^{11} is nondegenerate. Then (M, J, ω^{11}) is a Kähler manifold with Kähler metric $g = \omega^{11}(J\cdot, \cdot)$. $(M, J, \nabla, \omega^{11})$ is special Kähler if and only if $\nabla\omega^{11} = 0$.*

Proof: It is clear that g is a Hermitian metric on the complex manifold (M, J). By the previous proposition the Kähler form ω^{11} of g is closed and hence (M, J, g) is a Kähler manifold. The last statement is obvious. $\Box$

1.2 Special coordinates

A flat torsionfree connection ∇ on a manifold M defines on it an *affine structure*, i.e. an atlas with affine transition functions. A (local) function f on (M, ∇) is called affine if $\nabla df = 0$. A local coordinate system $(x^1, \ldots, x^m)$ on M, $m = \dim M$, is called affine if the x^i are affine functions. Any affine local coordinate system $(x^1, \ldots, x^m)$ defines a parallel local coframe $(dx^1, \ldots, dx^m)$. Conversely, since any parallel 1-form α is locally the differential of an affine function f, given a parallel coframe $(\alpha^1, \ldots, \alpha^m)$ defined on a simply connected domain $U \subset M$ there exist affine functions x^i on U such that $dx^i = \alpha^i$. The tuple $(x^1, \ldots, x^m)$ defines an affine local coordinate system near each point $p \in U$. This coordinate system is unique (as a germ, i.e. up to restrictions of the coordinate domain) up to translations in $\mathbb{R}^m$. If we require in addition that the coordinate system is centred at $p \in U$, i.e. that $x^i(p) = 0$, then it is uniquely determined.

Definition 2 *Let (M, J, ∇, ω) be a special symplectic manifold. A ∇–affine local coordinate system $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ on M is called a* real special coordinate system *if $\omega = 2\sum dx^i \wedge dy_i$. A* conjugate pair of special coordinates *is a pair of holomorphic local coordinates $(z^1, \ldots, z^n)$ and $(w_1, \ldots, w_n)$ such that $(x^1 = \operatorname{Re} z^1, \ldots, x^n = \operatorname{Re} z^n,\ y_1 = \operatorname{Re} w_1, \ldots, y_n = \operatorname{Re} w_n)$ is a real special coordinate system.*

Theorem 1 *(i) Any special symplectic manifold (M, J, ∇, ω) admits a real special coordinate system near any point $p \in M$. A real special coordinate system is unique up to an affine symplectic transformation.*

(ii) Any affine local coordinate system $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ on a special complex manifold admits a holomorphic extension, i.e. there exist holomorphic functions z^i and w_j with $\operatorname{Re} z^i = x^i$ and $\operatorname{Re} w_j = y_j$. The extension is unique up to (purely imaginary) translations.

(iii) Near any point of a special Kähler manifold there exists a real special coordinate system which admits a holomorphic extension to a conjugate pair of special coordinates.

Proof: The existence and uniqueness statements about real special coordinate systems are obvious. Let $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ be an affine local coordinate system on a special complex manifold. Then we define $\omega^i := dx^i - \sqrt{-1}J^*dx^i$. By Corollary 1 $J^*dx^i = dx^i \circ J$ is closed. This implies that ω^i are closed 1-forms of type $(1,0)$ and are hence closed holomorphic 1-forms. So there exist local holomorphic functions z^i such that $\omega^i = dz^i$. By adding real constants we can arrange that $\operatorname{Re} z^i = x^i$. Similarly, there exist local holomorphic functions w_j such that $\operatorname{Re} w_j = y_j$. The uniqueness statement concerning this holomorphic extension is obvious. We claim that in the case of special Kähler manifolds, real special coordinates can be chosen such that the dz^i, as well as the w_j, are linearly independent (over $\mathbb{C}$). To see this, let us first observe that the dx^i and dy_j define a Lagrangian splitting of T_p^*M with respect to ω^{-1} for any point p in the coordinate domain: $T_p^*M = L_x \oplus L_y$, where $L_x = \operatorname{span}\{dx^1, \ldots, dx^n\}$ and $L_y = \operatorname{span}\{dy_1, \ldots, dy_n\}$. The J-invariance of the symplectic (Kähler) form ω implies the existence of a Lagrangian splitting of the form $T_p^*M = L \oplus J^*L$. Since any two Lagrangian splittings of a symplectic vector space are related by a linear symplectic transformation, this shows that the real special coordinates $x^1, \ldots y_n$ near p can be chosen such that the corresponding Lagrangian subspaces L_x, L_y satisfy $L_x \cap J^*L_x = L_y \cap J^*L_y = 0$ at the point p and hence on a coordinate domain containing p. The equation $L_x \cap J^*L_x = 0$ forces the $dz^i = dx^i - \sqrt{-1}J^*dx^i$ to be linearly independent. So the z^i define local holomorphic coordinates on the special Kähler manifold. Similarly, as a consequence of the equation $L_y \cap J^*L_y = 0$, the w_j are local holomorphic coordinates. $\Box$

1.3 The extrinsic construction of special manifolds

As in [C2], we consider the following fundamental algebraic data: the complex vector space $V = T^*\mathbb{C}^n = \mathbb{C}^{2n}$ with canonical coordinates $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ and standard complex symplectic form $\Omega = \sum_{i=1}^n dz^i \wedge dw_i$, the standard real structure $\tau : V \to V$ with fixed point set $V^\tau = T^*\mathbb{R}^n$. Then $\gamma := \sqrt{-1}\ \Omega(\cdot, \tau\cdot)$ defines a Hermitian form of (complex) signature (n, n).

Let M be a connected complex n-fold. A holomorphic immersion $\phi : M \to V$ is called *nondegenerate* (respectively, *Lagrangian*) if $\phi^*\gamma$ is nondegenerate (respectively, if $\phi^*\Omega = 0$). If ϕ is nondegenerate, then $\phi^*\gamma$ defines a, possibly indefinite, Kähler metric g $(= \mathrm{Re}\,\phi^*\gamma)$ on the complex manifold M. The correponding Kähler form $g(\cdot, J\cdot)$ is a J-invariant symplectic form on M, where J denotes the complex structure of M. ϕ is called *totally complex* if $V^\tau \cap d\phi T_pM = 0$ for all $p \in M$.

