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Blowup in a chemotaxis model without symmetry assumptions

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Abstract: *In this paper we prove the existence of solutions of the so-called Keller-Segel model in chemotaxis, which blow up in finite or infinite time. This is done without assuming any symmetry properties of the solution.*

Keywords: *Chemotaxis, Keller-Segel model, blowup, nonlocal nonlinear elliptic boundary value problems, Neumann problem, Pohozaev's identity*

Mathematical subject classification numbers: 35J25, 35J60, 35K20, 35K55, 35K57, 49R99, 92B05, 92D25

1. Introduction

The collective behaviour of the myxamoebae of the cellular slime mold *Dictyostelium discoideum* has astonished many scientists since Dictyostelium was found in 1935. During its life cycle a Dictyostelium myxamoebae population grows by cell division as long as there is enough food. After the food resources are exhausted the myxamoebae spread over the whole domain that they can reach. Then a so-called founder cell starts to separate cyclic Adenosine Monophosphate (cAMP) which attracts the starving myxamoebae. They start to move chemotactically positive in direction of the founder cell and are also stimulated to separate cAMP. During this process the myxamoebae not only produce cAMP but also consume it and secrete a phosphodisterase, which converts the cAMP into chemotactically inactive AMP. According to this chemotactically positive movement to the founder cell the myxamoebae aggregate. At the end point of aggregation the myamoebae form a pseudoplasmoid, where every myxamoebae maintains its individual integrity. This pseudoplasmoid moves phototactically positive towards light. Finally a fruiting body is formed and spores are spread. When the spores become myxamoebae the life cycle is closed.

Since 1970 when E.F. Keller and L.A. Segel introduced their model for the aggregation of Dictyostelium discoideum, which is given in a simplified version by the equations

$$\left.\begin{array}{rcll} a_t & = & \nabla(\nabla a - \tilde{\chi} a \nabla c), & x \in \Omega,\ t > 0 \\ c_t & = & k_c \Delta c - \gamma c + \tilde{\alpha} a, & x \in \Omega,\ t > 0 \\ \frac{\partial a}{\partial n} & = & \frac{\partial c}{\partial n} \ = \ 0, & x \in \partial\Omega,\ t > 0 \\ a(0,x) & = & a_0(x),\ c(0,x) \ = \ c_0(x), & x \in \Omega, \end{array}\right\} \tag{1}$$

many authors were interested in the possible blowup of the solution of system (1). Here and in the following sections of the present paper n denotes the outer normal vector field on $\partial\Omega$.

In (1) the function $a(x,t)$ represents the Dictyostelium myxamoebae density in point $x \in \Omega$ at time t and the function $c(x,t)$ stands for the cAMP density, which attracts the myxamoebae to move positive chemotactically in direction of a higher cAMP concentration. $\tilde{\alpha}$, $\tilde{\chi}$, k_c and γ denote positive constants. For a detailed derivation of the equations see for instance [11, 14] or [18].

That there might exist solutions which blow up for $\Omega \subset \mathbb{R}^2$ has been expected in connection with the studies concerning the conjectures by V. Nanjundiah in [18] and by S. Childress and J.K. Percus in [4, 5], which say the following:

V. Nanjundiah [18] suggested in 1973 that "*the end-point (in time) of aggregation is such that the cells are distributed in form of $\delta-$function concentration*" (see [18, p. 102]).

S. Childress and J.K. Percus formulated in [4, 5] the following statement for space dimension $N = 2$:

- The myxamoebae density cannot form a $\delta-$function singularity, if the total myxamoebae density on $\Omega \subset \mathbb{R}^2$ is less than a critical number d_Ω.
- The myxamoebae density can form a $\delta-$function singularity, if the total myxamoebae density on Ω is larger than a critical value D_Ω.

In the following years one was led to believe that the equality $d_\Omega = D_\Omega$ should hold for the critical values mentioned in the conjecture.

If one uses the transformation

$$A(t,x) = \frac{|\Omega| a(t,x)}{\int\limits_\Omega a_0(x)dx},\ C(t,x) = \tilde{\chi}\left(c(t,x) - \frac{1}{|\Omega|}\int\limits_\Omega c(t,x)dx\right) \tag{2}$$

(see also [12, 14] and [18]) and the notation $\alpha\chi$ instead of $\tilde{\alpha}\tilde{\chi}\int\limits_\Omega a(x,t)dx/|\Omega|$ we get a transformed version of the Keller-Segel model. This transformed system is given by

$$\left.\begin{array}{rcll} A_t & = & \nabla\cdot(\nabla A - A\nabla C), & x\in\Omega, \quad t>0 \\ C_t & = & k_c\Delta C - \gamma C + \alpha\chi(A-1), & x\in\Omega, \quad t>0 \\ \frac{\partial A}{\partial n} & = & \frac{\partial C}{\partial n} \;=\; 0, & x\in\partial\Omega, \quad t>0 \\ A(0,x) & = & A_0(x) \;>\; 0,\; C(0,x) \;=\; C_0(x), & x\in\Omega \\ \int\limits_\Omega A_0(x)\;dx & = & |\Omega|,\; \int\limits_\Omega C(t,x) \;=\; 0, & t\geq 0. \end{array}\right\} \qquad (3)$$

In the present paper we will study the possibility that solutions of system (3) might blow up.

