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by

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Abstract

In the preceding papers [10]-[14], a class of matrices ($\mathcal{H}$-matrices) has been analysed which are data-sparse and allow an approximate matrix arithmetic of almost linear complexity. Several types of $\mathcal{H}$-matrices were shown to provide good approximation of nonlocal (integral) operators in FEM and BEM applications.

In the present paper, we develop special classes of $\mathcal{H}$-matrices with improved data sparsity to approximate elliptic problems posed in $\mathbb{R}^d$, $d = 1, 2, 3$. For the evaluation of integral operators on spatial domains in $\mathbb{R}^d$, the idea is to apply degenerate kernel expansions supported only on the boundaries of the geometrical clusters. This results in an algorithm of linear storage expenses, $O(n)$, which includes one call for an optimal Dirichlet solver (e.g., multi-grid method) on the involved cluster. In the case of a tensor product finite element ansatz space, we propose improved degenerate expansions of the kernel based on a separation with respect to the full set of one-dimensional variables..

For BEM applications applied to rather general elliptic operators, our approach reduces the order of expansion from $O(\log^d n)$ down to $O(\log^{d-1} n)$.

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1 Introduction

In the present paper, we develop special classes of $\mathcal{H}$-matrix approximations for elliptic problems which have linear memory requirements and allow a fast matrix-vector product of linear-logarithmic complexity under standard assumptions on the smoothness of fundamental solutions.

For the evaluation of potential fields as well as for the calculation of particular solutions in elliptic problems, we are interested in the data-sparse approximation of integral operators on a spatial domain in $\mathbb{R}^d$, $d = 1, 2, 3$, whose kernels satisfy the property (2.2). The idea is to build the degenerate kernel expansions only on the boundaries of certain geometrical clusters and to use their elliptic extension into the corresponding interior product domain. As a result, the domain integrals are transformed into equivalent surface integrals. This leads to an algorithm of linear complexity. It requires one call of an optimal Dirichlet solver (say, multi-grid method) on each geometrical cluster. We call the new approach the *wire-basket $\mathcal{H}$-matrix approximation*. For a general class of kernels, this approach essentially reduces the amount of data (memory requirements) for each $n_\tau \times n_\sigma$ matrix block from $O(n_\tau \log^d n)$ for the standard $\mathcal{H}$-matrices down to $O(n_\tau^{1/d} \log^{d-1} n + n_\tau)$, where $n_\tau \sim n_\sigma$ is the corresponding blocksize and n is the global dimension.

In the case of a tensor product finite element basis, we construct data-sparse expansions of the matrix blocks with *sublinear* complexity $O(n_\tau^{1/d} \log^{2d} n)$ and with the expansion order[1] $O(\log^{2d} n)$. It is based on the idea of data-sparse approximations of the matrix blocks with locally analytic kernels by *fully separated expansions* (see the smoothness requirements formulated in Assumption 2.4 of Section 2.3). In this way, we allow to increase the local rank, but at the same time, we gain additional sparsity of the approximation ansatz defined now on a low dimensional tensor product space. In the framework of the $\mathcal{H}^2$-matrix concept with fully separable expansions of variable order, we are led to schemes of linear complexity.

As a by-product, we derive degenerate expansions of the order $O(\log^{d-1} n)$ for kernels of boundary integral operators. The efficiency of the corresponding expansions in BEM does not depend on the smoothness of the computational boundary Γ. However, the constructive way to avoid this dependence and to retain expansions of the optimal order $O(\log^{d-1} n)$, needs to compute the restriction of elliptic extension on pieces of the boundary Γ located within associated "volume clusters" covering a part of the curve/surface in the spatial product domain.

[1] In this paper, the *order of expansion* is defined as the total number of terms in the separable approximation of the kernel.

2 $\mathcal{H}$-Matrix Approximation Revisited

2.1 Integral Operators in Elliptic Problems

We consider the case of scalar elliptic operators. Let

$$\mathcal{L} = -\sum_{j,k=1}^{d} \partial_j a_{jk} \partial_k + \sum_{j=1}^{d} b_j \partial_j + c_0 \qquad (\partial_j := \frac{\partial}{\partial x_j}) \tag{2.1}$$

be a second order elliptic operator with real constant coefficients a_{jk}, b_j and c_0. We assume that $\mathcal{L}$ has a fundamental solution S satisfying $\mathcal{L}S = \delta_0$, where δ_0 is the Dirac distribution at the origin. The associated weakly singular kernel $s(x,y) = S(x-y)$ defines a class of integral operators specified by their domain of definition. The function $(x,y) \to s(x,y)$ is C^∞ outside the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$ and satisfies Assumption 2.4 from Section 2.3. Moreover,

$$\mathcal{L}_x s(x,y) = -\mathcal{L}_y^* s(x,y) = \delta(x-y), \tag{2.2}$$

where $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$ with respect to the L^2 inner product $\langle \cdot, \cdot \rangle$. We further assume that $\mathcal{L}$ is $H_0^1(\Omega)$-elliptic which implies that the bilinear form

$$a_\Omega(u,v) := \int_\Omega (\sum_{j,k=1}^{d} a_{jk} \partial_k u \partial_j v + \sum_{j=1}^{d} b_j \, (\partial_j u)\, v + c_0 uv) dx$$

satisfies the inequality

$$a_\Omega(u,u) \geq c ||u||_{1,\Omega}^2 \qquad \text{for all } u \in H_0^1(\Omega) \text{ with } c > 0.$$

Note that the bilinear form $a_\Omega(\cdot,\cdot)$ is continuous on $H^1(\Omega) \times H^1(\Omega)$.

We consider the h-version of the Galerkin FE method for approximating the continuous integral operator $A : V \to V'$ defined in the Sobolev space $V = H^r(\Sigma)$ and deal with integral operators of the form

$$(Au)(x) = \int_\Sigma s(x,y) u(y) dy \qquad (x \in \Sigma), \tag{2.3}$$

with s being the kernel function mentioned above or with s replaced by a suitable directional derivatives Ds of s. We distinguish two different cases:

(**A**) Σ is a bounded $(d-1)$-dimensional manifold (BEM applications);

(**B**) Σ is a polyhedron in $\mathbb{R}^d$, $d = 2, 3$ (FEM applications).

In the latter case, we confine ourselves to the case of the unit cube $\Sigma = (0,1)^{d_\Sigma}$ and an ansatz space $V_h := \text{span}\{\varphi_i\}_{i \in I} \subset V$ of piecewise constant/linear basis functions with respect to a quasiuniform tensor-product mesh. Therefore, we specify $V = H_{00}^r(\Sigma)$ or $V = H^r(\Sigma)$, $r \in [-1/2, 1/2]$, and $d_\Sigma = d-1$ in Case (**A**), while $V = H^{-1}(\Sigma)$, $d_\Sigma = d$ for evaluation of the volume integral operators in Case (**B**). An extension of our approach to more general domains in $\mathbb{R}^d$, $d = 2, 3$, is possible.

2.2 Construction of $\mathcal{H}$-Matrices

We construct a data-sparse $\mathcal{H}$-matrix approximation for the integral operators A with asymptotically smooth kernels (cf. (2.11)), and with locally analytic kernels (cf. (2.10)). For the ease of presentation, we first consider the case of the piecewise constant Galerkin ansatz space $V_h \subset V$. Modifications for piecewise linear/bilinear elements will be discussed in Remark 2.2 below.

For the sake of completeness, we recall the important definition of an admissible block partitioning. Let I be the index set of unknowns (e.g., the FE-nodal points). For each $i \in I$, the support of the corresponding basis function φ_i is denoted by $X(i) := \mathrm{supp}(\varphi_i)$. The *cluster tree* $T(I)$ is characterised by the following properties: (i) all vertices of $T(I)$ are subsets of I, (ii) $I \in T(I)$ is the root; (iii) if $\tau \in T(I)$ contains more than one element, the set $S(\tau)$ of sons of τ consists of at least 2 disjoint subsets satisfying $\tau = \bigcup_{\sigma \in S(\tau)} \sigma$; (iv) the leaves of the tree are $\{i\}$ for all $i \in I$. For $\tau \in T(I)$ we extend the definition of the supports $X(\cdot)$ by $X(\tau) = \bigcup_{i \in \tau} X(i)$.

In the standard quasiuniform FE application, the cluster tree $T(I)$ is obtained by a recursive division of I into subsets of almost equal size having a diameter as small as possible. In the quasiuniform case, the term "almost equal size" can be understood in a geometrical sense (i.e., $\mathrm{diam}(X(\tau')) \approx \mathrm{diam}(X(\tau''))$ as well as with respect to the cardinality $\#\tau' \approx \#\tau''$. An appropriate construction of $T(I)$ will fulfil both criteria.

The matrix entries belong to the index set $I \times I$. In a canonical way (cf. [11]), a block-cluster tree $T(I \times I)$ can be constructed from $T(I)$, where all vertices $b \in T(I \times I)$ are of the form $b = \tau \times \sigma$ with $\tau, \sigma \in T(I)$. Given a matrix $M \in \mathbb{R}^{I \times I}$, the block-matrix corresponding to $b \in T(I \times I)$ is denoted by $M^b = (m_{ij})_{(i,j) \in b}$. A *block partitioning* $P_2 \subset T(I \times I)$ is a set of disjoint blocks $b \in T(I \times I)$, whose union equals $I \times I$. A block partitioning P_2 determines the $\mathcal{H}$-matrix format. We use the following explicit definition of $\mathcal{H}$-matrices.

Definition 2.1 *Let a block partitioning P_2 of $I \times I$ and $k \ll n$ be given. The set of real $\mathcal{H}$-matrices induced by P_2 and k is*

$$\mathcal{M}_{\mathcal{H},k}(I \times I, P_2) := \{M \in \mathbb{R}^{I \times I} : \textit{for all } b \in P_2 \textit{ there holds } \mathrm{rank}(M^b) \le k\}. \tag{2.4}$$

An admissibility condition is used to balance the size of matrix-blocks b and the distance between τ and σ (see [12] for more details). It takes into consideration the singularity location of the kernel function $s(x,y)$, $(x,y) \in \Sigma \times \Sigma$. We assume that the following *admissibility condition*

$$\min\{\mathrm{diam}(\sigma), \mathrm{diam}(\tau)\} \le 2\eta \, \mathrm{dist}(\sigma, \tau) \tag{2.5}$$

holds for all $\sigma \times \tau \in P_2$, where $\eta \le 1$ is a fixed threshold parameter. Here both dist and diam are defined with respect to the given norm $||\cdot||$ in $\mathbb{R}^d$. In general, this norm will depend on the coefficients of the elliptic operator. Specifically, let $\mathbf{L} = (a_{ij})_{i,j=1}^d \in \mathbb{R}^{d \times d}$ be the symmetric and positive definite coefficient matrix from (2.1). Define the matrix dependent scalar product and norm in $\mathbb{R}^d$ by $\langle u, v \rangle = (\mathbf{L}^{-1}u, v)$ and $||u|| = \langle u, u \rangle^{1/2}$, respectively, where $(\cdot,\cdot)$ is the Euclidean scalar product. In Section 4, we apply this construction to anisotropic elliptic equations.

For computational needs, we further use the splitting $P_2 = P_{far} \cup P_{near}$, where

$$P_{far} := \{\sigma \times \tau \in P_2 : \mathrm{dist}(X(\tau), X(\sigma)) > 0 \}. \tag{2.6}$$

Due to our assumption of a piecewise constant FE basis, the index set I is isomorphic to the disjoint supports X_i. While $\sigma \times \tau \in T(I \times I)$ indicates a matrix block, $X(\sigma) \times X(\tau) \subset \Sigma \times \Sigma$ is called the corresponding geometrical block. The standard $\mathcal{H}$-matrix approximation of the integral operator consists of three essential steps:

(a) the *admissible* block-partitioning $P_2 = P_{far} \cup P_{near}$ of the tensor product index set $I \times I$.

(b) the construction of an approximate integral operator $A_{\mathcal{H}} \in \mathcal{L}(V, V')$ with the kernel $s_{\mathcal{H}}(\cdot,\cdot)$ defined for each geometrical block $X(\sigma) \times X(\tau)$ with $\sigma \times \tau \in P_{far}$ by a *separable expansion* $s_{\tau,\sigma} = \sum_{i=1}^k a_i(x) c_i(y)$ of the order $k \ll n = \dim V_h$; In the near-field area, the kernel function is unchanged.

(c) the setup of the Galerkin $\mathcal{H}$-matrix $\mathbf{A}_{\mathcal{H}} = \langle A_{\mathcal{H}} \varphi_i, \varphi_j \rangle_{i,j \in I}$ for the operator $A_{\mathcal{H}}$, where $\{\varphi_i\}$ is the FE basis of V_h.