Lemma 2 *A holomorphic immersion $\phi : M \to V$ is totally complex if and only if its real part* $\mathrm{Re}\,\phi : M \to V^\tau$ *is an immersion.*

Proof: Let $\phi : M \to V$ be a totally complex holomorphic immersion. Restricting, i.e. pulling back via ϕ, the functions $x^i := \mathrm{Re}\, z^i$ and $y_j := \mathrm{Re}\, w_j$ to M we obtain $2n$ functions on M with everywhere linearly independent differentials. In fact, let $\alpha = \sum a_i dx^i + \sum b^j y_j$ be a real linear combination which vanishes on the complex n-dimensional linear subspace $d\phi T_pM \subset V$. Then, since α is real, it must also vanish on $\tau d\phi T_pM$. Now we can conclude that $\alpha = 0$ since, by our assumption on ϕ, $d\phi T_pM \cap \tau d\phi T_pM = V^\tau \cap d\phi T_pM \oplus iV^\tau \cap d\phi T_pM = 0$ and, therefore, $V = d\phi T_pM \oplus \tau d\phi T_pM$. This shows that the functions x^i and y_j restrict to local coordinates on M and, hence, that $\mathrm{Re}\,\phi$ is an immersion. Conversely, let $\phi : M \to V$ be a holomorphic immersion such that $\mathrm{Re}\,\phi : M \to V^\tau$ is an immersion. We have to show that $V^\tau \cap d\phi T_pM = 0$ for all $p \in M$. Suppose, that $X \in T_pM$ and $d\phi X \in V^\tau$. Then we have that $0 = \mathrm{Im}\, d\phi X = -\mathrm{Re}\sqrt{-1} d\phi X = -\mathrm{Re}\, d\phi JX$. This implies that $JX = 0$, because $d\mathrm{Re}\,\phi = \mathrm{Re}\, d\phi$ is injective. This shows that $X = 0$ proving $V^\tau \cap d\phi T_pM = 0$. □

A holomorphic totally complex immersion ϕ induces a flat torsionfree connection on the real tangent bundle of M as follows. Since $\mathrm{Re}\,\phi$ is an immersion, by Lemma 2, restricting the functions $x^i = \mathrm{Re}\, z^i$ and $y_j = \mathrm{Re}\, w_j$ to M we obtain local coordinates, which induce a flat torsionfree connection ∇ on M. Moreover, $2\sum dx^i \wedge dy_i$ restricts to a ∇-parallel symplectic form ω on M. We call ∇ and ω the induced connection and the induced symplectic form respectively. Now we can easily prove:

Theorem 2 *Let ϕ be a totally complex holomorphic immersion of a complex manifold (M, J) into $V = T^*\mathbb{C}^n$, $n = \dim_{\mathbb{C}} M$, ∇ the induced connection and $\omega = 2\phi^*(\sum dx^i \wedge dy_i)$ the induced symplectic form. Then the following hold:*

(i) (M, J, ∇, ω) *is a special symplectic manifold.*

(ii) The pull back via ϕ of the functions $(x^1 = \mathrm{Re}\, z^1, \ldots, x^n = \mathrm{Re}\, z^n, y_1 = \mathrm{Re}\, w_1, \ldots, y_n = \mathrm{Re}\, w_n)$ of V defines a real special coordinate system around each point of M.

Proof: We have to prove that $d^\nabla J = 0$. By Corollary 1, it is sufficient to check that the 1-forms $dx^i \circ J$ and $dy_j \circ J$ are closed. This follows immediately from the fact that the 1-forms $dz^i = dx^i - \sqrt{-1}dx^i \circ J$ and $dw_j = dy_j - \sqrt{-1}dy_j \circ J$ are closed. $\square$
The next proposition clarifies the relation between the three notions defined above.

Proposition 6 *Let ϕ be a holomorphic immersion of a complex n-fold M into $V = T^*\mathbb{C}^n$. The following conditions are equivalent:*

(i) ϕ is Lagrangian and nondegenerate.

(ii) ϕ is Lagrangian and totally complex.

Theorem 3 *Let ϕ be a holomorphic nondegenerate Lagrangian immersion of a complex manifold (M, J) into V inducing the Kähler metric g on M. The immersion ϕ is totally complex and hence induces also the data (∇, ω) on M. Moreover, the following hold:*

(i) (M, J, ∇, ω) *is a special Kähler manifold.*

(ii) ω coincides with the Kähler form of g, i.e. $\omega = g(\cdot, J\cdot)$.

(iii) The pull back via ϕ of the canonical coordinates $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ of V defines a conjugate pair of special coordinates around each point of M.

Proof: Thanks to Proposition 6 and Theorem 2 it is sufficient to prove that $g(\cdot, J\cdot) = \omega = 2\phi^*(\sum dx^i \wedge dy_i)$. A straightforward computation, which only uses the definition of g, shows that

$$2g(\cdot, J\cdot) = \omega + J^*\omega\,. \tag{4}$$

On the other hand, since ϕ is Lagrangian, we know also that

$$0 = 2\mathrm{Re}\,\phi^*\Omega = \omega - J^*\omega\,.$$

This implies that $g(\cdot, J\cdot) = \omega$. $\square$

Now we will show that any simply connected special (complex, symplectic or Kähler) manifold arises by the construction of Theorem 2 or Theorem 3.

Theorem 4 *(i) Let (M, J, ∇) be a simply connected special complex manifold of complex dimension n. Then there exists a holomorphic totally complex immersion $\phi : M \to V = T^*\mathbb{C}^n$ inducing the connection ∇ on M. Moreover, ϕ is unique up to an affine transformation of V preserving the real structure τ. Here the real structure is considered as a (constant) field of antilinear involutions on the tangent spaces of V. Finally, $\omega = 2\phi^*(\sum dx^i \wedge dy_i)$ is a ∇-parallel symplectic structure defining on (M, J, ∇) the structure of special symplectic manifold.*

(ii) Let (M, J, ∇, ω) be a simply connected special symplectic manifold of complex dimension n. Then there exists a holomorphic totally complex immersion $\phi : M \to V = T^*\mathbb{C}^n$ inducing the connection ∇ and the symplectic form ω on M. Moreover, ϕ is unique up to an affine transformation of V preserving the complex symplectic form Ω and the real structure τ.

