Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Conditions for equality of hulls in the calculus of variations

by

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Preprint no.: 1 2000

CONDITIONS FOR EQUALITY OF HULLS IN THE CALCULUS OF VARIATIONS

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ABSTRACT. We simplify and sharpen several results by K. Zhang concerning properties of quasiconvex hulls of sets and quasiconvex envelopes of their distance functions. The approach emphasizes the underlying geometry and in particular we show that $K^{pc} = K^c$ implies $K^{rc} = K^c$ if and only if $\min\{m, n\} \leq 2$ thus answering a question raised in [Z2].

This paper addresses a surprising relation between semiconvex hulls of compact sets $K \subset \mathbb{M}^{m\times n}$ first noted by Zhang [Z2], namely that $K^{qc} = K^c$ implies $K^{rc} = K^c$ and thus $K^{rc} = K^{qc}$. Our approach simplifies the original proof given by Zhang and emphasizes more the underlying geometry. As pointed out to us by D. Preiss [P], this implication constitutes a nonlinear version of the result that subspaces without rank-one directions do not support any nontrivial gradient Young measure (see [BFJK], Theorem 4.1). In particular, based on results by [T, Se, B], this equivalence allows us to show that Zhang's statement holds for the polyconvex hull if and only if $\min\{m, n\} \leq 2$, see Theorem 3 and the following remark. Thus the rank-one convex and the quasiconvex hulls agree for special sets K with $K^{qc} = K^c$, despite of the fact that rank-one convexity and quasiconvexity are known to be different concepts for $m \geq 3$ ([Sv1]).

In general the quasiconvex hull of a compact set is equal to the zero set of the quasiconvexification of the distance function $d_{K,p}(x) = \inf\{|x - y|^p : y \in K\}$, $p \in [1, \infty)$ (see [Z3]), and this motivates to study general properties of distance functions and their semiconvex envelopes. We show in Example 5 that one cannot expect a uniform growth of $d^{qc}_{K,p}(\cdot)$ close to K^{qc}. Moreover, Example 6 demonstrates that smooth sets need not have smooth quasiconvex hulls. Both results indicate that numerical schemes are likely to produce unreliable results, unless the boundary of the set is well resolved.

Our last result, Theorem 9, proves the surprising fact that the distance function itself cannot be used as an indicator of whether a given set is quasiconvex or not. In fact, $d_{K,p}(\cdot)$ is separately convex (i.e. convex in each argument, a property implied by quasiconvexity) on $\mathbb{M}^{m\times n} \setminus B(0, R)$, R arbitrarily large, if and only if K is convex. This was first proven in [Z3] for $p \leq 2$ under stronger assumptions on the set K. Our selfcontained proof relies entirely on the well-known fact that the metric projection onto K is unique if and only if K is convex.

Notation. Let $\mathbb{M}^{m\times n}$ be the space of all real $(m \times n)$–matrices. We use $|\cdot|$ for the Euclidean norm on $\mathbb{M}^{m\times n}$, i.e. $|x|^2 = \mathrm{tr}(x^T x)$. By $\dim(K)$ we understand the affine dimension of the set $K \subset \mathbb{M}^{m\times n}$. We write

$$B(K, \varepsilon) = \{x \in \mathbb{M}^{m\times n} : d_{K,1}(x) \leq \varepsilon\}, \quad U(K, \varepsilon) = \{x \in \mathbb{M}^{m\times n} : d_{K,1}(x) < \varepsilon\}.$$

A function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ is called polyconvex if there exists a convex function g of x and the vector $M(x)$ of all minors of x (i.e. all subdeterminants of x) such that $f(x) = g(x, M(x))$. It is

Date: December 31, 1999.

1991 *Mathematics Subject Classification.* 26B15, 74N15.

Key words and phrases. Gradient Young measures, quasiconvex hull, distance function.

GD, BK and JK were supported by the EU TMR project on 'Phase Transitions in Crystalline Solids' FMRX-CT98-0229 (DG 12-BDCN). JK gratefully acknowledges partial support from the Max Planck Institute for Mathematics in the Sciences, Leipzig, and from EPSRC through grant GR/L90262.

said to be quasiconvex if for all $x \in \mathbb{M}^{m\times n}$ and all smooth functions $\varphi : Q = [0,1]^n \to \mathbb{R}^m$ with $\varphi = 0$ on ∂Q the inequality

$$\int_Q f(x + D\varphi(z))dz \geq \int_Q f(x)dz$$

holds. Finally, f is rank-one convex if f is convex on rank-one lines, i.e., the functions $t \mapsto g(t) = f(x + tr)$ are convex in t for all $x, r \in \mathbb{M}^{m\times n}$, $\text{rank}(r) = 1$. A fundamental result in the calculus of variations states that a variational integral $I(u) = \int f(Du)dz$ is weakly sequentially lower semicontinuous in suitable function spaces if and only if f is quasiconvex. Polyconvexity is a sufficient condition for quasiconvexity while rank-one convexity is a necessary one.