For the sake of clarity we give the definition of solutions of system (3), which we will refer as blowup-solutions.

Definition 1 *We say that a solution of* (3) *blows up or is a blowup-solution of* (3)*, provided there is a time $T_{max}\leq\infty$ such that*

$$\limsup_{t\to T_{max}}||A(x,t)||_{L^\infty(\Omega)}=\infty \text{ or } \limsup_{t\to T_{max}}||C^+(x,t)||_{L^\infty(\Omega)}=\infty$$

where $C^+(x,t)$ denotes the positive part of the function $C(x,t)$. If $T_{max}<\infty$ we say that the solution of (3) *blows up in finite time and if $T_{max}=\infty$ we will call it blowup in infinite time.*

Up to now the existence of blowup-solutions of system (3) is only known under a radially symmetry assumption on the solution (see [9, 10] and [13] for existence results of blowup-solutions of system (3) in the radially symmetric case).

In this article we will prove the existence of blowup solutions of (3) for a smooth domain $\Omega\subset\mathbb{R}^2$ provided $4\pi k_c<\alpha\chi|\Omega|$ and $\alpha\chi|\Omega|/k_c\neq 4\pi m$, where $m\in\mathbb{N}$. The proof will be based on the same idea that has been used in [13] to prove the existence of blowup-solutions in the radially symmetric setting of (3) with $\gamma=0$ and a generalization of results by Brézis-Merle [1] and Li-Shafrir [16] which has been done in [22].

2. A summarizing section

In 1998 H. Gajewski and K. Zacharias proved the local existence of a weak solution of (3), where the definition of a weak solution is given as follows:

Definition 2 [8] *A pair of functions $(A(t,x),C(t,x))$ with*

$$\begin{array}{ll} A\in L^\infty(0,T;L^\infty_+(\Omega))\cap L^2(0,T;H^1(\Omega)), & A_t\in L^2(0,T;(H^1(\Omega))^*), \\ C\in L^\infty(0,T;L^\infty(\Omega))\cap C(0,T;H^1(\Omega)), & C_t\in L^2(0,T;L^2(\Omega)) \end{array}$$

is called a weak solution of (3) *if for all* $h \in L^2(0,T;H^1(\Omega))$ *the following identities hold:*

$$
\begin{aligned}
0 &= \int_0^T \langle A_t, h\rangle \, dt + \int_0^T \int_\Omega (\nabla A - A\nabla C)\cdot \nabla h \, dx \, dt, \\
0 &= \int_0^T \int_\Omega C_t h \, dx \, dt + \int_0^T \int_\Omega (k_c \nabla C \cdot \nabla h + (\gamma C - \alpha\chi(A-1))\cdot h) \, dx \, dt.
\end{aligned}
$$

Using the Lyapunov function

$$
\begin{aligned}
F(A(t),C(t)) &= \int_\Omega \frac{1}{2\alpha\chi}\left(k_c|\nabla C(t)|^2 + \gamma C(t)^2\right) + A(t)(\log A(t) - 1) + 1dx \\
&\quad - \int_\Omega (A(t)-1)C(t)dx
\end{aligned}
$$

and the lower estimate

$$
\begin{aligned}
F(A(t),C(t)) \;\geq\; \mathcal{F}(C(t)) &= \frac{1}{2\alpha\chi}\int_\Omega k_c|\nabla C(t)|^2 + \gamma C(t)^2 \, dx \\
&\quad -|\Omega|\log\left(\frac{1}{|\Omega|}\int_\Omega e^{C(t)}dx\right) \qquad (4)
\end{aligned}
$$

for $t \geq 0$, it is possible to show for a smooth domain $\Omega \subset \mathbb{R}^2$ that the Lyapunov function is bounded from below, provided

$$\frac{\alpha\chi|\Omega|}{4k_c\pi} < 1.$$

This fact is a simple consequence from a Moser-Trudinger type inequality by S.-Y.A. Chang and P. Yang [2, Proposition 2.3].

Remark 1 *If the boundary of* Ω *is piecewise* C^2 *then one can bound the Lyapunov function from below provided*

$$\frac{\alpha\chi|\Omega|}{4k_c\Theta} < 1,$$

where Θ *denotes the smallest interior angle of* $\partial\Omega$.

It results from the studies done by Nagai, Senba and Yoshida in [17] that in such a case the L^∞-norm of $A(x,t)$ and $C(x,t)$ remains uniformly bounded for all $t \geq 0$. Gajewski and Zacharias show in [8] that in this case the solution converges at least for subsequences $(t_k)_{k\in\mathbb{N}}$ with $t_k \to \infty$ to a stationary solution of (3). In [12] it was shown that this statement is in fact true for $t \to \infty$.

So we know from [8, 12] that for $\alpha\chi|\Omega| < 4k_c\pi$:

$$A(t) \to A^*$$

in $L^2(\Omega)$ and

$$C(t) \rightharpoonup C^*$$

in $H^1(\Omega)$ as $t \to \infty$, where

$$A^* = \frac{|\Omega| e^{C^*}}{\int\limits_\Omega e^{C^*} dx}$$

and C^* solves the nonlocal elliptic boundary value problem

$$\left.\begin{array}{rcll} -k_c\Delta v + \gamma v & = & \alpha\chi\left(\frac{|\Omega| e^v}{\int\limits_\Omega e^v\, dx} - 1\right), & \text{in } \Omega \\ \frac{\partial v}{\partial n} & = & 0, & \text{on } \partial\Omega. \end{array}\right\} \tag{5}$$

3. Existence of blowup-solutions for $\alpha\chi|\Omega| > 4k_c\pi$ and $\alpha\chi|\Omega|/k_c$ not equal to a multiple of an integer of 4π.

