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Arakelov type inequalities for Hodge bundles over algebraic varieties
Part I: Hodge bundles over algebraic curves

by

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. **Arakelov type inequalities for Hodge bundles over algebraic varieties**
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Abstract

We prove Arakelov inequalities for systems of Hodge bundles over algebraic varieties, generalizing the classical ones for families of semistable curves and abelian varieties. These inequalities are derived from the semistability of an associated Higgs bundle, a consequence of the existence of a Hermitian Yang-Mills metric.

Inspired by earlier works of J. Kollár [K], E. Viehweg [V1] [V2] and in particular a recent paper of E. Bedulev and E. Viehweg [BV], we shall prove Arakelov type inequalities for systems of Hodge bundles. These inequalities recover the original Arakelov inequality for families of semi-stable algebraic curves [A]. And they also improve the Faltings inequality for families of semi-stable abelian varieties [F].

The principal idea of our approach is quite simple. A variation of Hodge structure induces a Higgs bundle (E, ϑ) of degree 0, and since such a bundle carries a Hermitian Yang-Mills metric it is semistable, and so all ϑ-invariant subbundles have nonpositive degree. Rewriting this nonpositivity in concrete situations yields Arakelov type inequalities.
Nonpositivity is always related to curvature properties of natural metrics, here of Hermitian Yang-Mills metrics. In the situation studied here, namely of variations of Hodge structures, these metrics are explicitly given as natural metrics on period domains. In this sense, our construction can be considered as a more conceptual version of arguments based on Schwarz type lemmas that encode curvature bounds in analytic inequalities.

We shall point out that there is an inequality similar to our inequality for weight $k \geq 2$ in a paper of Peters ([P1], Proposition 3), but without proof. In contrast to his inequality our inequality counts the defect of the kernel of Kodaira-Spencer map. Peters had a sketch of proof for an inequality for the weight $k = 1$ case ([P1], Prop. 2). The proof is in principle correct. However, one needs an additional argument that uses the stability property of systems of Hodge bundles. After receiving a preprint of the first version of our paper, Peters provided a proof of Proposition 3 in [P1] in his recent preprint [P2]. The main idea is to use the stability property as we are doing here.

Acknowledgments We would like to thank E. Bedulev for interesting discussions on variations of Hodge structures and Hodge bundles and S-T Yau for pointing out an interesting paper of K-F Liu [L]. Liu uses Yau's Schwarz Lemma to bound the height of sections of families of curves. His original motivation is to prove Vojta's $1+\epsilon$ conjecture. It would be very interesting to know whether our technique can be used to bound heights of sections.
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Let V be an $\mathbb{R}-$variation of Hodge structure over an algebraic curve $C \setminus S$, of $g(C) = q$, and S a finite subset consisting of s points. If the monodromies around singularities are unipotent, W. Schmid has shown that the Hodge filtration has a canonical extension. We denote by

$$(E := \bigoplus_{p+q=k} E^{p,q}, \theta), \quad \theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_C(S)$$

the system of Hodge bundles corresponding to this extended Hodge filtration.
In general, given a system of Hodge bundles (E, θ) over $C \setminus S$, the extension also exists, but there is no canonical one. Simpson showed that there exists a decreasing left continuous filtration of extensions

$$\{(E, \theta)_\alpha\}_{-\infty<\alpha<+\infty}, \quad E_\alpha \subset E_\beta, \, \alpha \geq \beta.$$

Here, the real number α depends on the eigenvalues of the monodromy around S, and $E_{\alpha+1} = E_\alpha(-S)$. Since any $\alpha \in [r, r+1)$ really describes this filtration, we may consider, for example, $[-1, 0)$. It is very likely that extensions from geometric situations lie in this range. If the monodromy around S is unipotent, then all extensions in the range $[-1, 0)$ coincide with the canonical extension. Let $\bar{E}$ denote the system of Hodge bundles E_0 in the filtration.
In this note we shall show the following theorem

Theorem 1 (the unipotent case) *Let* $(\bigoplus_{p+q=k} E^{p,q} =: E, \theta)$ *be a system of Hodge bundles*

over an algebraic curve C of genus q, as above.
Let $h^{p,q} := \mathrm{rk} E^{p,q}$ and $h_0^{p,q} := \mathrm{rk}\ \mathrm{Ker}(\theta : E^{p,q} \to E^{p-1,q+1})$.