For any function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ we call

$$f^{qc}(x) = \sup\{g(x) : g : \mathbb{M}^{m\times n} \to \mathbb{R} \text{ quasiconvex and } g \leq f\}$$

the quasiconvex envelope of f; the rank-one convex and the polyconvex envelope are defined analogously. In analogy to one possible formula for the (closed) convex hull of a compact set K, denoted by K^c, we define its (closed) quasiconvex hull by

$$K^{qc} = \{x : f(x) \leq \sup_{y\in K} f(y) \text{ for all } f : \mathbb{M}^{m\times n} \to \mathbb{R} \text{ quasiconvex}\}.$$

A corresponding definition holds for the rank-one convex hull K^{rc} and the polyconvex hull K^{pc}. It is a useful fact that K^{qc} can also be defined as the set of barycenters of gradient Young measures supported on K. In addition to these hulls related to classes of semiconvex functions we define the closed lamination convex hull $L_c(K)$ of K as the closure of the set $L(K) = \bigcup_{i=0}^{\infty} L_i(K)$ where $L_0(K) = K$ and

$$L_{i+1}(K) = \big\{x = \lambda a + (1-\lambda)b : \lambda \in [0,1],\, a,\, b \in L_i(K),\, \text{rank}(a-b) \leq 1\big\}.$$

These definitions imply immediately that $L_c(K) \subset K^{rc} \subset K^{qc} \subset K^{pc} \subset K^c$ and it is an open question whether $K^{rc} = K^{qc}$ in $\mathbb{M}^{2\times 2}$. See [Mu] for a discussion of these notions and their relations.

Lemma 1. *Let $K \subset \mathbb{M}^{m\times n}$ be compact, contained in the affine subspace X and let H be a hyperplane in X supporting K^c. Then $(K \cap H)^{qc} = K^{qc} \cap H$ and $(K\cap H)^c = K^c \cap H$.*

Proof. By definition our assumption means that there exists a linear $f : X \to \mathbb{R}$ such that $H = \{x \in X : f(x) = \max f(K^c)\}$. Given any $x \in K^{qc} \cap H$, we find a gradient Young measure μ such that $\text{spt}(\mu) \subset K$ and $x = \int z\, d\mu(z)$. Since

$$f(x) = \int f(y)\, d\mu(y) \leq \int \max f(K^c)\, d\mu(y) = f(x),$$

we conclude $\text{spt}(\mu) \subset H \cap K$. Consequently, $x \in (K \cap H)^{qc}$. The reverse inclusion is obvious and the same argument with probability measures instead of gradient Young measures proves the result for the convex hull. ☐

The following result was first proven in [Z2]. We include a short proof for the convenience of the reader.

Theorem 2. *Let $K \subset \mathbb{M}^{m\times n}$ be compact. If $K^{qc} = K^c$ then $L_{\dim(K)}(K) = K^c$.*

Proof. Otherwise we find among all compact sets for which the conclusion fails a set K_0 of minimal affine dimension $d_0 \geq 1$. Let X be the d_0-dimensional affine subspace containing K_0. We choose

$$x_0 \in K_0^c \setminus L_{d_0}(K_0).$$

If X does not contain any rank-one line then $K_0 = K_0^{qc}$ by [BFJK, Theorem 4.1] and hence $K_0^c = K_0^{qc} = L_0(K_0)$, a contradiction. Therefore we find in X a rank-one line ℓ through x_0 and a point

$$x_1 \in (\ell \cap \partial_X K_0^c) \setminus L_{d_0-1}(K_0)$$

(if $x_0 \in \partial_X K_0^c$ then we may choose $x_1 = x_0$). Let $H \subset X$ be a supporting hyperplane of K_0^c through x_1. We have by Lemma 1 for $K_1 = H \cap K_0$ that

$$K_1^{qc} = H \cap (K_0)^{qc} = H \cap K_0^c = K_1^c,$$

but

$$x_1 \in K_1^c \setminus L_{d_0-1}(K_0) \subset K_1^c \setminus L_{d_0-1}(K_1).$$

This contradicts the minimality of d_0 and proves the proposition. □

Motivated by a remark by D. Preiss we notice the following. Theorem 2 was proved using the result from [BFJK] that in subspaces without rank-one directions all sets are quasiconvex. It turns out that this result is also implied by Theorem 2.

