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by

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1 Introduction

In paper [1] mathematical self-consistency of a new theory of interacting "electromagnetic field + pointlike particles" system was analyzed. It was proved that the the initial value problem for both field and particles is well posed. For that purpose the asymptotic behaviour of the field in a neighbourhood of the particle was necessary. It was obtained using symbolic calculus package based on MATHEMATICA.

For purposes of an extended theory, where particles may get polarization due to interaction, or may carry an angular momentum (spin) the above description of the asymptotic field near to particles is not sufficient. In the present paper we propose a different algorithm, which enables us to solve the problem in a much simpler way. To implement practically our algorithm we have used packages of symbolic calculus of the program MAPLE V.

2 Maxwell equations in 3+1 decomposition.

To eliminate all the unnecessary parameters, related to the special arrangements of measuring instruments, we are going to describe electromagnetic field of an accelerated particle with respect to its rest frame (Fermi frame – cf. [1] or [2]). This means that the time vector $\mathbf{e}_0$ of the *tetrad* $(\mathbf{e}_\mu)$ defining the system is always equal to the particle's four velocity. At each point $q(t)$ of the particle trajectory the space-like hypersurfaces Σ_t orthogonal to the trajectory are spanned by the remaining three elements of the tetrad: $(\mathbf{e}_k)$, where $k = 1, 2, 3$. The above triad defines uniquely the Cartesian (orthonormal) coordinate system (x^k) on Σ_t. The Fermi condition imposed on the system means that the covariant derivative $\nabla_{\mathbf{e}_0}\mathbf{e}_k$ has no space-like component, i. e. is proportional to $\mathbf{e}_0$. It is easy to show that this condition implies the following relations:

$$\nabla_{\mathbf{e}_0}\mathbf{e}_k = a_k\mathbf{e}_0 \ , \quad \nabla_{\mathbf{e}_0}\mathbf{e}_0 = \mathbf{a} \ , \tag{1}$$

where by $\mathbf{a} := a^k e_k$ (and $a_k = g_{kl}a^l$) we denote the acceleration (curvature) of the trajectory. It is a vector orthogonal to the trajectory. As the time parameter we take the proper time t along the trajectory.

We are going to solve Maxwell equations with respect to the above, degenerate coordinate system. It is degenerate, because different surfaces Σ_t may intersect, but this does not lead to any difficulties. Technically, we solve Maxwell equations in an abstract spacetime $\Sigma \times R^1$ equipped with time dependent metric (see [2]):

$$g_{\mu\nu} = \left(\begin{array}{c|c} N_k N^k - N^2 & N_l \\ \hline N_k & \mathbf{g}_{kl} \end{array} \right), \quad g^{\mu\nu} = \left(\begin{array}{c|c} -\frac{1}{N^2} & \frac{N^l}{N^2} \\ \hline \frac{N^k}{N^2} & \mathbf{g}^{kl} - \frac{N^k N^l}{N^2} \end{array} \right), \tag{2}$$

where the lapse function N and the shift vector N^m encode information about particular split. In case of the Fermi – Walker transport (1) the shift vector N^m vanishes and the lapse function equals $N = 1 + a_i x^i$. We rewrite Maxwell equations

$$f_{[\mu\nu,\gamma]} = 0, \tag{3}$$

$$\partial_\nu \mathcal{F}^{\mu\nu} = \mathcal{J}^\mu \ , \tag{4}$$

with $\mathcal{F}^{\mu\nu} = \sqrt{g} g^{\mu\alpha} g^{\nu\beta} f_{\alpha\beta}$, in the above 3+1 decomposition. For this purpose we use the corresponding $3+1$ decomposition of the Maxwell tensor: (see [3]):

$$f_{kl} = \sqrt{\det g_{kl}} \epsilon_{klm} B^m = \epsilon_{klm} \mathcal{B}^m \tag{5}$$

