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Branched microstructures: scaling and asymptotic self-similarity

by

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Abstract

We address some properties of a scalar 2D model which has been proposed to describe microstructure in martensitic phase transformations, consisting in minimizing the bulk energy

$$I[u] = \int_0^l \int_0^h u_x^2 + \sigma |u_{yy}|$$

where $|u_y| = 1$ a.e. and $u(0, \cdot) = 0$. Kohn and Müller [R. V. Kohn and S. Müller, Comm. Pure and Appl. Math. **47**, 405 (1994)] proved the existence of a minimizer for $\sigma > 0$, and obtained bounds on the total energy which suggested self-similarity of the minimizer. Building upon their work, we derive a local upper bound on the energy and on the minimizer itself, and show that the minimizer u is asymptotically self-similar, in the sense that the sequence

$$u^j(x, y) = \theta^{-2j/3} u(\theta^j x, \theta^{2j/3} y)$$

$(0 < \theta < 1)$ has a strongly converging subsequence in $W^{1,2}$.

1 Introduction

Martensitic phase transitions lead to mixtures of distinct phases or phase variants with characteristic fine scale structures. An almost universal phenomenon is that of twinning, whereby distinct variants of the martensite phase occur in long thin lamellae. Some gross features of the microstructure, such as the volume fractions and the direction

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of the domain walls, can be succesfully explained in terms of a comprimise between elastic energy and some given boundary conditions, arising e.g. from an interface with underformed austenite, or with other grains in polycrystals, or with other materials. This idea allows one to determine the large-scale behaviour of the material, and has been analyzed by Khachaturyan, Roitburd and Shatalov [9, 10, 14, 15, 11] in a geometrically linear framework, and by Ball and James [2, 3] with a geometrically nonlinear approach.

However, elastic energy alone does not predict fine-scale features such as the characteristic length scale for twinning, which are mainly determined by surface energy. Indeed, in a simple one-dimensional picture one finds for the average twin width d the scaling $d \sim \rho^{1/2} L^{1/2}$, where L is the length of the sample and ρ a material parameter (see e.g. [1, 2, 4, 8, 9, 10, 11, 14]).

Kohn and Müller (KM) have shown [12, 13] that, if the surface energy is small and/or the austenite is much harder than the martensite the above one-dimensional picture is incorrect, and the domains of the true minimizer branch near the interface. Indeed, fine twinning is preferred near the austenite, and a coarser one away from it. The twin width at distance l from the interface is then $d \sim \rho^{1/3} l^{2/3}$, where ρ is another material parameter. The work of KM was based on a scalar variational model, which neglected the wall width including only a wall energy proportional to the wall length, in addition to the the elastic energy [see Eq. (1.1)]. In a different setting, a similar domain branching has been recently demonstrated for uniaxial ferromagnets by Choksi, Kohn and Otto[5, 6].

This paper addresses the limiting model introduced by KM for small surface energy (or infinitely hard austenite), which is presented in Section 1.1. The minimizers are studied in some detail, obtaining a local bound on the minimizer and on its local energy (Section 2). Using this result, one can then prove that any minimizer is asymptotically self-similar around any point of the interface (Section 3). A brief account of the main results has been given in [7].

1.1 Model

We consider the minimizers of the functional introduced in [12] by Kohn and Müller,

$$I_{l_x, l_y}[u] = \int_0^{l_x} \int_0^{l_y} u_x^2 + \sigma |u_{yy}| dx dy \,, \tag{1.1}$$

where u is in the set of admissible functions

$$\begin{aligned} A_{l_x, l_y} \quad = \quad & \{u \in W^{1,2}[(0, l) \times (0, l_y)]; \\ & |u_y| = 1 \; a.e.; \\ & u_{yy} \text{ is a Radon measure with finite mass}\}. \end{aligned} \tag{1.2}$$

The function u represents a *scalar* deformation of a crystal, and can be thought of e.g. as the relevant shear component of the deformation [its gradient (u_x, u_y) then represents the strain, and the relevant elastic constant is taken to be 1]. The two phases, $u_y = 1$

and $u_y = -1$, represent two martensite variants. The width of the domain wall is zero, and σ is its energy per unit length. The boundary conditions will be denoted by

$$\begin{aligned} u(0,y) &= u_L(y) \qquad & u(x,0) &= u_B(x) \\ u(l_x,y) &= u_R(y) \qquad & u(x,l_y) &= u_T(x)\,; \end{aligned} \tag{1.3}$$

we will always assume the they are compatible [i.e. they join continuously at the corners, $u_L(0) = u_B(0)$ etc., $|u_T(x) - u_B(x)| \le l_y$, $|u_{(L,R)y}| \le 1$ – otherwise, no u with finite energy exists]. The planar interface with the austenite is represented by

$$u_L(y) = u(0,y) = 0\,. \tag{1.4}$$

We will also assume that the surface energy σ is small enough, i.e. that $l_y \ge 2c_1\sigma^{1/3}l_x^{2/3}$, where c_1 is a numerical constant specified later.

An example of a candidate minimizer is depicted in Figure 1.1. The above model is essentially geometric in nature, in the sense that the minimizer u is fully determined by its value on one boundary (e.g. $y = 0$) and by the position of the domain boundaries.

In [13], after proving existence of the minimizer, Kohn and Müller considered the problem with free $u_{B,T,R}$ boundary conditions and proved that

$$d_1\sigma^{2/3}l_x^{1/3}l_y < \min I < d_2\sigma^{2/3}l_x^{1/3}l_y \tag{1.5}$$

where d_1 and d_2 are numerical constants. The proof of existence is essentially based on the following compactness result, which we will need in Section 3:

Lemma 1 (compactness) *Let $\Omega \subset \boldsymbol{R}^2$ be open and let $u^j : \Omega \to \boldsymbol{R}$ be a sequence such that $u^j \rightharpoonup u$ in $W^{1,2}(\Omega)$, and such that u^j_{yy} lies in a compact subset of $W^{-1,2}(\Omega)$. Then*

$$u^j_y \to u_y \qquad \text{in } L^2_{\text{loc}}(\Omega) \tag{1.6}$$

Proof. By compensated compactness applied to $v^j = (u^j_x, u^j_y)$, or see [13] for a self-contained argument.

The upper bound of Eq. (1.5) follows from Lemma 3 below, which uses the construction proposed by Kohn and Müller. The proof of the lower bound is based on the fact that, if $|f_y| = 1$, one has

$$\int_0^1 f^2 dy \ge \frac{1}{3}\left(\int_0^1 |f_{yy}|dy + 2\right)^{-2} \tag{1.7}$$

(see [13] for details).

Let us now discuss qualitatively the behaviour of u, to understand the origin of the $l_x^{1/3}$ scaling in Eq. (1.5). Consider the x-dependent average twin width $d(x) = 2l_y/\int |u_{yy}|dy$. The average value of $|u(x,y)|$ is also of order $d(x)$, since $|u_y| = 1$ (see Figure 1.2). Therefore the functional $I[u]$ can be approximately rewritten as a functional of d alone, in the form

$$I[u] \sim \tilde{I}[d] = l_y \int_0^{l_x} d'(x)^2 + \frac{\sigma}{d(x)} dx \tag{1.8}$$

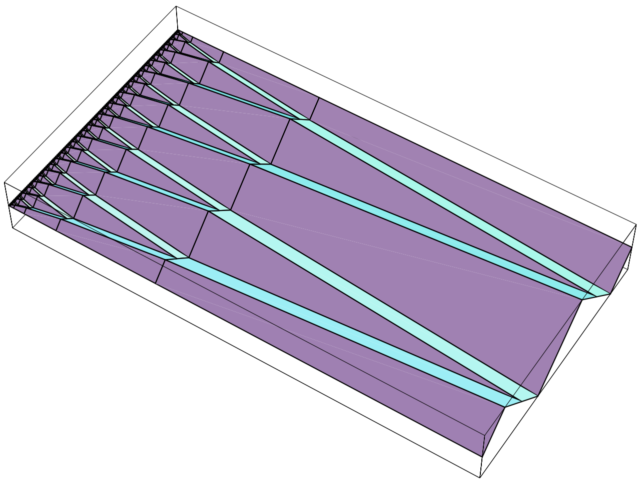

FIGURE 1.1: 3D plot of one function with finite energy. The plotted function is the one used in the construction below in the case of free boundary conditions.

with the condition $d(0) = 0$. Minimization of $\tilde{I}[d]$ gives $d(x) \propto \sigma^{1/3} x^{2/3}$, i.e.

$$|u|(x, y) \sim d(x) \propto \sigma^{1/3} x^{2/3} \tag{1.9}$$

and

$$I[u] \sim \tilde{I}[d] \propto \sigma^{2/3} l_x^{1/3} l_y \,. \tag{1.10}$$

These equations indicate that the boundary condition $u_L = 0$ forces the twin width d to vanish for $x = 0$, whereas the surface energy term favours large values of d at finite x. The $x^{2/3}$ scaling arises from a balance between elastic and surface energy, with elastic energy favouring slow variation of d. For the shape of the minimizers, these results already indicate that with decreasing x, as the number of twins increases, branching must occur. Further, one can see that the functional $\tilde{I}[d]$ and the minimizing $d(x)$ are unchanged under the mapping $d(x) \to \theta^{2/3} d(\theta x)$, for any θ. In terms of the original u, this mapping reads

$$u(x, y) \to \theta^{-2/3} u(\theta x, \theta^{2/3} y) \,. \tag{1.11}$$

Whereas these estimates have no mathematical rigour, the results of KM show that at least Eq. (1.10) holds for the total energy of the true minimizer. In the following, rigorous versions of all the mentioned scalings are formulated and proved (see Theorem 1 in Section 2 and Theorem 2 in Section 3).