Theorem 3. *Assume that X is a linear subspace of $\mathbb{M}^{m\times n}$ not containing any rank-one line. Suppose that for every $K \subset X$ compact $K^{pc} = K^c$ ($K^{qc} = K^c$) implies $L_c(K) = K^c$. Then every compact subset of X is polyconvex (quasiconvex).*

Proof. We consider the case of polyconvexity only, the argument in the quasiconvex situation is similar. Assume that the conclusion fails. Then we can fix $K \subset X$ compact and $x \in K^{pc} \setminus B(K, 3\varepsilon)$ for some $\varepsilon > 0$. Let $\tilde{K} = (B(K,\varepsilon) \cap X)^c \setminus U(x,\varepsilon)$; then $\tilde{K}$ is compact, and obviously $\tilde{K} \supset B(K,\varepsilon) \cap X$. Since taking the polyconvex hull commutes with translations we obtain

$$\tilde{K}^c \subset (B(K,\varepsilon) \cap X)^c \subset \tilde{K} \cup (B(x,\varepsilon) \cap X) \subset \tilde{K} \cup (B(K,\varepsilon) \cap X)^{pc} \subset \tilde{K}^{pc}.$$

The opposite inclusion is true in general, hence $\tilde{K}^c = \tilde{K}^{pc}$. On the other hand, $x \in \tilde{K}^c \setminus \tilde{K}$ shows that $\tilde{K}^c \neq \tilde{K}$ but by our assumption on X we have $\tilde{K} = L_c(\tilde{K}) = \tilde{K}^c$, a contradiction. □

Remark. This equivalence allows us to answer a question raised in [Z2] whether the polyconvex version of Theorem 2 holds: if $K^{pc} = K^c$, then $L_{\dim(K)}(K) = K^c$.

The paper [B] contains an example of a subspace of $\mathbb{M}^{3\times 3}$ without rank-one matrices but with a subset that is not polyconvex. On the other hand, as pointed out in [B] the results from [T] (see also [Se]) imply the following: Whenever a linear subspace X of $\mathbb{M}^{m\times 2}$ or $\mathbb{M}^{2\times m}$ does not contain any rank-one line then all compact subsets of X are polyconvex. This shows that the polyconvex variant of Theorem 2 is true if and only if $\min(m,n) \le 2$. □

As in [Z2] we obtain easily a comparision principle for distance functions. By definition, $d_K^{qc} \le d_{K\cap H}^{qc}$, but there holds also a reversed verion of this inequality.

Proposition 4. *Let $p \in [1,\infty)$, $K \subset \mathbb{M}^{m\times n}$ be compact and let H be a supporting hyperplane of K. Then there is a non-decreasing, continuous function $W_H : [0,\infty) \to [0,\infty)$ such that $W_H(0) = 0$ and*

$$d_{K\cap H,p}^{qc}(x) \le W_H(d_{K,p}^{qc}(x)) \text{ for all } x \in H.$$

Proof. Assume otherwise. Then there exists a positive ε and a sequence $\{x_k\}_{k=1}^{\infty} \in H$ such that $d_{K\cap H,p}^{qc}(x_k) > \varepsilon$ but $d_{K,p}^{qc}(x_k) < 1/k$ for all $k \ge 1$. Since $d_{K,p}^{qc}(\cdot)$ majorises the convex function $x \to (\max\{0, |x| - \max\{|y| \; ; \; y \in K\}\})^p$, we can suppose that $x_k \to x_0 \in H$. Then obviously $d_{K,p}^{qc}(x_0) = 0$ but $d_{K\cap H,p}^{qc}(x_0) \ge \varepsilon$, which implies (see [Z1]) that $x_0 \in K^{qc} \cap H \setminus (K \cap H)^{qc}$. This contradiction to Lemma 1 finishes our proof. □

The argument just used also proves that for all quasiconvex, compact sets K and all $p \in [1,\infty)$ there exist non-decreasing, continuous functions $W_{K,p} : [0,\infty) \to [0,\infty)$ such that $W_{K,p}(0) = 0$ and

$$d_{K,p}(x) \le W_{K,p}(d_{K,p}^{qc}(x)) \text{ for all } x \in \mathbb{M}^{m\times n}.$$

Our next example shows that the quantifiers cannot be interchanged.