$$f_{0k} = -N D_k + N^m f_{mk} \ , \tag{6}$$

where ϵ_{klm} denotes the standard Levi-Civitta tensor density and $\mathcal{B} = \sqrt{\det g_{kl}} B$ is the vector-density of magnetic induction. Using a similar notation (i. e. $\mathcal{D} = \sqrt{\det g_{kl}} D$) for the electric induction field we obtain:

$$\mathcal{F}^{kl} = \mathcal{D}^k N^l - \mathcal{D}^l N^k - \epsilon^{klm} B_m \ , \tag{7}$$

$$\mathcal{F}^{0k} = \mathcal{D}^k \ . \tag{8}$$

Rewritten in terms of these quantities, equations (3) and (4) read ("dot" is the time derivative):

$$\partial_k \mathcal{D}^k = \mathcal{J}^0 \ , \tag{9}$$

$$\partial_k \mathcal{B}^k = 0 \ , \tag{10}$$

$$\dot{\mathcal{D}}^k = \partial_l (N^l \mathcal{D}^k - N^k \mathcal{D}^l - N \epsilon^{lkm} B_m) - \mathcal{J}^k \ , \tag{11}$$

$$-\dot{\mathcal{B}}^k = \partial_l (N^l \mathcal{B}^k - N^k \mathcal{B}^l + N \epsilon^{lkm} D_m). \tag{12}$$

In this paper we analyze the behaviour of solutions of these equations in a neighbourhood of a time dependent electric n-pole particle. The charge distribution of such a particle is given by a corresponding n-th derivative of the Dirac delta: $\mathcal{J}^0 = -P^{I_n} \partial_{I_n} \delta^{(3)}$, where P^{I_n} is a totally symmetric, traceless tensor ($I_n = \{i_1, \ldots, i_n\}$ is a multi-index). The charge conservation law implies that the changes in time of the above charge distribution must be matched by the following electric current: $\mathcal{J}^k = \dot{P}^{\{k,I_{n-1}\}} \partial_{I_{n-1}} \delta^{(3)}$. Using specific values of the lapse function and the shift vector, corresponding to the Fermi – Walker system we obtain the final set of equations that govern our problem:

$$\partial_k \mathcal{D}^k = -P^{I_n} \partial_{I_n} \delta^{(3)}, \tag{13}$$

$$\partial_k \mathcal{B}^k = 0, \tag{14}$$

$$\dot{\mathcal{D}}^k = \epsilon^{klm} \partial_l ((1 + a_i x^i) B_m) - \dot{P}^{\{k,I_{n-1}\}} \partial_{I_{n-1}} \delta^{(3)}, \tag{15}$$

$$-\dot{\mathcal{B}}^k = \epsilon^{klm} \partial_l ((1 + a_i x^i) D_m). \tag{16}$$

The degenerate case $n = 0$ (just a simple monopole particle) also fits into this scheme if we put $\mathcal{J}^0 = e\delta^{(3)}$ and $\mathcal{J}^k = 0$. Here, the electric current conservation implies that the "0-pole" (i. e. the electric charge e) must be constant during the evolution.

The equations for the magnetic multipole may be easily obtained by the duality transformation: $D \to B$ and $B \to -D$. More precisely, we define:

$$B^k_{\text{new}} := D^k_{\text{old}} + P^{\{k,I_{n-1}\}}\partial_{I_{n-1}}\delta^{(3)} , \tag{17}$$

$$D^k_{\text{new}} := -B^k_{\text{old}} , \tag{18}$$

and obtain correct equations for the new fields from equations (13) – (16) for the old ones. In the most interesting case of a magnetic dipole ($n = 1$) this gives us $\mathcal{J}^0 = 0$ and $\mathcal{J}^k = \epsilon^{klm}P_l\partial_m\delta^{(3)}$.