2 Local bounds

This Section aims at understanding quantitatively the estimates of Eqs. (1.9) and (1.10) for the scaling of the minimizer u and of its local energy. The result by KM of equation

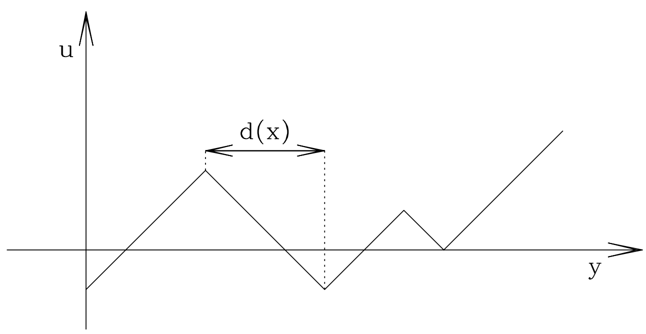

FIGURE 1.2: Schematic representation of the meaning of $d(x)$ (see Eqs. (1.8–1.10).

(1.5) by itself does not prove anything about the local behaviour of a minimizer in a small region near the interface, it only bounds the total energy. We intend to prove a local bound with the same scaling – i.e., we prove that if we fix a minimizer, and compute its energy in a small region of size $l'_x \times l'_y \ll l_x \times l_y$, the result again obeys a bound of the form $\sigma^{2/3} {l'_x}^{1/3} l'_y$. Furthermore, we prove that $|u|(x,y) \leq c\sigma^{1/3}x^{2/3}$, which implies that at distance x from the interface the twin width scales as $\sigma^{1/3}x^{2/3}$ (since $|u_y| = 1$, the maximum twin width at a given x is bounded by $2\sup|u|$ at the same x). With respect to the hypothesis of the KM theorem, we need to have additional assumptions on the boundary conditions, which can be understood as saying that the boundary conditions are no worse than the expected average behaviour of the minimizer. Finally, let us mention that we cannot expect to get the scaling of Eq. (1.10) for the energy in a region of height h below the characteristic length scale of twinning, therefore we assume $l_y \geq 2c_1\sigma^{1/3}l_x^{2/3}$, and the same for l'_y and l'_x. The main result is the following

Theorem 1 (local bounds) *Let u be a minimizer of the problem defined in Sec. 1.1 on $R = (0, l_x) \times (0, l_y)$, with boundary conditions u_T, u_B, u_L, u_R (on the top, bottom, left and right side respectively) satisfying $u^L = 0$, $|u^R| \leq c_2c_1\sigma^{1/3}l_x^{2/3}$, $|u^{T,B}|(x) \leq c_2c_1\sigma^{1/3}x^{2/3}$, $|u_x^{T,B}|(x) \leq \sqrt{6}c_0^{1/2}x^{-1/3}\sigma^{1/3}$. Then,*

(i) for any $(x,y) \in R$,

$$|u(x,y)| < d_3\sigma^{1/3}x^{2/3}\,, \tag{2.1}$$

(ii) for any $R' = (0,l) \times (0,h) \subset R$ with $h \geq 2c_1\sigma^{1/3}l^{2/3}$,

$$I_{R'}[u] < d_4\sigma^{2/3}l^{1/3}h\,. \tag{2.2}$$

Numerical values of the constants c_i and d_i are given in Definition 2 at the end of this Section.

The proof, which is given in Section 2.2, is composed by two main ingredients: first we obtain, via an explicit construction, a bound on the energy of u inside any rectangle, in terms of the boundary conditions on the four sides (Section 2.1, aiming to Proposition 1). Then, we show with a "reverse-bootstrap" argument how a global bound implies the desired local bound.

2.1 Explicit construction

This Section presents the basic construction, which is used to obtain a function with small energy in a rectangle with four given boundary conditions. The aim is to bound the energy of the minimizer in any subrectangle. The bound will of course depend on the regularity of the boundary conditions, which in turn depend on the minimizer itself: the next section will deal with the appropriate inductive procedure.

We start with the case of only two boundary conditions imposed (on the left and right sides of the rectangle). In Lemma 2 we consider the simple case where u^L and u^R are piecewise linear with ± 1 slopes, then in Lemma 3 we generalize to arbitrary $u^{L,R}$, obeying only $|u_y^{L,R}| \leq 1$. Then, Proposition 1 gives the final result of this Section, including the $u^{T,B}$ boundary conditions.

In the following we will often use "linearized" functions. Given the nature of the set of admissible functions A_{l_x,l_y}, it is clear that no linearization in y is possible within A_{l_x,l_y} - indeed, "linear" will always be referred *only* to the dependence on x, with y - when present - kept fixed. The linear interpolation between $c(0)$ and $c(a)$ is denoted by $c_l^{[0,a]}(x)$, or by $c_l(x)$ when there is no ambiguity. It is also natural to subtract from the functional $I[u]$ the linear contribution which depends only on the boundary conditions, and consider the modified functional

$$I_R^l[u] = \int_R (u-u_l)_x^2 + \sigma|u_{yy}| = I_R[u] - \int_R u_{lx}^2 = I_R[u] - \int_0^{l_y} \frac{[u_L(y)-u_R(y)]^2}{l_x}. \quad (2.3)$$

Lemma 2 (piecewise linear $u_{L,R}$) *Let the boundary conditions on the left and right sides of the rectangle $R = (0,l_x) \times (0,l_y)$ be given functions $u^L(y)$ and $u^R(y)$, piecewise linear with slopes ± 1. For any partition of $(0,l_y)$ in N intervals $\{y_0,\ldots,y_N\}$ ($y_0 = 0$, $y_N = l_y$) which incorporates all discontinuity points of u_{Ry} and u_{Ly}, there is a function $u(x,y)$ such that*

- *$u \in W^{1,2}(R)$, with $|u_y| = 1$ a.e.;*
- *for every $x \in (0,l_x)$, the number of discontinuity points of $u_y(x,\cdot)$ is no larger than N, i.e. $\int |u_{yy}|dy \leq 2N$;*
- *If $h = \max_i(y_i - y_{i-1})$, one has*

$$\int_R (u-u_l)_x^2 \leq \frac{2}{3}\frac{h^2 l_y}{l_x}, \quad (2.4)$$

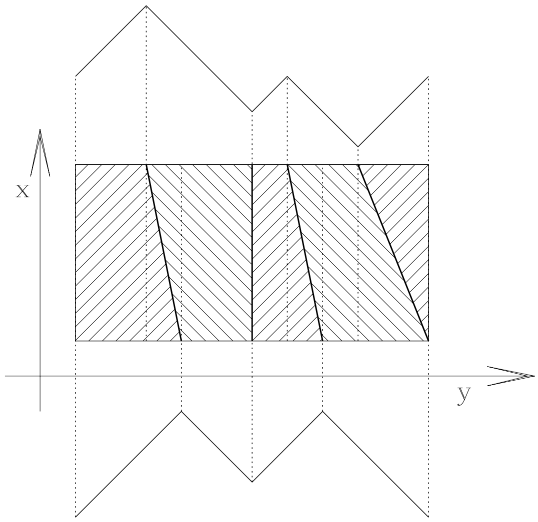

FIGURE 2.1: Example of the construction for two piecewise linear boundary conditions used in Lemma 2. The 7 sub rectangles are delimited by dotted lines. The four thicker lines represent the internal interfaces. Note that the x-axis is vertical in this figure.

where u_l is the linear interpolation between u_L and u_R;

•
$$|u - u_l| \leq \frac{h}{2}. \tag{2.5}$$

Proof: The constructed function is depicted, for one particular choice of u_R and u_L, in Figure 2.1. Consider a rectangle $R_i = [0, l_x] \times [y_i, y_{i+1})$. If $u_y^L = u_y^R$ in (y_i, y_{i+1}), we can take

$$u(x,y) = u_l(x,y) = u^L(y)\frac{x}{l_x} + u^R(y)\frac{l_x - x}{l_x} \tag{2.6}$$

within R_i. Otherwise, one has $u(0,y) = u^L(y_i) + (y - y_i)$, $u(l_x, y) = u^R(y_i) - (y - y_i)$ (or vice versa). Then, there exist two admissible u with a single jump in u_y inside R (along either diagonal). Such u coincide with the linear interpolation on the top and bottom sides of the rectangle, and have a jump in the derivative along a diagonal. Consider for definiteness one of them, given by

$$u(x,y) = \begin{cases} \frac{x}{l_x}u^R(y_i) + \frac{l_x - x}{l_x}u^L(y_i) - (y - y_i) & \frac{y - y_i}{y_{i+1} - y_i} < \frac{x}{l_x} \\ \frac{x}{l_x}u^R(y_{i+1}) + \frac{l_x - x}{l_x}u^L(y_{i+1}) + (y - y_{i+1}) & \frac{y - y_i}{y_{i+1} - y_i} > \frac{x}{l_x} \end{cases} \tag{2.7}$$

Direct integration gives

$$\int_{R_i} (u-u_l)_x^2 = \frac{2}{3}\frac{(y_{i+1}-y_i)^3}{l_x}\,, \tag{2.8}$$

and comparing with u_l one gets (2.5).

To evaluate the total number of jumps in u_y, we start from one extremum (e.g. $y=0$) and show that to add each rectangle at most one jump is needed. Indeed, if the rectangle is of the first type, there is at most a single jump (on the bottom side). If it is of the second, one of the two choices for the diagonal gives no jump on the bottom, and there is only one jump along the diagonal. It follows that the total number of jumps is no bigger than the total number of rectangles, i.e. N.