Example 5. *For all $p \in [1,\infty)$ and all non-decreasing, continuous functions $W : [0,\infty) \to [0,\infty)$ with $W(0) = 0$ there exist a compact quasiconvex set K and a sequence $\{x_k\}$ in $\mathbb{M}^{2\times 2}$ such that*

$$d_{K,p}(x_k)/W(d^{qc}_{K,p}(x_k)) \to \infty \text{ and } d_{K,p}(x_k) \to 0.$$

Proof. In the construction we use the isometric embedding of $\mathbb{R}^2$ into $\mathbb{M}^{2\times 2}$ which identifies x with $\operatorname{diag}(x_1, x_2)$. Choose a sequence $\{b_k\}_{k=1}^{\infty}$ strictly decreasing to zero such that $W(b_k^p) < 2^{-p(k+2)}/k$. Now we define

$$K = \{(2^{-k}, b_k) \ : \ k \geq 1\} \cup \{(0,0)\} \text{ and } x_k = (3 \cdot 2^{-k-2}, b_{k+1}).$$

Obviously, $d_{K,p}(x_k) \geq \operatorname{dist}(3 \cdot 2^{-k-2}, \operatorname{proj}_1(K))^p \geq 2^{-(k+2)p}$, where proj_1 denotes the projection $x \mapsto x_{11}$. Since $(1,0)$ is a rank-one direction, we can estimate

$$d^{rc}_{K,p}(x_k) \leq \frac{1}{2}\Big(d_{K,p}(2^{-k-1}, b_{k+1}) + d_{K,p}(2^{-k}, b_{k+1})\Big) \leq (b_k - b_{k+1})^p \leq b_k^p,$$

and hence $d_{K,p}(x_k)/W(d^{rc}_{K,p}(x_k)) \geq 2^{-(k+2)p}/W(b_k^p) \geq k$. On the other hand K is quasiconvex (even polyconvex) since $\det(x-y) > 0$ for any $x, y \in K$, $x \neq y$, see [Sv2]. □

Example 6. *There exists a smooth set $U \subset \mathbb{M}^{2\times 2}$ such that U^{qc} is not smooth, in the sense that $Tan(U^{qc}, x)$ (see e.g.* [F, Definition 4.3]*), is not a linear subspace for a suitable $x \in U^{qc}$. This contrasts with the situation for convex hulls, where it is easily seen that the convex hull of a smooth compact set is again smooth.*

To keep the calculations simple, we use conformal and anticonformal coordinates, i.e., we make the identification

$$x \in \mathbb{R}^4 \cong \begin{pmatrix} x_1 + x_3 & -x_2 + x_4 \\ x_2 + x_4 & x_1 - x_3 \end{pmatrix} \in \mathbb{M}^{2\times 2}.$$

Then $\det(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$ and $|x|^2 = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2)$. We define on $\mathbb{R}^4$ the function

$$f(x) = \Big(\sqrt{x_1^2 + x_2^2} - 2\Big)^2 + x_3^2 + x_4^2 - 2$$

and define $U = \{x \ : \ f(x) < 0\}$. Note that even on the larger set of all x with $f(x) < 1$ the gradient

$$\nabla f(x) = 2\frac{\sqrt{x_1^2 + x_2^2} - 2}{\sqrt{x_1^2 + x_2^2}}(x_1 e_1 + x_2 e_2) + 2(x_3 e_3 + x_4 e_4)$$

is a non-vanishing C^∞ function. Hence, ∂U is a smooth set containing all points x with $|(x_1, x_2)| = |(x_3, x_4)| = 1$. This implies of course that $0 \in (\bar{U})^{rc}$ and therefore $x \in (\bar{U})^{rc}$ whenever $\det(x) = 0$ and $|x| \leq 2$. On the other hand, we have for $v = (1, \pm 1) \in \mathbb{R}^2$ and all $x \in U$

$$\langle v, (|(x_1,x_2)|, |(x_3,x_4)|)\rangle \geq \langle v, (2,0)\rangle - |v| \cdot \big|(|(x_1,x_2)|, |(x_3,x_4)|) - (2,0)\big| > 2 - 2 = 0,$$

consequently $\det(x) > 0$ whenever $x \in U$. This shows that $(\bar{U})^{pc} \subset \{x : \ \det(x) \geq 0\}$ and in particular none of the sets $(\bar{U})^{rc}, (\bar{U})^{qc}$, and $(\bar{U})^{pc}$ is smooth at the origin. An easy, but slightly longer calculation also shows that the same holds true for the (boundary of) the hulls U^{rc}, U^{qc}, and U^{pc} of the open set U itself. □