3 The algebra of X and Y fields.

Any divergence-free vector field may be expanded with respect to the following basic fields:

$$X^i(m,n,Q) = \frac{1}{r^m}\left((m-n-1)Q^i{}_{i_2\ldots i_n}x^{i_2}\cdots x^{i_n} - m\frac{x^i}{r^2}Q_{i_1\ldots i_n}x^{i_1}\cdots x^{i_n}\right) , \tag{19}$$

$$Y^i(m,n,Q) = \frac{1}{r^m}\epsilon^{ijk}x_jQ_{k\ i_2\ldots i_n}x^{i_2}\cdots x^{i_n}, \tag{20}$$

where $Q_{i_1\ldots i_n}$ is a totally symmetric, traceless tensor of rank n. The following may be easily verified:

$$\operatorname{curl} Y(m,n,Q) = X(m,n,Q) , \tag{21}$$

$$\operatorname{curl} X(m,n,Q) = m(2n-m+1)Y(m+2,n,Q) . \tag{22}$$

We conclude that, restricted to the family of singular vector fields (i. e. for $m > 0$), the "curl" operator is reversible, provided $2n - m + 1 \neq 0$.

To handle Maxwell equations (13) – (16) we need the following operators, which – given a vector a and a tensor Q of rank n – produce the (totally symmetric, traceless) tensors: $a \vee Q$ of rank $n+1$, $a \times Q$ of rank n and $a\rfloor Q$ of rank $n-1$:

$$(a \vee Q)_{i_1\ldots i_n i_{n+1}} = \frac{1}{n}\left(\sum_{k=1}^{n+1} Q_{i_1\overset{\wedge}{\underset{k}{\ldots}} i_{n+1}}a_{i_k} - \frac{2}{2n+1}\sum_{k<m} Q_{i_1\overset{\wedge\wedge}{\underset{k\,l}{\ldots}} i_{n+1}j}\, a^j g_{i_k i_l}\right) , \tag{23}$$

$$(a \times Q)_{i_1\ldots i_n} = \sum_{k=1}^{n} \epsilon_{i_k}^{\ jl}a_j\, Q_{li_1\overset{\wedge}{\underset{k}{\ldots}} i_n} , \tag{24}$$

$$(a\rfloor Q)_{i_1\ldots i_n} = Q_{i_1\ldots i_{n-1}l}\, a^l . \tag{25}$$

Following identities for two vectors a and d may be easily proved:

$$d \vee (a \vee Q) = a \vee (d \vee Q) \tag{26}$$

$$d \times (a \vee Q) = (d \times a) \vee Q + (d \times Q) \vee a \tag{27}$$

$$d\rfloor(a \vee Q_n) = \frac{1}{n}\left((a_id^i)Q_n + (n-1)(a \vee (d\rfloor Q_n)) - \frac{2(n-1)}{2n+1}(d \vee (a\rfloor Q_n))\right) . \tag{28}$$

With help of the above algebra we can find the lapse-deformed version of relations (21) - (22), which are necessary for further analysis:

$$\begin{aligned}\text{curl }(a_i x^i X(m,n,Q)) = {} & \frac{n-1}{n+1}\left(\frac{nm(2n-m+2)}{2n+1} - \frac{(m-n-1)^2}{n}\right) Y(m,n-1,a\rfloor Q) + \\ & + \frac{n+1-m}{n(n+1)} X(m,n,a\times Q) + \frac{mn(2n-m+2)}{n+1} Y(m+2,n+1,a\vee Q),\end{aligned} \tag{29}$$

$$\begin{aligned}\text{curl }(a_i x^i Y(m,n,Q)) = {} & \frac{n}{n+1} X(m,n+1,a\vee Q) - \frac{m-n-2}{n(n+1)} Y(m,n,a\times Q) + \\ & + \frac{(n^2-1)}{n(2n+1)} X(m-2,n-1,a\rfloor Q).\end{aligned} \tag{30}$$