Remark 2.2 *In the case of piecewise linear/bilinear elements there is a minor difference in the definition of the local separable expansions at Step (**b**). Since now the supports $X(\sigma) \times X(\tau)$ for different $\sigma \times \tau$ may overlap, the kernel function $s_{\mathcal{H}} : \Sigma \times \Sigma \to \mathbb{R}$ is defined by a multi-valued mapping in the overlap. This allows to use a smooth version of $s_{\mathcal{H}}$ for all a_{ij}, $(i,j) \in \sigma \times \tau$. The error analysis can easily be modified.*

2.3 Complexity and Approximation

A bound of the solution error caused by Step (**b**) as well as the computational complexity of the $\mathcal{H}$-matrix formats for quasiuniform meshes were considered in [11]. An almost linear complexity bound was proven in [12], which is valid in both cases (**A**) and (**B**).

Proposition 2.3 *Let $d \in \{1, 2, 3\}$, $A \in \mathcal{M}_{\mathcal{H},k}(I \times I, P_2)$, and $\eta = \frac{\sqrt{d}}{2\mu}$, $\mu = 1, 2, \ldots$ Then the storage and matrix-vector multiplication expenses are bounded by*

$$\mathcal{N}_{st} \le (2^d - 1)(\sqrt{d}\eta^{-1} + 1)^d \, pkn, \qquad \mathcal{N}_{MV} \le \mathcal{N}_{st}, \tag{2.7}$$

where the cost unit of $\mathcal{N}_{MV}$ is one addition and one multiplication. Both estimates are asymptotically sharp.

The main goal of the present paper is the essential improvement of inequality (2.7) in Case (**B**) described in Sections 3 and 4.

The perturbation of the matrix induced by $A_{\mathcal{H}} - A$ yields a perturbed discrete solution of the original variational equation

$$\langle (\lambda I + A)u, v\rangle = \langle f, v\rangle \qquad \text{for all } v \in V := H^r(\Sigma),\ r \le 1, \tag{2.8}$$

where $\lambda \in \mathbb{R}$ is a given parameter. For the given ansatz space $V_h \subset V$ of piecewise constant/linear FEs, consider the perturbed Galerkin equation for $u_{\mathcal{H}} \in V_h$,

$$\langle (\lambda I + A_{\mathcal{H}})u_{\mathcal{H}}, v\rangle = \langle f, v\rangle \qquad \text{for all } v \in V_h. \tag{2.9}$$

Our error analysis can be based on either of the two following smoothness assumptions. The first one requires analyticity of the kernel for $x \neq y$.

Assumption 2.4 *For any $x_0, y_0 \in \Omega$, $x_0 \neq y_0$, the kernel function $s(x, y)$ is analytic with respect to x and y at least in the domain*

$$|x - x_0| + |y - y_0| < |x_0 - y_0|. \tag{2.10}$$

A similar condition is used in [18] for the analysis of the Galerkin wavelet approximations in BEM.

An alternative assumption requires that the singularity function s is asymptotically smooth, i.e.,

Assumption 2.5 *For all $x, y \in \mathbb{R}^d, x \neq y$, and all multi-indices α, β with $|\alpha| = \alpha_1 + \ldots + \alpha_d$ let*

$$|\partial_x^\alpha \partial_y^\beta s(x, y)| \le c(|\alpha|, |\beta|)|x - y|^{1-|\alpha|-|\beta|-d-2r} \qquad \text{for all } |\alpha|, |\beta| \le m. \tag{2.11}$$

Here $2r \in \mathbb{R}$ is the order of the integral operator $A : H^r(\Sigma) \to H^{-r}(\Sigma)$ in Case (**A**) with the possible choice $r \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$. In Case (**B**), we specify $r = -\frac{1}{2}$, such that the operator $A : H^{-1}(\Omega) \to H^1(\Omega)$ is continuous. Similar smoothness prerequisites are common in the wavelet or multi-resolution techniques [2, 3, 20], in the multipole expansion method (cf. [6] and references therein) as well as in the related mosaic-skeleton approach (cf. [23]).

Theorem 2.6 *Assume that (2.11) is valid and that V_h allows the standard inverse inequality. Suppose that the operator $\lambda I + A \in \mathcal{L}(V, V')$ is V-elliptic and let $r \in [-1/2, 1/2]$. Then there holds*

$$||u - u_{\mathcal{H}}||_V \lesssim \inf_{v_h \in V_h} ||u - v_h||_V + \frac{c(0, m)}{m!}\, \eta^m\, N^{d_\Sigma/2}\, ||u||_V. \tag{2.12}$$

Proof. The proof is a minor modification of the arguments from [12]. ∎

2.4 Cluster Tree on the Tensor-Product Index Set

To design "super-fast" matrix operations (see Section 4 for fully separable expansions), we will be interested in the $\mathcal{H}$-matrix approximation of an integral operator A defined in (2.3) with $\Sigma = (0, 1)^d$ and $d = 1, 2, 3$. Here we briefly recall the recursive construction of $\mathcal{H}$-matrices (cf. [12]) for a special index set. Consider the regular grid

$$I = \{\mathbf{i} = (i_1, \ldots, i_d) : 1 \le\ i_k \le N,\, k = 1, \ldots, d\}, \quad N = 2^p, \tag{2.13}$$

and define the norm $|\mathbf{i}|_\infty = \max_{1\le k\le d} |i_k|$. The cardinality of I is $n = N^d = 2^{pd}$.

The cluster tree $T_1 = T(I)$ of I is based on a division of the underlying cubes into 2^d subcubes. The blocks

$$t_{\mathbf{j}}^\ell = \{\mathbf{i} :\ 2^{p-\ell} j_1 + 1 \le i_1 \le 2^{p-\ell}(j_1+1),\ \ldots,\ 2^{p-\ell} j_d + 1 \le i_d \le 2^{p-\ell}(j_d+1)\} \quad \text{for } \mathbf{j} \in \{0,\ldots,2^\ell - 1\}^d$$

belong to level ℓ. $S(t_{\mathbf{j}'}^{\ell-1}) := \{t_{\mathbf{j}}^\ell : \mathbf{j} \text{ with } 0 \le 2j_k' - j_k \le 1 \text{ for } 1 \le k \le d\}$ defines the set of sons of the cluster $t_{\mathbf{j}'}^{\ell-1}$. Hence, the tree T_1 consisting of all blocks at all levels $\ell \in \{0,\ldots,p\}$ is a binary, quad- or octree for $d = 1, 2, 3$, respectively. The number of clusters on level ℓ equals $O(2^{d\ell})$.

Each index $\mathbf{i} \in I$ is associated with the d-dimensional cube

$$X_{\mathbf{i}} := \{(x_1, ..., x_d) : (i_1 - 1)h \le x_1 \le i_1 h, \ldots, (i_d - 1)h \le x_d \le i_d h\}, \tag{2.14}$$

which is the support of a piecewise constant function for the index $\mathbf{i}$. Using the Euclidean norm, we obtain the diameter $\operatorname{diam}(\tau) = \sqrt{d}\, 2^{p-\ell} h = \sqrt{d}/2^\ell$ for blocks of level ℓ. Let $\tau = t_{\mathbf{j}}^\ell$, $\sigma = t_{\mathbf{j}'}^\ell$ be two blocks of level ℓ characterised by $\mathbf{j}$ and $\mathbf{j}'$. Then

$$\operatorname{dist}(\tau,\sigma) = 2^{-\ell}\sqrt{\delta(j_1 - j_1')^2 + \ldots + \delta(j_d - j_d')^2} \tag{2.15}$$

with $\delta(\xi) := \max\{0, |\xi| - 1\}$. Let $T_2 = T(I \times I)$ be the block-cluster tree corresponding to the cluster tree $T_1 = T(I)$. The definition of T_2 implies the following remark (cf. [10]).

Remark 2.7 *Let $\tau \times \sigma \in T(I \times I)$. Then $\tau, \sigma \in T(I)$ belong to the same level $\ell \in \{0,\ldots,p\}$.*

The set of clusters $\tau \in T(I)$ from level ℓ is called T_1^ℓ. In view of Remark 2.7, for $\ell \in \{0,\ldots,p\}$, we denote by T_2^ℓ the set of clusters $\tau \times \sigma \in T_2$ such that blocks τ, σ belong to level ℓ. In particular, $T_2^0 = \{I \times I\}$ is the root of T_2 and $T_2^p = \{\{(x,y)\} : x, y \in I\}$ is the set of leaves. Correspondingly, we define $P_2^\ell := P_2 \cap T_2^\ell$.

3 $\mathcal{H}$-Matrices via Wire-Basket Expansions

3.1 Basic Idea: Description on the Continuous Level

The basic idea of the wire-basket approach is the interface representation of the scalar product $\langle A_{\mathcal{H}} u, v\rangle$, $u, v \in V_h$, for the hierarchical approximation to the operator A from (2.3). By definition, there holds

$$\langle A_{\mathcal{H}} v, u\rangle = \sum_{\tau\times\sigma\in P_2} \int_{X(\tau)\times X(\sigma)} s_{\tau,\sigma}(x,y) u(x) v(y) dy dx. \tag{3.1}$$

First, we consider the exact Galerkin ansatz $\langle Av, u\rangle$ with the kernel-function $s(x,y)$ instead of $s_{\tau,\sigma}$ and transform each domain integral for $\tau \times \sigma \in P_{far}$ into its boundary form. For notational convenience, define a set of geometrical blocks (product subdomains) $X_{far} := \{X(\tau) \times X(\sigma) : \tau \times \sigma \in P_{far}\}$. In view of (2.2), we have

$$\mathcal{L}_x s(x,y) = \mathcal{L}_y^* s(x,y) = 0 \qquad \text{for } (x,y) \in X(\tau) \times X(\sigma) \in X_{far}. \tag{3.2}$$

In the following, the symbol Ω is used as variable for a domain. Let $a_\Omega(\cdot,\cdot)$ be the bilinear form associated with the operator $\mathcal{L}$ as above. Then the first Green formula holds:

$$\langle v, \mathcal{L}u\rangle_\Omega = a_\Omega(u,v) - \langle \partial_\nu u, v\rangle_{0,\partial\Omega} \qquad \text{for all } u \in H^2(\Omega,\mathcal{L}),\ v \in H^1(\Omega), \tag{3.3}$$

where $\langle\cdot,\cdot\rangle_{0,\Sigma}$ with $\Sigma := \partial\Omega$ is the $L^2(\Sigma)$-scalar product and ∂_ν is the conormal derivative

$$\partial_\nu := \sum_{j,k=1}^d n_j a_{jk} \partial_k - \sum_{j=1}^d n_j b_j$$

with $\partial_j = \partial/\partial x_j$ and n_j being the components of the outward unit normal vector.

Denote $W^0 = H^2(\Omega) \cap H_0^1(\Omega)$. For any $z \in L^2(\Omega)$, introduce a function $g_z \in W^0$ (note that convexity of the domain Ω implies the full elliptic regularity, $||g_z||_{2,\Omega} \le c||z||_{0,\Omega}$) such that

$$a_\Omega(g_z, \eta) = \langle z, \eta \rangle_{0,\Omega} \qquad \text{for all } \eta \in H_0^1(\Omega). \tag{3.4}$$

The continuous operator $\mathcal{L}_\Omega^{-1} : L^2(\Omega) \to W^0$ is defined by $\mathcal{L}_\Omega^{-1} z = g_z$, which has a continuous extension as a mapping $\mathcal{L}_\Omega^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$. In the case of a hierarchical cluster tree T_1 of depth p, the far-field component P_{far} from (2.6) may be specified by the choice of parameter $p_0 \in \mathbb{N}$ of size $O(1)$ yielding the alternative definition

$$P_{far} = \cup_{\ell=2}^{p-p_0} P_2^\ell, \qquad P_{near} = \cup_{\ell=p-p_0+1}^{p} P_2^\ell.$$

We recall that on block-clusters from P_{far}, we approximate the kernel function by degenerate expansions, while matrix entries corresponding to P_{near} are computed exactly (we assume that the computation of one matrix entry costs $O(1)$ arithmetical operations). Note that (3.2) implies the Galerkin orthogonality

$$a_\tau(g_u(x), s(x,y)) = a_\sigma(g_v(y), s(x,y)) = 0 \qquad \text{for } (x,y) \in X(\tau) \times X(\sigma) \in X_{far}. \tag{3.5}$$

Lemma 3.1 *For any $u, v \in L^2(\Omega)$ there holds*

$$\begin{aligned} \langle Av, u \rangle = &\sum_{\tau \times \sigma \in P_{far}} \int_{\partial X(\tau)} \int_{\partial X(\sigma)} s(x,y) \partial_\nu g_v(y) \partial_\nu g_u(x) dx dy \\ &+ \sum_{\tau \times \sigma \in P_{near}} \int_{X(\tau) \times X(\sigma)} s(x,y) u(x) v(y) dx dy. \end{aligned} \tag{3.6}$$