In the last part of this note we extend results from [Z3] concerning the square of the distance function to the case of arbitrary powers. We apply a more elementary and selfcontained reasoning which does not rely on the precise knowledge of the quasiconvex envelope of the squared distance to a double-well (see [K]). Instead we use the following estimate the proof of which we leave to the reader.

Lemma 7. *Let $K \subset \mathbb{M}^{m\times n}$ be a compact set and $p \in [1,\infty)$. Suppose $x \in \mathbb{M}^{m\times n}$ and $y \in K$ are such that $|x-y| = d_{K,1}(x) > 0$. Then we have for all $v \in \mathbb{M}^{m\times n}$ that*

$$\limsup_{t\searrow 0} \frac{d_{K,p}(x+tv)-d_{K,p}(x)}{t} \le p\cdot d_{K,p-1}(x)\lim_{t\searrow 0}\frac{|x+tv-y|-|x-y|}{t} = p\cdot d_{K,p-1}(x)\langle v, \frac{x-y}{|y-x|}\rangle.$$

As in [Z3], the crucial point in our proof is the nonuniqueness of the metric projection onto nonconvex sets. For the convenience of the reader we give a short proof of this fact following [F].

Theorem 8. *Let $K \subset \mathbb{M}^{m\times n}$ be compact and suppose that for every $x \in \mathbb{M}^{m\times n}$ there exists a unique point $\eta(x) \in K$ such that $|\eta(x)-x| = d_{K,1}(x)$. Then K is convex.*

Proof. Throughout the proof we write $d(x) = d_{K,1}(x)$. As $d(y) \le d(x)+|x-y|$, d is 1-Lipschitz. Hence, if $x_k \to x$ and $\eta(x_k) \to y$ then $\eta(x) = y$ which together with the compactness of K proves that η is continuous.

We first assert that for all $x \notin K$, and all t nonnegative $\eta\big(\eta(x)+t(x-\eta(x))\big) = \eta(x)$. Otherwise there exists an $x \notin K$ for which the assertion fails. Using uniqueness of the nearest point it is easy to check that the set of t for which the required equality holds is a closed interval $[0,T]$, $T \ge 1$. Let $\tilde{x} = \eta(x)+T(x-\eta(x))$. The continuity of η and Peano's existence theorem ([W], §7) allow us to choose a positive $\varepsilon < d(\tilde{x})$ and a curve $\varphi : (-\varepsilon,\varepsilon) \to \mathbb{M}^{m\times n}$ such that $\varphi(0) = \tilde{x}$ and $\varphi'(t) = (\eta(\varphi(t))-\varphi(t))/|\eta(\varphi(t))-\varphi(t)|$. Now consider the real-valued function $t \mapsto (d\circ\varphi)(t)$. From Lemma 7 we infer that its derivative from the right, $(d\circ\varphi)'_+(t)$, is bounded from above by -1 for all $t \in (-\varepsilon,\varepsilon)$ and hence $|\varphi(p)-\varphi(q)| \ge d(\varphi(p))-d(\varphi(q)) \ge q-p$ if $-\varepsilon < p < q < \varepsilon$. On the other hand, as $|\varphi'(t)| \le 1$ on $(-\varepsilon,\varepsilon)$, we also obtain $|\varphi(p)-\varphi(q)| \le |p-q|$ for all p,q. Both estimates together imply

$$d(\varphi(p))-d(\varphi(q)) = |\varphi(p)-\varphi(q)| = q-p \text{ whenever } -\varepsilon < p < q < \varepsilon.$$

The strict convexity of the Euclidean norm implies that $\varphi((-\varepsilon,\varepsilon))$ is a segment, which is due to the direction of $\varphi'(0)$ contained in the ray $\{\eta(x)+t(x-\eta(x)) \;:\; t>0\}$. This shows that $d(\varphi(p)) = d(\varphi(0))-p = |\varphi(p)-\eta(x)|$ for all $p \in (-\varepsilon,\varepsilon)$ and by uniqueness of the metric projection $\eta(\varphi(p)) = \eta(x)$. This contradicts the maximality of T, and the claim is established.