Lemma 3 (Arbitrary $u_{L,R}$) *Let the boundary conditions on the left and right sides of the rectangle $R=(0,l_x)\times(0,l_y)$ be given, with $|u_y^{L,R}|\le 1$ and $l_y\ge 2c_1\sigma^{1/3}l_x^{2/3}$. Then, there is an admissible u such that*

$$\int_R (u-u_l)_x^2+\sigma|u_{yy}|\le c_0 l_y l_x^{1/3}\sigma^{2/3}-8\sigma l_x \tag{2.9}$$

and

$$|u-u_l|\le \sigma^{1/3}l_x^{2/3}\,, \tag{2.10}$$

where $u_l=(xu_L(y)+(l_x-x)u_R(y))/l_x$ denotes the global *linear interpolation (in x).*

Remark: Lemma 2.5 of [13], which is based on a similar construction, states that for all $\delta>0$ there is a C_δ such that

$$\int_R (u-u_l)_x^2+\sigma|u_{yy}|\le \delta\int_R u_{lx}^2+C_\delta l_y l_x^{1/3}\sigma^{2/3}\,. \tag{2.11}$$

The present result is stronger, since it gives the case $\delta=0$, with $C_0=c_0$.

Proof. To construct u, consider the geometric subdivision shown in Figure 2.2. The subdivision points are

$$x_i=\begin{cases} l_x\left(1-\frac{1}{2}\theta^{|i|}\right) & i\ge 0\\ l_x\frac{1}{2}\theta^{|i|} & i<0\end{cases} \tag{2.12}$$

where $\theta\in(0,1)$ is to be specified. We shall focus our attention on the region $x\ge l_x/2$, the other one being symmetric. For $i=0,1,2,\ldots$, divide the segment $\{x_i\}\times(0,l_y)$ into $N_i=N_0 2^{|i|}$ equal parts (N_0 is a large integer to be chosen later), each of length $h_i=l_y/N_i$. Next, for each i choose a function $\tilde u^i$ which agrees with the linear interpolation u_l at the points (x_i,kh_i) $(0\le k\le N_i)$, which satisfies $|\tilde u^i_y|=1$ a.e, and with $\tilde u^i_y$ changing sign at most once between two successive points (this is possible, since $|u_y^{L,R}|\le 1$).

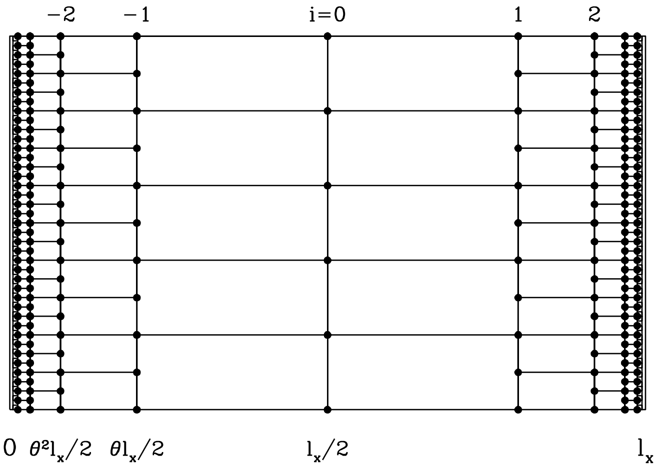

FIGURE 2.2: Geometric subdivision used in the construction of Lemma 3. The marked points are those where $\tilde{u} = u_l$.

We apply then Lemma 2 to the region $R_i = (x_{i-1}, x_i) \times (0, l_y)$ $(i = 1, 2, \dots)$, with $N = N_i + N_{i-1} = \frac{3}{2}N_i$, $u^L = \tilde{u}^{i-1}$, $u^R = \tilde{u}^i$, $h = 2h_i$, and obtain a function u such that

$$\int_{R_i} |u_{yy}| \leq 3N_i(x_i - x_{i-1}) \,, \tag{2.13}$$

$$\int_{R_i} (u - \tilde{u}_l^i)_x^2 \leq \frac{8}{3} \frac{l_y h_i^2}{x_i - x_{i-1}} \,, \tag{2.14}$$

and $|u - \tilde{u}_l^i| \leq h_i$, where $\tilde{u}_l^i$ is the linear interpolation between $\tilde{u}^{i-1}$ and $\tilde{u}^i$. Since $|\tilde{u}_l^i - u_l| \leq 2h_i$, we get $|u - u_l| \leq 3h_i$, which – with the choice done below for N_0 – gives Eq. (2.10).

Since $u = \tilde{u}_l^i$ at $x = x_i$ and $x = x_{i-1}$, one has

$$\int_{R_i} (u - u_l)_x^2 = \int_{R_i} (u - \tilde{u}_l^i)_x^2 + \int_{R_i} (u_l - \tilde{u}_l^i)_x^2 \tag{2.15}$$

To estimate the last term we observe that both functions are linear in x, and by construction they agree on all points selected above, i.e. on three corners of each rectangle of the form $(x_{i-1}, x_i) \times (jh_i, (j+1)h_i)$ $(j = 1 \dots N_i)$. Then, Lemma 4 below applied to $f = u_l - \tilde{u}_l^i$ gives

$$\int_{R_i} (u_l - \tilde{u}_l^i)_x^2 \leq \frac{8}{3} \frac{l_y h_i^2}{x_i - x_{i-1}} \tag{2.16}$$

In summary, one has

$$\int_{R_i} (u - u_l)_x^2 \le \frac{l_y h_i^2}{x_i - x_{i-1}} \left(\frac{8}{3} + \frac{8}{3} \right) \tag{2.17}$$

It follows that for $2\theta \le 1 \le 4\theta$,

$$\sum_i \int_{R_i} (u - u_l)_x^2 = \frac{16}{3} \frac{l_y^3}{l_x N_0^2} \frac{4\theta}{(1-\theta)(4\theta - 1)} \tag{2.18}$$

with

$$\int_R \sigma |u_{yy}| \le 2\sigma \sum_{i>0} (x_i - x_{i-1}) 3N_i = \sigma 6 N_0 l_x \frac{1-\theta}{1-2\theta} \tag{2.19}$$

We now choose $\theta = 1/3$, and obtain the bound on the energy

$$\int (u - u_l)_x^2 + \sigma |u_{yy}| \le \frac{32}{N_0^2} \frac{l_y^3}{l_x} + 12 N_0 \sigma l_x \tag{2.20}$$

By choosing N_0 to be the smallest integer larger or equal to $(16/3)^{1/3} l_y \sigma^{-1/3} l_x^{-2/3}$, we finally get the thesis, provided that

$$c_0 \ge 12 \cdot 18^{1/3} + \frac{20}{c_1} \tag{2.21}$$

Lemma 4 *Let $f(x, y)$ be linear in x, $f = 0$ at $(0,0)$, $(l_x, 0)$, and (l_x, h), with $|f_y|(x, \cdot) \le 2$ for $x = 0$ and $x = l_x$. Then*

$$\int_{(0,l_x)\times(0,h)} f_x^2 \le \frac{8}{3} \frac{h^3}{l_x}. \tag{2.22}$$

Proof.

$$\int_{(0,l_x)\times(0,h)} f_x^2 = \frac{1}{l_x} \int_0^h [f(l_x, y) - f(0, y)]^2 \, dy \tag{2.23}$$

For $0 < y < h/2$, the integrand is dominated by $(4y)^2$; for $h/2 < y < h$ it is dominated by $(2h)^2$. The conclusion follows by direct integration.

The following lemma will be useful for the construction including the top and bottom boundary conditions.

Lemma 5 *Let v, Ψ_i, η_i be in $W^{1,2}([0, l])$, for $1 \le i \le n$, such that $\Psi_i \le v \le \eta_i$ at $x = 0$ and $x = l$ for any i. Let $w = \min [\{\eta_i\}, \max(\{\Psi_i\}, v)]$. Then,*

$$\int_0^l (w - w_l)_x^2 \le \int_0^l (v - v_l)_x^2 + \sum_i \int_0^l \left[(\Psi_i - \Psi_{il})_x^2 \chi_{\{\Psi_i > v\}} + (\eta_i - \eta_{il})_x^2 \chi_{\{\eta_i < v\}} \right] \tag{2.24}$$

where χ_E denotes the characteristic function of the set E.

Proof. First we show that if $h(x) = \max(f(x), g(x))$, with $h = g$ for both $x = 0$ and $x = l$ (i.e. $h_l = g_l$), one has

$$\int_0^l (h - h_l)_x^2 \le \int_0^l (g - g_l)_x^2 + \int_0^l (f - f_l)_x^2 \, \theta\,(f > g) \tag{2.25}$$

Let A be the set where $f > g$. In A, expanding $(h - h_l)_x^2 = [(f - f_l)_x + (f_l - g_l)_x]^2$ one gets

$$\int_A (h - h_l)_x^2 = \int_A \left[(f - f_l)_x^2 + (f_l - g_l)_x^2 + 2(f - f_l)_x (f_l - g_l)_x\right] \tag{2.26}$$

Since $(f_l - g_l)_x$ is a constant, and $f = g$ on ∂A, in the last term one can replace f with g. Comparing with the expansion of $[(g - f_l)_x + (f_l - g_l)_x]^2$, one gets

$$\int_A (h - h_l)_x^2 = \int_A \left[(f - f_l)_x^2 + (g - g_l)_x^2 - (g - f_l)_x^2\right] \tag{2.27}$$

which immediately gives (2.25). The same is clearly true if one replaces max with min. Further, one can have multiple max and min, with the sole condition that the linearization of the whole function coincides with the linearization of the first one of the starting functions. This gives (2.24).

Proposition 1 (Construction with four boundary conditions) *Let the boundary conditions on the four sides of the rectangle $R = (0, l_x) \times (0, l_y)$, u_T, u_B, u_L, u_R (top, bottom, left and right respectively), be given, with $l_y \ge 2c_1 \sigma^{1/3} l_x^{2/3}$. Then there is an admissible u such that*

$$\int_R (u - u_l)_x^2 + \int_R \sigma |u_{yy}| \le \omega \int \left[(u^T - u_l^T)_x^2 + (u^B - u_l^B)_x^2\right] + c_0 l_y l_x^{1/3} \sigma^{2/3} - 4 l_x \sigma \tag{2.28}$$

where $\omega = \sup_x |u_{T,B}| + \sup_y |u_{L,R}| + \sigma^{1/3} l_x^{2/3}$.