Proof. (3.3) and (3.5) with $u = \mathcal{L}_x g_u$ and $v = \mathcal{L}_y g_v$ lead to

$$\begin{aligned} &\int_{X(\tau) \times X(\sigma)} s(x,y) u(x) v(y) dx dy \\ &= \int_{X(\tau)} \left\{ a_\sigma(g_v(y), s(x,y)) - \int_{\partial X(\sigma)} s(x,y) \partial_\nu g_v(y) dy \right\} \mathcal{L}_x g_u(x) dx \\ &= \int_{\partial X(\sigma)} \left\{ -a_\tau(g_u(x), s(x,y)) + \int_{\partial X(\tau)} s(x,y) \partial_\nu g_u(x) dx \right\} \partial_\nu g_v(y) dy \\ &= \int_{\partial X(\sigma)} \int_{\partial X(\tau)} s(x,y) \partial_\nu g_v(y) \partial_\nu g_u(x) dx dy \end{aligned}$$

for all $\tau \times \sigma \in P_{far}$. Hence, (3.6) follows. ∎

Assume we are given a degenerate expansion

$$s_{\partial\tau, \partial\sigma} = \sum_{\alpha=1}^{k_1} a_\alpha(x) c_\alpha(y), \qquad (x,y) \in \partial X(\tau) \times \partial X(\sigma), \tag{3.7}$$

for each $\tau \times \sigma \in P_{far}$ such that

$$|s(x,y) - s_{\partial\tau,\partial\sigma}(x,y)| \le c\, \eta^m \{\mathrm{dist}(\tau, \sigma)\}^{2-d} \qquad \text{for } (x,y) \in \partial X(\tau) \times \partial X(\sigma),$$

where $k_1 = O(m^{d-1})$. Then we introduce the *wire-basket representation* of the operator $A_{\mathcal{H}}$ by

$$\begin{aligned} \langle A_{\mathcal{H}} v, u \rangle \quad = \quad &\sum_{\tau \times \sigma \in P_{near}} \int_{X(\tau) \times X(\sigma)} s(x,y) u(x) v(y) dx dy \\ &+ \sum_{\tau \times \sigma \in P_{far}} \sum_{\alpha} \int_{\partial X(\tau)} a_\alpha(x) \partial_\nu g_u(x) dx \cdot \int_{\partial X(\sigma)} c_\alpha(y) \partial_\nu g_v(y) dy, \qquad u, v \in L^2(\Omega). \end{aligned} \tag{3.8}$$

The second sum will be abbreviated by $\langle v, A_{\mathcal{H}} u \rangle^{far}$. Particular constructions of $s_{\partial\tau,\partial\sigma}$ will be considered in Section 4.

Below, we introduce the Galerkin approximation to the second sum in (3.8). To this end, we represent each integral over $\partial X(\tau)$ (or $\partial X(\sigma)$) in terms of domain integrals using an easily computable extension of $a_\alpha(x)$ and $b_\alpha(y)$ into the interior of $X(\tau)$ and $X(\sigma)$, respectively.

Remark 3.2 *Let $E_x : H^{1/2}(\partial X(\tau)) \to H^1(X(\tau))$ be any continuous extension operator defined for each $\tau \in T_1$. Due to (3.4) there holds*

$$\int_{\partial X(\tau)} a_\alpha(x)\partial_\nu g_u(x)dx = -a_\tau(g_u, E_x a_\alpha) + (u, E_x a_\alpha)_{0,X(\tau)}. \tag{3.9}$$

The same extension with respect to the y-variable is denoted by E_y.

With the given ansatz space $W_h \subset H^1(X(\tau))$, let $g_{h,z} \in W^0_{h,\tau} := W_h \cap H^1_0(X(\tau))$ be the Ritz projection of g_z defined by

$$a_\tau(g_{h,z}, \eta) = (z, \eta)_{0,X(\tau)} \qquad \text{for all } \eta \in W^0_{h,\tau}. \tag{3.10}$$

The FE Galerkin approximation to the far-field contribution in (3.8) is defined for any $u, v \in V_h$ by substituting $g_{h,u}$ (resp. $g_{h,v}$) into the right-hand side of (3.9) and choosing E_x (resp. E_y) as the extension into W_h by FE functions with minimal support inside $X(\tau)$.

Corollary 3.3 *For any $u, v \in V_h$, the FE Galerkin approximation $A_{\partial\mathcal{H},h} = A_h$ for the operator $A_\mathcal{H}$ is defined by*

$$\langle A_h v, u\rangle \quad = \quad \sum_{\tau\times\sigma\in P_{near}} \int_{X(\tau)\times X(\sigma)} s(x,y)u(x)v(y)dxdy \tag{3.11}$$

$$+ \sum_{\tau\times\sigma\in P_{far}} \sum_{\alpha=1}^{k_1} \left((u, E_x a_\alpha)_{0,X(\tau)} - a_\tau(g_{h,u}, E_x a_\alpha)\right)\left((v, E_y c_\alpha)_{0,X(\sigma)} - a_\sigma(g_{h,v}, E_y c_\alpha)\right).$$

Remark 3.4 *Note that the Galerkin ansatz space $V_h \subset L^2(\Omega)$ restricted to the geometrical clusters $X(\tau)$ may differ from W_h defined above. However, for the ease of presentation, we further assume $V_h = W_h$ for each $\tau \in T_1(I)$.*

3.2 Matrix Representation and Complexity Bound

The representation (3.11) defines the generalised $\mathcal{H}$-matrices which inherit the standard hierarchical block structure from the P_2-partitioning, but now the rank-k structure of the blocks $b \in P_2^\ell$ is given implicitly based on the factorisation by local Schur-complement matrices. Such a factorisation allows to reduce the amount of data for the storage and matrix arithmetic essentially. To build the explicit representation of the matrix block $\mathbf{A}_h^{\tau\times\sigma}$ of the resulting $\mathcal{H}$-matrix, we introduce the local Schur-complement operator

$$S_{\tau,h} : W^0_{h,\tau} \to \Gamma_{h,\tau} := W_{h\,|\partial X(\tau)},$$

by

$$\langle S_{\tau,h} z, w\rangle := (z, E_x w)_{0,X(\tau)} - a_\tau(g_{h,z}, E_x w), \qquad z \in W^0_{h,\tau},\ w \in \Gamma_{h,\tau}, \tag{3.12}$$

which, in fact, provides a FE variational approximation of the operator $S = \partial_\nu \mathcal{L}_\tau^{-1} : L^2(X(\tau)) \to H^{1/2}(\partial X(\tau))$. The construction is independent of the extension operator E_x due to the Galerkin orthogonality (see (3.10)).

The L^2-projection operator $\mathcal{Q}_{h,\tau}$ onto $\Gamma_{h,\tau}$ is given by

$$(\mathcal{Q}_h u, v)_{0,\partial X(\tau)} = (u, v)_{0,\partial X(\tau)} \qquad \text{for all } v \in \Gamma_{h,\tau}.$$

Let $\mathbf{a}_{h,\alpha}, \mathbf{c}_{h,\alpha}$ be the vector representations of $\mathcal{Q}_h a_\alpha$ and $\mathcal{Q}_h c_\alpha$, while $\mathbf{S}_{\tau,h}$ and $\mathbf{S}_{\sigma,h}$ are the matrix representations of $S_{\tau,h}$ and $S_{\sigma,h}$. The matrix block $\mathbf{A}_h^{\tau\times\sigma}$ defined by (3.11) for the product index-set $\tau\times\sigma \in P_{far}$ has the factorisation

$$\mathbf{A}_h^{\tau\times\sigma} = \sum_{\alpha=1}^{k_1} (\mathbf{S}^T_{\tau,h}\cdot\mathbf{a}_{h,\alpha}) * (\mathbf{c}^T_{h,\alpha}\cdot\mathbf{S}_{\sigma,h}) = \mathbf{S}^T_{\tau,h}\left(\sum_{\alpha=1}^{k_1} \mathbf{a}_{h,\alpha} * \mathbf{c}^T_{h,\alpha}\right)\mathbf{S}_{\sigma,h}, \tag{3.13}$$

where $\mathbf{S}_{\tau,h} : \mathbb{R}^{n_\tau} \to \mathbb{R}^{n_{\partial\tau}}$, $\mathbf{a}_{h,\alpha} \in \mathbb{R}^{n_{\partial\tau}}$, $n_\tau := \dim W_{h,\tau}$, $n_{\partial\tau} = \dim\Gamma_{h,\tau}$ and the same for $\mathbf{S}_{\sigma,h}$, $\mathbf{c}_{h,\alpha}$. Clearly, (3.13) defines a matrix block with rank $\le k_1$.

Denote by $E_{\tau,h}$ the discrete $\mathcal{L}$-harmonic extension operator in $X(\tau)$. Assume that the Ritz projection $g_{h,u}$ in (3.11) and the extension operator $E_{\tau,h}$ (the same for $g_{h,v}$ and $E_{\sigma,h}$) can be evaluated on each geometrical cluster $X(\tau)$ with linear cost $c_{RP}n_\tau$ (say, by the multi-grid method; see also the algorithm in Section 3.3 based on the reuse of particular solutions).

The advantage of the presented method is the reduction of the order of expansion, on the one hand, and the linear bound for $Q_{\tau,\sigma}$ with respect to the block-size n_τ, on the other hand. The latter is due to reduction to the boundary. Moreover, the constant in the asymptotical complexity is essentially dominated by c_{RP} which may be smaller then the corresponding constant in the $\mathcal{H}$-matrix arithmetic, especially for $d = 3$.

Lemma 3.5 *Let our construction be based on a hierarchical cluster tree of the depth p. For the variable order approximation with $k_1(\ell) = (a_1(p-\ell)+b_1)^{d-1}$ (as used for the $\mathcal{H}^2$-matrices in [10, 14]) the storage and matrix-vector multiplication expenses are dominated by*

$$\mathcal{N}_{st} = O(pn), \qquad \mathcal{N}_{MV} = O(n) + c_{RP}O(pn).$$

Proof. The storage for the implementation of the Schur-complement operator from (3.12) is dominated by $\sum_{\tau\times\sigma\in P_2^{far}} \#\tau = O((p-p_0)n)$. On the other hand, a simple estimate $n_\tau = O(2^{d(p-\ell)}) = 2^{p-\ell}O(n_{\partial\tau})$ implies that the coefficients of Rk-matrix blocks from (3.13) need only a storage size of $O(k_1 n_{\partial\tau}) = O(k_1 n_\tau 2^{\ell-p})$. This proves the linear-logarithmic bound for $\mathcal{N}_{st}$.

The matrix-vector product for each block has the complexity

$$Q_{\tau,\sigma} = ck_1(n_{\partial\tau} + n_{\partial\sigma}) + 2c_{RP}(n_\tau + n_\sigma). \tag{3.14}$$

In fact, due to (3.12), the implementation of $\mathbf{S}_{\sigma,h}$ is of linear cost. Furthermore, the matrix-vector product by $\mathbf{S}^T_{\tau,h}$ is equivalent to the implementation of the elliptic extension $E_{\tau,h}$ due to the relation (3.12) and the Galerkin orthogonality,

$$\langle z, S^T_{\tau,h}w\rangle = (z, E_{\tau,h}w)_{0,X(\tau)} - a_\tau(g_{h,z}, E_{\tau,h}w) = (z, E_{\tau,h}w)_{0,X(\tau)} \tag{3.15}$$

for $z \in W^0_{h,\tau}$, $w \in \Gamma_{h,\tau}$. Thus, the matrix-vector product for all blocks on level ℓ has the complexity

$$Q_{\tau,\sigma} = ck_1 n_\tau 2^{\ell-p} + 2c_{RP}n_\tau. \tag{3.16}$$

Summation over all the blocks $\tau\times\sigma \in P_2$ completes our proof. ∎

3.3 Further Optimisation by Reusing Particular Solutions

Further, we briefly consider an optimised construction of the wire-basket scheme based on the reuse of particular solutions. Assume we are given a balanced hierarchical cluster tree T_1 of the depth p. The idea is that subdomain solvers are used only on few coarse levels $\ell = 2,\dots,\ell_0$ while the restriction of these solutions to the smaller domains corresponding to the levels $\ell > \ell_0$ yield particular solutions (with wrong Dirichlet data). Assuming a correction scheme for the Dirichlet data of the cost $O(n_{\partial\tau}\log^q n_{\partial\tau})$, $q = O(1)$ (see Remark 3.7), linear complexity of the overall scheme holds.