Suppose that the monodromy around each point in S is unipotent. Then we have
If $k = 2l+1$, we have

$$\deg E^{k,0} \le (\frac{1}{2}(h^{k-l,l} - h_0^{k-l,l}) + \sum_{j=0}^{l-1}(h^{k-j,j} - h_0^{k-j,j}))(2q-2+s)$$

If $k = 2l$, then

$$\deg E^{k,0} \le (\sum_{j=0}^{l-1}(h^{j,k-j} - h_0^{j,k-j}))(2q-2+s).$$

In general, we shall bound the degree of $E_\alpha^{k,0}$ for any $\alpha \in [-1,0)$.

Theorem 2 (the general case) *Set $(\bar{E},\theta) := (E,\theta)_0$.*
If $k = 2l+1$, then

$$\deg \bar{E}^{k,0} \le (\frac{1}{2}(h^{k-l,l} - h_0^{k-l,l}) + \sum_{j=0}^{l-1}(h^{k-j,j} - h_0^{k-j,j}))(2q-2+s).$$

If $k = 2l$, then

$$\deg \bar{E}^{k,0} \le (\sum_{j=0}^{l-1}(h^{j,k-j} - h_0^{j,k-j}))(2q-2+s)$$

Let $\alpha \in [-1,0)$. Then $E_0 \subset E_\alpha \subset E_0(S)$. By Theorem 2 we obtain

Cor 1 *For any $\alpha \in [-1,0)$,*
if $k = 2l+1$, then

$$\deg E_\alpha^{k,0} \le (\frac{1}{2}(h^{k-l,l} - h_0^{k-l,l}) + \sum_{j=0}^{l-1}(h^{k-j,j} - h_0^{k-j,j}))(2q-2+s) + sh^{k,0}.$$

If $k=2l,$ *then*

$$\deg E_{\alpha}^{k,0}\leq (\sum_{j=0}^{l-1}(h^{j,k-j}-h_0^{j,k-j}))(2q-2+s)+sh^{k,0}$$

The idea of the proof is to use the Kodaira-Spencer map, and to use the stability of Hodge bundles (Lemma 1 below) successively to estimate the degrees of the kernel and the image of the Kodaira-Spencer map. In fact, some time ago, E. Viehweg had arrived also at considering the Kodaira-Spencer map and realized the importance of controlling its kernel and image. We thank him very much for explaining that to us.

In view of harmonic maps from Kähler manifolds into spaces of non-positive sectional curvature, such inequalities exist for the following reason. Let $f:C\to \bar{M}_g$ be a holomorphic map into the Teichmüller space of curves of genus g. By Yau's Schwarz Lemma the pull back of the Weil-Petersson metric on $\bar{M}_g$ via f is bounded above by a multiple of the Poincaré metric on C [JY]. In the VHS case, one uses the horizontal Schwarz Lemma of Griffiths-Schmid [GS].

We should view Lemma 1 below as some sort of Schwarz Lemma in algebraic geometry. Here we can make the estimate more precise than in differential geometry.

Applications of Theorem 1: 1) Let $f:X\to C$ be a semi-stable family of algebraic curves of genus g with s singular fibres. By taking $H=R^1f_*(\mathbb{Z})$, we get a $\mathbb{Z}$-VHS of $k=1$ and with Hodge bundles $H^{1,0}=R^0f_*(\Omega^1_{X/C})=f_*(\omega_{X/C}),\quad H^{0,1}=R^1f_*\mathcal{O}_X$. It is well known that the semi-stability condition implies that the local monodromies around the singular fibres are unipotent. We also notice that the number g_0 in the Arakelov inequality is exactly the dimension of the space of those holomorphic 1-forms on the fibre X_c that come from restrictions of holomorphic 1-forms on X. Since they are vanishing under the Kodaira-Spencer map, $g_0\leq h_0^{1,0}$. Hence by taking $k=1$ in i) in Theorem 1, we obtain the following inequality, which is slightly stronger than the Arakelov inequality

$$\deg f_*(\omega_{X/C})\leq \frac{1}{2}(h^{1,0}-h_0^{1,0})(2q-2+s)\leq \frac{1}{2}(g-g_0)(2q-2+s)$$

.

Remark By using the Miyaoka-Yau inequality and an inequality of Xiao, Vojta has deduced the following inequality

$$\deg f_*(\omega_{X/C})\leq \frac{1}{2}g(2q-2+s),$$

which is slightly weaker than ours.

2) Let $f : A \to C$ be a semi-stable family of abelian varieties of dimension g and with s singular fibres. So, $H := R^1 f_*(\mathbb{Z})$ is a $\mathbb{Z}-$VHS of weight $k = 1$, and with the Hodge bundles $H^{1,0} = R^0 f_*(\Omega^1_{A/C})\, H^{0,1} = R^1 f_*(\mathcal{O}_A)$. Again, the semi-stability implies that the local monodromies around the singular fibres are unipotent. Applying Theorem 1, we have

$$\deg f_*(\Omega^1_{A/C}) \leq \frac{1}{2}(g - h_0^{1,0})(2q - 2 + s).$$

This inequality is slightly better than Faltings' inequality.