Proof. Let $\tilde{u}$ be the function constructed in Lemma 3, and define

$$u = \max(u^B - y, u^T - (l_y - y), \min(u^B + y, u^T + (l_y - y), \tilde{u})) \tag{2.29}$$

(see Figure 2.3 for an illustration of this construction). Since at most two new interfaces are generated, the surface energy increases by at most $4\sigma l_x$. As results from Figure 2.3, $\tilde{u} = u^B \pm y$ for $y \le y_1(x) \le \sup_y |u_B(x) - \tilde{u}(x, y)|$, and analogously $\tilde{u} = u^T \pm (l_y - y)$ for $l_y - y \le y_2(x) \le \sup_y |u_T(x) - \tilde{u}(x, y)|$. Since $|\tilde{u} - u_l| \le \sigma^{1/3} l_x^{2/3}$, and u_l is an interpolation between u_L and u_R, one has

$$|u_{B(T)}(x) - \tilde{u}(x, y)| \le \sup |u_{B(T)}(x) - u_{L(R)}(y)| + \sigma^{1/3} l_x^{2/3} \tag{2.30}$$

where the sup is taken over y and over the pair of indexes $L(R)$, $B(T)$. Therefore the width of the strips where $\tilde{u}$ is modified is less than ω (as defined in the statement of this proposition), and $u = \tilde{u}$ for $\omega \le y \le l_y - \omega$. The elastic energy increase in the two strips $y < \omega$ and $y > l_y - \omega$ is finally estimated from Lemma 5, obtaining the thesis.

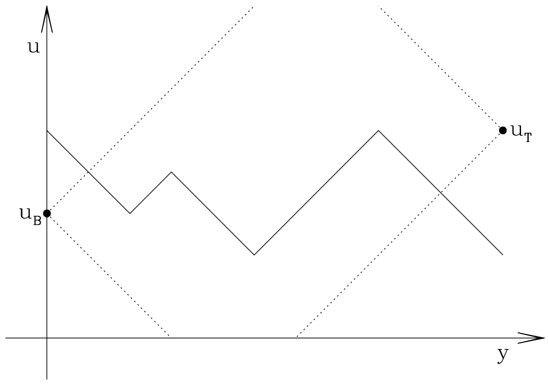

FIGURE 2.3: Sketch of the construction used in Proposition 1, for fixed x. The function $\tilde{u}$ (see Eq. (2.29)) is modified in order to comply to the boundary conditions u_T and u_B. The new function u agrees with $u_B \pm y$ for small y, with $\tilde{u}$ in the central area, and with $u_T \pm (l_y - y)$ for large y.

2.2 Proof of Theorem 1

Before we give a detailed proof of let us outline the main ideas. The proof is based on the fact that u is a local minimizer (i.e. a minimizer with respect to its own boundary conditions) therefore its local energy is bounded by Proposition 1. It is therefore natural to consider the energy as a function of the domain, and to subtract the linear term, leading to the following

Definition 1 *Let u be a minimizer in a given rectangle $R = (0, l_x) \times (0, l_y)$. For all subrectangles $R' = (0, l) \times (a-h, a+h) \subset R$, we define*

$$\beta_u(l, a, h) = \int_0^l \int_{a-h}^{a+h} (u - u_l)_x^2 + \sigma |u_{yy}| \tag{2.31}$$

where u_l is the linearization in x, in the interval $(0, l)$.

Note that β is the same as $I^l[u]$ as defined in (2.3) except that I^l is seen as a functional on u with a fixed domain, whereas β is seen as a function of the domain with fixed u.

Consider first l to be fixed, and consider the statement of Proposition 1 for all h. In the case of the total rectangle R the boundary term is controlled by hypothesis, and for smaller rectangles we use the estimate

$$\int_0^l \left[(u^T - u_l^T)_x^2 + (u^B - u_l^B)_x^2 \right] \le \frac{d}{dh} \beta_u(l, a, h) \tag{2.32}$$

[in differentiating Eq. (2.31) only the integration domain changes]. This leads to a differential inequality of the form

$$\beta_u(h) \le \omega \frac{d\beta_u}{dh} + 2c_0 h l_x^{1/3} \sigma^{2/3} \tag{2.33}$$

(the factor 2 comes from the fact that the height is $2h$) with a boundary condition at large h (entire rectangle). Assume for now that one can show that $\omega \leq h/2$. Qualitatively, Eq. (2.33) says that either $\beta(h)$ is small, or it is rapidly decreasing with decreasing h (i.e. has a big derivative). This allows one to obtain an estimate at small h from the given result at large h. A precise statement is given in Lemma 6. Note that one only needs to estimate $\omega(h)$ *given the estimate of* $\beta_u(h)$ *for the same* h. The relevant relation between $\sup|u|$ (which is strictly related to ω) and β_u is given in Lemma 9. This concludes the argument for fixed l and a. The same argument can be applied to the derivative with respect to a (see below), obtaining the thesis for all a and h at fixed l.

Estimates for smaller l are obtained by induction: the above result on $\sup|u|$ allows one to control the boundary conditions on a smaller rectangle $(0, \theta l) \times (0, l_y)$, etc.

We now present the lemmas mentioned above, then give the full argument for the part of the proof for fixed l in Proposition 2, and finally conclude with the inductive step.

Lemma 6 (Differential inequality) *Suppose that for* $t_1 < t < t_0$ *one has*

$$f(t) \leq t\left[a(t)\frac{df(t)}{dt} + b\right], \tag{2.34}$$

and suppose that there are constants $\alpha < 1$ *and* $k \geq b/(1-\alpha)$ *such that, for each* $t \in [t_1, t_0]$, $f(t) \leq kt$ *implies* $a(t) < \alpha$. *It follows that if* $f(t_0) < kt_0$, *then* $f(t) < kt$ *for all* $t \in [t_1, t_0]$.

Proof. Let $h(t) = f(t) - kt$. Since $h(t_0) < 0$, if the thesis is false there is a $t_2 \in [t_1, t_0]$ such that $h(t_2) = 0$, $h'(t_2) \leq 0$ (this is defined as the largest t_2 such that $h(t_2) = 0$). It follows

$$kt_2 < \alpha k t_2 + bt_2 \tag{2.35}$$

which gives the contradiction $k < b/(1-\alpha)$.

The following three lemmas deal with estimates of $|u|$ given an estimate for the energy.

Lemma 7 *If* $|u_y| \leq \alpha$, *then*

$$\int_0^{l_y} u^2 \geq \min\left(\frac{1}{3\alpha}(\sup|u|)^3, \frac{\alpha^2}{3}l_y^3\right) \tag{2.36}$$

Proof. Let $s = |u|(y_0) = \sup|u|$. Then, in the set $A = (y_0 - s/\alpha, y_0 + s/\alpha) \cap (0, l_y)$ one has $|u|(y) \geq s - \alpha|y - y_0|$. Direct integration, separately in the two cases $s > \alpha l_y$ and $s \leq \alpha l_y$, leads to the conclusion.

Lemma 8 *Let* $x_1 = \phi x_0$*, with* $0 < \phi \le 1$*, and let* $\gamma_0 = \int_0^{l_y} \int_0^{x_0} (u - u_l)_x^2$*. If*

$$\gamma_0 \le \frac{1}{3} \frac{l_y^3}{x_0} \tag{2.37}$$

it follows that

$$\sup_{0 \le y \le l_y} |u(x_1, y)| \le \phi \sup_{0 \le y \le l_y} |u(x_0, y)| + (3\phi(1 - \phi^2) x_0 \gamma_0)^{1/3} . \tag{2.38}$$

Proof. Let $\psi(y) = u(x_1, y) - \phi u(x_0, y)$. Since

$$\int_0^{x_0} (u - u_l)_x^2 \ge \frac{\psi(y)^2}{x_1} + \frac{\psi(y)^2}{x_0 - x_1} = \frac{\psi^2}{x_0 \phi (1 - \phi)} \tag{2.39}$$

we get

$$\int \psi^2 \le x_0 \phi (1 - \phi) \gamma_0 \tag{2.40}$$

Since $|d\psi/dy| \le 1 + \phi$, we can apply Lemma 7 to ψ; from Eqs. (2.37) and (2.40) we see that $\int \psi^2 \le l_y^3/3$, hence

$$|\psi(y)| \le \left[3(1 + \phi) \int \psi^2 \right]^{1/3} \le [3\phi(1 - \phi^2) x_0 \gamma_0]^{1/3} . \tag{2.41}$$

the conclusion follows via the triangular inequality.

Lemma 9 (Estimate of $\sup |u|$**)** *If, for a given* $h \ge c_1 \sigma^{1/3} l_x^{2/3}$*, one has*

$$\sup_{y \in [a-h, a+h]} |u(l_x, y)| \le c_2 h , \tag{2.42}$$

and

$$\beta_u(l_x, a, h) \le 4 c_0 h l_x^{1/3} \sigma^{2/3} \tag{2.43}$$

then, in $(0, l_x) \times (a - h, a + h)$*,*

$$|u| \le c_3 h \tag{2.44}$$

Proof. Use Lemma 8, and $c_3 \ge c_2 + (6 c_0 / c_1^2)^{1/3}$ $[\phi(1 - \phi) \le 1/4$ for all $\phi \in [0, 1]]$. The condition of Eq. (2.37) is satisfied, provided that $c_0 \le 2 c_1^2$, which is always the case for large enough c_1 (see Definition 2).

We are now ready to complete the first part of the proof of Theorem 1, i.e. the case where l is fixed. The following Proposition gives this result, and Lemma 10 will then give the hypothesis needed for the inductive step.