We conclude that for any $x \notin K$ and t arbitrarily large $K\cap U(\eta(x)+t(x-\eta(x)), t|x-\eta(x)|) = \emptyset$. In the limit $t\to\infty$ we obtain $\langle x, x-\eta(x)\rangle > \langle \eta(x), x-\eta(x)\rangle = \max\{\langle y, x-\eta(x)\rangle \;:\; y \in K\}$ which represents K as the intersection of halfspaces and finishes our proof. ☐

We are now in a position to generalize the results in [Z3] for the squared distance function to arbitrary powers under weaker assumptions on K.

Theorem 9. *Let $K \subset \mathbb{M}^{m\times n}$ be compact, let $p \in [1,\infty)$, and assume that $d_{K,p}$ is a separately convex function outside of some compact set. If $\mathbb{M}^{m\times n}\setminus K$ is connected, then K is convex.*

Proof. By assumption we may choose a finite R such that $d_{K,p}$ is separately convex whenever restricted to a ball disjoint with $\tilde{K} = B(K,R)$. We first show that the metric projection onto K is unique in each point $x \notin \tilde{K}$. Otherwise we find $x \notin \tilde{K}$ and $y_1, y_2 \in K$, $y_1 \ne y_2$, such that $|y_i - x| = d_{K,1}(x)$, and we may assume that $\langle y_2-y_1, e_j\rangle < 0$ for some canonical basis vector e_j. We estimate using Lemma 7 that

$$\begin{aligned}\limsup_{t\searrow 0}\; & \frac{d_{K,p}(x+te_j)+d_{K,p}(x-te_j)-2d_{K,p}(x)}{t} \\ & \le p\cdot d_{K,p-1}(x)\left(\langle e_j, \frac{x-y_1}{|x-y_1|}\rangle + \langle -e_j, \frac{x-y_2}{|x-y_2|}\rangle\right) = p\cdot d_{K,p-1}(x)\langle e_j, \frac{y_2-y_1}{|y_1-x|}\rangle < 0.\end{aligned}$$

This implies that $d_{K,p}$ is not separately convex near x, a contradiction.

Next, note that the metric projection onto the enlarged set $\tilde{K}$ is unique on the whole space $\mathbb{M}^{m\times n}$. Uniqueness is trivial for points in $\tilde{K}$, so we may suppose that $x \notin \tilde{K}$. Consider $z_1, z_2 \in \tilde{K}$ with

$|z_i - x| = d_{\tilde{K},1}(x) \le d_{K,1}(x) - R$, and $y_i \in K$ with $|z_i - y_i| = R$. This implies $|x - y_i| \le d_{K,1}(x)$, and thus $y_1 = y_2 = \eta_K(x)$ by the claim already established. Moreover, as $|x - \eta_K(x)| = |x - z_i| + |z_i - \eta_K(x)|$ we see that both z_1, z_2 are in the segment $[\eta_K(x), x]$ and because furthermore $|z_1 - \eta_K(x)| = R = |z_2 - \eta_K(x)|$, $z_1 = z_2$, so $\eta_{\tilde{K}}(x)$ is unique. In particular, due to the foregoing theorem, $\tilde{K}$ is a convex set.

Now, we show that $\partial K^c \subset K$. In fact, given any $x \in \partial K^c$ there is a unit vector $v \in \mathbb{M}^{m\times n}$ such that $\langle x, v\rangle = \max\{\langle y, v\rangle \ : \ y \in K\}$. The convexity of $\tilde{K}$ implies that $x + Rv \in \tilde{K}$ as well, but it is easy to check that

$$B(x + Rv, R) \cap K \subset B(x + Rv, R) \cap \{y \ : \ \langle y, v\rangle \le \langle x, v\rangle\} = \{x\}.$$

By definition of $\tilde{K}$ this shows that $x \in K$ as required.

We finish our proof by showing that $\partial K^c \subset K$ and $\mathbb{M}^{m\times n} \setminus K$ connected imply that K is convex. This statement is similar to Lemma 2.1 in [Z3], but slightly more general. To verify our assertion, choose any $y \notin K$ and connect it by an arc φ inside $\mathbb{M}^{m\times n} \setminus K$ to a point x sufficiently far away from the origin to ensure that $x \notin K^c$. By assumption, φ does not intersect ∂K^c, and we thus infer that $y \notin K^c$. This concludes the proof of the theorem. □

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