In the case where $\alpha\chi|\Omega| > 4k_c\pi$ we know from Lemma 2 in [12] or Lemma 2.2 in [22] that there exists a sequence $(v_\varepsilon)_{\varepsilon>0} \subset \mathcal{D} \equiv \{v \in H^1(\Omega) \mid \int\limits_\Omega v\, dx = 0\}$ such that

$$\mathcal{F}(v_\varepsilon) \to -\infty \tag{6}$$

and

$$||\nabla v_\varepsilon||_{L^2(\Omega)} \to \infty \tag{7}$$

as $\varepsilon \to 0$. A consequence of these observations is that the Lyapunov function $F(A(t), C(t))$ might become unbounded from below as $t \to T_{max}$. If we now can find a constant $\hat{K}$ such that

$$\mathcal{F}(v) > \hat{K} \tag{8}$$

holds true for all solutions of (5), we can construct initial data for (3), for which the corresponding solution of (3) has to blow up in finite or infinite time. For the radially symmetric case of (3) (with $\gamma = 0$) this was possible for $\alpha\chi|\Omega| > 8k_c\pi$ and $\alpha\chi|\Omega|/k_c$ not equal $8\pi m$ for $m \in \mathbb{N}$ (see [13] for details).

That there are nontrivial solutions of (5) has been proved independently in [12] and [22]. We will use results similar to those used in [22, Section 3] to show the existence of a constant $\hat{K}$ such that (8) holds true for all solutions of (5), provided $\alpha\chi|\Omega|/k_c$ is not equal to $4\pi m$, $m \in \mathbb{N}$.

This claim will be shown by contradiction. Therefore let $\alpha\chi|\Omega|/k_c > 4\pi$ and not be equal to $4\pi m$, $m \in \mathbb{N}$. If there is no constant $\hat{K}$ such that (8) holds true, then there exists a sequence $(v_k)_{k\in\mathbb{N}}$ of solutions of (5) such that

$$||\nabla v_k||_{L^2(\Omega)} \rightarrow \infty, \tag{9}$$

$$\int_\Omega e^{v_k} dx \rightarrow \infty \tag{10}$$

and

$$\max_{x\in\overline{\Omega}} v_k(x) \rightarrow \infty \tag{11}$$

as $k \to \infty$. If (11) does not hold, we get a uniform L^∞-bound of the righthand side of (5) for all k, which gives us the existence of the constant $\hat{K}$.

We now use the transformation

$$u_k = v_k + \frac{\alpha\chi}{\gamma}.$$

So each u_k solves the problem

$$\left.\begin{array}{rcll} -\Delta u_k + \frac{\gamma}{k_c} u_k & = & \mu_k e^{u_k}, & \text{in } \Omega \\ \frac{\partial u_k}{\partial n} & = & 0, & \text{on } \partial\Omega \\ \int_\Omega u_k dx & = & \frac{\alpha\chi|\Omega|}{\gamma}, & \end{array}\right\} \tag{12}$$

where

$$\mu_k = \frac{\alpha\chi|\Omega|}{k_c \int_\Omega e^{u_k} dx} \tag{13}$$

and $\mu_k \to 0$ as $k \to \infty$.

According to the maximum principle for elliptic operators we notice that $u_k > 0$ in Ω. In the following we will show in the same way as it has been done in [22] that the $(u_k)_{k\in\mathbb{N}}$ contain a subsequence (for the sake of simplicity again denoted by $(u_k)_{k\in\mathbb{N}}$) such that

$$\mu_k \int_\Omega e^{u_k} dx \rightarrow 4\pi m \tag{14}$$

for some integer m as $k \to \infty$.
But this would contradict the fact that $\alpha\chi|\Omega|/k_c \neq 4\pi m$, $m \in \mathbb{N}$.

To show (14) we make use of the following lemma:

Lemma 1 [3] *Let* $L = \sum\limits_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ *be a uniformly elliptic operator, namely*

$$\nu_0 I \leq (a_{ij})_{1 \leq i,j \leq 2} \leq \nu_1 I.$$

Then there exists a constant $\beta = \beta(\nu_0, \nu_1)$ *such that for any solution* u *of the following problem*

$$Lu = f(x) \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega$$

we have

$$\int\limits_{\Omega} \exp\left(\frac{\beta |u(x)|}{||f||_{L^1(\Omega)}}\right) dx \leq K.$$

Let us define the following set:

$$\mathcal{BS} \equiv \left\{ x \in \overline{\Omega} \;\middle|\; \begin{array}{l} \text{there exists a sequence } \mu_k \to 0, \text{ with corresponding} \\ \text{solutions } u_k \text{ of } (12), \text{ and a sequence } (x_k)_{k \in \mathbb{N}} (x_k \in \overline{\Omega}), \\ \text{such that } u_k(x_k) \to \infty, \; x_k \to x \text{ as } k \to \infty \end{array} \right\} \tag{15}$$

By our assumption $\mathcal{BS} \neq \emptyset$ holds true.