Proposition 2 *Let u be a minimizer in a rectangle R, with boundary conditions satisfying $u^L = 0$, $|u^R| \le c_2 c_1 \sigma^{1/3} l_x^{2/3}$, $|u^{T,B}(x)| \le c_3 c_1 \sigma^{1/3} x^{2/3}$,*

$$\int_0^{l_x} (u^T - u_l^T)_x^2 \le 2c_0 l_x^{1/3} \sigma^{2/3} , \tag{2.45}$$

and

$$\int_0^{l_x} (u^B - u_l^B)_x^2 \le 2c_0 l_x^{1/3} \sigma^{2/3}. \tag{2.46}$$

Then, for all intervals $(a-h, a+h) \subset (0, l_y)$ with $h \ge c_1 \sigma^{1/3} l_x^{2/3}$,

$$\beta_u(l_x, a, h) \le 4c_0 h l_x^{1/3} \sigma^{2/3} , \tag{2.47}$$

and for all $(x, y) \in R$,

$$|u(x,y)| \le c_3 c_1 \sigma^{1/3} l_x^{2/3} . \tag{2.48}$$

Proof. The proof is based on Proposition 1. First consider the case $a = h = l_y/2$. The hypothesis on the boundary conditions, using the constraint on the domain $l_y \ge 2c_1 \sigma^{1/3} l_x^{2/3}$, give for the quantity ω defined in Proposition 1,

$$\omega \le [c_2 c_1 + c_3 c_1 + 1] \, \sigma^{1/3} l_x^{2/3} \le \frac{1}{4} l_y , \tag{2.49}$$

since $c_2 + c_3 + 1/c_1 \le 1/2$. Then, from Proposition 1, using Eqs. (2.45-2.46), we get

$$\beta_u(l_x, l_y/2, l_y/2) < \frac{1}{4} l_y 4c_0 l_x^{1/3} \sigma^{2/3} + c_0 l_y l_x^{1/3} \sigma^{2/3} \le 2c_0 l_y l_x^{1/3} \sigma^{2/3} \tag{2.50}$$

which implies the thesis in this case. Now, consider the case $a = h < l_y/2$ (the interval is here $(0, 2h)$). The condition on u_T is obtained from the relation

$$\int_0^{l_x} [u(x,y) - u_l(x,y)]_x^2 \, dx \le \frac{1}{2} \left. \frac{d\beta_u(l_x, h, h)}{dh} \right|_{h=y/2} . \tag{2.51}$$

Using the estimate $|u| \le c_3 h$ from Lemma 9, as well as the initial boundary conditions, we get $\omega \le h/2$, and then Proposition 1 gives

$$\beta_u(l_x, h, h) \frac{1}{2} h \left(2c_0 l_x^{1/3} \sigma^{2/3} + \frac{1}{2} \frac{d\beta_u}{dh} \right) + 2c_0 h l_x^{1/3} \sigma^{2/3} \le \frac{1}{4} h \frac{d\beta_u}{dh} + 3c_0 h l_x^{1/3} \sigma^{2/3} . \tag{2.52}$$

Now apply Lemma 6, with $t = h$, $\alpha = 1/4$, $t_0 = l_y/2$, $t_1 = c_1 \sigma^{1/3} l_x^{2/3}$, and $k = 4c_0 h l_x^{1/3} \sigma^{2/3}$. This gives the thesis for all rectangles of the form $(0, l_x) \times (0, 2h)$. By symmetry, the same argument gives the thesis for all rectangles of the form $(0, l_x) \times (l_y - 2h, l_y)$.

Consider finally a generic interval $(a-h, a+h) \subset (0, l_y)$. For definiteness, assume that $a \leq l_y/2$ (the other case is identical). We fix a, and observe that the previous result gives a bound on $\beta_u(l_x, a, a)$, which will serve as initial condition for the differential inequality. For $h < a$ Lemma 9 still gives $\omega \leq h/2$, and since

$$\frac{d\beta_u(l_x, a, h)}{dh} = \int_0^{l_x} (u-u_l)_x^2(x, a+h) + (u-u_l)_x^2(x, a-h)dx \tag{2.53}$$

Proposition 1 in this case gives

$$\beta_u(l_x, a, h) \leq \frac{1}{2} h \frac{d\beta_u}{dh} + 2c_0 h l_x^{1/3} \sigma^{2/3} \,. \tag{2.54}$$

The thesis follows again from Lemma 6, with $t = h$, $\alpha = 1/2$, $t_0 = a$, $t_1 = c_1 \sigma^{1/3} l_x^{2/3}$, and $k = 4c_0$.

Lemma 10 *Under the hypothesis of Prop. 2, one has*

$$\sup_y |u(\phi l_x, y)| \leq c_2 c_1 \sigma^{1/3} (\phi l_x)^{2/3} \,, \tag{2.55}$$

where $\phi = 1/8$.

Proof. We first derive the result for any $\phi \in (0,1)$, for large enough c_1, and then specialize to $\phi = 1/8$. Consider a rectangle of the form $(a_0 - h_0, a_0 + h_0)$ containing the point where the sup is reached, with $h_0 = c_1 \sigma^{1/3} l_x^{2/3}$. Then, Lemma 8 and the result for β of Prop. 2 give the result, provided that

$$\phi c_2 c_1 + (12\phi(1-\phi^2)c_0 c_1)^{1/3} \leq c_2 c_1 \phi^{2/3} \tag{2.56}$$

which is true for

$$\frac{c_1^2 c_2^3}{c_0} \geq \frac{12(1-\phi^2)}{(1-\phi^{1/3})^3 \phi} \,, \tag{2.57}$$

which for $\phi = 1/8$ gives the condition $c_1^2 c_2^3 \geq 756 c_0$ (see Definition 2). Other values of ϕ can also be used, provided that the constants are modified accordingly.

Proof of Theorem 1. Consider a sequence of rectangles $R_i = (0, \phi^i l_x) \times (0, l_y)$ (as in the above Lemma, we deal specifically with $\phi = 1/8$). The proof is done by induction on i. For $i = 0$, $x \in (\phi l_x, l_x)$, $y \in (0, l_y)$, Proposition 2 gives

$$|u|(x,y) \leq c_3 c_1 \sigma^{1/3} l_x^{2/3} \leq 4 c_3 c_1 \sigma^{1/3} x \tag{2.58}$$

which is Eq. (2.1) with $d_3 = 4c_3 c_1$, and for $(a, a+h) \subset (0, l_y)$,

$$\begin{aligned} \int_0^x \int_a^{a+h} u_x^2 + \sigma |u_{yy}| &\leq \beta_u(l_x, a+h/2, h/2) + \frac{1}{l_x} \int_a^{a+h} u^2(l_x, y) dy \\ &\leq 2c_0 h l_x^{1/3} \sigma^{2/3} + c_1^2 c_3^2 \sigma^{2/3} l_x^{1/3} h \end{aligned} \tag{2.59}$$

which, since $l_x \leq 8x$, gives the upper bound of Eq. (2.2) with $d_4 = 4c_0 + 2c_1^2 c_3^2$. This concludes the proof for all $x \in (\phi l_x, l_x)$. Finally, observe that Lemma 10 gives the hypothesis needed to apply the same reasoning to the next rectangle, and therefore the inductive proof is completed. It remains to show that the numerical constants can be chosen so that all required relations between them are satisfied. This is done in the following

Definition 2 (Constants) *The constants c_0, c_1, c_2 and c_3 are any positive numbers which satisfy*

$$c_0 \geq 12 \cdot 18^{1/3} + \frac{20}{c_1} \tag{2.60}$$

(from Lemma 3),

$$c_3 \geq c_2 + \left(\frac{6c_0}{c_1^2}\right)^{1/3} \tag{2.61}$$

(from Lemma 9).

$$c_2 + c_3 + \frac{1}{c_1} \leq \frac{1}{2}\,, \tag{2.62}$$

(from Proposition 2),

$$\frac{c_1^2 c_2^3}{c_0} \geq 756 \tag{2.63}$$

(from Lemma 10).

For example, one can take $c_0 = 32$, $c_1 = 1500$, $c_2 = 0.225$, $c_3 = 0.27$; the constants entering Theorem 1 are then $d_3 = 1620 \geq 4c_3 c_1$ and $d_4 = 33 \times 10^4 \geq 4c_0 + 2c_1^2 c_3^2$.

In concluding the present Section we derive a bound on the local energy of u in rectangles which do not contain the L border, i.e. where the boundary condition $u_L = 0$ is lost. Consider the restriction of a minimizer u obeying the hypothesis of Theorem 1, to a region $D = (x - l, x) \subset (0, l_x)$, and define

$$t(y) = \int_D (u - u_l)_x^2 (x, y) dx\,. \tag{2.64}$$

The next Proposition, which is based on the result of Proposition 1 for the local upper bound on the energy in a domain with four arbitrary boundary conditions, allows one to control the deviation of the local energy from the linear part. The T and B boundary conditions stated in the hypothesis of Theorem 1 give immediately

$$t(l_y) \leq \int_D (u_x^T)^2 \leq 6c_0 l x^{-2/3} \sigma^{2/3} \leq 6c_0 l^{1/3} \sigma^{2/3} \tag{2.65}$$

(since $l < x$) and the same for $l(0)$. In the following we will need a weaker control of $t(0)$, $t(l_y)$, but we are going to use that u obeys the bound of Eq. (2.1) everywhere, to control the ω factor. Our main result in this respect is the following

Proposition 3 *For a given region $D = (x - l, x)$, and for any integer $n \geq 5$, if $t(y_0)$, $t(y_0 + h) \leq 2nc_0 l^{1/3}\sigma^{2/3}$, with $h \geq 4nc_6\sigma^{1/3}x^{2/3}$, then for all subsets $B = (a, a + d) \subset A = (y_0, y_0 + h)$ with $d \geq 2nc_6\sigma^{1/3}x^{2/3}$ one has*

$$\int_B t(y) + \int_{B\times D} \sigma|u_{yy}| \leq 2nc_0 l_x^{1/3}\sigma^{2/3} d\,, \tag{2.66}$$

where $c_6 = 2d_3 + 1$.