(iii) Let (M, J, ∇, ω) be a simply connected special Kähler manifold of complex dimension n then there exists a holomorphic nondegenerate Lagrangian (and hence totally complex) immersion $\phi : M \to V = T^*\mathbb{C}^n$ inducing the Kähler metric g, the connection ∇ and the symplectic form $\omega = 2\phi^*(\sum dx^i \wedge dy_i) = g(\cdot, J\cdot)$ on M. Moreover ϕ is unique up to an affine transformation of V preserving the complex symplectic form Ω and the real structure τ. Here the real structure is considered as a field of antilinear involutions on the tangent spaces of V.

Proof: We prove (ii) and (iii). The proof of (i) is similar. By Theorem 1 there exist real special coordinates near each point of M. Since M is simply connected, we can choose these local coordinates in a compatible way obtaining globally defined functions x^i and y_j on M such that $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ is a real special coordinate system near each point of M. Then again by Theorem 1 and the simple connectedness of M we can holomorphically extend these functions, i.e. there exist globally defined holomorphic functions z^i and w_j such that $\operatorname{Re} z^i = x^i$ and $\operatorname{Re} w_j = y_j$. Moreover, if (M, J, ∇, ω) is special Kähler we can assume that $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ form a conjugate pair of special coordinates. We define the holomorphic map

$$\phi := (z^1, \ldots, z^n, w_1, \ldots, w_n) : M \to \mathbb{C}^{2n} = V\,.$$

The fact that ϕ is a totally complex immersion follows from the linear independence of $(dx^1, \ldots, dx^n, dy_1, \ldots, dy_n)$. This proves the existence statement in (ii). To prove (iii) we need to check that ϕ is Lagrangian, i.e. that the holomorphic 2-form $\Omega := \sum dz^i \wedge dw_i = 0$. This follows from the J-invariance of $\omega = 2\sum dx^i \wedge dy_i$, since $2\operatorname{Re}\Omega = \omega - J^*\omega$ and

$2\mathrm{Im}\,\Omega = J \cdot \omega = 2\sum J dx^i \wedge dy_i + 2\sum J dy_i \wedge dx^i$. Here the $\cdot$ stands for the natural action of $\mathfrak{gl}(E)$ on $\wedge^2 E^*$, where $E = T_p M$, $p \in M$. The uniqueness statement is a consequence of the uniqueness statement in Theorem 1. $\Box$

We will call a holomorphic 1-form $\sum F_i dz^i$ on an open subset $U \subset \mathbb{C}^n$ *regular* if the real matrix $\mathrm{Im}\,\partial F_i/\partial z^j$ is invertible. A holomorphic function F on U is called *nondegenerate* if its differential dF is a regular holomorphic 1-form. Any holomorphic 1-form ϕ on a domain $U \subset \mathbb{C}^n$ can be considered as a holomorphic immersion

$$\phi : U \to V = T^*\mathbb{C}^n .$$

So it makes sense to speak of totally complex or Lagrangian holomorphic 1-forms.

Lemma 3 *Let ϕ be a holomorphic 1-form. Then the following hold:*

(i) ϕ is totally complex if and only if it is regular.

(ii) ϕ is Lagrangian if and only if it is closed.

Proof: (ii) is a well known fact from classical mechanics. To see (i) let $\phi = \sum F_i dz^i$ be a holomorphic 1-form on a domain $U \subset \mathbb{C}^n$. It is totally complex if and only if the form $\frac{1}{2}\omega = \phi^*(\sum dx^i \wedge dy^i)$ is nondegenerate on U. We compute

$$\frac{1}{2}\omega = \sum dx^i \wedge d\mathrm{Re}\, F_i = \sum (\mathrm{Re}\, \frac{\partial F_i}{\partial z^j}) dx^i \wedge dx^j - \sum (\mathrm{Im}\, \frac{\partial F_i}{\partial z^j}) dx^i \wedge du^j .$$

From this it is easy to see that ω is nondegenerate if and only if the matrix $\mathrm{Im}\,\partial F_i/\partial z^j$ is invertible, i.e. if and only if ϕ is regular. $\Box$

The following is a corollary of Lemma 3, Theorem 2 and Theorem 4.

Corollary 3 *Any regular local holomorphic 1-form ϕ on $\mathbb{C}^n$ defines a special symplectic manifold of complex dimension n. Conversely, any special symplectic manifold of complex dimension n can be locally obtained in this way.*

Corollary 4 *Any nondegenerate local holomorphic function on $\mathbb{C}^n$ defines a special Kähler manifold of complex dimension n. Conversely, any special Kähler manifold of complex dimension n can be locally obtained in this way.*

Proof: A nondegenerate holomorphic function F defines a regular and closed holomorphic 1-form dF. The corresponding holomorphic immersion $\phi = dF$ is totally complex and Lagrangian (by Lemma 3) and, by Proposition 6, nondegenerate. So it defines a special Kähler manifold by Theorem 3. The converse statement follows from Theorem 4 and the fact that any holomorphic nondegenerate Lagrangian immersion into V is locally defined by a regular closed holomorphic 1-form (after choosing an appropriate isomorphism $V = T^*\mathbb{C}^n$). Notice that every regular closed holomorphic 1-form on a simply connected domain is the differential of a nondegenerate holomorphic function. $\Box$

2 Projective special geometry

2.1 Conic and projective special manifolds

We recall that a *local holomorphic* $\mathbb{C}^*$*-action* on a complex manifold M is a holomorphic map

$$\mathbb{C}^* \times M \ni (\lambda, p) \mapsto \varphi_\lambda(p) \in M$$

defined on an open neighbourhood W of $\{1\} \times M$ such that

(i) $\varphi_1(p) = p$ for all $p \in M$ and

(ii) $\varphi_\lambda(\varphi_\mu(p)) = \varphi_{\lambda\mu}(p)$ if both sides are defined, i.e. if $(\lambda, \varphi_\mu(p)) \in W$ and $(\lambda\mu, p) \in W$.