Again, we start from the continuous case. Given $\ell_0 \in \mathbb{N}$, $\ell_0 \geq 2$, we assume the covering property

$$\bigcup_{\ell=2}^{\ell_0}\bigcup_{\tau\times\sigma\in P_2^\ell}\tau = \bigcup_{\ell=2}^{\ell_0}\bigcup_{\tau\times\sigma\in P_2^\ell}\sigma = I, \tag{3.17}$$

where $\ell_0 = O(1)$. In most of the cases, (3.17) holds with either $\ell_0 = 2$ or $\ell_0 = 3$. For the ease of exposition, we assume $\ell_0 = 2$. We denote the $\mathcal{L}$-harmonic extension operator by $E_\Omega : H^{1/2}(\partial\Omega) \to H^1(\Omega)$

Assumption 3.6 *We assume furthermore, that for all $\tau \in T_1^\ell$ with $\ell \geq 3$ the Poincaré-Steklov operator $S_\mathcal{L} = \partial_\nu E_\tau$ can be approximated accurately with the cost $O(n_\tau^q)$, $q < 1$.*

Remark 3.7 *For most of the common elliptic operators of the form (2.1), the discrete Poincaré-Steklov operator $\mathbf{S}_\tau$ defined by (3.20) can be evaluated with the complexity $O(n_{\partial\tau}\log^q n_{\partial\tau})$, $q = O(1)$. In fact, for the Laplace, biharmonic, Stokes and Lamé operators on rectangular domains, sparse approximation of linear-logarithmic complexity $O(n_{\partial\tau}\log^2 n_{\partial\tau})$ are known (cf. [16, 17]). An extension of these results to the case of polygonal domains is possible.*

Let $T_1(\tau)$ be the largest subtree of $T_1(I)$ having the root $\tau \in T_1(I)$. Denote by $E_\Omega : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ the $\mathcal{L}$-harmonic extension operator in Ω. Then, we obtain the representation

$$\mathcal{L}_{\tau'}^{-1} r_{X(\tau')} u = r_{X(\tau')} \mathcal{L}_\tau^{-1} u - E_{\tau'} r_{\partial X(\tau')} \mathcal{L}_\tau^{-1} u \qquad \text{for all } \tau' \in T_1(\tau) \subset T_1(I), \tag{3.18}$$

where $u \in L^2(X(\tau))$ and r_Ω is the restriction operator onto the given domain/manifold Ω.

Substitution of (3.18) into (3.8) instead of g_u (the same for g_v) results in the fast Schur-complement scheme which may be discretised in the variational framework. In this way, we need an optimal elliptic solver only on each cluster of the low level $\ell = 2$ (for coarse subdomains). Now, the far-field contribution in the right-hand side of (3.8) takes the form

$$\langle v, A_\Omega u \rangle^{far} = \sum_{\tau \times \sigma \in P_2^2} \sum_\alpha \int_{\partial X(\tau)} a_\alpha \partial_\nu g_u dx \int_{\partial X(\sigma)} c_\alpha \partial_\nu g_v dy \tag{3.19}$$

$$+ \sum_{\ell=3}^{p-p_0} \sum_{\tau' \times \sigma' \in P_2^\ell} \sum_\alpha \int_{\partial X(\tau')} a_\alpha(x) \partial_\nu (r_{X(\tau')} - E_{\tau'} r_{\partial X(\tau')}) g_{u,\tau} dx$$

$$\cdot \int_{\partial X(\sigma')} c_\alpha(y) \partial_\nu (r_{X(\sigma')} - E_{\sigma'} r_{\partial X(\sigma')}) g_{v,\sigma} dy,$$

where τ, σ are the clusters from T_1 such that $\tau' \in T_1(\tau)$, $\sigma' \in T_1(\sigma)$ and $g_{u,\tau} = \mathcal{L}^{-1} r_{X(\tau)} u$. For computational needs, the boundary integrals in (3.19) may be rewritten, as above, in terms of domain integrals using "minimal support" extensions of the elements a_α and c_α into $X(\tau')$ and $X(\sigma')$, respectively. As above, we assume a linear cost $c_{RP} n_\tau$ for the implementation of $\mathcal{L}_\tau^{-1}$.

In the first step of the matrix-vector multiplication, we compute the action of all matrix blocks $\mathbf{A}_{\partial\mathcal{H}}^{\tau\times\sigma}$ on the level $\ell = 2$ which amounts to $O(n)$ operations. In the second step, we treat the blocks $\tau' \times \sigma' \in P_2^\ell$, $\ell \geq 3$. Its implementation uses the data from the level $\ell = 2$, i.e., the Schur-complement matrices are multiplied by the vector via

$$\mathbf{S}_{\tau',h}\mathbf{u} := (\mathbf{S}_\nu \mathbf{R}_{\tau,\tau'} - \mathbf{S}_{\tau'} \mathbf{R}_{\tau,\partial\tau'}) \mathbf{g}_{u,\tau}, \quad \mathbf{S}_{\sigma',h}\mathbf{v} := (\mathbf{S}_\nu \mathbf{R}_{\sigma,\sigma'} - \mathbf{S}_{\sigma'} \mathbf{R}_{\sigma,\partial\sigma'}) \mathbf{g}_{v,\sigma},$$

where $\mathbf{R}_{\tau,\tau'}$ ($\mathbf{R}_{\tau,\partial\tau'}$) denotes a matrix representation of the restriction operator $r_{X(\tau')}$ ($r_{X(\partial\tau')}$) and $\mathbf{S}_\nu$ is the FE approximation to the operator ∂_ν. In turn, $\mathbf{S}_{\tau'}$ is the Schur-complement FE approximation to the elliptic Poincaré-Steklov operator $S_\mathcal{L} = \partial_\nu E_\Omega : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ now defined for $\Omega = X(\tau')$,

$$\langle S_{\tau'} z, w \rangle := a_{\tau'}(E_{\tau',h} z, E_x w) \qquad \text{for all } z, w \in \Gamma_{h,\tau'}. \tag{3.20}$$

Furthermore, $\mathbf{g}_{u,\tau}$ (resp. $\mathbf{g}_{v,\sigma}$) is the vector representation of the Ritz projection $g_{h,u}$ on τ (resp. $g_{h,v}$ on σ). Denote by $\mathbf{E}_{\tau,h}$ the matrix representation of $E_{\tau,h}$, then the action of $\mathbf{S}_{\tau',h}^T$ is performed using the matrix $\mathbf{E}_{\tau,h}$, see (3.15). The quadratic form and matrix-vector product in question have now the representations

$$(\mathbf{A}_{\partial\mathcal{H}}^{\tau'\times\sigma'} \mathbf{v}, \mathbf{u}) = \sum_{\alpha=1}^{k_1} \left((\mathbf{a}_{h,\alpha} * \mathbf{c}_{h,\alpha}^T) \mathbf{d}_\sigma, \mathbf{d}_\tau \right), \tag{3.21}$$

and

$$\mathbf{A}_{\partial\mathcal{H}}^{\tau'\times\sigma'} \mathbf{v} = \mathbf{E}_{\tau',h} \sum_{\alpha=1}^{k_1} \mathbf{a}_{h,\alpha} \cdot (\mathbf{c}_{h,\alpha}, \mathbf{d}_\sigma), \tag{3.22}$$

respectively, where $\mathbf{d}_\tau = (\mathbf{S}_\nu \mathbf{R}_{\tau,\tau'} - \mathbf{S}_{\tau'} \mathbf{R}_{\tau,\partial\tau'}) \mathbf{g}_{u,\tau}$ (similar for $\mathbf{d}_\sigma$) with $\mathbf{d}_\tau \in \mathbb{R}^{n_\tau}$, $\mathbf{d}_\sigma \in \mathbb{R}^{n_\sigma}$.

Lemma 3.8 *The reuse of the particular solutions from level ℓ_0 in the matrix operations from (3.21) and (3.22) leads to the linear complexity $\mathcal{N}_{st} = O(n)$ and $\mathcal{N}_{MV} = O(p\,n)$, for the variable order approximation. The bilinear form $(\mathbf{A}_{\partial\mathcal{H}}\mathbf{v}, \mathbf{u})$ can be evaluated with $O(n)$ operations.*

Proof. For the right-hand sides in (3.21) and (3.22) the matrix $\mathbf{S}_\nu$ can be multiplied by the vector with $O(n_{\partial\tau'})$ arithmetical operations. The assertion follows from the representations (3.21), (3.22) and Remark 3.7. ∎

3.4 Approximation Error

In the following, we estimate the approximation error of the scheme defined above. The optimal error bound $O(h)$ is based on the full elliptic regularity of the local problems (3.4) on $\Omega = X(\tau)$, $\tau \in T_1(I)$, as well as on the technical assumption concerning the "stability" of expansion coefficients (see (3.23) below). In the case of non-convex clusters, we arrive at an accuracy $O(h^\beta)$, $0 < \beta < 1$, depending on the elliptic regularity of the subproblems.

Assumption 3.9 *For each $\tau \times \sigma \in P_{far}$, the coefficients $a_\alpha(x)$, $c_\alpha(y)$ for $(x,y) \in \partial X(\tau) \times \partial X(\sigma)$ from (3.7) satisfy the estimates*

$$\sum_{\alpha=1}^{k_1} \Big(||a_\alpha||_{1/2,\partial X(\tau)} ||c_\alpha||_{1/2,\partial X(\sigma)} + C_0 \gamma^\alpha_{0,1/2} + C_1 \gamma^\alpha_{1/2,-1/2} \Big) \le c \ \mathrm{dist}(\tau,\sigma)^{-2r}, \tag{3.23}$$

where

$$\gamma^\alpha_{0,1/2} = ||a_\alpha||_{1/2,\partial X(\tau)} ||c_\alpha||_{0,\partial X(\sigma)} + ||a_\alpha||_{0,\partial X(\tau)} ||c_\alpha||_{1/2,\partial X(\sigma)},$$
$$\gamma^\alpha_{1/2,-1/2} = ||a_\alpha||_{1/2,\partial X(\tau)} ||c_\alpha||_{-1/2,\partial X(\sigma)} + ||a_\alpha||_{-1/2,\partial X(\tau)} ||c_\alpha||_{1/2,\partial X(\sigma)}.$$

Assumption 3.10 *The operator $\mathcal{L}$ satisfies the* Maximum-Minimum Principle *([9]) .*

Lemma 3.11 *Let $k_1 = O(p^{d-1})$ and $r = -1/2$. Under Assumptions 3.9 and 3.10,*

$$|\langle v, (A - A_{\partial\mathcal{H}})u\rangle| \le (c\eta^m n^q + C_0^{-1} h^{3/2} + C_1^{-1} h) ||u||_0 ||v||_0 + ch(||u||_0 ||v||_{-1} + ||u||_{-1} ||v||_0), \qquad q = O(1),$$

holds for all $u, v \in L^2(\Omega)$.

Proof. First, we use the representation

$$\langle v, (A - A_{\partial\mathcal{H}})u\rangle = \langle v, (A - A_{\mathcal{H}})u\rangle + \langle v, (A_{\mathcal{H}} - A_{\partial\mathcal{H}})u\rangle \tag{3.24}$$

indicating that the total error contains the standard consistency error $\langle v, (A - A_{\mathcal{H}})u\rangle$, as well as the error involved by the local Ritz projections. The first term in the right-hand side in (3.24) is estimated using the stability of problem (3.2) with respect to the Dirichlet data on $\partial X(\tau) \times \partial X(\sigma)$. In fact, the Maximum-Minimum Principle with respect to both the x- and y-variables leads to

$$\begin{aligned} &\max_{(x,y)\in X(\tau)\times X(\sigma)} \left| s(x,y) - \sum_\alpha E_\tau a_\alpha(x) E_\sigma c_\alpha(y) \right| \\ &\le \max_{(x,y)\in \partial X(\tau)\times \partial X(\sigma)} \left| s(x,y) - \sum_\alpha a_\alpha(x)\, c_\alpha(y) \right| \quad \le \quad c\, \eta^m dist(\tau,\sigma)^{2-d}. \end{aligned}$$

Then, similarly to the proof of Theorem 2.6, the following estimate holds

$$|\langle v, (A - A_{\mathcal{H}})u\rangle| \le c\, \eta^m n^q ||v||_{0,\Omega} ||u||_{0,\Omega}.$$

To estimate the second term in (3.24), we first note that the choice $\mathrm{image}(E_x) = \mathrm{image}(E_y) \subset W_h$ implies that the terms $(u, E_x a_\alpha)_{0,X(\tau)}$ and $(v, E_y c_\alpha)_{0,X(\sigma)}$ arising in (3.11) can be evaluated exactly. Then, it is sufficient to consider the bound of

$$\langle v, (A_{\mathcal{H}} - A_{\partial\mathcal{H}})u\rangle = \sum_{\ell=2}^{p-p_0} \sum_{\tau\times\sigma\in P_2^\ell} \sum_{\alpha\le k_1} (e^1_\alpha + e^2_\alpha + e^3_\alpha + e^4_\alpha), \tag{3.25}$$

where

$$\begin{aligned} e^1_\alpha &= a_\tau(g_{h,u} - g_u, E_x a_\alpha) \cdot a_\sigma(g_{h,v}, E_y c_\alpha), \\ e^2_\alpha &= a_\tau(g_{h,u}, E_x a_\alpha) \cdot a_\sigma(g_{h,v} - g_v, E_y c_\alpha), \\ e^3_\alpha &= -(u, E_x a_\alpha)_{0,X(\tau)} \cdot a_\sigma(g_{h,v} - g_v, E_y c_\alpha), \\ e^4_\alpha &= -a_\tau(g_{h,u} - g_u, E_x a_\alpha) \cdot (v, E_y c_\alpha)_{0,X(\sigma)}. \end{aligned}$$