We now set

$$\Sigma_k \equiv \int\limits_{\Omega} \mu_k e^{u_k} dx \;\; \left(= \frac{\alpha \chi |\Omega|}{k_c}\right).$$

Since

$$\int\limits_{\Omega} \frac{\mu_k e^{u_k}}{\Sigma_k} dx = 1$$

for all k, we can extract a subsequence of the u_k (still denoted by u_k as mentioned above) such that there exists a finite measure μ in the set of all real bounded Borel measures on $\overline{\Omega}$ (denoted by $\mathcal{M}(\overline{\Omega})$), such that

$$\int\limits_{\Omega} \frac{\mu_k e^{u_k}}{\Sigma_k} \varphi \; dx \to \int\limits_{\Omega} \varphi \; d\mu \tag{16}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ as $k \to \infty$.

For each boundary point $x_0 \in \partial\Omega$ we can strengthen the boundary (see [22] and [19] for more details about this fact) and the Laplacian becomes

$$L_{x_0} + \sum_{k=1}^{2} b_l \frac{\partial}{\partial x_l}$$

with a uniformly elliptic operator L_{x_0} and $|b_l| \leq C = const$. Using the compactness of the boundary we can choose a uniform $\beta = \beta_0$ in Lemma 1 for all L_{x_0}, $x_0 \in \partial\Omega$. Now we define δ-regular points of $\overline{\Omega}$.

Definition 3 *For any $\delta > 0$, we call $x_0 \in \overline{\Omega}$ a δ-regular point if there is a function $\varphi \in C_0^\infty(\mathbb{R}^2)$, $0 \le \varphi \le 1$, with $\varphi = 1$ in a neighbourhood of x_0 such that*

$$\int_\Omega \varphi \, d\mu < \frac{\beta_0}{1+3\delta}. \tag{17}$$

We also define the set $\Sigma(\delta)$ of all points in $\overline{\Omega}$ which are not δ-regular:

$$\Sigma(\delta) \equiv \{x_0 \in \overline{\Omega} \mid x_0 \text{ is not a } \delta-\text{regular point}\} \tag{18}$$

We remark the following:

Lemma 2 *For any $1 < q < 2$, there is a constant C_q independent of k such that $\|\nabla u_k\|_q \le C_q$.*

Proof:
Let $q' = \frac{q}{q-1} > 2$. We know

$$\|\nabla u_k\|_q \le \sup \left\{ \, |\int_\Omega \nabla u_k \cdot \nabla \varphi \, dx| \mid \varphi \in L_1^{q'}(\Omega), \int_\Omega \varphi \, dx = 0, \|\varphi\|_{L_1^{q'}(\Omega)} = 1 \right\}.$$

By the Sobolev embedding theorem we have

$$\|\varphi\|_{L^\infty(\Omega)} \le C_1.$$

It is clear that

$$\begin{aligned} |\int_\Omega \nabla u_k \cdot \nabla \varphi \, dx| &= |\int_\Omega \Delta u_k \varphi \, dx| \\ &= |\int_\Omega \left(\frac{\gamma}{k_c} u_k - \mu_k e^{u_k} \right) \varphi \, dx| \\ &\le C_1 \int_\Omega (u_k + \mu_k e^{u_k}) dx \\ &\le C_2. \end{aligned}$$

Here we have used the fact that $u_k > 0$. □

Now we can use exactly the same arguments as in [22, Proof of Lemma 3.2 and Proof of Lemma 3.3] and [1] to see that

1. if x_0 is a δ-regular point, then $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\overline{\Omega} \cap B_{R_0}(x_0))$, where $B_{R_0}(x_0)$ denotes a ball with radius R_0 centered in x_0.

2. $\mathcal{BS} = \Sigma(\delta)$ for any $\delta > 0$.

These two statements imply that

$$1 \le card(\mathcal{BS}) < \infty.$$

Let $\mathcal{BS} = \{P_1, ..., P_N\}$. We decompose $\mathcal{BS}$ into a boundary blowup set $\mathcal{BS}_{\partial\Omega} = \mathcal{BS} \cap \partial\Omega$ and an interior blowup set $\mathcal{BS}_\Omega = \mathcal{BS} \cap \Omega$. For a small constant $r > 0$ we set

$$\sigma_j^k(r) = \int\limits_{B_r(P_j)} \mu_k e^{u_k} dx.$$

We now see that for all small r the following equality holds true:

$$\lim_{k\to\infty} \int\limits_\Omega \mu_k e^{u_k} dx = \sum_{j=1}^N \lim_{k\to\infty} \sigma_j^k(r). \tag{19}$$

This implies the equality of

$$\lim_{k\to\infty} \int\limits_\Omega \mu_k e^{u_k} dx = \sum_{j=1}^N \lim_{r\to 0} \lim_{k\to\infty} \sigma_j^k(r), \tag{20}$$

which would give us (14) and thus a contradiction to the value of $\alpha\chi|\Omega|/k_c$ provided

$$\lim_{r\to 0} \lim_{k\to\infty} \sigma_j^k(r) = 4q\pi \tag{21}$$

for some $q \in \mathbb{N}$. But this is true as one can see in the following lemma, which is similar to Lemma 3.4 in [22].