From this definition it follows that for every $p \in M$ there exist open neighbourhoods U_1 of $1 \in \mathbb{C}^*$ and U_p of p such that $U_1 \times U_p \subset W$ and $\varphi_\lambda|U_p$ is a diffeomorphism onto its image for all $\lambda \in U_1$. We will say that an equation involving φ_λ holds *locally* if it holds on any open set $U \subset M$ on which φ_λ is defined and on which it is a diffeomorphism onto its image. Of course, even if it is not explicitly mentioned, an equation involving φ_λ is always meant to hold only locally.

We use polar coordinates (r, θ) to parametrize $\mathbb{C}^* = \{\lambda = re^{i\theta} | r, \theta \in \mathbb{R}, r > 0\}$ and consider θ as a map from $\mathbb{C}^*$ to $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 3 *(i) Let (M, J, ∇) be a complex manifold with a flat torsionfree connection. It is called a conic complex manifold if it admits a local holomorphic $\mathbb{C}^*$-action φ_λ such that locally $d\varphi_\lambda X = re^{\theta J}X = r(\cos\theta)X + r(\sin\theta)JX$ for all ∇-parallel vector fields X, where $\lambda = re^{i\theta}$.*

(ii) A conic symplectic manifold *is a conic complex manifold (M, J, ∇) together with a parallel symplectic form ω.*

(iii) A conic symplectic manifold (M, J, ∇, ω) is called a conic Kähler manifold *if ω is J-invariant.*

Notice that the condition $d\varphi_\lambda X = re^{\theta J}X$ for all ∇-parallel vector fields X implies that $\varphi_\lambda^* \nabla = \nabla^\theta$.

Proposition 7 *(i) Any conic complex manifold is a special complex manifold.*

(ii) Any conic symplectic manifold is a special symplectic manifold.

(iii) Any conic Kähler manifold is a special Kähler manifold.

Proof: Let (M, J, ∇) be a conic complex manifold and φ_λ the corresponding local action. Since $d^\nabla J = 0$ is a local condition, it is sufficient to prove that any point $p \in M$ has an open neighbourhood U such that (U, J, ∇) is a special complex manifold. By Proposition 1 it is sufficient to check that for any point $p \in M$ there exist open neighbourhoods U_1 of $1 \in \mathbb{R}/2\pi\mathbb{Z}$ and U_p of p such that ∇^θ is a torsionfree connection on U_p for all $\theta \in U_1$. From the definition of local action it follows that for any $p \in M$ there exist open neighbourhoods U_1 of $1 \in \mathbb{R}/2\pi\mathbb{Z}$ and U_p of p such that φ_λ is defined on U_p and $\varphi_\lambda|U_p$ is a diffeomorphism onto its image for all $\lambda = e^{i\theta}$ with $\theta \in U_1$. Since (M, J, ∇) is a conic complex manifold we have $\nabla^\theta = \varphi_\lambda^* \nabla$ on U_p for all $\theta \in U_1$. Thus ∇^θ is a torsionfree connection on U_p, proving (i). Statements (ii) and (iii) follow easily from (i). □

Theorem 5 *(i) Let (M, J, ∇) be a complex manifold with a flat torsionfree connection. Then (M, J, ∇) is a conic complex manifold if and only if there exists a local holomorphic $\mathbb{C}^*$-action φ_λ and for every $p \in M$ holomorphic functions $z^1, \ldots, z^n$ and $w_1, \ldots, w_n$ defined near p such that*

(a) $z^i \circ \varphi_\lambda = \lambda z^i$ and $w_j \circ \varphi_\lambda = \lambda w_j$ near p and

(b) $x^1 := \operatorname{Re} z^1, \ldots, x^n := \operatorname{Re} z^n, y_1 := \operatorname{Re} w_1, \ldots, y_n := \operatorname{Re} w_n$ are affine local coordinates near p.

(ii) Let (M, J, ∇, ω) be a complex manifold with a flat torsionfree connection and a parallel symplectic form. Then (M, J, ∇, ω) is a conic symplectic manifold if and only if there exists a local holomorphic $\mathbb{C}^*$-action φ_λ and for every $p \in M$ holomorphic functions $z^1, \ldots, z^n$ and $w_1, \ldots, w_n$ defined near p such that

(a) $z^i \circ \varphi_\lambda = \lambda z^i$ and $w_j \circ \varphi_\lambda = \lambda w_j$ near p and

(b) $x^1 := \operatorname{Re} z^1, \ldots, x^n := \operatorname{Re} z^n, y_1 := \operatorname{Re} w_1, \ldots, y_n := \operatorname{Re} w_n$ are affine local coordinates near p.

Moreover, if (M, J, ∇, ω) is a conic (special) symplectic manifold then the local holomorphic functions z^i and w_j can be chosen such that their real parts x^i and y_j form a real special coordinate system.

(iii) Let (M, J, ∇, ω) be a complex manifold with a flat torsionfree connection and a parallel J-invariant symplectic form. Then (M, J, ∇, ω) is a conic Kähler manifold if and only if there exists a local holomorphic $\mathbb{C}^*$-action φ_λ and for every $p \in M$ holomorphic functions $z^1, \ldots, z^n$ and $w_1, \ldots, w_n$ defined near p such that

(a) $z^i \circ \varphi_\lambda = \lambda z^i$ and $w_j \circ \varphi_\lambda = \lambda w_j$ near p and

(b) $x^1 := \mathrm{Re}\, z^1, \ldots, x^n := \mathrm{Re}\, z^n, y_1 := \mathrm{Re}\, w_1, \ldots, y_n := \mathrm{Re}\, w_n$ *are affine local coordinates near* p.