Using the Galerkin orthogonality for $g_{h,z}$ and g_z, the a priori estimate for the Dirichlet boundary value problem in $X(\tau)$ and the standard H^1-error bound for the Ritz projection yield

$$|a_\tau(g_{h,u}-g_u, E_x a_\alpha)| = |a_\tau(g_{h,u}-g_u, E_{\tau,h} a_\alpha)| \le c\,h||u||_{0,X(\tau)}||a_\alpha||_{1/2,\partial X(\tau)}.$$

The L^2-estimate of the discrete $\mathcal{L}$-harmonic function

$$||E_{\tau,h}c_\alpha||_{0,X(\sigma)} \le c||c_\alpha||_{-1/2,\partial X(\sigma)} \tag{3.26}$$

is valid in the case of full elliptic regularity, the proof is similar to the case of Laplace equation, considered in [1]. This implies

$$\begin{aligned}|a_\sigma(g_{h,v}, E_y c_\alpha)| &\le c(||g_{h,v}||_{1,X(\sigma)}||E_{\sigma,h}c_\alpha||_{1,X(\sigma)} + ||v||_{0,X(\sigma)}||E_{\sigma,h}c_\alpha||_{0,X(\sigma)})\\ &\le c||v||_{-1,X(\sigma)}||c_\alpha||_{1/2,\partial X(\sigma)} + c||v||_{0,X(\sigma)}||c_\alpha||_{-1/2,\partial X(\sigma)},\end{aligned}$$

where $||\cdot||_{-1,\Omega}$ is the norm of $H^{-1}(\Omega) = \left(H_0^1(\Omega)\right)'$. Therefore,

$$\begin{aligned}|e_\alpha^1| &\le c\,h||u||_{0,X(\tau)}||a_\alpha||_{1/2,\partial X(\tau)}(||v||_{-1,X(\sigma)}||c_\alpha||_{1/2,\partial X(\sigma)} + ||v||_{0,X(\sigma)}||c_\alpha||_{-1/2,\partial X(\sigma)}),\\ |e_\alpha^2| &\le c\,h||v||_{0,X(\sigma)}||c_\alpha||_{1/2,\partial X(\sigma)}(||u||_{-1,X(\tau)}||a_\alpha||_{1/2,\partial X(\tau)} + ||u||_{0,X(\tau)}||a_\alpha||_{-1/2,\partial X(\tau)}).\end{aligned}$$

Using similar arguments and applying the bound

$$||E_y c_\alpha||_{0,X(\sigma)} \le ch^{1/2}||c_\alpha||_{0,\partial X(\sigma)}, \tag{3.27}$$

but now to both a_α and c_α, we obtain

$$\begin{aligned}|e_\alpha^3| &\le c\,h^{3/2}||u||_{0,X(\tau)}||v||_{0,X(\sigma)}||a_\alpha||_{0,\partial X(\tau)}||c_\alpha||_{1/2,\partial X(\sigma)},\\ |e_\alpha^4| &\le c\,h^{3/2}||u||_{0,X(\tau)}||v||_{0,X(\sigma)}||a_\alpha||_{1/2,\partial X(\tau)}||c_\alpha||_{0,\partial X(\sigma)}.\end{aligned}$$

Finally, the substitution of these estimates into (3.25) yields

$$\begin{aligned}&|\langle v, (A_{\mathcal{H}} - A_{\partial\mathcal{H}})u\rangle|\\ &\le c_1\,h\sum_{\tau\times\sigma\in P_2}(||u||_{0,X(\tau)}||v||_{-1,X(\sigma)} + ||u||_{-1,X(\tau)}||v||_{0,X(\sigma)})\cdot\sqrt{|\tau||\sigma|}\sum_{\alpha\le k_1}||a_\alpha||_{1/2,\partial X(\tau)}||c_\alpha||_{1/2,\partial X(\sigma)}\\ &+ c_1\,h\sum_{\tau\times\sigma\in P_2}||u||_{0,X(\tau)}||v||_{0,X(\sigma)}\sqrt{|\tau||\sigma|}\sum_{\alpha\le k_1}\gamma^\alpha_{1/2,-1/2}\\ &+ c_2\,h^{3/2}\sum_{\tau\times\sigma\in P_2}||u||_{0,X(\tau)}||v||_{0,X(\sigma)}\sqrt{|\tau||\sigma|}\sum_{\alpha\le k_1}\gamma^\alpha_{0,1/2}\\ &\le c_1\,h\left(\sum_{\ell=2}^{p-p_0}||u||_{0,\Omega}\sum_{\tau\times\sigma\in P_2^\ell}|\sigma|\;||v||_{-1,X(\sigma)} + \sum_{\ell=2}^{p-p_0}||v||_{0,\Omega}\sum_{\tau\times\sigma\in P_2^\ell}|\tau|\;||u||_{-1,X(\tau)}\right)\\ &+ (c_2\,h^{3/2}+c_1 h)||u||_{0,\Omega}\sum_{\ell=2}^{p-p_0}||v||_{0,X(\sigma)}\sum_{\tau\times\sigma\in P_2^\ell}\sqrt{|\tau||\sigma|}.\end{aligned}$$

Hence, the assertion follows. ∎

Remark 3.12 *Lemma 3.11 guarantees a hierarchical approximation of the exact Galerkin stiffness matrix with an error $O(h)$ with respect to the spectral norm.*

4 Construction of Kernel Expansions (Case B)

4.1 Polynomial Approximation of Multivariate Functions

We assume that our kernel function $s(x,y)$ satisfies Assumption 2.4 (cf. Section 2.3). For deriving the desired low order expansions, we use classical approximation results for functions which are analytic in the interval $I_1 = [-1,1]$.

Definition 4.1 *A function $f \in C^\infty(I_1)$ has Bernstein's regularity ellipse $\mathcal{E}_H(I_1)$ if it admits an analytic extension to the closed ellipse $\mathcal{E}_H(I_1) \subset \mathbb{C}$ with foci in $z = \pm 1$ and the sum of semi-axes equal to $H > 1$.*

The definition of $\mathcal{E}_H(I_1)$ for other intervals than $[-1,1]$ is obvious.

The following statement goes back to the classical result of S.N. Bernstein (see also [22] for more details). In particular, we apply the result from [21].

Proposition 4.2 *Assume that the function $f \in C^\infty(I_1)$ has the regularity ellipse $\mathcal{E}_H(I_1)$, according to Definition 4.1. Let $[I_N f](x) \in P_N[I_1]$ on $[-1,1]$ be the interpolation polynomial with respect to the Chebyshev-Gauss-Lobatto nodes $\xi_j = cos\frac{\pi j}{N}$, $j = 0,\dots,N$. Then the following approximation property holds:*

$$||f - I_N f||_{L^\infty(I)} \le cN \frac{H^{-N}}{H-1} \max_{z \in \mathcal{E}_H} |f(z)|. \tag{4.1}$$

For multivariate functions $f = f(x_1,...,x_d) : \mathbb{R}^d \to \mathbb{R}$, we use the tensor product interpolant

$$\mathbf{I}_N f = I_N^1 \cdots I_N^d f \in P_N[I_1^d],$$

where $I_N^i f$ denotes the interpolation polynomial with respect to x_i, $i = 1,\dots,d$, at the Chebyshev-Gauss-Lobatto nodes. The interpolation points ξ_α, $\alpha = (i_1,...,i_d) \in \mathbb{N}_0^d$, in I_1^d are obtained by the Cartesian product of the one-dimensional nodes,

$$\xi_\alpha := \left(\cos\frac{\pi i_1}{N}, \dots, \cos\frac{\pi i_d}{N}\right).$$

Denote by X_{-i} the subset $X_{-i} := \{x_1,\dots,x_{i-1},x_{i+1},\dots,x_d\}$ of $d-1$ spatial variables. Related to Definition 4.1, we are interested in polynomial approximations of the following class of functions.

Assumption 4.3 *For a given function $f \in C^\infty(I_1^d)$, assume that there is an $H_0 > 1$ such that for each of the subset $z_i \in X_{-i}$, $i = 1,...,d$, there exists an analytic extension with respect to $x_i \in \mathcal{E}_{H_0}(I_i) \subset \mathbb{C}$.*

Proposition 4.4 *Let Assumption 4.3 be valid. Then, for $1 < H < H_0$ there holds*

$$\begin{aligned} ||f - \mathbf{I}_N f||_{L^\infty(I_1^d)} &\le cN \log^{d-1} N \frac{H^{-N}}{H-1} M_H(f), \\ M_H(f) &= \max_{j \le d}\{\max_{X_{-j}} \max_{x_j \in \mathcal{E}_H(I_j)} |f(x_1,...,x_d)|\}. \end{aligned} \tag{4.2}$$

Proof. The multiple use of (4.1) and the triangle inequality lead to

$$\begin{aligned} |f - \mathbf{I}_N f| &\le |f - I_N^1 f| + |I_N^1 (f - I_N^2 \dots I_N^d f)| \\ &\le |f - I_N^1 f| + |I_N^1 (f - I_N^2 f)| + |I_N^1 I_N^2 (f - I_N^3 f)| + \dots + |I_N^1 I_N^2 \dots I_N^{d-1}(f - I_N^d f)| \\ &\le c \left(\max_{X_{-1}} \max_{x_1 \in \mathcal{E}_H(I_1)} |f(x)| + \log N \max_{X_{-2}} \max_{x_2 \in \mathcal{E}_H(I_2)} |f(x)| + \right. \\ &\left. \dots + \log^{d-1} N \max_{X_{-d}} \max_{x_d \in \mathcal{E}_H(I_d)} |f(x)| \right) N \frac{H^{-N}}{H-1}, \end{aligned}$$

where, similar to [18], we apply the L^∞-estimate of the scalar interpolant I_N^i with respect to each space variable x_i, $i = 1,...,d$,

$$||I_N^i f||_{L^\infty(I_i)} \le c \log N ||f||_{L^\infty(I_i)} \qquad \text{for } f \in C^0(I_i).$$

Hence (4.2) follows. ∎

Remark 4.5 *In the case of a scaled domain I_δ^d, $I_\delta = [-\delta,\delta]$, $\delta > 0$, the corresponding exponent in the error estimates (4.2), (4.4) is equal to $(H/\delta)^{-1}$.*

4.2 Application to Volume Integral Operators (Case B)

Consider the kernel function $s(x_\tau, x_\sigma) = S(x_\tau - x_\sigma)$, $(x_\tau, x_\sigma) \in X(\tau) \times X(\sigma)$ associated with the fundamental solution S of (2.1) satisfying Assumption 2.4.

In the case $d = 2$ and with the standard Euclidean metric, let $\tau \times \sigma \in P_2^\ell$ be a block satisfying the admissibility condition (2.5). In the following, we use the notation $x_\tau = (x_{1\tau}, x_{2\tau})$, $x_\sigma = (x_{1\sigma}, x_{2\sigma})$.

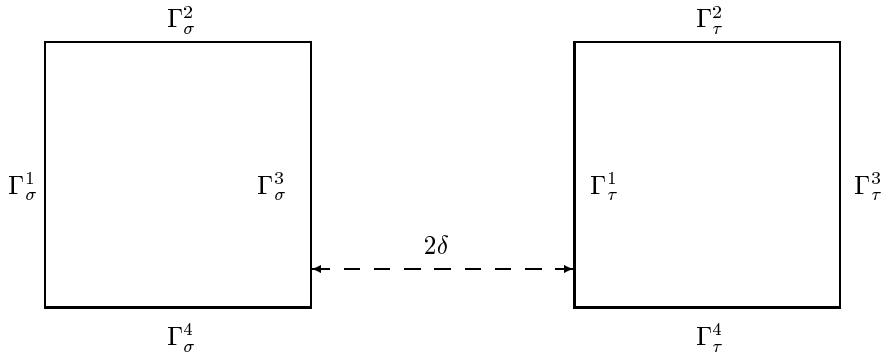


Figure 1: Location of the geometrical clusters $X(\tau)$ and $X(\sigma)$.

We assume that $X(\tau)$ is a rectangle with the boundary $\partial X(\tau) = \cup_{i=1}^4 \Gamma_\tau^i$ and $\partial X(\sigma) = \cup_{i=1}^4 \Gamma_\sigma^i$ with $|\Gamma_\sigma^i| = |\Gamma_\tau^i| = 2\delta$, see Fig. 1. Suppose that the edges Γ_σ^3 and Γ_τ^1 are parallel to the x_2-axis and satisfy $\mathrm{dist}(\Gamma_\sigma^3, \Gamma_\tau^1) = 2\delta$. Construct a kernel expansion on the subset $\Gamma_\sigma^3 \times \Gamma_\tau^1 \subset \partial X(\sigma) \times \partial X(\tau)$.