Proof. By Proposition 1,

$$\int_A t(y) + \int_{A\times D} \sigma|u_{yy}| \leq 2c_0 l^{1/3}\sigma^{2/3} h \tag{2.67}$$

since $\omega \leq (2d_3 + 1)\sigma^{1/3}x^{2/3} \leq h/4n$.

Clearly $\int_B \cdots \leq \int_A \ldots$, therefore if $d \geq h/n$ there is nothing to prove. Otherwise, divide A in n equal intervals, labelled A_i $(1 \leq i \leq n)$. The set B is contained in two consecutive ones, call them A_k and A_{k+1}. If $k > 1$, there is $y \in A_{k-1}$ such that

$$t(y) \leq \frac{n}{h}\int_A t(y) \leq 2nc_0\sigma^{2/3}l^{1/3}\,. \tag{2.68}$$

If $k = 1$, let $y = y_0$. Analogously, define $y' \in A_{k+2}$, or $y' = y_0 + h$. It is easily seen that the hypothesis of the present proposition are satisfied if the segment A is replaced by $A' = (y, y')$, and that $2h/n \leq h' = |y' - y| \leq 4h/n$. Therefore in a finite number of steps one reaches the case $h \leq nd$, and the proof is complete.

Note that under the hypothesis of Theorem 1, the present Proposition with $n = 5$ implies that for any $R' = (x - l, x) \times (a, a + h) \in R$, with $h \geq 10c_6\sigma^{1/3}x^{2/3}$, one has

$$\int_{R'} (u - u_l)_x^2 + |u_{yy}| \leq 10c_0\sigma^{2/3}l^{1/3}h \tag{2.69}$$

where only the constraint on h, and *not* the bound on the energy, depend on x. Here, u_l refers to the local linearization, i.e. within R', the bound on the full energy includes the additional term $\int_{R'} u_{lx}^2 \leq 2h \sup|u|^2/l$.

3 Self-similarity

In this Section we show that the minimizers are asymptotically self-similar around any point on the left boundary $(0, y_0)$, as suggested by Eq. (1.11). Fix u^0 to be a minimizer in a rectangle $R_0 = (0, l_x) \times (-l_y, l_y)$, satisfying the hypothesis of Theorem 1, and for a given sequence $\theta_j \to 0$, and $y_0 \in (-l_y, l_y)$ define

$$u^j(x, y) = \theta_j^{-2/3}u^0\left(\theta_j x, y_0 + \theta_j^{2/3}(y - y_0)\right)\,. \tag{3.1}$$

Note that u^j is, in general, defined on a domain bigger than R_0. To make the notation easier, we restrict any u^j to the largest symmetric rectangle of the form $R_j = (0, l_x) \times$

$(-h_j, h_j)$ which is included in its domain. For $y_0 \geq 0$, one gets $h_j = y_0 + (l_y - y_0)\theta_j^{-2/3}$. It is clear that $h_j \to \infty$, and that $R_0 \subset R_j$ for all j. Since we are in any case dealing with subsequences, we can assume that θ_j is nonincreasing (or, equivalently, h_j is nondecreasing).

The results of Theorem 1 are invariant under the scaling of Equation (3.1), and therefore for any $R' = (0, l) \times (a, a+h) \subset R_j$, with the usual condition $h \geq 2c_1\sigma^{1/3}l^{2/3}$, we have

$$I_{R'}[u^j] \leq d_4 l^{1/3}\sigma^{2/3}h \tag{3.2}$$

and

$$|u^j|(x,y) \leq d_3\sigma^{1/3}x^{2/3} \tag{3.3}$$

for all j. Since $|u_y| = 1$ it follows that $\{u^j\}$ has a uniform bound in $W^{1,2}(R')$. The following Lemma collects all the direct consequences of these bounds. Let $R_\infty = (0, l_x) \times \mathbf{R}$.

Lemma 11 *Given the sequence defined in (3.1), there is a subsequence $\tilde{u}_j$, and a function $u : R_\infty \to \boldsymbol{R}$ such that:*

(i) $\tilde{u}_j$ is a local minimizer in R_j (i.e. it minimizes I_{R_j} with respect to its own boundary conditions);

(ii) for any k, $\tilde{u}_j \rightharpoonup u$ in $W^{1,2}(R_k)$;

(iii) for any k, the restriction of u to R_k is in the set of admissible functions A_{R_k};

(iv) for any k, $(\tilde{u}_j)_y \to u_y$ in $L^2(R_k)$;

(v) for any k, $\tilde{u}_j \to u$ in $L^\infty(R_k)$.

We first construct a subsequence $\{\tilde{u}_k\}$ and a function $u : R_\infty \to \mathbf{R}$ such that for any j, $\tilde{u}_k \rightharpoonup u$ in $W^{1,2}(R_j)$ (u_k is defined in R_j for $k \geq j$). Fix k. Since the sequence is bounded in $W^{1,2}(R_k)$ there is a subsequence that weakly converges to some $u^{(k)} : R_k \to \mathbf{R}$. Then consider $k+1$. Within the previously defined subsequence, extract a further subsequence that weakly converges to $u^{(k+1)} : R_{k+1} \to \mathbf{R}$. Note that, within the common part of the domain (i.e. R_k), $u^{(k)} = u^{(k+1)}$. The sequence $\tilde{u}_k$ is then the diagonal subsequence obtained from this procedure, and the limit u is defined, for all k, in R_k by $u^{(k)}$. [In general, $\tilde{u}_k$ is defined on R_{j_k}, with $j_k \geq k$. By the monotonicity of h_j, $R_k \subset R_{j_k}$, and in the following we consider only its restriction to R_k.] This concludes the proof of points (i) and (ii).

The compactness result of Lemma 1 gives then points (iii) and (iv).

It remains to prove point (v), i.e. that $\tilde{u}_k$ converges strongly in L^∞. Let $a_j(x,y) = \tilde{u}_j(x,y) - u(x,y)$ and $b_j(x) = \int_{-h_k}^{h_k} a_j^2 dy$. Since $|(a_j)_y| \leq 2$, by Lemma 7 we only need to show that $b_j \to 0$ uniformly in x. First, note that b_j is uniformly Lipschitz. This follows from the estimate

$$\begin{aligned} |b_j(x_1)-b_j(x_2)| &\leq 2\sup|a_j|\int dy|a_j(x_1,y)-a_j(x_2,y)| \\ &\leq 2\sup|a_j||x_1-x_2|\int dy\int_{x_1}^{x_2}(a_j)_x^2 dx \end{aligned} \tag{3.4}$$

since both $|a_j|$ and the last integral have a uniform bound by Theorem 1. But we know that $b_j \to 0$ in L^1 (this is equivalent to L^2 convergence of u_j, which follows from weak $W^{1,2}$ convergence), and therefore $b_j \to 0$ in L^∞. This concludes the proof of the Lemma.

The proof of self-similarity is based on an explicit construction, whose basic ingredients are given in Lemmas 12, 13 and 14 below. The outcome of the construction is the following

Proposition 4 *Let u and v be two admissible functions satisfying the bounds of Theorem 1 and Proposition 3, defined in a region containing the rectangle $R=(0,l_x)\times(-h,h)$, with $h \geq 2c_4\sigma^{1/3}l_x^{2/3}$ and $s=\sup_R|u-v|\leq l_x^2$. Then, if v is a local minimizer, one has*

$$I_R[v] \leq I_R[u]+\frac{\sigma}{2}A_y+CA_x^{1/2}+CA_x+Cs^{1/6} \tag{3.5}$$

where C depends only on l_x and σ, and

$$A_x=\frac{d}{dh}\int_{-h}^{h}\int_0^{l_x}(u-v)_x^2 dxdy\,, \qquad A_y=\frac{d}{dh}\int_{-h}^{h}\int_0^{l_x}(u-v)_y^2 dxdy\,. \tag{3.6}$$

Proof. The proof is done by explicitly constructing a function $\bar u$ so that it equals u in the interior part of R, except for a thin boundary region, and $\bar u = v$ on the boundary of R, and comparing its energy with that of v. First apply Lemma 14 below to the two regions $(0,l_x)\times(h-2l_y,h)$ and $(0,l_x)\times(-h,-h+2l_y)$, with $l_y=c_4\sigma^{1/3}l_x^{2/3}$. This gives a function $\tilde u$ such that $\tilde u = v$ for $y=\pm h$, $\tilde u = u$ for $|y|\leq h-2l_y$, with energy bounded by

$$I_{R_h}[\tilde u] \leq I_{R_h}[u]+c_5(A_x\sigma^{2/3}l_x^{1/3})^{1/2}l_y+2A_xl_y\,. \tag{3.7}$$

Now consider a narrow region $R'=(l_x-l_0)\times(-h,h)$, with l_0 to be chosen later, and apply Lemma 13, with $u_1=v$ and $u_2=\tilde u$. The energy of v is in this region bounded by Eq. (2.69). We therefore obtain a function $\bar u$ such that

$$I_{R_h}[\bar u] \leq I_{R_h}[u]+c_5A_x^{1/2}\sigma^{1/3}l_x^{1/6}l_y+2A_xl_y+\frac{8hs^2}{l_0}+4\sigma l_0+8\frac{hs}{l}d_3\sigma^{1/3}l_x^{2/3}+10c_0\sigma^{2/3}l_0^{1/3} \tag{3.8}$$

with $\bar u = v$ along the entire boundary of R_h. It follows that

$$I_{R_h}[v] \leq I_{R_h}[\bar u]+\frac{\sigma}{2}A_y \tag{3.9}$$

where the last term corresponds to the surface energy cost of the matching, and is bounded by $4\sigma l_x$.