Moreover, if (M, J, ∇, ω) *is a conic (special) Kähler manifold then the local holomorphic functions* z^i *and* w_j *can be chosen such that they form a conjugate pair of special coordinates.*

Proof: We prove only (i). Parts (ii) and (iii) are proven similarly. Let (M, J, ∇) be a conic complex manifold and $x^1, \ldots, x^n, y_1, \ldots, y_n$ affine local coordinates on it. By Proposition 7 and Theorem 1 it is a special complex manifold and the affine local coordinates admit a holomorphic extension $z^1, \ldots, z^n, w_1, \ldots, w_n$. From $d\varphi_\lambda X = re^{\theta J} X$ for all ∇-parallel vector fields it follows that $z^i \circ \varphi_\lambda = \lambda z^i + c(\lambda)$, where $c : \mathbb{C}^* \to \mathbb{C}^n$ is a smooth map. Since φ_λ is a local action, the map c must satisfy the functional equation

$$c(\lambda\mu) = \lambda c(\mu) + c(\lambda)$$

for all $\lambda, \mu \in \mathbb{C}^*$ near $1 \in \mathbb{C}^*$ and $c(1) = 0$. It is easy to see that this implies $c(\lambda) = (1 - \lambda) z_0$ for some constant vector $z_0 \in \mathbb{C}^n$. Up to adding (real) constants to the x^i, we can assume that the vector z_0 has purely imaginary components. Then changing the holomorphic extensions z^i by adding purely imaginary constants, we can arrange that $c = z_0 = 0$ and hence that $z^i \circ \varphi_\lambda = \lambda z^i$. Similarly, we can show that by adding constants one can arrange that $w_j \circ \varphi_\lambda = \lambda w_j$. This shows that a conic complex manifold admits a local holomorphic $\mathbb{C}^*$-action and local holomorphic functions with the properties (a) and (b). Next we prove the converse statement of (i). So let φ_λ be a local holomorphic $\mathbb{C}^*$-action on (M, J, ∇) and $z^1, \ldots, z^n, w_1, \ldots, w_n$ local holomorphic functions satisfying (a) and (b). From (a) and (b) it follows that $d\varphi_\lambda X = re^{\theta J} X$ for all ∇-parallel vector fields X, by differentiation. This shows that (M, J, ∇) is a conic complex manifold. $\square$

Next we are going to define the notion of projective special (complex, symplectic or Kähler) manifold. These manifolds arise as orbit spaces of conic special (complex, symplectic or Kähler) manifolds. Let φ_λ be a local holomorphic $\mathbb{C}^*$-action on a complex manifold M. To any point $p \in M$ we associate the holomorphic curve $\varphi(p) : \lambda \mapsto \varphi_\lambda(p)$ in M defined on an open neighbourhood of $1 \in \mathbb{C}^*$. If φ_λ is the local $\mathbb{C}^*$-action associated to a conic complex manifold then $\varphi(p)$ is an immersion and $\mathcal{D}_p := \varphi(p)_* T_1 \mathbb{C}^* \subset T_p M$ defines an integrable complex 1-dimensional holomorphic distribution on M. Its leaves are by definition the *orbits* of the local $\mathbb{C}^*$-action φ_λ. We denote by $\overline{M} = M/\mathbb{C}^*$ the set of orbits with the the quotient topology. $\overline{M}$ will be called the orbit space of M. If M is a conic (complex, symplectic or Kähler) manifold and the projection $M \to \overline{M}$

is a holomorphic submersion onto a Hausdorff complex manifold, then $\overline{M}$ is called a *projective special* (complex, symplectic or Kähler) *manifold.*

2.2 Conic special coordinates

Definition 4 *An affine local coordinate system* $(x, y) := (x^1, \ldots, x^n, y_1, \ldots, y_n)$ *on a conic complex manifold* (M, J, ∇) *with corresponding local* $\mathbb{C}^*$*-action* φ_λ *is called a* conic affine local coordinate system *if it admits a holomorphic extension* $(z, w) := (z^1, \ldots, z^n, w_1, \ldots, w_n)$ *such that locally* $(z, w) \circ \varphi_\lambda = \lambda(z, w)$*. Such a holomorphic extension is called a* conic holomorphic extension.

In view of Definition 4 we will freely speak of conic real special coordinate systems (x, y) on conic symplectic manifolds and of conic conjugate pairs of special coordinates (z, w) on conic Kähler manifolds. The following theorem is a corollary of Theorem 5.

Theorem 6 *(i) Any conic complex manifold admits a conic local affine coordinate system near any point* $p \in M$*. A conic local affine coordinate system is unique up to a linear transformation.*

(ii) Any conic symplectic manifold admits a conic real special coordinate system near any point $p \in M$*. A conic real special coordinate system is unique up to a linear symplectic transformation.*

(iii) Any conic Kähler manifold admits a conic conjugate pair of special coordinates. A conic conjugate pair of special coordinates is unique up to a (complex) linear symplectic transformation.

2.3 The extrinsic construction of conic and projective special manifolds

Let us consider the same fundamental data V, Ω and τ as in 1.3. On V we have the standard (global) holomorphic $\mathbb{C}^*$-action $\mathbb{C}^* \times V \ni (\lambda, v) \mapsto \lambda v \in V$. A holomorphic immersion ϕ of a complex manifold M into V is called *conic* if for every point $p \in M$ and every neighbourhood U of p there exist neighbourhoods U_1 of $1 \in \mathbb{C}^*$ and U_p of p such that $\lambda\phi(U_p) \subset \phi(U)$ for all $\lambda \in U_1$. Notice that we do not require the image $\phi(M)$ to be a complex cone, i.e. (globally) invariant under the $\mathbb{C}^*$-action on V.

Theorem 7 *Let* ϕ *be a conic totally complex holomorphic immersion of a complex manifold* (M, J) *into* $V = T^*\mathbb{C}^n$*,* $n = \dim_{\mathbb{C}} M$*,* ∇ *the induced connection and* $\omega = 2\phi^*(\sum dx^i \wedge dy_i)$ *the induced symplectic form. Then the following hold:*

(i) (M, J, ∇, ω) *is a conic symplectic manifold.*

(ii) The pull back via ϕ *of the functions* $(x^1 = \mathrm{Re}\, z^1, \ldots, x^n = \mathrm{Re}\, z^n, y_1 = \mathrm{Re}\, w_1, \ldots, y_n = \mathrm{Re}\, w_n)$ *of* V *defines a conic real special coordinate system around each point of* M*.*

Proof: Since ϕ is a conic holomorphic immersion, the holomorphic $\mathbb{C}^*$-action on V induces a local holomorphic $\mathbb{C}^*$-action φ_λ on M. One can easily check that φ_λ defines on (M, J, ∇, ω) the structure of a conic symplectic manifold with conic real special coordinates $x^1 \circ \phi, \ldots, x^n \circ \phi, y_1 \circ \phi, y_n \circ \phi$. $\square$

Theorem 8 *Let* ϕ *be a conic holomorphic nondegenerate Lagrangian immersion of a complex manifold* (M, J) *into* V *inducing the Kähler metric* g *on* M*. The immersion* ϕ *is totally complex and hence induces also the data* (∇, ω) *on* M*. Moreover, the following hold:*

(i) (M, J, ∇, ω) *is a conic Kähler manifold.*

(ii) ω *coincides with the Kähler form of* g*, i.e.* $\omega = g(\cdot, J\cdot)$*.*

(iii) The pull back via ϕ *of the canonical coordinates* $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ *of* V *defines a conic conjugate pair of special coordinates around each point of* M*.*

Proof: This follows from Theorem 3 and Theorem 7. $\square$

Now we will show that any simply connected conic (complex, symplectic or Kähler) manifold arises by the construction of Theorem 7 or Theorem 8.