Due to assumptions from above, the coordinates $x_{1\sigma}$ for $x_\sigma \in \Gamma_\sigma^3$ and $x_{1\tau}$ for $x_\tau \in \Gamma_\tau^1$ are both fixed. Hence, in Euclidean distance, the function of two variables $f(x_{2\sigma}, x_{2\tau}) = s(x_{1\tau}, x_{2\tau}, x_{1\sigma}, x_{2\sigma})$ has the family of regularity ellipses $\mathcal{E}_{H_0}$ in the sense of Assumption 4.3 and with $H_0 = a + b$, where $a^2 = b^2 + \delta^2$ and the small semiaxis b is bounded by $b < \mathrm{dist}(\tau, \sigma)$. Due to (2.5), there holds $\mathrm{dist}(\tau, \sigma) \geq \sqrt{d}\delta\eta^{-1}$ implying the upper bound $b < \sqrt{d}\delta\eta^{-1}$ and also $a < \sqrt{1 + d\eta^{-2}}\,\delta$. This yields

$$H_0 < \left(\sqrt{d}\eta^{-1} + \sqrt{1 + d\eta^{-2}}\right)\delta. \tag{4.3}$$

In particular, for the choice $\eta = \frac{\sqrt{d}}{2}$ with $d = 1, 2, 3$, we obtain

$$H_0 < \left(2 + \sqrt{5}\right)\delta. \tag{4.4}$$

Applying Proposition 4.4 with the scaling argument from Remark 4.5 leads to the following estimate on the exponent in a convergence rate of the polynomial approximation

$$\left(\frac{H}{\delta}\right)^{-1} \leq \eta / \left(\sqrt{d} + \sqrt{d + \eta^2}\right).$$

The constant $M_H(f)$ may be estimated by

$$M_H(f) \leq c \max_{|\xi| \geq \sqrt{d}\delta\eta^{-1} - b} |S(\xi)|.$$

Assume that $|S(\xi)| \leq c|\log|\xi||$. Solving the simple optimisation problem

$$\min_{0 < b < \sqrt{d}\delta\eta^{-1}} \frac{H^{-m}}{H - 1} M_H(f) \qquad \text{with } H = b + \sqrt{\delta^2 + b^2}$$

leads to an error estimate of the form

$$||f - \mathbf{I}_m f||_{L^\infty(I_1^2)} \leq c m^q \log^{q_1} m\, (c_1\eta)^m, \qquad q_1 = O(1) \tag{4.5}$$

with $c_1 = 1/\left(\sqrt{d} + \sqrt{d + \eta^2}\right)$, uniformly with respect to n and $\eta \leq 1$. The bound (4.5) implies the uniform L^∞-estimate

$$||f - \mathbf{I}_m f||_{L^\infty} \leq c h^\alpha = c 2^{-\alpha p}$$

with the polynomial degree $m = O(p)$ for all $\alpha = O(1)$. Let $\{T_i(x_{2\tau})\}$ and $\{T_j(x_{2\sigma})\}$ be the bases of the corresponding univariate polynomial sets P_N defined on Γ^1_τ and Γ^3_σ, respectively (say, Chebyshev polynomials). The number of terms k in the product-polynomial interpolant

$$\mathbf{I}_m f := \sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij} T_i(x_{2\tau}) T_j(x_{2\sigma}),$$

where coefficients a_{ij} are the linear combinations of the values $f(\xi_\alpha)$ in the tensor product set of nodes ξ_α, is then estimated by $k = m^2 + 1 = O(p^2)$. Let $|\Gamma^1_\tau| \le |\Gamma^3_\sigma|$ for definiteness. Then combining all terms having the factor $T_i(x_{2\tau})$, $i = 0, 1, \dots, m$, we obtain the desired expansion of the order $k = O(p)$,

$$[\mathbf{I}_m f](x_\tau, x_\sigma) := \sum_{i=0}^{m} T_i(x_{2\tau}) \cdot G_i(x_{2\sigma}), \qquad G_i(x_{2\sigma}) = \sum_{j=0}^{m} a_{ij} T_j(x_{2\sigma}). \tag{4.6}$$

A similar construction for $d = 3$ leads to expansions of the order $O(p^{d-1})$, using Proposition 4.4. The result is based on the polynomial approximations for multivariate analytic functions on a rectangular product piece $\Gamma^i_\tau \times \Gamma^j_\sigma \in \mathbb{R}^4$ of $\partial X(\tau) \times \partial X(\sigma)$, $i, j = 1, \dots, 6$.

Remark 4.6 *Note that the corresponding Taylor interpolant with respect to the Chebyshev centre of Γ^1_τ may be also applied for the construction of the wire-basket expansion. It has the same order $k = O(p^{d-1})$ but involves a larger constant $c_1 \simeq 1.0$ in (4.5) (cf. [11]). This is consistent with Chebyshev's classical result that the best polynomial approximation of an analytic function is by far more accurate than the Taylor interpolant. Moreover, (4.6) is based only on the pointwise evaluation of the kernel at the Chebyshev-Gauss-Lobatto points.*

Remark 4.7 *The global degenerate expansion of the order $O(p^{d-1})$ on $\partial X(\tau) \times \partial X(\sigma)$ is constructed in two steps: First, we obtain an expansion on $\Gamma^1_\tau \times \partial X(\sigma)$ by composing (agglomerating) the expansions for $\Gamma^j_\sigma \times \Gamma^1_\tau$, $j = 1, \dots, 4$, based on a fixed polynomial basis $\{T_\ell\}_{\ell=0}^m$ on Γ^1_τ, and then by assembling the corresponding representations constructed for each Γ^i_τ, $i = 1, ..., 4$, separately, as above. This approach is suited for the FE approximation of the elliptic extension operator in (3.15), (3.22).*

To estimate the computational complexity of such an approximation, we proceed as follows. Instead of one global Rk_1-matrix $\mathbf{S} := \sum_{\alpha=1}^{k_1} \mathbf{a}_{h,\alpha} * \mathbf{c}^T_{h,\alpha}$ corresponding to (3.7), we consider the 4×4 blockwise Rk-approximation of the form

$$\widehat{\mathbf{S}} := \begin{pmatrix} C & B & A & TBT \\ D & F & TBT & G \\ H & TDT & C & TDT \\ TDT & G & TBT & F \end{pmatrix} =: \{\widehat{S}_{ij}\}_{i,j=1}^4,$$

with respect to the degrees of freedom located on the product pieces $\Gamma^i_\tau \times \Gamma^j_\sigma$, $i, j = 1, \dots, 4$, see Fig. 1. Here, T denotes the proper permutation matrix of the size $\dim \Gamma^i_{h,\tau}$. We introduce the *wire-basket rank* $R_{wb} = R_{wb}(\widehat{\mathbf{S}})$ and the *reduced wire-basket rank* $R_{rwb} = R_{rwb}(\widehat{\mathbf{S}})$ of the matrix-block $\widehat{\mathbf{S}}$ by

$$R_{wb}(\widehat{\mathbf{S}}) := \frac{1}{4} \sum_{i,j=1}^{4} R(\widehat{S}_{ij}), \tag{4.7}$$

and

$$R_{rwb}(\widehat{\mathbf{S}}) := \frac{1}{4} \left(R(A) + R(B) + R(C) + R(D) + R(F) + R(G) + R(H) \right), \tag{4.8}$$

respectively, where $R(A)$ is the rank of A. The value R_{wb} characterises the complexity of matrix-vector multiplication by the block $\widetilde{\mathbf{S}}$, while R_{rwb} specifies the memory requirements. The numerical results estimating R_{wb} and R_{rwb} for the harmonic kernel in $2D$ will be presented at the end of this section.

Remark 4.8 *Another alternative construction of the wire-basket expansions is designed for using the exact $\mathcal{L}$-harmonic extensions instead of $\mathbf{E}_{\tau,h}$. For this purpose, we apply generalised harmonic polynomials in*

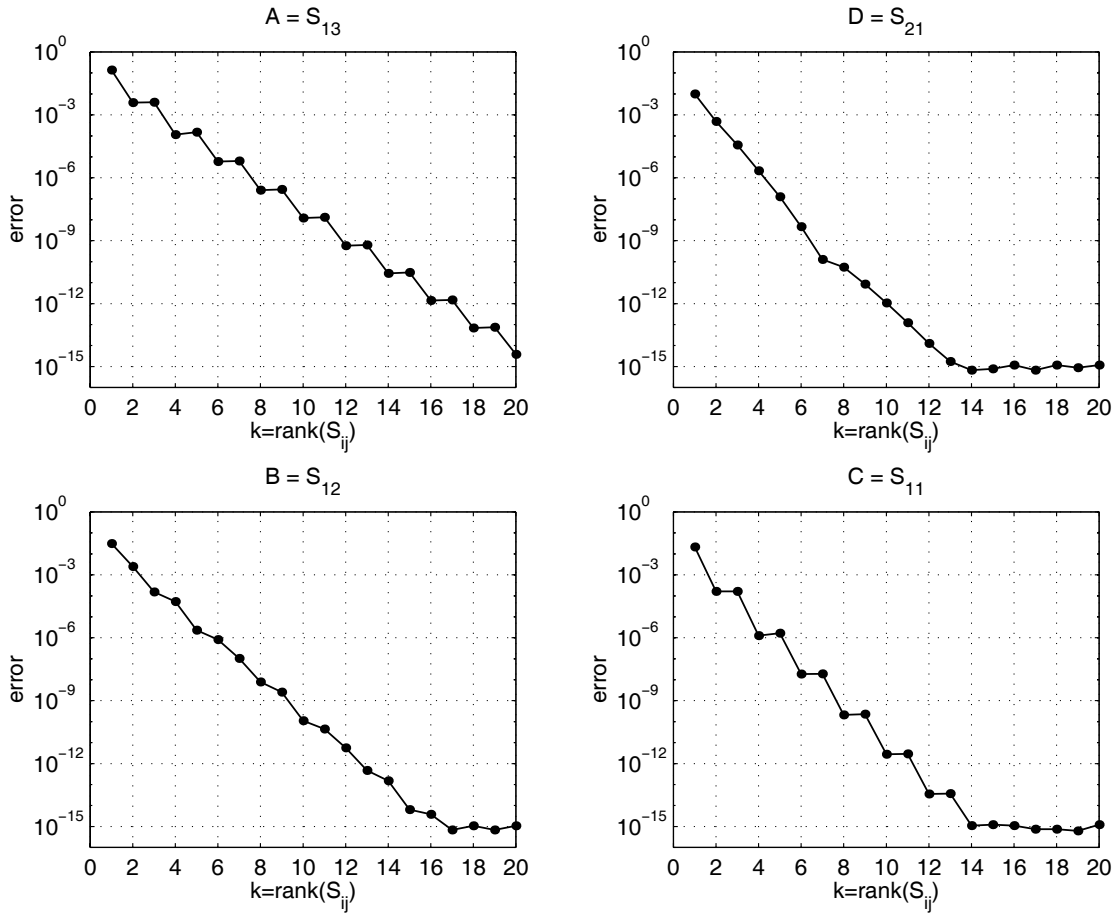


Figure 2: Approximation of the blocks A, B, C and D in $\widehat{\mathbf{S}}$.

the tensor-product domain, see also [19], concerning the "operator adapted spectral element methods". For example, in the case of the Laplace equation, we apply the system of trigonometric harmonic polynomials $\sin(k\,\pi\,x)\exp(\pm k\,\pi\,y)$, $k = 1, 2, \ldots$, *in a rectangle. Then the approximation of the kernel by trigonometric polynomials on the edges* Γ_τ^i *of a computational cluster allows the exact harmonic extensions into the interior by the harmonic polynomials defined above. The details will be discussed in a forthcoming paper.*

It is worth to note that in the particular case of *harmonic kernel* $s(x,y) = \frac{1}{4\pi|x-y|}$ for $d = 3$, we obtain an expansion with almost the same number of terms compared with the familiar *multipole expansion* of the optimal order $k = O(p^2)$. In fact, let x and y have spherical coordinates (r, θ, ϕ) and (ρ, α, β), respectively. Define spherical harmonics $Y_\nu^\mu(\theta,\phi)$, $\nu = 0, 1, 2, \ldots$ and $\mu = -\nu, \ldots, \nu$, by

$$Y_\nu^\mu(\theta,\phi) := \sqrt{\frac{2\nu+1}{4\pi}\frac{(\nu-|\mu|)!}{(\nu+|\mu|)!}}\,P_\nu^{|\mu|}(\cos\theta)e^{i\mu\phi},$$

where P_ν^μ are the associated Legendre functions,

$$P_\nu^\mu(x) := (-1)^\mu(1-x^2)^{\mu/2}\frac{d^\mu}{dx^\mu}P_\nu(x),$$

and $P_\nu(x)$ is the Legendre polynomial of degree ν. It is shown in [6, 4] that the multipole expansion of the form

$$\frac{1}{|x-y|} = \frac{1}{r}\sum_{\nu=0}^{m}\sum_{\mu=-\nu}^{\nu}\left(\frac{\rho}{r}\right)^\nu Y_\mu^\nu(\alpha,\beta)Y_\mu^{-\nu}(\theta,\phi) + R_m \tag{4.9}$$