By choosing $l_0 = s^{1/2}$ we can see that all terms after the first two in the RHS of Eq. (3.8) are infinitesimal, and since s is uniformly bounded by l_x their sum is controlled by $Cs^{1/6}$, with C depending on l_x and σ only. This gives the thesis.

We are now ready to state the main result of this Section,

Theorem 2 (self-similarity) *Given a minimizer u^0, and a sequence $\theta_j \to 0$, the sequence*

$$u^j(x,y) = \theta_j^{-2/3} u^0 \left(\theta_j x, y_0 + \theta_j^{2/3}(y - y_0) \right) \tag{3.10}$$

has a strongly converging (in $W^{1,2}(R)$) subsequence, $u^j \to u^*$. The limit u^* is a local minimizer, and $I[u^*] = \lim I[u^j]$.

Proof. We start from the subsequence $\{\tilde{u}_j\}$ obtained in Lemma 11, and take a further subsequence, that we will denote again by $\{u^j\}$, such that

$$|u - u^j| \leq \frac{l_x^{1/2}}{j} \tag{3.11}$$

and

$$\int_{R_j} (u^j - u)_y^2 \leq \frac{1}{j}\,. \tag{3.12}$$

Since $h_j \to \infty$, we can safely assume $h_j \geq 15(2d_3 + 1)$ (this will be needed to apply Proposition 3).

In order to obtain a bound on $\int_R (\nabla u^k - \nabla u)^2$ (for $k \geq j$) from the result of Proposition 4, we write

$$\int_{R_h} (\nabla u^k - \nabla u)^2 = \int_{R_h} (u_y^k - u_y)^2 + I_{R_h}[u^k] - I_{R_h}[u] - b_k(h)\,, \tag{3.13}$$

where

$$b_k(h) = \int_{-h}^{h} \int_0^{l_x} 2u_x(u_x^k - u_x) + \sigma(|u_{yy}^k| - |u_{yy}|) dx dy\,. \tag{3.14}$$

Consider the limit of $b_k(h)$ for $k \to \infty$, with j fixed. The first term converges to zero, because of weak convergence of u_x^k. The second term has a nonnegative lim inf, because it is lower semicontinuous. Hence the sequence $f_k(h) = \min(0, b_k(h))$ converges pointwise to 0, and therefore in measure. Let k_j and $\Omega_1 \subset (2h_j/3, h_j)$, $\Omega_2 \subset (h_j/3, 2h_j/3)$ be such that $|\Omega_{1,2}| \geq h_j/6$, and

$$b_{k_j}(h) \geq f_{k_j}(h) \geq -\frac{1}{j} \tag{3.15}$$

for $h \in \Omega_{1,2}$. In the following we denote k_j simply by k.

Now define $A_x^{(k)}(h)$ and $A_y^{(k)}(h)$ as in Eq. (3.6). Since by Theorem 1 $\int_0^{h_j} A_x^{(k)}(h) \leq 2d_4 l_x^{1/3}\sigma^{2/3}h_j$, we can find $h^{(1)} \in \Omega_1$ such that

$$A_x^{(k)}(h^{(1)}) \leq 12 d_4 l_x^{1/3}\sigma^{2/3} \tag{3.16}$$

which is a bound depending only on l_x, not on k.

From Proposition 4, applied to the domain $R' = (0, l_x) \times (-h^{(1)}, h^{(1)})$ with $v = u_{k_j}$, and using the bound $A_y \leq 8l_x$, we get

$$I_{R'}[u_{k_j}] - I_{R'}[u] \leq C \tag{3.17}$$

where C depends only on l_x and σ. Combining this with Eqs. (3.15) and (3.12) we get

$$\int_0^{l_x}\int_{-h^{(1)}}^{h^{(1)}} (\nabla u^k - \nabla u)^2 \leq C + \frac{2}{j} \leq C + 2\,. \tag{3.18}$$

This implies that we can choose $h^{(2)} \in \Omega_2$ such that

$$A_x(h^{(2)}) + A_y(h^{(2)}) = \frac{d}{dh}\int_{-h}^{h} (\nabla u^k - \nabla u)^2 \Big|_{h=h^{(2)}} \leq \frac{6C+12}{h_j} \tag{3.19}$$

which is infinitesimal for $j \to \infty$. From Proposition 4 applied to the domain $R'' = (0, l_x) \times (-h^{(2)}, h^{(2)})$, we have

$$I_{R''}[u_{k_j}] - I_{R''}[u] \leq \xi_j \tag{3.20}$$

with $\xi_j \leq C'(h_j^{-1/2} + j^{-1/6})$ infinitesimal. Using (3.15), we finally obtain

$$\int_R (\nabla u^k - \nabla u)^2 \leq \int_{R''} (\nabla u^k - \nabla u)^2 \leq \xi_j + \frac{1}{j} \tag{3.21}$$

which proves that $u^{k_j} \to u$ strongly in $W^{1,2}(R)$.

It remains to be shown that $I_R[u] = \lim I_R[u_{k_j}]$. Define, in analogy with (3.14),

$$c_k(h) = I_{R_h \setminus R}[u_k] - I_{R_h \setminus R}[u]\,. \tag{3.22}$$

Again, it is clear that $\liminf c_k(h) \geq 0$ for all h, therefore $\min(0, c_k(h))$ converges in measure, i.e. we can find $\Omega_2' \subset \Omega_2$ such that $|\Omega_2'| \geq h_j/8$ and

$$c_k(h) \geq -\zeta_k \tag{3.23}$$

for all $h \in \Omega_2'$, with $\zeta_k \to 0$. Then, by choosing $h^{(3)} \in \Omega_2'$ as above, we get

$$I_{R'''}[u_{k_j}] - I_{R'''}[u] \leq \xi_{k_j} \tag{3.24}$$

with ξ_{k_j} infinitesimal and $R''' = (0, l_x) \times (-h^{(3)}, h^{(3)})$. We finally conclude that

$$I_R[u_{k_j}] - I_R[u] \leq \xi_{k_j} + \zeta_{k_j} \tag{3.25}$$

which implies that $\limsup I_R[u_{k_j}] \leq I_R[u]$. Combining this with lower semicontinuity we get the thesis. With similar arguments one then proves that u is a local minimizer. This concludes the proof of Theorem 2.

The next lemmas deal with the construction which we used in the proof of Proposition 4, to match the limiting u to a function u_j across the boundary of some rectangle R_k. We first consider the matching on the R boundary (Lemmas 12 and 13) and then the T boundary (Lemma 14).

Lemma 12 *Assume u_1 and u_2 are two admissible functions on a rectangle $R = (0, l) \times (0, h)$, and let $u = \max(u_1, u_2)$. Then,*

$$\int u_x^2 \leq \int (u_1 - u_{1l})_x^2 + \int u_{2x}^2 + \frac{4hs^2}{l} + \frac{4hs}{l} \sup_R |u_{1,2}|\,, \tag{3.26}$$

where $s = \sup |u_1 - u_2|$, and

$$\int |u_{yy}| \leq \int |u_{1yy}| + \int |u_{2yy}| + 2\sigma l\,. \tag{3.27}$$

Proof. Since

$$\int (u - u_l)_x^2 \leq \int (u_1 - u_l)_x^2 + (u_2 - u_l)_x^2 \tag{3.28}$$

using $|u_{(1,2)l} - u_l| \leq s$, $\int (u_l - u_{1l})_x^2 \leq 2hs^2/l$, $|\int u_{2lx}^2 - u_{lx}^2| \leq 4hs \sup |u_{1,2}|/l$ we get the first part of the thesis. The result for the surface energy follows from the fact that for any fixed x, the set $J_{1,2}$ of jumps in $u_y = \max(u_1, u_2)_y$ is composed by some points of the sets J_1 (J_2) where u_{1y} (u_{2y}) jumps, plus points where u_1 crosses u_2 (in the (y, u) plane). Given the constraints $|u_{1y}| = |u_{2y}| = 1$, the latter are at most one plus the number of points in $(J_1 \cup J_2) \setminus J_{1,2}$, from which the conclusion follows.

Lemma 13 *Assume u_1 and u_2 are two admissible functions on a rectangle $R = (0, l) \times (0, h)$ with*

$$|u_1 - u_2| \leq s \tag{3.29}$$

on R, and that $u_1 = u_2$ on T and B. Then there is a function u such that $u = u_1 = u_2$ on T and B; $u = u_1$ on L, $u = u_2$ on R, with energy

$$I_R[u] \leq 2I_R^l[u_1] + I_R[u_2] + 8\frac{hs^2}{l} + 4\sigma l + 8\frac{hs}{l} \sup_R |u_{1,2}| \tag{3.30}$$

where $I_R^l[u]$ was defined in (2.3).

Proof. The function u is

$$u(x,y) = \min\left[u_1 + s\frac{x}{l}, \max\left(u_1 - s\frac{x}{l}, u_2\right)\right] . \tag{3.31}$$

On the T, B and R boundaries $u = u_2$. On the L boundary $u = u_1$, since $\min(f, \max(f,g)) = f$. The result follows from Lemma 12, since

$$I_R^l\left[u - s\frac{x}{l}\right] = I_R^l[u] . \tag{3.32}$$

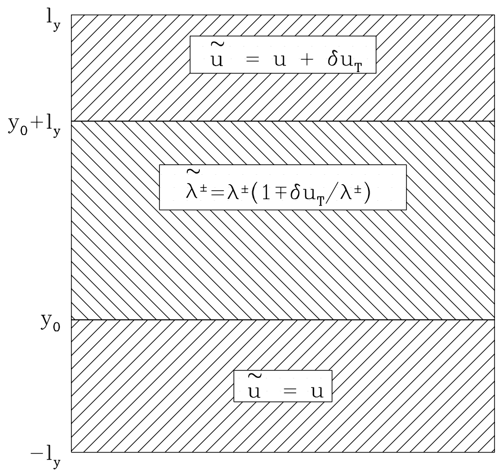

FIGURE 3.1: Subdivision of the rectangle used in Lemma 14.