Lemma 3 *Suppose* $P_j \in \mathcal{BS}_{\partial\Omega}$ *then* $\lim\limits_{r\to 0} \lim\limits_{k\to\infty} \sigma_j^k(r) = 4\pi$. *If* $P_j \in \mathcal{BS}_\Omega$ *then* $\lim\limits_{r\to 0} \lim\limits_{k\to\infty} \sigma_j^k(r) = 8\pi$.

Proof:
We first prove the case when $P \in \mathcal{BS}_{\partial\Omega}$. Recall that the Pohozaev identity for a function u satisfying

$$\Delta u - \beta u + f(u) = 0, \text{ in } U \subset R^2$$

is given by

$$\int\limits_U (-\beta u^2 + 2F(u))dx = \int\limits_{\partial U} \left[(x \cdot \nabla u)\frac{\partial u}{\partial n} - (x \cdot n)\frac{|\nabla u|^2}{2}\right] dS$$

$$+ \int\limits_{\partial U} (x \cdot n) \left(-\beta \frac{u^2}{2} + F(u)\right) dS, \tag{22}$$

where $F(u) = \int\limits_0^u f(s)ds$.

Let $f(u) = \mu_k e^u$ and $\beta = \frac{\gamma}{k_c}$. We may assume without loss of generality that $P = 0$. Now we set $U_r = B_r(0) \cap \bar{\Omega}$ and consider the function w_k which is a solution of the following problem

$$\left.\begin{array}{rcll} \Delta w - \beta w & = & 0 & \text{in } U_r, \\ \frac{\partial w}{\partial n} & = & \frac{\partial u_k}{\partial n} & \text{on } \partial U_r. \end{array}\right\} \tag{23}$$

It is easy to see that $w_k = O(1)$ in $C^2(U_r)$ since $|\frac{\partial u_k}{\partial n}| \le C$ on ∂U_r.

If we put $h_k = (u_k - w_k)/(\sigma_j^k(r))$, we have that $h_k \to G(\cdot, 0)$ in $C^2_{loc}(B_r(0) \cap \bar{\Omega} \setminus \{0\})$, where $G(\cdot, 0)$ satisfies

$$-\Delta G + \beta G = \delta_0 \;\text{ in } U_r, \frac{\partial G}{\partial n} = 0 \;\text{ on } \partial U_r.$$

See a proof in [7] for this claim. By potential theory, it is easy to see that for $|x|$ small

$$G(\cdot, 0) = -\frac{1}{\pi} \log |x| + O(1).$$

Hence we have

$$u_k = -\frac{\sigma_j^k(r)}{\pi} \log |x| + O(1)$$

in $C^1(\partial U_r)$ (here $O(1)$ may depend on r but is uniform in k).

By Pohozaev's identity we have

$$\int\limits_{U_r} (-\beta u_k^2 + 2\mu_k e^{u_k} - 2\mu_k) dx$$

$$=$$

$$\int\limits_{\partial U_r} \left[(x \cdot \nabla u_k) \frac{\partial u_k}{\partial n} - (x \cdot n) \frac{|\nabla u_k|^2}{2} + (x \cdot n) \left(-\beta \frac{u_k^2}{2} + \mu_k e^{u_k} - \mu_k \right) \right] dS. \tag{24}$$

We now estimate each term on both sides of (24):

$$\begin{array}{rcl} \int\limits_{U_r} u_k^2 dx & = & O(r^{1/2} \|u_k\|_{L^4(U_r)}) = O(r^{1/2} \|u_k\|_{W^{1,3/2}(\Omega)}) = O(r^{1/2}), \\ \int\limits_{U_r} 2\mu_k e^{u_k} dx & = & 2\sigma_j^k(r) + O(\mu_k), \end{array}$$

$$\begin{aligned}
\int\limits_{U_r} 2\mu_k dx &= O(\mu_k r^2), \\
\int\limits_{\partial U_r} (x\cdot\nabla u_k)\frac{\partial u_k}{\partial n} dS &= \left(\frac{\sigma_j^k(r)}{\pi}\right)^2 \int\limits_{\partial U_r} \left(\frac{(x\cdot n)}{|x|^2} + O(1)\right) \\
&= \left(\frac{\sigma_j^k(r)}{\pi}\right)^2 (\pi + O(r)), \\
\int\limits_{\partial U_r} (x\cdot n)\frac{|\nabla u_k|^2}{2} dS &= \left(\frac{\sigma_j^k(r)}{\pi}\right)^2 \left(\frac{\pi}{2} + O(r)\right), \\
\int\limits_{\partial U_r} u_k^2 dS &= O(r), \\
\int\limits_{\partial U_r} (x\cdot n)\mu_k e^{u_k} dS &= O(\mu_k \max_{x\in\partial U_r} e^{u_k}) = O(\mu_k), \\
\int\limits_{\partial U_r} (x\cdot n)\mu_k dS &= O(\mu_k r).
\end{aligned}$$

Here we used the statement of Lemma 2.