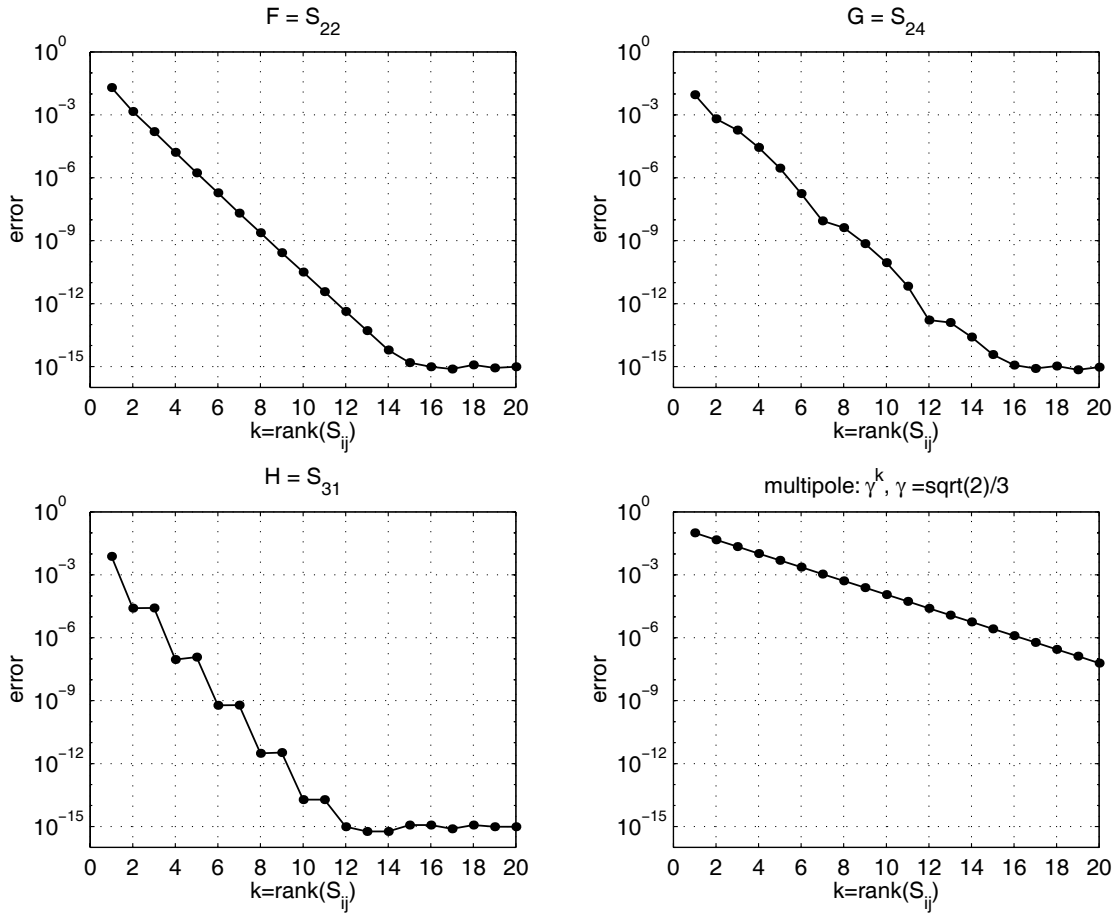


Figure 3: Approximation of the blocks F, G and H in $\widehat{\mathbf{S}}$ versus the multipole expansion.

provides the error estimate

$$|R_m| \le \frac{c}{|x-y|}\frac{\gamma^{m+1}}{(1-\gamma)^2}, \qquad \gamma = \frac{1}{2}\frac{\operatorname{diam}\tau}{\operatorname{dist}(\sigma,\tau_*)} < 1, \tag{4.10}$$

where τ_* is the Chebyshev centre of τ. With the choice $\eta = \frac{\sqrt{3}}{2}$ in the admissibility condition, we obtain $\gamma = \frac{1}{\sqrt{3}} \sim 0.58$, while for $d = 3$, there holds $c_1\eta = (2+\sqrt{5})^{-1} \sim 0.24$, by virtue of (4.4). Therefore, a bound like (4.5) provides better asymptotic convergence rate than (4.10). However, both the wire-basket and multipole expansions have faster convergence than the Taylor interpolant.

For the $2D$ harmonic potential $s(x,y) = \frac{1}{2\pi}\log|x-y|$, the convergence rate for the multipole expansion with $\eta = \frac{\sqrt{2}}{2}$ is estimated by $\gamma = \frac{\sqrt{2}}{3} \sim 0.47$, while the wire-basket expansion again yields $c_1\eta \sim 0.24$. Applying the degenerate polynomial approximation of kernels on the product domains $\Gamma_\tau^i \times \Gamma_\sigma^j$ by interpolation at the Chebyshev-Gauss-Lobatto points, we obtain the following results for R_{wb} and R_{rwb}, depending on ε, see Fig. 2 and 3. Here we present the maximal approximation error of all blocks in $\widehat{\mathbf{S}}$ versus the degree of the interpolation polynomials. The last picture in Fig. 3 presents the accuracy provided by the 2D multipole expansion corresponding to the exponent $\gamma = \frac{\sqrt{2}}{3}$. Fig. 4 presents the corresponding rank $R_{wb}(\widehat{\mathbf{S}})$ and $R_{rwb}(\widehat{\mathbf{S}})$ defined by (4.7) and (4.8), respectively, depending on the approximation accuracy achieved. Here *Rmp* corresponds to the multipole expansion, while *RToepl* stands for the factor $c\log n$ characterising the linear-logarithmic complexity of matrix-vector multiplication by a Toeplitz matrix associated with any pair of parallel edges from $\Gamma_\tau \times \Gamma_\sigma$. This confirms our theoretical estimates.

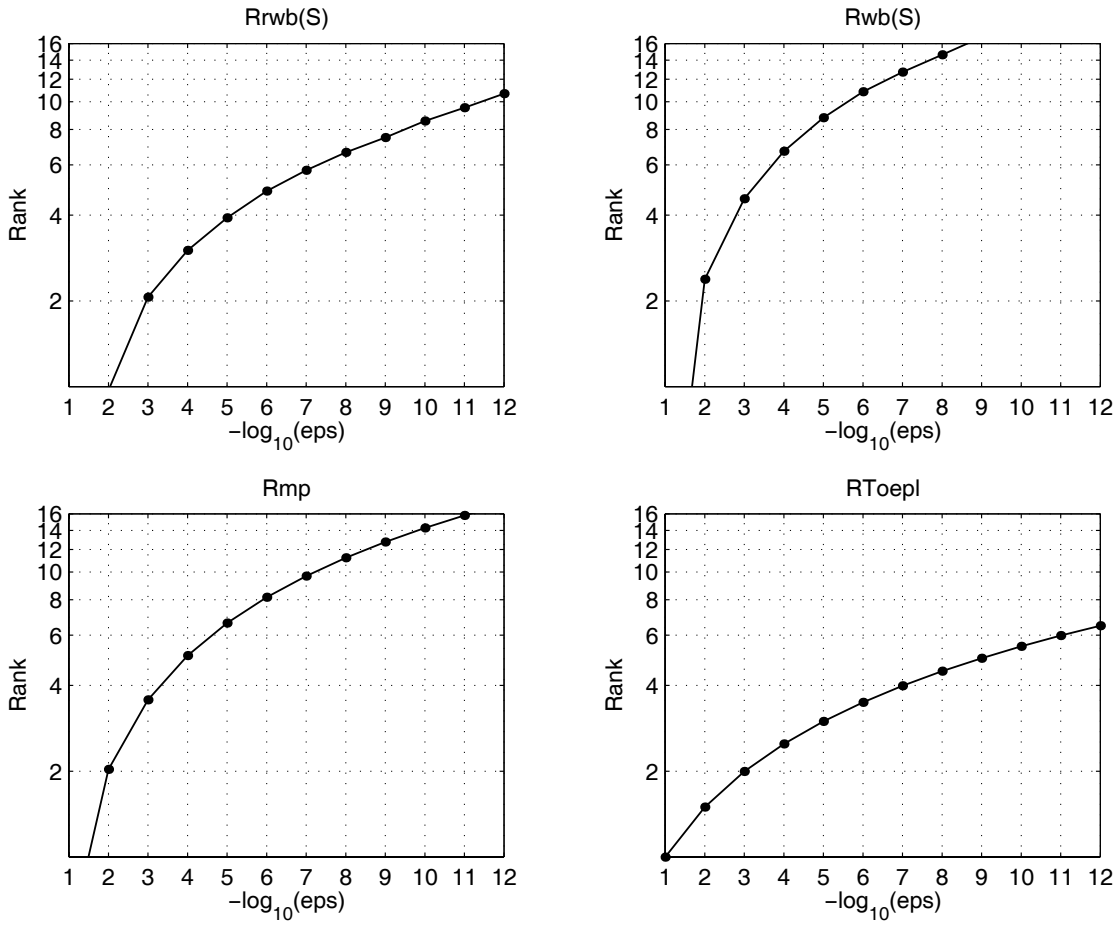


Figure 4: Computational rank for different approximations of $\widehat{\mathbf{S}}$.

4.3 Fully Separable Expansions

Now we describe *fully separable domain-based expansions* and the corresponding wire-basket counterparts reducing the asymptotic complexity considerably. The idea is to use separable expansions for the kernel $s(x,y)$ not only with respect to the two multi-dimensional variables x and y, but also with respect to the $2d$ vairables $x_1,\dots,x_d$, $y_1,\dots,y_d$. We further assume that the FE ansatz space V_h has the tensor-product basis $\{\varphi_{\mathbf{i}}\} = \{\varphi_{i_1} \otimes \dots \otimes \varphi_{i_d}\}$ with $\varphi_{i_q} \in \mathrm{span}\{\varphi_i\}_{i=1}^{n_\ell}$, $q = 1,\dots,d$, such that for the restriction on cluster of level ℓ there holds $|\mathbf{i}|_\infty = 2^{p-\ell}$.

We start from an expansion of the order $k = O(p^{2d})$ defined on the product domain $X(\tau) \times X(\sigma)$ (the corresponding parametric domain is given by $I_\delta^d \times I_\delta^d$) using the tensor-product polynomial approximation from Proposition 4.4 (for $d = 3$),

$$s_{\tau,\sigma} = \sum_{|\mathbf{j}|=0}^{m} \sum_{|\mathbf{i}|=0}^{m} a_{\mathbf{i}}^{\mathbf{j}} \, (T_{j_1} \otimes T_{j_2} \otimes T_{j_3}) \otimes (T_{i_1} \otimes T_{i_2} \otimes T_{i_3}),$$

where $\{T_\nu\}$ stands for the basis set of polynomials (say Chebyshev polynomials) which is used to represent corresponding univariate polynomial interpolants, analysed in Proposition 4.4. Here, the tensor $a_{\mathbf{i}}^{\mathbf{j}}$ with multiindices $\mathbf{i},\mathbf{j} \in \mathbb{N}^d$, represents expansion coefficients of the Lagrange interpolant in the product-polynomial basis. Such an expansion allows an asymptotically optimal data-sparse approximation of the smooth kernel with sublinear storage size $O(p^{2d} \cdot 2^{p-\ell})$ for the blocks on level $\ell = 2,3,\dots,p-p_0$. Indeed, in the case $d = 3$,

the corresponding matrix blocks have the following tensor-product form

$$\mathbf{A}^{\tau\times\sigma} = \sum_{|\mathbf{j}|=0}^{m}\sum_{|\mathbf{i}|=0}^{m} a_{\mathbf{i}}^{\mathbf{j}}\,(\mathbf{T}_{j_1}\otimes\mathbf{T}_{j_2}\otimes\mathbf{T}_{j_3}) * (\mathbf{T}_{i_1}\otimes\mathbf{T}_{i_2}\otimes\mathbf{T}_{i_3}), \tag{4.11}$$

with $m = O(p)$, where $\mathbf{T}_\nu \in \mathbb{R}^{n_\ell}$, $\nu = 1,\dots,m$, $n_\ell = 2^{p-\ell}$. The vector $\mathbf{T}_\nu = \{(T_\nu, \varphi_i)\}_{i=1}^{n_\ell}$ represents the component of tensor product Galerkin stiffness matrix with respect to the nodal basis $\{\varphi_i\}$.