Lemma 14 *Consider a box* $R = (0, l_x) \times (-l_y, l_y)$*, an admissible function* u *obeying the bounds of Theorem 1, and an additional "top" boundary condition* u^T *such that* $u^T(0) = u(0, l_y) = 0$*, with* $l_y \ge c_4 \sigma^{1/3} l_x^{2/3}$ *(we need* $c_4 = \max(4d_3, c_1)$*). Let*

$$A = \int_0^{l_x} \left[u^T(x) - u(x, l_y)\right]_x^2 dx , \tag{3.33}$$

and assume that, as in Theorem 1, $|u|$ *and* $|u^T|$ *are bounded by* $d_3\sigma^{1/3}l_x^{2/3}$*. Then, there is an admissible function* $\tilde{u}$ *such that* $\tilde{u} = u$ *and* $\tilde{u}_y = u_y$ *for* $y = -l_y$*,* $\tilde{u} = u$ *for* $y = l_y$*,* $\sup|u - \tilde{u}| \le A^{1/2} l_x^{1/2}$*, and*

$$I_R[\tilde{u}] \le I_R[u] + c_5 A^{1/2} \sigma^{1/3} l_x^{1/6} l_y + 2A l_y , \tag{3.34}$$

where c_5 *is a numerical constant.*

Proof. We construct $\tilde{u}$ explicitly. If $a = A\sigma^{-2/3} l_x^{-1/3} \geq 1$ we use the same construction as in Proposition 1, and let

$$\tilde{u} = \max(u^T - (l_y - y), \min(u^T + (l_y - y), u)) \,. \tag{3.35}$$

This obviously gives $I[\tilde{u}] \leq I[u] + 2Al_y + 2\sigma l_x$, hence the thesis.

Consider now the case $a < 1$. Since $\int_R u_x^2 \leq 2d_4 \sigma^{2/3} l_x^{1/3} l_y$, we can pick $y_0 \in [-l_y, 0]$ such that

$$\int_0^{l_x} [u_x(x, y_0)]^2 + [u_x(x, l_y + y_0)]^2 dx \leq 2d_4 \sigma^{2/3} l_x^{1/3} \,. \tag{3.36}$$

For $y \in [-l_y, y_0]$ we take $\tilde{u} = u$, and for $y \in [y_0 + l_y, l_y]$ we take $\tilde{u}(x,y) = u(x,y) + \delta u_T(x)$, with $\delta u_T = u_T(x) - u(x, l_y)$ (see Figure 3.1). Let $a = A\sigma^{-2/3} l_x^{-1/3}$. From the relation $(\alpha + \beta)^2 \leq (1 + \eta)\alpha^2 + 2\beta^2/\eta$, valid for $\eta \in (0,1)$, we get for $\eta = a^{1/2}$

$$\int_{y_0 + l_y}^{l_y} \int_0^{l_x} \tilde{u}_x^2 \leq (1 + a^{1/2}) \int_{y_0 + l_y}^{l_y} \int_0^{l_x} u_x^2 + 2a^{1/2} l_y \sigma^{2/3} l_x^{1/3} \,. \tag{3.37}$$

Consider now the region $y_0 \leq y \leq y_0 + l_y$. The construction exploits the geometric nature of the problem. In particular, we construct $\tilde{u}$ stretching the domains where $u_y = 1$ (-1) by an amout proportional to δu^T $(-\delta u^T)$. This allows one to join u to u^T without changing $\int |u_{yy}|$, i.e. the number of domains. To be precise, consider a fixed x, and label λ_i^+ the widths of the regions where $u_y = 1$, and λ_i^- those with $u_y = -1$ (see Figure 3.2). The total number of intervals is given by $N(x) + 1$, where $N(x) = \frac{1}{2}\int |u_{yy}| dy$ is the number of discontinuity points of u_y. It is clear that u is fully determined by its value for $y = y_0$ and by the set of functions $\{\lambda_i^\pm(x)\}$. We define $\tilde{u}$ such that $\tilde{u}(x, y_0) = u(x, y_0)$, and

$$\tilde{\lambda}_i^\pm(x) = \lambda_i^\pm(x) \left(1 \pm \frac{\delta u^T(x)}{2\lambda^\pm(x)}\right) \tag{3.38}$$

where $\lambda_\pm = \sum_i \lambda_i^\pm$. Since $\tilde{\lambda}_\pm = \lambda_\pm \pm \frac{1}{2}\delta u^T$, we have $\tilde{u}(x, y_0 + l_y) = u(x,y) + \delta u_T(x)$. Note that since $\lambda^+ + \lambda^- = l_y$, $\lambda_+ - \lambda_- = u(x, y_0 + l_y) - u(x, y_0)$, and $l_y \geq 4 \sup |u|$, we have $\lambda^\pm \geq l_y/4$.

We now estimate $\int \tilde{u}_x^2$ in this region. Define $f_i^\pm(x)$ as the position where the (prolongation of the) line segment corresponding to $\lambda_i^\pm$ intersects the $u = 0$ axis in the (y, u) plane, so that within the interval $(i, \pm)$ one has $u(x,y) = f_i^\pm(x) \pm y$ (see Figure 3.2). Straightforward computations show that

$$f_i^\pm = \mp u(x, y_0) + 2 \sum \lambda_j^\mp(x) \tag{3.39}$$

where the sum is done only over intervals located at smaller y. Then, in the i-th interval u_x is given by

$$\tilde{f}_{i,x}^\mp = f_{i,x}^\mp \pm 2 \frac{d}{dx} \left[\frac{\delta u^T}{2\lambda^\pm} \sum_{j=0}^{i} \lambda_j^\pm\right] \tag{3.40}$$

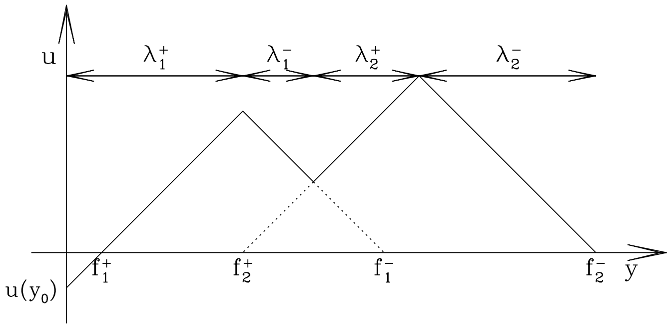

FIGURE 3.2: Representation in the (y, u) plane illustrating the definition of $\lambda_i^{\pm}$ and $f_i^{\pm}$.

and the elastic energy takes the form

$$\int_R (\tilde{u}_x)^2 = \int_0^{l_x} dx \sum_{i,\pm} \tilde{\lambda}_i^{\pm}(x)(\tilde{f}_{i,x}^{\pm})^2(x) \tag{3.41}$$

where the sum is done both over i and the sign. The square is expanded as before, $\tilde{f}_x^2 \leq (1 + a^{1/2}) f_x^2 + 2a^{-1/2}(\tilde{f} - f)_x^2$. The three terms coming from expansion of the derivative are bounded using the estimate $|\delta u_T| \leq (l_x A)^{1/2}$, which is obtained from Eq.(3.33) and the condition $\delta u_T(0) = 0$, and the results $\lambda_+ + \lambda_- = l_y$, $\lambda_+ - \lambda_- = u(x, y_0 + l_y) - u(x, y_0)$, as well as Eqs. (3.33), (3.36). The remaining is just straightforward algebra.

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References

[1] G. Arlt, *Twinning in ferroelectric and ferroelastic ceramics: stress relief*, J. Mater. Sci. **22** (1990), 2655–2666.

[2] J. Ball and R. D. James, *Fine phase mixtures as minimizers of the energy*, Arch. Ration. Mech. Analysis **100** (1987), 13–52.

[3] ———, *Proposed experimental tests of a theory of fine microstructure and the two-well problem*, Phil. Trans. R. Soc. Lond. A **338** (1992), 389–450.

[4] G. R. Barsch, B. Horovitz, and J. A. Krumhansl, *Dynamics of twin boundaries in martensites*, Phys. Rev. Lett. **59** (1987), 1251–1254.

[5] R. Choksi and R. V. Kohn, *Bounds on the micromagnetic energy of a uniaxial ferromagnet*, Comm. Pure Appl. Math. **51** (1998), 259–289.

[6] R. Choksi, R. V. Kohn, and F. Otto, *Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy*, Comm. Math. Phys. **201** (1999), 61–79.

[7] S. Conti, *Asymptotic self similarity in a model of branching in microstructured materials*, Proceedings of the Equadiff 99 conference held in Berlin, August 1999, World Scientific, 2000.

[8] B. Horovitz, G. R. Barsch, and J. A. Krumhansl, *Twin bands in martensites: Statics and dynamics*, Phys. Rev. B **43** (1991), 1021–1033.

[9] A. G. Khachaturyan, *Some questions concerning the theory of phase transformations in solids*, Soviet Phys. Solid State **8** (1967), 2163–2168.

[10] ———, *Theory of structural transformations in solids*, Wiley, New York, 1983.

[11] A. G. Khachaturyan and G. A. Shatalov, *Theory of macroscopic periodicity for a phase transition in the solid state*, Soviet Phys. JETP **29** (1969), 557–561.

[12] R. V. Kohn and S. Müller, *Branching of twins near an austenite-twinned-martensite interface*, Phil. Mag. A **66** (1992), 697–715.

[13] ———, *Surface energy and microstructure in coherent phase transitions*, Comm. Pure Appl. Math. **47** (1994), 405–435.

[14] A. L. Roitburd, *The domain structure of crystals formed in the solid phase*, Soviet Phys. Solid State **10** (1969), 2870–2876.

[15] ———, *Martensitic transformation as a typical phase transformation in solids*, Solid State Physics, vol. 33, Academic Press, New York, 1978, pp. 317–390.