3) Let $f : X \to C$ be a family of $K3-$surfaces with s singular fibres. The 2^{nd} direct image $R^2 f_*(\mathbb{Z})$ is a VHS of weight $k = 2$, and $H^{2,0} = f_*(\omega_{X/C})$. Suppose that the local monodromies aroud the singular fibres are unipotent. This can always be achieved by a base change. Applying Theorem 1 for $k = 2$, we get

$$\deg f_*(\omega_{X/C}) \leq (h^{2,0} - h_0^{2,0})(2q - 2 + s) = (1 - h_0^{2,0})(2q - 2 + s).$$

Proof of Theorem 1 We consider first the unipotent case. Let $k = 2l + 1$. We shall prove the inequality inductively.
Let $k = 1$. We consider the Kodaira-Spencer map

$$\theta : E^{1,0} \to E^{0,1}\Omega^1_C(S).$$

Let

$$E_0^{1,0} := \mathrm{Ker}(\theta : E^{1,0} \to E^{0,1}\Omega^1_C(S))$$

$$I := \mathrm{Im}(\theta : E^{1,0} \to E^{0,1}\Omega^1_C(S)).$$

We may split the Kodaira-Spencer map into the following two short exact sequences:

$$0 \to E_0^{1,0} \to E^{1,0} \to F \to 0$$

and

$$0 \to I \to E^{0,1} \otimes \Omega^1_C(S) \to Q \to 0.$$

Furthermore, we may enlarge the subsheaf $I \subset I' \subset E^{0,1} \otimes \Omega^1_C(S)$ by a subsheaf I' so that the quotient Q' of the exact sequence

$$0 \to I' \to E^{0,1} \otimes \Omega^1_C(S) \to Q' \to 0$$

is torsion free. Hence it is locally free over our curve.

We have corresponding calculations for the degrees of the bundles in the first and third exact sequence and notice that $E^{1,0^{dual}} = E^{0,1}$.

$$\deg E_0^{1,0} + \deg I = \deg E^{1,0}$$

and

$$\deg I' + \deg Q' = \deg(E^{0,1} \otimes \Omega^1_C(S) = -\deg E^{1,0} + h^{1,0}(2q-2+s).$$

This implies the following equality and inequality

$$\begin{aligned}\deg E^{1,0} &= \frac{1}{2}(h^{1,0}(2q-2+s) + \deg I - \deg I' + \deg E_0^{1,0} - \deg Q') \\ &\le \frac{1}{2}(h^{1,0}(2q-2+s) + \deg E_0^{1,0} - \deg Q'),\end{aligned}$$

since the quotient of $I \subset I'$ is a torsion sheaf and we have $\deg I \le \deg I'$.

We shall use the following lemma to bound degrees of $E_0^{1,0}$ and Q'^{dual}.

Lemma 1 *Consider the Kodaira-Spencer map*

$$\theta : E^{i,k-i} \to E^{i-1,k-i+1} \otimes \Omega^1_C(S).$$

Let

$$E_0^{i,k-i} := \mathrm{Ker}\ (\theta : E^{i,k-i} \to E^{i-1,k-i+1}\Omega^1_C(S))$$

and

$$Q' := \mathit{Quotient}(\theta : E^{i,k-i} \to E^{i-1,k-i+1}\Omega^1_C(S))/\mathit{Torsion}.$$

Then

$$\deg E_0^{i,k-i} \le 0$$

and

$$\deg Q'^{dual} \otimes \Omega^1_C(S) \le 0.$$

Proof of Lemma 1 Since $\theta(E_0^{i,k-i}) = 0$, in particular $E_0^{i,k-i} \subset E$ is a $\theta-$invariant subsheaf. In view that (E,θ) is a Higgs bundle, the Hodge metric is the Hermitian-Yang-Mills metric. By the work of E. Cattani, A. Kaplan and W. Schmid [CKS] for Hodge bundles and the work of J. Kollár [K] in the general case, the Chern form of a subbundle $F \subset E$ of this singular metric calculates the degree of F, if the monodromies around the singularities are unipotent. In particular, one gets $\deg(E) = 0$ (see [S]). By a calculation of the curvature form of the Hermitian-Yang-Mills metric on (E,θ) (see [UY] or [S]) one sees that (E,θ) is semi-stable. In the flat case we get

$$\frac{\deg E_0^{i,k-i}}{h_0^{i,k-i}} \leq \frac{\deg E}{\mathrm{rk} E} = 0.$$

So, we obtain $\deg E_0^{i,k-i} \leq 0$.