4 Asymptotic expansion of the Maxwell field

Consider the Maxwell field in a neighbourhood of a dipole-like ($n=1$) particle (it will be obvious that a similar analysis works also for $n=0$). We will use asymptotic expansion of the field with respect to powers of the distance r. Electric field starts with a r^{-3}-term because it must match the charge density $-P^k\partial_k\delta$ in equation (13)):

$$D = \underset{(-3)}{D} + \underset{(-2)}{D} + \underset{(-1)}{D} + \cdots \,, \tag{31}$$

$$B = \underset{(-2)}{B} + \underset{(-1)}{B} + \underset{(0)}{B} + \cdots \,. \tag{32}$$

Plugging this *Ansatz* into the second part of Maxwell equations (15) – (16) we get:

$$\begin{aligned} -\underset{(-2)}{\dot B} - \underset{(-1)}{\dot B} - \underset{(0)}{\dot B} - \cdots = {} & \underbrace{\text{curl }\underset{(-3)}{D}}_{(-4)} + \underbrace{\text{curl}(a_i x^i \underset{(-3)}{D}) + \text{curl }\underset{(-2)}{D}}_{(-3)} + \\ & + \underbrace{\text{curl}(a_i x^i \underset{(-2)}{D}) + \text{curl }\underset{(-1)}{D}}_{(-2)} + \underbrace{\text{curl}(a_i x^i \underset{(-1)}{D}) + \text{curl }\underset{(0)}{D}}_{(-1)} + \cdots \end{aligned} \tag{33}$$

$$\begin{aligned} \underset{(-3)}{\dot D} + \underset{(-2)}{\dot D} + \underset{(-1)}{\dot D} + \cdots = {} & \underbrace{\text{curl }\underset{(-2)}{B}}_{(-3)} + \underbrace{\text{curl}(a_i x^i \underset{(-2)}{B}) + \text{curl }\underset{(-1)}{B}}_{(-2)} + \\ & + \underbrace{\text{curl}(a_i x^i \underset{(-1)}{B}) + \text{curl }\underset{(0)}{B}}_{(-1)} + \cdots \end{aligned} \tag{34}$$

The above equality implies equalities in every order. Hence, we have:

- $\text{curl }\underset{(-3)}{D} = 0$ and $\text{div }\underset{(-3)}{D} = -P^k\partial_k\delta$ – that part we have to solve manually and translate into language of fields X and Y. For a dipole field $\underset{(-3)}{D} = (3n_i P^i n^k - P^k)r^{-3} = -X(3,1,P)$.
- $\text{curl}(a_i x^i \underset{(-3)}{D}) + \text{curl }\underset{(-2)}{D} = 0 \Leftrightarrow \underset{(-2)}{D} = -\,\text{curl}^{-1}\left(\text{curl}(a_i x^i \underset{(-3)}{D})\right)$. To calculate $\text{curl}(a_i x^i \underset{(-3)}{D})$ we use formulae (29) - (30). The inverse of the curl operator will be calculated from (21) and (22).

Finally, for $k \geq -1$:

$$\operatorname{curl}(a_i x^i \underset{(k)}{D}) + \operatorname{curl} \underset{(k+1)}{D} = -\underset{(k)}{\dot{B}} \Leftrightarrow \underset{(k+1)}{D} = -\operatorname{curl}^{-1}\left(\operatorname{curl}(a_i x^i \underset{(k)}{D}) + \underset{(k)}{\dot{B}}\right), \tag{35}$$

$$\operatorname{curl}(a_i x^i \underset{(k)}{B}) + \operatorname{curl} \underset{(k+1)}{B} = \underset{(k)}{\dot{D}} \Leftrightarrow \underset{(k+1)}{B} = \operatorname{curl}^{-1}\left(-\operatorname{curl}(a_i x^i \underset{(k)}{B}) + \underset{(k)}{\dot{D}}\right). \tag{36}$$

Note that for any n-pole source the entire procedure would be similar: three preliminary steps would produce $\underset{(-n-2)}{D}, \underset{(-n-1)}{D}, \underset{(-n-1)}{B}$ and recurrence relations (35) - (36), for $k \geq -n$ would provide the rest. The duality transformation (17) – (18) gives immediately the corresponding results for the magnetic dipole. The algorithm outlined here was implemented for the MAPLE V r.5. The result exists as a program XY'99 and may be downloaded from the Internet site of the Department of Mathematical Methods in Physics, University of Warsaw (http://info.fuw.edu.pl/∼kmmf/marcin/startxy.html).