Now let $k \to +\infty$ first and then $r \to 0$. We see that

$$2\lim_{r\to 0}\lim_{k\to\infty} \sigma_j^k(r) = \frac{1}{\pi^2}\frac{\pi}{2}(\lim_{r\to 0}\lim_{k\to+\infty}\sigma_j^k(r))^2,$$

$$\lim_{r\to 0}\lim_{k\to+\infty}\sigma_j^k(r) = 4\pi.$$

The case when $P \in \mathcal{BS}_\Omega$ can be proved similarly. For convenience, we give a sketch of the proof. Instead of (23), in this case we consider w_k satisfying

$$\left.\begin{aligned}
\Delta w - \beta w &= 0 \quad \text{in } U_r, \\
w &= u_k \quad \text{on } \partial U_r.
\end{aligned}\right\} \tag{25}$$

We put $h_k = (u_k - w_k)/(\sigma_j^k(r))$ and assume that $P = 0 \in \Omega$. Similarly, $h_k \to G(\cdot, 0)$ in $C^2_{loc}(B_r(0)/\{0\})$, where G now is a Green function with Dirichlet boundary data:

$$-\Delta G + \beta G = \delta_0 \ \text{ in } B_r,\ G = 0 \ \text{ on } \partial U_r.$$

In this case, the Green function has following expansion near 0:

$$G(\cdot, 0) = -\frac{1}{2\pi}\log|x| + O(1).$$

We obtain the same estimates as in the first case when $P \in \mathcal{BS}_{\partial\Omega}$ except

$$\begin{aligned}
\int\limits_{\partial U_r} (x \cdot \nabla u_k) \frac{\partial u_k}{\partial n} dS &= \left(\frac{\sigma_j^k(r)}{2\pi} \right)^2 \int\limits_{\partial U_r} \left(\frac{(x \cdot n)}{|x|^2} + O(1) \right) \\
&= \left(\frac{\sigma_j^k(r)}{2\pi} \right)^2 (2\pi + O(r)), \\
\int\limits_{\partial U_r} (x \cdot n) \frac{|\nabla u_k|^2}{2} dS &= \left(\frac{\sigma_j^k(r)}{2\pi} \right)^2 (\pi + O(r)).
\end{aligned}$$

Now applying Pohozaev's identity again, we have in this case

$$2 \lim_{r \to 0} \lim_{k \to \infty} \sigma_j^k(r) = \frac{1}{4\pi^2} \pi (\lim_{r \to 0} \lim_{k \to +\infty} \sigma_j^k(r))^2,$$

$$\lim_{r \to 0} \lim_{k \to +\infty} \sigma_j^k(r) = 8\pi \text{ for } P \in \mathcal{BS}_\Omega.$$

This completes the proof. □

From Lemma 3, we get the following lemma:

Lemma 4 *Suppose $\alpha\chi|\Omega|/4k_c\pi > 1$ and $\alpha\chi|\Omega|/k_c \neq 4\pi m$ for $m \in \mathbb{N}$, then there exists a constant $\hat{K} \in \mathbb{R}$ ($\hat{K} \leq 0$), such that for all solutions v of* (5)

$$\mathcal{F}(v) \geq \hat{K} > -\infty$$

holds.

A direct consequence of this lemma, (6) and (7) is the following theorem, which also collects some known facts concerning blowup-solutions:

Theorem 1 *Let $\Omega \subset \mathbb{R}^2$ be a smooth domain and let $\hat{K}$ denote the constant from Lemma 4. Furthermore assume that $4k_c\pi < \alpha\chi|\Omega|$ and that*

$$\frac{\alpha\chi|\Omega|}{k_c} \neq 4\pi m$$

for $m \in \mathbb{N}$, then there exist initial data (A_0, C_0), such that

$$\hat{K} > F(A_0, C_0)$$

and the corresponding solution of (3) *blows up in finite or infinite time. For these blowup-solutions the following statements hold true:*

1. $\lim_{t \to T_{max}} ||A(x,t)||_{L^2(\Omega)} = \infty$

2. $\lim\limits_{t\to T_{max}} \int\limits_\Omega A(x,t)C(x,t)\ dx = \infty$

3. $\lim\limits_{t\to T_{max}} ||\nabla C(x,t)||_{L^2(\Omega)} = \infty$

4. $\lim\limits_{t\to T_{max}} \int\limits_\Omega e^{C(x,t)}\ dx = \infty$

5. $\lim\limits_{t\to T_{max}} ||A(x,t)||_{L^\infty(\Omega)} = \lim\limits_{t\to T_{max}} ||C(x,t)||_{L^\infty(\Omega)} = \infty$

6. If $4\pi k_c < \alpha\chi|\Omega| < 8\pi k_c$ *and* Ω *is a simply connected domain, then*

$$\lim_{t\to T_{max}} \int\limits_{\partial\Omega} e^{C(x,t)/2} dS = \infty.$$

Proof:
The existence of a blowup-solution follows from Lemma 4, (6) and (7).