Similarly, we dualize the above sequence and tensor it with $\Omega^1_C(S)$, and get

$$Q'^{dual} \otimes \Omega^1_C(S) \subset E^{i,k-i}{}^{dual}$$

and $\theta^{dual}(Q'^{dual} \otimes \Omega^1_C(S)) = 0$.
Since $(E^{dual}, \theta^{dual})$ comes from the dual of the original VHS, it is again semi-stable with respect to the $\theta^{dual}-$invariant subsheaves. So, we prove also $\deg Q'^{dual} \otimes \Omega^1_C(S) \leq 0$. Lemma 1 is proved.

Now we apply Lemma 1 for $i=1$, and notice that $\mathrm{rk} Q' = h_0^{1,0}$, so the second inequality in Lemma 1 can be reformulated as

$$-\deg Q' \leq -h_0^{1,0}(2q-2+s).$$

So, we get the inequality in Theorem for $k=1$.

Now we are going to prove the inequality in Theorem for $k=3$. We will also see how to prove the inequality inductively for any $k=2l+1$.
We consider a system of Hodge bundles of weight $k=3$.

$$\theta : E^{3,0} \oplus E^{2,1} \oplus E^{1,2} \oplus E^{0,3} \to (E^{3,0} \oplus E^{2,1} \oplus E^{1,2} \oplus E^{0,3}) \otimes \Omega^1_C(S),$$

with $\theta(E^{p,q}) \subset E^{p-1,q+1} \otimes \Omega^1_C(S)$.
First we look at the Kodaira-Spencer map

$$\theta : E^{3,0} \to E^{2,1} \otimes \Omega^1_C(S).$$

As in the case $k=1$, it splits into two short exact sequences

$$0 \to E_0^{3,0} \to E^{3,0} \to I \to 0$$

and

$$0 \to I' \to E^{2,1} \otimes \Omega^1_C(S) \to Q' \to 0$$

such that $I \subset I'$. The corresponding quotient is a torsion sheaf and Q' is torsion free. We calculate the degrees of the subbundles and get

$$\deg E_0^{3,0} + \deg I = \deg E^{3,0},$$

and

$$\deg I' + \deg Q' = \deg E^{2,1} + h^{2,1}(2q-2+s).$$

We derive

$$\deg E^{3,0} = h^{2,1}(2q-2+s) + \deg E^{2,1} + \deg E_0^{3,0} - \deg Q' + \deg I - \mathrm{I}'.$$

Since the quotient of $I \subset I'$ is a torsion sheaf, we get

$$\deg E^{3,0} \leq h^{2,1}(2q-2+s) + \deg E^{2,1} + \deg E_0^{3,0} - \deg Q'.$$

We apply now Lemma 1 for $E_0^{3,0}$ and Q' here and notice that

$$\mathrm{rk} Q' = h^{2,1} + h_0^{3,0} - h^{3,0},$$

to obtain

$$\deg E^{3,0} \leq (h^{3,0} - h_0^{3,0})(2q-2+s) + \deg E^{2,1}.$$

Now we consider the Kodaira-Spencer map

$$\theta : E^{2,1} \to E^{1,2}.$$

We just repeat the same procedure as above again, and obtain

$$\deg E^{2,0} \leq \frac{1}{2}(h^{2,0} - h^{2,0})(2q-2+s).$$

We put the last two inequalities together and get

$$\deg E^{3,0} \leq ((h^{3,0} - h_0^{3,0}) + \frac{1}{2}(h^{2,1} - h^{2,1}))(2q-2+s).$$

So, we proved the inequality for $k=3$. And we prove also the inequality for any $k = 2l+1$ inductively.

Now we shall prove the inequality for $k = 2l$. We first will prove that for $l = 2, 4$, then for any $2l$ inductively. First of all, since the VHS here is real, $E^{l,l} = E^{l,l^{dual}}$. Hence $\deg E^{l,l} = 0$.

Let $k=2$. We consider the Kodaira-Spencer map

$$\theta : E^{2,0} \to E^{1,1} \otimes \Omega^1 C(S).$$

Let

$$E_0^{2,0} := \mathrm{Ker}(\theta : E^{2,0} \to E^{1,1} \otimes \Omega^1 C(S))$$
$$I := \mathrm{Im}(\theta : E^{2,0} \to E^{1,1} \otimes \Omega^1 C(S))$$

we get

$$0 \to E_0^{2,0} \to E^{2,0} \to I$$

and enlarging the subsheaf $I \subset I\prime \subset E^{1,1} \otimes \Omega^1 C(S)$ we obtain

$$0 \to I\prime \to E^{1,1} \otimes \Omega^1 C(S) \to Q' \to 0$$

with Q' is locally free.