A An example of solutions

With the help of XY'99 the singular part of Maxwell field any n-pole sources can be computed in any order of r. Below, we present the results for the dipole source in language of X and Y fields. We restrict electric and magnetic field to first order in r.

Notation: uppercase letters denote vectors, $M := \frac{\vec{x}}{|\vec{x}|}$, any pair of lowercase letters stands for the scalar product of two vectors i.e. $pa\,ma = p_i a^i\, m_i a^i$, dots over letters indicate time derivative i.e. $\dot{a}, \ddot{a}, \dot{P}, \ddot{P}, \dddot{P}, \ddddot{P}$. The symmetric, traceless tensors that appear in fields X and Y should be decoded from right to left i.e. $(A \times A \rfloor A \vee \dot{P}) = A \times \left(A\rfloor \left(A \vee \dot{P}\right)\right)$

$$\underset{-3}{D} = \frac{-P + 3\,mpM}{r^3} \tag{37}$$

$$\underset{-2}{D} = \frac{(-3/2\,mamp + 1/2\,pa)\,M + 1/2\,maP - 1/2\,mp\,A}{r^2} \tag{38}$$

$$\begin{aligned}\underset{-1}{D} = \Bigg[&\left(-\frac{1}{4}\,pa\,ma + \frac{9}{8}\,ma^2mp - \frac{3}{8}\,aa\,mp - \frac{1}{2}\,m\ddot{p}\right) M + \left(\frac{3}{4}\,ma\,mp - \frac{1}{4}\,pa\right) A+ \\ &+ \left(-\frac{3}{8}\,ma^2 + \frac{3}{8}\,aa\right) P - \frac{1}{2}\,\ddot{P}\Bigg]\, r^{-1}\end{aligned} \tag{39}$$

$$\begin{aligned}\underset{0}{D} = \Bigg(&\frac{3}{16}\,pa\,aa + \frac{9}{16}\,ma\,aa\,mp - \frac{3}{4}\,\ddot{p}a + \frac{1}{2}\,m\dot{a}\,m\dot{p} - +\frac{15}{16}\,ma^3\,mp - \frac{1}{2}\,\dot{p}\dot{a} + \\ &+ \frac{3}{16}\,ma^2\,pa - \frac{1}{8}\,p\ddot{a} + \frac{3}{4}\,ma\,m\ddot{p} + \frac{1}{8}\,m\ddot{a}\,mp\Bigg) M + \\ &+ \left(-\frac{15}{16}\,ma^2\,mp + \frac{3}{8}\,pa\,ma + \frac{3}{4}\,m\ddot{p} - \frac{3}{16}\,aa\,mp\right) A + m\dot{p}\,\dot{A} + \frac{3}{8} mp\,\ddot{A} + \\ &+ \left(\frac{5}{16}\,ma^3 - \frac{1}{8}\,m\ddot{a} - \frac{9}{16}\,aa\,ma\right) P + \frac{3}{4}\,ma\,\ddot{P}\end{aligned} \tag{40}$$

$$\begin{aligned}\underset{1}{D} = \Bigg[\Bigg(&\frac{1}{8}\,m\dddot{p} + \frac{1}{4}\,m\dot{a}\,\dot{p}a + \frac{9}{8}\,ma\,\ddot{p}a + \frac{5}{24}\,mp\,\dot{a}\dot{a} - \frac{5}{16}\,ma\,m\ddot{a}\,mp + \frac{3}{16}\,ma\,p\ddot{a}+ \\ &+ \frac{3}{4}\,ma\,\dot{p}\dot{a} - \frac{1}{16}\,m\ddot{a}\,pa + \frac{5}{16}\,mp\,a\ddot{a} - \frac{15}{128}\,aa^2\,mp + \frac{7}{8}\,m\dot{p}\,a\dot{a} - \frac{45}{64}\,ma^2\,mp\,aa - \\ &+ \frac{5}{4}\,m\dot{a}\,ma\,m\dot{p} - \frac{5}{24}\,ma^2\,mp + \frac{7}{16}\,m\ddot{p}\,aa - \frac{15}{16}\,ma^2m\ddot{p} + \frac{1}{24}\,m\dot{a}\,p\dot{a} -\end{aligned} \tag{41}$$