Theorem 9 *(i) Let* (M, J, ∇) *be a simply connected conic complex manifold of complex dimension* n*. Then there exists a conic holomorphic totally complex immersion* $\phi : M \to V = T^*\mathbb{C}^n$ *inducing the connection* ∇ *on* M*. Moreover,* ϕ *is unique up to a linear transformation of* V *preserving the real structure* τ*. Here the real structure is considered as a (constant) field of antilinear involutions on the tangent spaces of* V*. Finally,* $\omega = 2\phi^*(\sum dx^i \wedge dy_i)$ *is a* ∇*-parallel symplectic structure defining on* (M, J, ∇) *the structure of conic symplectic manifold.*

(ii) Let (M, J, ∇, ω) *be a simply connected conic symplectic manifold of complex dimension* n*. Then there exists a conic holomorphic totally complex immersion* $\phi : M \to V = T^*\mathbb{C}^n$ *inducing the connection* ∇ *and the symplectic form* ω *on* M*. Moreover,* ϕ *is unique up to a linear transformation of* V *preserving the complex symplectic form* Ω *and the real structure* τ*.*

(iii) Let (M, J, ∇, ω) be a simply connected conic Kähler manifold of complex dimension n then there exists a conic holomorphic nondegenerate Lagrangian (and hence totally complex) immersion $\phi : M \to V = T^*\mathbb{C}^n$ inducing the Kähler metric g, the connection ∇ and the symplectic form $\omega = 2\phi^*(\sum dx^i \wedge dy_i) = g(\cdot, J\cdot)$ on M. Moreover ϕ is unique up to a linear transformation of V preserving the complex symplectic form Ω and the real structure τ. Here the real structure is considered as a field of antilinear involutions on the tangent spaces of V.

Proof: The proof is completely analogous to that of Theorem 4. To prove (ii), for instance, it is essentially sufficient to replace real special coordinates by conic real special coordinates in the proof of Theorem 4 (ii). □

We will call a holomorphic 1-form $\sum F_i dz^i$ on an open subset $U \subset \mathbb{C}^n$ *conic* if the corresponding holomorphic immersion $U \ni z \mapsto \sum F_i(z)dz^i \in T_z^*\mathbb{C}^n \subset T^*\mathbb{C}^n = V$ is conic. This is the case if and only if the functions F_i are locally homogeneous of degree one, i.e. if $F_i(\lambda z) = \lambda F_i(z)$ for all $z \in U$ and all λ near $1 \in \mathbb{C}^*$.

A holomorphic function F on U is called *conic* if its differential dF is conic. This is the case if and only if F is locally homogeneous of degree 2, i.e. if $F(\lambda z) = \lambda^2 F(z)$ for all $z \in U$ and all λ near $1 \in \mathbb{C}^*$.

We have the following analogues of Corollary 3 and Corollary 4.

Corollary 5 *Any conic regular local holomorphic* 1*-form ϕ on $\mathbb{C}^n$ defines a conic symplectic manifold of complex dimension n. Conversely, any conic symplectic manifold of complex dimension n can be locally obtained in this way.*

Corollary 6 *Any conic nondegenerate local holomorphic function on $\mathbb{C}^n$ defines a conic Kähler manifold of complex dimension n. Conversely, any conic Kähler manifold of complex dimension n can be locally obtained in this way.*

Remark 3: Let $\overline{M} = M/\mathbb{C}^*$ be a projective special (complex, symplectic or Kähler) manifold, with M simply connected. Then the holomorphic immersion $\phi : M \to V$ constructed in Theorem 9 induces a holomorphic immersion $\overline{\phi} : \overline{M} \to P(V)$ into the complex projective space of complex dimension $2n - 1$. The holomorphic immersion $\overline{\phi}$ is unique up to a projective transformation induced by a linear symplectic transformation of V preserving the real structure τ. To construct $\overline{\phi}$ it is sufficient to assume that $\overline{M}$ is simply connected.

3 Geometric structures on the cotangent bundle of special symplectic manifolds

In this section we prove that the cotangent bundle of a special symplectic manifold carries two canonical complex structures J_1, J_2. Moreover, if the (1,1)-part of the symplectic form ω is nondegenerate it also carries an almost hyper-Hermitian structure. This almost hyper-Hermitian structure is hyper-Kähler if and only if ω^{11} is parallel. If the $(2,0)$-part of ω is nondegenerate we obtain an almost para-hypercomplex structure. It is para-hypercomplex if and only if ω^{20} is parallel. This generalises the known construction of a hyper-Kähler metric on the cotangent bundle of a special Kähler manifold [CFG, C2, F, H].

Let M be a manifold and denote by $N = T^*M$ its cotangent bundle. A connection ∇ on M defines a decomposition

$$T_\xi N = \mathcal{H}^\nabla_\xi \oplus T^v_\xi N \cong T_pM \oplus T^*_pM \ , \ \xi \in N \ , \ p = \pi(\xi) \ , \tag{5}$$

where $\pi : N = T^*M \to M$, $T^v_\xi N$ is the vertical subspace and $\mathcal{H}^\nabla_\xi$ is the horizontal subspace defined by the connection ∇. Here we have a natural identification of $T^v_\xi N$ with T^*_pM and an identification of $\mathcal{H}^\nabla_\xi$ with T_pM defined by the projection π. If M is a complex manifold with complex structure J, then N carries a natural complex structure J_N. We note that the vertical subspace $T^v_\xi N$ is J_N-invariant, but the horizontal subspace $\mathcal{H}^\nabla_\xi$ is in general not. We denote by J^∇ the almost complex structure on N defined with respect to the decomposition (5) by

$$J^\nabla = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} \tag{6}$$

In general J^∇ is not integrable.