The statements of the theorem can be shown by using the Lyapunov function $F(A(x,t), C(x,t))$. We know that from Proposition 2 in [12] that *3.* and *4.* are true for a blowup-solution of system (3). Furthermore we see by the properties of $F(A, C)$ that

$$\begin{aligned} \hat{K} &\geq F(A_0(x), C_0(x)) \\ &\geq F(A(x,t), C(x,t)) \end{aligned}$$

and thus

$$\frac{1}{2\alpha\chi} \int\limits_\Omega k_c|\nabla C(x,t)|^2 + \gamma C(x,t)^2 dx \leq \int\limits_\Omega (A(x,t) - 1)C(x,t)dx + \hat{K} \tag{26}$$

holds true. This inequality gives us statement *2.* and using Cauchy's inequality we also derive *1.* Statement *5.* is a direct consequence of *1.* and *4.*

We still have to show the last statement of the theorem. Therefore we remark the following. Using Lemma 3 in [12] we can estimate the Lyapunov function $F(A, C)$ for an arbitrary but fixed $p \in (1, 8\pi k_c/\alpha\chi|\Omega|)$ from below by

$$\begin{aligned} F(A(t), C(t)) &\geq \frac{1}{2\alpha\chi} \int\limits_\Omega k_c|\nabla C(t)|^2 + \gamma C^2(t)\ dx - |\Omega| \log\left(\frac{1}{|\Omega|}\int\limits_\Omega e^{C(t)} dx\right) \\ &\geq \int\limits_\Omega \left(\frac{k_c}{2\alpha\chi} - \frac{p|\Omega|}{16\pi}\right) |\nabla C(t)|^2 + \frac{\gamma}{2\alpha\chi} C^2(t)\ dx \\ &\quad - \frac{2|\Omega|}{q} \log\left(\int\limits_{\partial\Omega} e^{qC(t)/2}\ dS\right) + K_1(p, q, \alpha\chi, k_c, |\Omega|) \end{aligned}$$

where $q = p/(p-1)$ and $K_1(p, q, \alpha\chi, k_c, |\Omega|)$ is a constant depending on the parameters in the brackets. So in view of *3.* we get that

$$\lim_{t \to T_{max}} \int\limits_{\partial\Omega} e^{qC(x,t)/2} dS = \infty$$

for every $q \in (8\pi k_c/(8\pi k_c - \alpha\chi|\Omega|), \infty)$. But it is possible to improve this result.

Independently from [12] Senba and Suzuki improved the statement of [12, Lemma 3] and showed in [20, Proposition 2] that for $v \in H^1(\Omega)$ and a simply connected, smooth domain $\Omega \subset \mathbb{R}^2$ the following estimate holds:

$$\begin{aligned} \log\left(\frac{1}{|\Omega|}\int\limits_{\Omega} e^v dx\right) &\leq \frac{1}{16\pi}\int\limits_{\Omega} |\nabla v|^2 dx + \frac{1}{2|\partial\Omega|}\int\limits_{\partial\Omega} v\ dS \\ &\quad + \log\left(\frac{1}{|\partial\Omega|}\int\limits_{\partial\Omega} e^{v/2} dS\right) + K \end{aligned} \tag{27}$$

Here K is an absolute constant. Using this inequality instead of [12, Lemma 3] we get the estimate

$$\begin{aligned} F(A(t), C(t)) &\geq \frac{1}{2\alpha\chi}\int\limits_{\Omega} k_c|\nabla C(t)|^2 + \gamma C^2(t)\ dx - |\Omega| \log\left(\frac{1}{|\Omega|}\int\limits_{\Omega} e^{C(t)} dx\right) \\ &\geq \int\limits_{\Omega} \left(\frac{k_c}{2\alpha\chi} - \frac{|\Omega|}{16\pi}\right)|\nabla C(t)|^2 + \frac{\gamma}{2\alpha\chi} C^2(t)\ dx \\ &\quad - \frac{|\Omega|}{2|\partial\Omega|}\int\limits_{\partial\Omega} C(t)\ dS - |\Omega|\log\left(\frac{1}{|\partial\Omega|}\int\limits_{\partial\Omega} e^{C(t)/2} dS\right) - K|\Omega| \end{aligned}$$

which finally leads us to *6.* □

Lemma 3 and Lemma 4 also imply the following corollary for the radially symmetric case of system (3).

Corollary 1 *Suppose $\Omega \subset \mathbb{R}^2$ is a disk of radius R, which is centered in point $x_0 \in \mathbb{R}^2$. Furthermore assume that $\gamma > 0$. If*

$$\alpha\chi|\Omega| > 8\pi k_c,$$

then there exist radially symmetric blowup-solutions for system (3).

Remark 2 *We know that* $1 \leq card(\mathcal{BS}) < \infty$ *holds true. Since*

$$\lim_{r\to 0}\lim_{k\to\infty}\sigma_j^k(r) = 8\pi,$$

we see in the radially symmetric case that the claim of Lemma 4 *is true provided* $\alpha\chi|\Omega| > 8\pi k_c$. *This implies the statement of Corollary* 1.

Remark 3 *If* $\gamma = 0$ *we still have to exclude in the radially symmetric case that* $\alpha\chi|\Omega|/k_c$ *is equal to a multiple of an integer of* 8π. *Let us briefly compare the present paper with the results and proofs in* [12]. *In the present paper we used the transformation* $u_k = v_k + (\alpha\chi/\gamma)$. *We concluded via maximum principle that* $u_k > 0$. *This property was then used several times in the present paper.*

However in [12] *we used the transformation*

$$\tilde{u}_k = v_k - \log\left(\frac{1}{|\Omega|}\int\limits_{\Omega} e^{v_k}\,dx\right) - \frac{\alpha\chi}{4k_c}|x|^2.$$

Unfortunately we can not apply the maximum principle in this case. We also get some problems with Pohozaev's identity for our transformed problem. Thus we still have to exclude in the radially symmetric case of system (3) *with* $\gamma = 0$ *that* $\alpha\chi|\Omega|/k_c$ *is equal to a multiple of an integer of* 8π. *See* [12] *for more details about this case.*

Under the assumption that $T_{max} < \infty$ it is known that for the solution of system (1)

$$\lim_{t\to T_{max}} ||a(t)\log a(t)||_{L^1(\Omega)} = \infty$$

is true (see [20, Theorem 1]).