$$
\begin{aligned}
&+ \frac{5}{32}\, ma^3\, pa + \frac{105}{128}\, ma^4\, mp - \frac{9}{32}\, pa\, aa\, ma \Big) M + \\
&+ \Big(\frac{1}{16}\, p\ddot{a} + \frac{1}{4}\, \dot{p}\dot{a} - \frac{15}{8}\, ma\, m\ddot{p} + \frac{3}{8}\, \ddot{p}a - \frac{5}{4}\, m\dot{a}\, m\dot{p} - \frac{15}{32}\, ma^2\, pa + \frac{15}{32}\, ma\, aa\, mp + \\
&- \frac{5}{16}\, m\ddot{a}\, mp - \frac{3}{32}\, pa\, aa + \frac{35}{32}\, ma^3\, mp \Big) A + \Big(-\frac{5}{2}\, ma\, m\dot{p} - \frac{5}{6}\, m\dot{a}\, mp + \frac{1}{2}\, \dot{p}a + \\
&+ \frac{7}{24}\, p\dot{a} \Big) \dot{A} + \Big(\frac{3}{16}\, pa - \frac{15}{16}\, ma\, mp \Big) \ddot{A} + \Big(\frac{5}{24}\, m\dot{a}^2 + \frac{5}{16}\, ma\, m\ddot{a} + \frac{5}{24}\, \dot{a}\dot{a} - \\
&+ \frac{15}{128}\, aa^2 + \frac{5}{16}\, a\ddot{a} - \frac{35}{128}\, ma^4 + \frac{45}{64}\, aa\, ma^2 \Big) P + \frac{9}{8}\, a\dot{a}\, \dot{P} + \\
&+ \Big(-\frac{15}{16}\, ma^2 + \frac{9}{16}\, aa \Big) \ddot{P} - \frac{3}{8}\, \dddot{P} \Big] r
\end{aligned}
\tag{42}
$$

$$
\underset{-2}{B} = -\frac{M \times \dot{P}}{r^2} \tag{43}
$$

$$
\underset{-1}{B} = \frac{1/2\, ma\, M \times \dot{P} + m\dot{p}\, M \times A + 1/2\, mp\, M \times \dot{A} - 1/2\, Px\dot{A} - 1/2\, \dot{P} \times A}{r} \tag{44}
$$