Proposition 8 *Let ∇ be a connection on a complex manifold (M, J). The horizontal distribution $\mathcal{H}^\nabla \subset TN$ is J_N-invariant if and only if there exists a torsionfree complex (i.e. $DJ = 0$) connection D on M such that the tensor field $A := \nabla - D$ satisfies the condition*

$$A^\xi_X \circ J = A^\xi_{JX} \quad \forall\ X \in TM\,, \tag{7}$$

where $A^\xi_X Y = \xi(A_X Y)$.

For the proof we need two lemmas. The first one is well known.

Lemma 4 *Let D and ∇ be connections on a manifold M and $A = \nabla - D$. Then the corresponding horizontal distributions $\mathcal{H}^D$ and $\mathcal{H}^\nabla$ are related by:*

$$\mathcal{H}^\nabla_\xi = A^\xi \mathcal{H}^D_\xi = \{\hat{v} = v + A^\xi_v | v \in \mathcal{H}^D_\xi \cong T_pM\} ,$$

where $\xi \in N = T^*M$ and $p = \pi(\xi)$.

Lemma 5 *Let D be a torsionfree complex connection on a complex manifold (M, J) then the horizontal distribution $\mathcal{H}^\nabla \subset TN$ is J_N-invariant and hence $J^D = J_N$.*

Proof: Let $(x^1, \dots, x^n, y^1, \dots, y^n, u_1, \dots, u_n, v_1, \dots, v_n)$ be the local coordinate system on $N = T^*M$ associated to a holomorphic local coordinate system $(z^1, \dots, z^n)$ on M, i.e. $z^i = x^i + \sqrt{-1}y^i$ and $\omega = \sum dx^i \wedge du_i + \sum dy^j \wedge dv_j$ is the canonical symplectic structure on N. Note that

$$T^vN = \mathrm{span}\left\{\frac{\partial}{\partial \mathrm{u}_1}, \dots \frac{\partial}{\partial \mathrm{u_n}}, \frac{\partial}{\partial \mathrm{v}_1}, \dots, \frac{\partial}{\partial \mathrm{v_n}}\right\} .$$

We denote by D the local connection on M with horizontal space

$$\mathcal{H}^D := \mathrm{span}\left\{\frac{\partial}{\partial \mathrm{x}^1}, \dots \frac{\partial}{\partial \mathrm{x^n}}, \frac{\partial}{\partial \mathrm{y}^1}, \dots, \frac{\partial}{\partial \mathrm{y^n}}\right\} .$$

This connection is flat and torsionfree, with affine local coordinates $x^1, \dots, x^n, y^1, \dots, y^n$. It is also complex because the complex structure J is constant in these coordinates:

$$J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i} \quad , \quad J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i} .$$

In terms of the induced coordinate system on N, J_N is given by

$$J_N\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^i} \quad , \quad J_N\frac{\partial}{\partial u_j} = -\frac{\partial}{\partial v_j}$$

$$J_N\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i} \quad , \quad J_N\frac{\partial}{\partial v_j} = \frac{\partial}{\partial u_j}$$

This clearly shows that $J^D = J_N$. Now let ∇ be any torsionfree complex connection on (M, J). This means that the $(1,2)$ tensor $A = \nabla - D$ is symmetric and J-linear, i.e.

$$A_XY = A_YX \quad , \quad [A_X\, , \, J] = 0 \quad \forall\, X, Y \in TM .$$

The latter equation can also be written in the form $J^*A^\xi_X = A^{J^*\xi}_X$ for all $\xi \in T^*M$. We claim that this implies the J_N-invariance of $\mathcal{H}^\nabla = A\mathcal{H}^D$. In fact we have

$$\begin{aligned} J_N\hat{v} &= J_N(v + A^\xi_v) = Jv + J^*A^\xi_v = Jv + A^{J^*\xi}_v \\ &= Jv + A^{J^*\xi}v = Jv + J^*(A^\xi)v = Jv + A^\xi Jv \\ &= Jv + A^\xi_{Jv} = \widehat{Jv} \quad \forall v \in \mathcal{H}^D_\xi \cong T_pM \ , \ p = \pi(\xi) \in M . \quad \square \end{aligned}$$

Proof (of Proposition 8): Let D be a torsionfree complex connection on M and ∇ a connection on M such that $A = \nabla - D$ satisfies (7). To prove that $\mathcal{H}^\nabla$ is J_N-invariant it suffices to check that $J_N\hat{v} = \widehat{Jv}$ for all $v \in \mathcal{H}^D_\xi \cong T_pM$. Using the identification (5) and the identity (7) we compute:

$$J_N\hat{v} = J_N(v + A^\xi_v) = Jv + J^*A^\xi_v = Jv + A^\xi_v \circ J = Jv + A^\xi_{Jv} = \widehat{Jv}\,.$$

Conversely, let ∇ be a connection on M such that $\mathcal{H}^\nabla$ is J_N-invariant. From the integrability of J it follows that there exists a torsionfree complex connection D on M. Now we check that $J_N\mathcal{H}^\nabla = \mathcal{H}^\nabla$ implies (7). For $\hat{v} = v + A^\xi_v \in \mathcal{H}^\nabla_\xi$, we have by Lemma 5: $J_N\hat{v} = Jv + J^*A^\xi_v$. This shows that $J_N\hat{v} \in \mathcal{H}^\nabla_\xi$ if and only if $J_N\hat{v} = \widehat{Jv}$. The latter equation is equivalent to $J^*A^\xi_v = A^\xi_{Jv}$, which is precisely (7). $\square$

Now let ω be a field of nondegenerate bilinear forms on a manifold M, considered as a map $TM \to T^*M$, and ∇ a connection on M. Using the identification (5) we define an almost complex structure J^ω on $N = T^*M$ by

$$J^\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{8}$$

Lemma 6 *If ∇ is flat and torsionfree and ω is ∇-parallel then J^ω is integrable.*