However it is absolutely not clear if either $T_{max} < \infty$ or $T_{max} = \infty$ is true for a blowup-solution of system (3) (resp. system (1)). There is only one example of a blowup-solution known, which blows up in finite time. This has been constructed for the radially symmetric case by M.A.Herrero and J.J.L. Velázquez in [10].

Suppose $T_{max} < \infty$, then we also do not know if either

1. $\inf\limits_{0\leq t<T_{max}} F(A(t), C(t)) > -\infty$ or
2. $\lim\limits_{t\to T_{max}} F(A(t), C(t)) = -\infty$.

In fact a numerical example for a blowup-solution of system (3) given by H. Gajewski and K. Zacharias behaves in such a way that

$$\lim_{t\to T_{max}} F(A(t),C(t)) = -\infty$$

(see [8, Remark 4.5, page 94 & 95]), while one can also think about the possibility that

$$\inf_{0\le t<T_{max}} F(A(t),C(t)) > -\infty,$$

since we are talking about finite time blow up.

Nevertheless we can formulate the following lemma, which gives us another result for a blowup-solution, which is independent from the questions mentioned above. (The statements from Theorem 1 are also independent from these facts, as one can easily see from the proof given in the present paper.)

Lemma 5 *Suppose the solution* $(A(t),C(t))$ *of system* (3) *blows up. Then*

$$\lim_{t\to T_{max}} ||A(t)\log A(t)||_{L^1(\Omega)} = \infty. \tag{28}$$

The proof of Lemma 5 is similar to [21, Proof of Proposition 3.2].

Proof of Lemma 5
Let $(A(t),C(t))$ denote a blowup-solution of system (3). Since

$$\int_\Omega A(t)\log A(t)dx \ge -\frac{|\Omega|}{e}$$

and

$$F(A(t),C(t)) \ge -\frac{|\Omega|}{e} - \int_\Omega (A(t)-1)C(t)dx + \frac{k_c}{2\alpha\chi}||\nabla C(t)||^2_{L^2(\Omega)}$$

we get with the help of [6, Theorem 2 & 3] that

$$||C(t)||^2_{L^\Phi(\Omega)} \le \tilde{K}||\nabla C(t)||^2_{L^2(\Omega)},$$

where $\Phi(s) \equiv e^s - s - 1$ (remember

$$\int_\Omega C(t)\ dx = 0$$

for all $t \ge 0$).

Here and in the following $L^\Phi(\Omega)$ denotes the Orlicz space which corresponds to the Young function $\Phi(s)$ and $||\cdot||_{L^\Phi(\Omega)}$ its norm. With Ψ we will denote the

Young function complementary to Φ and consequently with $L^{\Psi}(\Omega)$ the Orlicz space with norm $||\cdot||_{L^{\Psi}(\Omega)}$ which corresponds to the Young function Ψ. It is known that $\Psi(s) \equiv (s+1)\log(s+1) - s$ (see [15, Example 3.3.5. (iii)]). For more details on Orlicz spaces we refer once again to [15].

Using Hölder's inequality for Orlicz spaces [15, Theorem 3.7.5, p. 152] we see that

$$\begin{aligned} F(A(t), C(t)) &\geq -\frac{|\Omega|}{e} - \int_{\Omega} (A(t) - 1)C(t)dx + \frac{k_c}{2\alpha\chi}||\nabla C(t)||^2_{L^2(\Omega)} \\ &\geq -\frac{|\Omega|}{e} - ||C(t)||_{L^{\Phi}(\Omega)}||A(t) - 1||_{L^{\Psi}(\Omega)} + \frac{k_c}{2\alpha\chi}||\nabla C(t)||^2_{L^2(\Omega)} \\ &\geq -\frac{|\Omega|}{e} - \frac{\tilde{K}}{4\varepsilon}||A(t) - 1||^2_{L^{\Psi}(\Omega)} + \left(\frac{k_c}{2\alpha\chi} - \varepsilon\right)||\nabla C(t)||^2, \end{aligned}$$

where $\varepsilon < \frac{k_c}{2\alpha\chi}$. This however gives us together with [8, Lemma 6.3] and

$$\int_{\Omega} A(t)\ dx = |\Omega| \text{ (for all } t \geq 0)$$

the claim of Lemma 5. $\square$

Remark 4 *We get that*

$$||A(t)\log A(t)||_{L^1(\Omega)} \leq K||A(t)||_{L^p(\Omega)} \tag{29}$$

for every $1 < p$ (*see* [15, Theorem 3.17.1, page 185]) *and consequently*

$$\lim_{t \to T_{max}} ||A(t)||_{L^p(\Omega)} = \infty$$

for a blowup solution of system (3)*.*

Remark 5 *With exception of statement 6. the statements of Theorem* 1 *and Lemma* 5 *are also true for blowup solutions of* (3) *if* $\Omega \subset \mathbb{R}^2$ *has a boundary, which is piecewise* C^2*.*

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