Now we calculate the degrees of vector bundles in those two exact sequences as in the case $k=1$ and note that $\deg E^{1,1} = 0$

$$\deg E^{2,0} = h^{1,1}(2g-2+s) + \deg E_0^{2,0} + \deg I - \deg I' - \deg Q'$$
$$\leq h^{1,1}(2g-2+s) + \deg E_0^{2,0} - \deg Q'$$

and apply Lemma 1 for $\deg E_0^{2,0}$ and $\deg Q'$,

$$\leq \mathrm{rk} I(2g-2+s) = (h^{2,0} - h_0^{2,0})(2g-2+s).$$

So, we proved the inequality for $k=2$.

Now let $k=4$. By the same calculation of the degrees of vector bundles in the following exact sequences

$$0 \to E_0^{4,0} \to E^{4,0} \to I \to 0,$$
$$0 \to I' \to E^{3,1} \otimes \Omega_C^1(S) \to Q' \to 0,$$

and by Lemma 1 we get

$$\deg E^{2,0} \leq \deg E^{3,1} + (h^{4,0} - h_0^{4,0})(2g-2+s).$$

Then we consider the Kodaira-Spencer map

$$\theta : E^{3,1} \to E^{2,2} \otimes \Omega_C^1(S).$$

We just repeat the argument for $k=2$, and obtain

$$\deg E^{3,1} \leq (h^{3,1} - h3,1_0)(2g-2+s).$$

The inequality for $k=4$ is implied by those two inequalities.

For any even number $k=2l$ we prove the inequality inductively.

Now we are in the position to prove the inequality in the general case without the unipotent condition.

Given a system of Hodge bundles over $C \setminus S$, or, in general, a Higgs bundle (E,θ) satisfying a growth condition near S, Simpson showed that the extension as an algebraic vector bundle exists. (The Hodge bundle case goes back to W. Schmid.) There exists, in general, no canonical extension. However, there exists a filtration of extensions

$$\{(E,\theta)_\alpha\}_{-\infty<\alpha<+\infty}$$

The real number α depends on the eigenvalues of the monodromy around S (see [S] for more details). Let $\bar{E}$ denote the vector bundle E_0 in the above extensions. The degree of this filtered bundle is defined as

$$\deg(E) = \deg(\bar{E}) + \sum_{x\in S}\sum_{0\leq\alpha<1} \alpha \dim \mathrm{Gr}_\alpha(\bar{E}(x)).$$

Simpson showed that for any $\theta-$invariant subbundle $F \subset E$

$$\frac{\deg F}{\mathrm{rk} F} \leq \frac{\deg E}{\mathrm{rk} E}.$$

If (E,θ) comes from a reductive representation of $\pi_1(C \setminus S)$, then $\deg(E) = 0$.

Now let (E,θ) be a system of Hodge bundles coming from a real VHS of weight k over

$C \setminus S$. We shall prove the same bounds for the degree of $\bar{E}^{k,0}$. Using the stability argument we used in the proof of Lemma 1 we have

Lemma 2 *In the same notation as in Lemma 1, one has*

$$\deg \bar{E}_0^{i,k-i} \leq -\sum_{x\in S}\sum_{0\leq\alpha<1} \alpha \dim Gr_\alpha(\bar{E}_0^{i,k}(x)) \leq 0$$

and

$$\deg Q'^{dual} \otimes \Omega^1_C(S) \leq ---\sum_{x\in S}\sum_{0\leq\alpha<1} \alpha \dim Gr_\alpha(\bar{Q'}^{dual} \otimes \Omega^1_C(S)(x)) \leq 0.$$

By applying Lemma 2 and the same argument as in the proof of Theorem 1 we obtain the following inequalities.

If $k = 2l+1$, then

$$\deg E^{k,0} \leq (\frac{1}{2}(h^{k-l,l} - h_0^{k-l,l}) + \sum_{j=0}^{l-1}(h^{k-j,j} - h_0^{k-j,j}))(2q-2+s).$$

If $k = 2l$, then

$$\deg E^{k,0} \leq (\sum_{j=0}^{l-1}(h^{j,k-j} - h_0^{j,k-j}))(2q-2+s).$$

So we complete the proof of Theorem 2.

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