Proof: If we express J^ω in terms of the canonical coordinates on $N = T^*M$ induced by local affine coordinates on M, then it has constant coefficients. This shows that J^ω is integrable. $\square$

Theorem 10 *Let (M, J, ∇, ω) be a special symplectic manifold. Then the cotangent bundle $N = T^*M$ carries two natural complex structures*

$$J_1 = J^\nabla = \begin{pmatrix} J & 0 \\ 0 & J^* \end{pmatrix} \quad \textit{and} \quad J_2 = J^\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad .$$

The commutator and anticommutator of J_1 and J_2 are given by

$$[J_1, J_2] = 2J_1 \begin{pmatrix} 0 & -(\omega^{-1})^{11} \\ \omega^{11} & 0 \end{pmatrix} = -2 \begin{pmatrix} 0 & -(\omega^{-1})^{11} \\ \omega^{11} & 0 \end{pmatrix} J_1\,,$$

$$\{J_1, J_2\} = 2J_1 \begin{pmatrix} 0 & -(\omega^{-1})' \\ \omega' & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & -(\omega^{-1})' \\ \omega' & 0 \end{pmatrix} J_1\,,$$

where $\omega' = \omega^{20} + \omega^{02}$.

Proof: The integrability of J_2 follows from Lemma 6. To prove the integrability of J_1, by Proposition 8, it is sufficient to check the identity (7) for $A = \nabla - D = -\frac{1}{2}J\nabla J$, where $D = \frac{1}{2}(\nabla + \nabla^{(J)})$ is the torsionfree complex connection of Proposition 3. Using the fact that ∇J is symmetric we compute:

$$-2A_X \circ J = J(\nabla_X J) \circ J = \nabla_X J = (\nabla_. J)X = J(\nabla_. J)JX = J(\nabla_{JX} J) = -2A_{JX} \quad . \quad \square$$

Theorem 11 *Let (M, J, ∇, ω) be a special symplectic manifold.*

(i) Assume that ω^{11} is nondegenerate. Then the cotangent bundle $N = T^*M$ carries a canonical almost hyper-Hermitian structure $(J_1, J_2, J_3 = J_1J_2 = -J_2J_1, g_N)$ given by

$$J_1 = J^{\nabla}\,, \quad J_2 = J^{\omega^{11}}\,, \quad g_N = diag(g, g^{-1})\,,$$

where $g = \omega^{11}(J\cdot, \cdot)$ is the Kähler metric on M, see Proposition 5. J_1 is the standard (integrable) complex structure on the cotangent bundle of the complex manifold (M, J). The almost hyper-Hermitian manifold (M, J_1, J_2, J_3, g_N) is hyper-Hermitian (i.e. the almost complex structures J_1, J_2, J_3 are integrable) if and only if $\nabla\omega^{11} = 0$. In this case (M, J_1, J_2, J_3, g_N) is a hyper-Kähler manifold.

(ii) Assume that $\omega' = \omega^{20}+\omega^{02}$ is nondegenerate. Then the cotangent bundle $N = T^*M$ carries a canonical almost para-hypercomplex structure (J_1, J_2), i.e. a commuting pair of almost complex structures, given by

$$J_1 = J^{\nabla}\,, \quad J_2 = J^{\omega'}\,.$$

J_1 is again the standard (integrable) complex structure and (J_1, J_2) is an (integrable) para-hypercomplex structure (i.e. J_1 and J_2 is integrable) if and only if $\nabla\omega^{20} = 0$.

Note that in the second case $J_3 = J_1J_2$ is not an almost complex structure but an almost product structure, i.e. an involution.

Proof: Using the identities

$$J^* \circ \omega^{11} = -\omega^{11} \circ J\,, \quad J^* \circ \omega' = \omega' \circ J\,,$$

where the two-forms ω^{11} and ω' are considered as linear maps $TM \to T^*M$, one can check that J_1 and J_2 are anticommuting or commuting almost complex structures in case (i) and (ii) respectively. To check that g_N is Hermitian with respect to the almost complex structures (J_1, J_2, J_3) in case (i) we compute $\omega_\alpha := g_N \circ J_\alpha$ as follows:

$$\omega_1 = -\sum \omega_{ij} dq^i \wedge dq^j + \sum \omega^{ij} dp_i \wedge dp_j\,,$$

where $\omega^{11} = \sum \omega_{ij}(q)dq^i \wedge dq^j$ is the expression of the symplectic form ω^{11} in ∇-affine coordinates q^i on M, $(\omega^{ij}) = (\omega_{ij})^{-1}$ and the p_i are the conjugate momenta correponding to the q^i.

$$\omega_2 = -\sum (J^* dq^j) \wedge dp_j\,, \quad \omega_3 = \sum dq^j \wedge dp_j\,.$$

From these formulas we see that the ω_α are skew-symmetric and therefore that the J_α are g_N-orthogonal. This shows that (J_1, J_2, J_3, g_N) is an almost hyper-Hermitian structure. The form ω_3 is closed. The form ω_2 is closed since $dJ^*\eta = 0$ for any parallel 1-form η. The form ω_3 is closed if and only if the coefficients ω_{ij} are constant, i.e. if and only if ω^{11} is parallel. If this is the case, then the almost hyper-Hermitian structure (J_1, J_2, J_3, g_N) is hyper-Kähler by a Lemma of Hitchin.

Assume now that J_2 is integrable, i.e. the Nijenhuis tensor $N_{J_2} = 0$. A direct calculation shows that

$$J_2 N_{J_2}(\partial_{q^i}, \partial_{q^j}) = \sum_k (\rho_{jk,i} - \rho_{ik,j})\partial_{p_k}\,,$$

where $\rho_{ij}(q)$ are the coefficients of $\rho = \omega^{11}$ or ω' in cases (i) or (ii) respectively. Notice that $\rho_{ik,j} - \rho_{jk,i}$ are the coefficients of the 2-form $d(i_{\partial_{q^k}}\rho) = L_{\partial_{q^k}}\rho$. This shows that $N_{J_2}(\partial_{q^i}, \partial_{q^j}) = 0$ implies that Lie derivative of ρ in the direction of any parallel vector field on M vanishes and hence that ρ is parallel. $\square$

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