$$
\begin{aligned}
\underset{0}{B} = &\Big(-\frac{3}{2}\, ma\, m\dot{p} - \frac{1}{2}\, m\dot{a}\, mp \Big) M \times A + \Big(-\frac{3}{4}\, ma\, mp - \frac{1}{4}\, pa \Big) M \times \dot{A} + \\
&+ \Big(-\frac{1}{8}\, aa - \frac{3}{8}\, ma^2 \Big) M \times \dot{P} + \frac{1}{2}\, M \times \dddot{P} + \frac{3}{4}\, ma\, P \times \dot{A} + \frac{3}{4}\, ma\, \dot{P} \times A + \\
&+ \frac{1}{4}\, mp\, A \times \dot{A} + \frac{1}{2}\, m\dot{a}\, P \times A
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
\underset{1}{B} = &\Big[\Big(\frac{1}{8}\, aa\, m\dot{p} + \frac{5}{4}\, ma\, mp\, m\dot{a} + \frac{1}{18}\, pa\, m\dot{a} - \frac{1}{2}\, m\dddot{p} + \frac{7}{36}\, a\dot{a}\, mp + \frac{15}{8}\, ma^2\, m\dot{p} \Big) M \times A + \\
&+ \Big(\frac{15}{16}\, ma^2\, mp + \frac{41}{72}\, pa\, ma - \frac{1}{144}\, aa\, mp - \frac{3}{4}\, m\ddot{p} \Big) M \times \dot{A} - \frac{1}{2}\, m\dot{p}\, M \times \ddot{A} - \\
&+ \frac{1}{8}\, mp\, M \times \dddot{A} + \Big(\frac{7}{36}\, aa\, m\dot{a} - \frac{7}{36}\, a\dot{a}\, ma \Big) M \times P + \Big(\frac{3}{16}\, aa\, ma - \frac{1}{8}\, m\ddot{a} + \frac{5}{16}\, ma^3 \Big) M \times \dot{P} - \\
&+ \frac{1}{2}\, m\dot{a}\, M \times \ddot{P} - \frac{3}{4}\, ma\, M \times \dddot{P} + \Big(-\frac{19}{18}\, ma\, m\dot{a} - \frac{7}{36}\, a\dot{a} \Big) P \times A + \\
&+ \Big(-\frac{15}{16}\, ma^2 - \frac{1}{16}\, aa \Big) \dot{P} \times A + \frac{1}{4}\, \dddot{P} \times A + \Big(\frac{55}{144}\, aa - \frac{163}{144}\, ma^2 \Big) P \times \dot{A} - \\
&+ \frac{1}{4}\, \ddot{P} \times \dot{A} - \frac{3}{8}\, \dot{P} \times \ddot{A} - \frac{1}{8}\, P \times \dddot{A} + \Big(-\frac{31}{72}\, ma\, mp - \frac{5}{72}\, pa \Big) A \times \dot{A} \Big] r
\end{aligned}
\tag{46}
$$

$$\underset{-3}{D} = -X\left(3,1,(P)\right), \tag{47}$$

$$\underset{-2}{D} = -\frac{1}{2}Y(3,1,(A\times P)) + \frac{1}{4}X\left(3,2,(A\vee D)\right), \tag{48}$$

$$\begin{aligned}
\underset{-1}{D} &= X(1,1,(Q_1)) + \frac{1}{4}Y(3,2,(A\vee A\times P)) - \frac{1}{8}X(3,3,(A\vee A\vee P)),\\
&\quad Q_1 = \frac{1}{2}\,\ddot{P} - \frac{3}{20}(A\rfloor A\vee P)
\end{aligned} \tag{49}$$

$$\begin{aligned}
\underset{0}{D} &= Y(1,1,(Q_1)) + X(1,2,(Q_2)) + Y(3,3,(Q_3)) + \frac{5}{64}X(3,4,(A\vee A\vee A\vee P)),\\
&\quad Q_1 = \frac{1}{4}\,(\ddot{A}\times P) + \frac{1}{2}\,(\dot{A}\times\dot{P}) + \frac{3}{40}\,(A\times A\rfloor A\vee P) - \frac{3}{40}\,(A\rfloor A\vee A\times P),\\
&\quad Q_2 = -\frac{3}{8}\,(A\vee\ddot{P}) - \frac{1}{16}\,(\ddot{A}\vee P) - \frac{1}{4}\,(\dot{A}\vee\dot{P}) - \frac{1}{96}\,(A\times A\vee A\times P) +\\
&\qquad + \frac{9}{160}\,(A\vee A\rfloor A\vee P) + \frac{2}{21}\,(A\rfloor A\vee A\vee P),\\
&\quad Q_3 = \frac{1}{96}\,(A\times A\vee A\vee P) - \frac{1}{6}\,(A\vee A\vee A\times P),
\end{aligned} \tag{50}$$