For the given trial vector $\mathbf{u} \in \mathbb{R}^{n_\tau}$ compute the ℓ_2-ortho-projection $\mathbf{u}_0 = \sum_{|\mathbf{i}|=0}^{m} u_{\mathbf{i},0}\mathbf{T}_{i_1}\otimes\mathbf{T}_{i_2}\otimes\mathbf{T}_{i_3}$ onto the (m^d+1)-dimensional subspace $\mathrm{span}\{\mathbf{T}_{i_1}\otimes\mathbf{T}_{i_2}\otimes\mathbf{T}_{i_3}\}_{|\mathbf{i}|=0}^{m}$. We denote by $\overline{\mathbf{u}} = \{u_{\mathbf{i},0}\}$ the coefficient vector of $\mathbf{u}_0$. Then, the matrix-vector multiplication has the representation

$$\mathbf{A}^{\tau\times\sigma}\mathbf{u} = \sum_{|\mathbf{j}|=0}^{m}(\sum_{|\mathbf{i}|=0}^{m} a_{\mathbf{i}}^{\mathbf{j}} u_{\mathbf{i},0})(\mathbf{T}_{j_1}\otimes\mathbf{T}_{j_2}\otimes\mathbf{T}_{j_3}),$$

The scalar product has the even simpler representation

$$(\mathbf{z}, \mathbf{A}^{\tau\times\sigma}\mathbf{u}) := (\overline{\mathbf{z}}, \mathbf{Z}\overline{\mathbf{u}}) \quad \text{with } \mathbf{Z} = \{a_{\mathbf{i}}^{\mathbf{j}}\}.$$

In both cases, the cost $O(m^{2d}) = O(\log^{2d} n)$ grows only logarithmically with n. We assume that the projection $\mathbf{u}_0$ can be computed by a fast trasform (e.g., by FFT in the case of trigonometric polynomials) with the cost $O(n_\tau \log n_\tau)$ arithmetical operations. As a result, we prove the following complexity bound.

Lemma 4.9 *The storage and matrix-vector multiplication expenses for the block with a fully separable expansion are bounded by*

$$\mathcal{N}_{st}(\mathbf{A}^{\tau\times\sigma}) = O(p^{2d} n_\tau^{1/d}), \qquad \mathcal{N}_{MV}(\mathbf{A}^{\tau\times\sigma}) = O(n_\tau \log n_\tau + p^{2d}).$$

Now we discuss the *wire-basket version* of fully separable expansions. For definiteness, consider the case $d = 3$. We analyse one typical block defined on the parametric domain $I_1^{d-1}\times I_1^{d-1}$ involved in the overall expansion on $\partial X(\tau)\times\partial X(\sigma)$, see Remark 4.7, and corresponding to the product domain $\Gamma_\tau^1\times\Gamma_\sigma^3$. For this purpose, we build the expansion of the order $k_1 = O(p^{2(d-1)})$ which yields an asymptotically optimal approximation of the kernel with complexity $O(p^{2(d-1)}\cdot 2^{p-\ell})$ of storage for blocks of level $\ell \in \{2,3,\dots,p-p_0\}$. The corresponding contribution $\mathbf{A}_{13}^{\tau\times\sigma}$ to the whole matrix block $\mathbf{A}^{\tau\times\sigma}$ has the form

$$\mathbf{A}_{13}^{\tau\times\sigma} = \mathbf{S}_{\tau,h}^{T}\left(\sum_{|\mathbf{i}|=1}^{m}\sum_{|\mathbf{j}|=1}^{m} a_{\mathbf{i}}^{\mathbf{j}} * (\mathbf{T}_{j_1}\otimes\mathbf{T}_{j_2}) * (\mathbf{T}_{i_1}\otimes\mathbf{T}_{i_2})\right)\mathbf{S}_{\sigma,h} := \mathbf{S}_{\tau,\mathbf{h}}^{\mathbf{T}}\mathbf{A}_{\gamma}\mathbf{S}_{\sigma,h}, \tag{4.12}$$

with the same notations as above. The corresponding matrix operations with the rank-k matrix $\mathbf{A}_\gamma$, $k = m^2$, are analysed similarly as in Lemma 4.9.

Remark 4.10 *A sparse approximation may be also applied to a more general class of kernels with bounded mixed derivatives, see (4.13). If the Galerkin ansatz space is defined on the same sparse grid as the approximation of kernel function, we obtain an algorithm of complexity $O(n)$ for Case (**B**). Note that the $\mathcal{H}$-matrix technique using sparse grids requires the weakened smoothness condition*

$$\left|\frac{\partial^{|\alpha|+|\beta|}s(x,y)}{\partial x_1^{\alpha_1}\dots\partial x_d^{\alpha_d}\partial y_1^{\beta_1}\dots\partial y_d^{\beta_d}}\right| \le c\,|x-y|^{1-3d-2r} \qquad (|\alpha|_\infty, |\beta|_\infty \le 2, x\ne y) \tag{4.13}$$

for the kernel function $s(x,y)$, where $2r$ is the order of integral operator (see Assumption 2.5).

4.4 Complexity of Matrix Addition and Multiplication

The class of $\mathcal{H}$-matrices with the block structure of the special form (4.11) corresponding to the fully separable expansions for $d = 2, 3$, may be regarded as a special case of standard $\mathcal{H}$-matrices but with an improved representation of the rank-k-blocks (Rk-blocks). Obviously, the matrix block (4.11) defines an Rk-matrix with $k = p^d$ and needs a storage size of $O(p^{2d} n_\tau^{1/d})$, see Lemma 4.9. Note that the matrix block of the standard $\mathcal{H}$-matrix based on the kernel expansion of the same order has the complexity $O(p^d\, n_\tau)$ for both the storage and matrix-vector multiplication.

It is easily seen that the matrix addition of two blocks (4.11) (using the same polynomial system $\{T_\nu\}$) has the complexity $O(p^{2d})$. Full complexity analysis of the truncated matrix-matrix multiplication is quite involved. However, the product of two blocks like (4.11) is performed exactly in the given format (similar to the case of Rk–matrices). It is equivalent to the calculation of the product $c_{\mathbf{i}}^{\mathbf{l}}$ of two tensor coefficients $a_{\mathbf{i}}^{\mathbf{j}}$ and $b_{\mathbf{k}}^{\mathbf{l}}$ characterising the corresponding matrix blocks. For the ease of discussion, we assume that the system $\{T_\nu\}$ is ℓ_2-orthogonal. Then we obtain

$$\sum_{|\mathbf{j}|=0,|\mathbf{i}|=0}^{m} a_{\mathbf{i}}^{\mathbf{j}}\,(\mathbf{T}_{j_1}\otimes\mathbf{T}_{j_2}\otimes\mathbf{T}_{j_3})*(\mathbf{T}_{i_1}\otimes\mathbf{T}_{i_2}\otimes\mathbf{T}_{i_3})\sum_{|\mathbf{l}|=0,|\mathbf{k}|=0}^{m} b_{\mathbf{k}}^{\mathbf{l}}\,(\mathbf{T}_{\ell_1}\otimes\mathbf{T}_{\ell_2}\otimes\mathbf{T}_{\ell_3})*(\mathbf{T}_{k_1}\otimes\mathbf{T}_{k_2}\otimes\mathbf{T}_{k_3})$$

$$=\sum_{|\mathbf{l}|=0,|\mathbf{i}|=0}^{m}\left(\sum_{|\mathbf{j}|=0,|\mathbf{k}|=0}^{m} a_{\mathbf{i}}^{\mathbf{j}} b_{\mathbf{k}}^{\mathbf{l}}\right)(\mathbf{T}_{\ell_1}\otimes\mathbf{T}_{\ell_2}\otimes\mathbf{T}_{\ell_3})*(\mathbf{T}_{i_1}\otimes\mathbf{T}_{i_2}\otimes\mathbf{T}_{i_3}).$$

Hence, it has a logarithmic complexity $O(p^{3d})$. The full analysis of the formatted matrix-matrix product will be considered separately.

5 BEM Applications

5.1 General Remarks

Consider the case of a polygonal/polyhedral boundary $\Gamma = \partial\Omega$, $\Omega \in \mathbb{R}^d$, $d = 2, 3$. To apply the wire-basket expansions in BEM, we assume that for each pair of admissible clusters $\tau \times \sigma \in P_2^\ell$ there exist two parallelepipeds $\widehat{X}(\tau), \widehat{X}(\sigma) \in \mathbb{R}^d$ which contain $X(\tau)$ and $X(\sigma)$, respectively, and are admissible considered as spatial domains. Specifically, they satisfy the geometrical admissibility condition, similar to (2.5),

$$\min\{\mathrm{diam}(\widehat{X}(\sigma)), \mathrm{diam}(\widehat{X}(\tau))\} \le 2\eta\ \mathrm{dist}(\widehat{X}(\sigma), \widehat{X}(\tau)). \tag{5.1}$$

This condition specifies the choice of $\widehat{X}(\tau)$ and $\widehat{X}(\sigma)$. Then the desired expansion for $s(x, y)$, $(x, y) \in X(\tau) \times X(\sigma)$, is the restriction of the separable $\mathcal{L}$-harmonic extension

$$s_{\tau,\sigma} := \sum_\alpha r_{X(\tau)} E_{\widehat{X}(\tau)} a_\alpha(x) \cdot r_{X(\sigma)} E_{\widehat{X}(\sigma)} c_\alpha(y), \tag{5.2}$$

where a_α and c_α are the coefficients from (3.7) describing the kernel expansion on the product boundary of spatial domain $\partial\widehat{X}(\tau) \times \partial\widehat{X}(\sigma)$ and $E_{\widehat{X}(\tau)}$, $E_{\widehat{X}(\sigma)}$ denote the elliptic extension operators onto $\widehat{X}(\tau)$ and $\widehat{X}(\sigma)$, respectively. The complexity of performing (5.2) may depend on the geometry of the pieces $X(\tau)$ and $X(\sigma)$.

If a planar piece $X(\tau)$ (resp. $X(\sigma)$) is parallel to some facet of $\partial\widehat{X}(\tau)$ (resp. $\partial\widehat{X}(\sigma)$) then the evaluation of restrictions in (5.2) needs $O(n_\Gamma \log^2 n_\Gamma)$ arithmetical operations. Here n_Γ is the number of degrees of freedom on $X(\tau)$. For arbitrary plane section of $\widehat{X}(\tau)$ (resp. $\widehat{X}(\sigma)$) this expense is estimated by $O(n_\Gamma \log^3 n_\Gamma)$, see [16, 17] for more details. In the case of curvilinear patches $X(\tau), X(\sigma)$, the linear-logarithmic complexity may be achieved by multilevel extension procedure or by using the $\mathcal{L}$-harmonic polynomials (see Remark 4.8) which will be discussed separately. However, regardless of the particular extension procedure, we arrive at expansions of the order $O(p^{d-1})$ in rather general BEM applications.

5.2 Anisotropic Laplacian

We consider the example of the kernel expansion in FEM/BEM for $d = 2$ corresponding to the anisotropic Laplace operator. We will be especially interested in the robust and accurate approximations for the singularly perturbed equation.

For the most common operators of the form (2.1), the fundamental solution is known in the explicit form (see [9] and references therein, as well as [5]).

Consider the *anisotropic* elliptic operator with

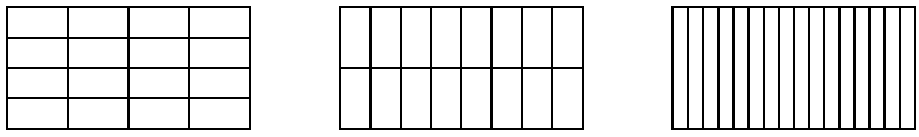
$$\mathbf{L} = \mathrm{diag}\{a_1, \ldots, a_d\}, \quad a_i = \varepsilon_i^2 > 0, \quad \mathbf{b} = 0,\ c_0 = 0; \tag{5.3}$$

$$S(x) := \begin{cases} \frac{1}{2\pi\sqrt{det\,\mathbf{L}}} \log \frac{1}{|x|_{\mathbf{L}}} & \text{for } d = 2 \\ \frac{1}{4\pi\sqrt{det\,\mathbf{L}}} \frac{1}{|x|_{\mathbf{L}}} & \text{for } d = 3 \end{cases}$$

in the case $d = 2$, $a_1 = \varepsilon^2$, $a_2 = 1$, $0 < \varepsilon \leq 1$. Let $\tau \times \sigma \in P_2^\ell$ be an admissible (rectangular) block satisfying condition (2.5). The suitable norm involved in the definitions of diam := $\mathrm{diam}_{\mathbf{L}}$ and dist is now the *anisotropic norm* $||x|| := ||x||_{\mathbf{L}} := \langle x, \mathbf{L}^{-1}x \rangle^{1/2}$. The coefficient dependent separation scheme is based on the criteria

$$\#\tau' \simeq \#\tau'', \qquad \mathrm{diam}\,\tau' \simeq \mathrm{diam}\,\tau''$$

for all sons $\tau', \tau'' \in \tau$ of the parent cluster τ. Fig. 1 shows those clusters from the resulting tree $T_1(I)$ which correspond to level $\ell = 2$ dependent on the singular perturbation parameter ε. For small enough ε, we have the decomposition into stripes like in the semi-coarsening variant of the multi-grid method. The arising $\mathcal{H}$-matrix becomes close to the block-diagonal one with the blocks corresponding to the vertical grid-lines.



$\varepsilon = 1$ $\qquad$ $\varepsilon = 2^{-2}$ $\qquad$ $\varepsilon = 2^{-4}$

Figure 5: Anisotropy-dependent separation strategies

The error and complexity analysis is identical to the case of the Laplace operator after applying the coordinate transform $y' = y$, $x' = \frac{1}{\varepsilon}x$. Therefore, we skip the details.

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