$$\begin{aligned}
\underset{1}{D} &= X(-1,1,(Q_1)) + Y(1,2,(Q_2)) + X(1,3,(Q_3)) + Y(3,4,(Q_4)) + X(3,5,(Q_5)),\\
&\quad Q_1 = \frac{1}{8}\,\dddot{P} + \frac{21}{160}\,(\dot{A}\rfloor A\vee\dot{P}) - \frac{3}{20}\,(A\rfloor\dot{A}\vee\dot{P}) - \frac{3}{32}\,(A\rfloor A\vee\ddot{P}) - \frac{3}{32}\,(\dot{A}\rfloor\dot{A}\vee P) +\\
&\qquad - \frac{9}{160}\,(A\rfloor\ddot{A}\vee P) - \frac{3}{80}\,(\ddot{A}\rfloor A\vee P) + \frac{3}{16}\,(A\times\dot{A}\times\dot{P}) + \frac{1}{32}\,(\dot{A}\times A\times\dot{P}) +\\
&\qquad + \frac{3}{32}\,(A\times\ddot{A}\times P) + \frac{1}{32}\,(\dot{A}\times\dot{A}\times P) + \frac{1}{32}\,(A\times\ A\times\ddot{P}) +\\
&\qquad + \frac{3}{160}\,(A\times A\times A\rfloor A\vee P) - \frac{3}{160}\,(A\times A\rfloor A\vee A\times P),\\
&\quad Q_2 = \frac{1}{32}\,(\dot{A}\times A\vee\dot{P}) + \frac{1}{8}\,(A\times\dot{A}\vee\dot{P}) + \frac{5}{32}\,(A\times A\vee\ddot{P}) + \frac{1}{96}\,(\dot{A}\times\dot{A}\vee P) +\\
&\qquad + \frac{1}{32}\,(A\times\ddot{A}\vee P) - \frac{7}{32}\,(\dot{A}\vee\dot{A}\times P) - \frac{9}{32}\,(A\vee\ddot{A}\times P) - \frac{9}{16}\,(A\vee\dot{A}\times\dot{P}) +\\
&\qquad - \frac{7}{32}\,(\dot{A}\vee A\times\dot{P}) - \frac{5}{32}\,(A\vee A\times\ddot{P}) - \frac{1}{16}\,(\ddot{A}\vee A\times P) +\\
&\qquad - \frac{2}{63}\,(A\times A\rfloor A\vee A\vee P) - \frac{1}{252}\,(A\rfloor A\times A\vee A\vee P) - \frac{3}{160}\,(A\times A\vee A\rfloor A\vee P) +\\
&\qquad - \frac{3}{80}\,(A\vee A\times A\rfloor A\vee P) + \frac{4}{63}\,(A\rfloor A\vee A\vee A\times P) + \frac{3}{80}\,(A\vee A\rfloor A\vee A\times P) +\\
&\qquad + \frac{1}{288}\,(A\times A\times A\vee A\times P),\\
&\quad Q_3 = \frac{5}{48}\,(\ddot{A}\vee A\vee P) + \frac{5}{12}\,(A\vee\dot{A}\vee\dot{P}) + \frac{5}{16}\,(A\vee A\vee\ddot{P}) +\\
&\qquad + \frac{5}{72}\,(\dot{A}\vee\dot{A}\vee P) - \frac{25}{384}\,(A\rfloor A\vee A\vee A\vee P) +\\
&\qquad - \frac{10}{189}\,(A\vee A\rfloor A\vee A\vee P) - \frac{1}{3456}\,(A\times A\times A\vee A\vee P) +\\
&\qquad + \frac{1}{216}\,(A\times A\vee A\vee A\times P) + \frac{5}{864}\,(A\vee A\times A\vee A\times P) - \frac{1}{32}\,(A\vee A\vee A\rfloor A\vee P),\\
&\quad Q_4 = -\frac{1}{128}\,(A\times A\vee A\vee A\vee P) - \frac{1}{128}\,(A\vee A\times A\vee A\vee P) + \frac{1}{8}\,(A\vee A\vee A\vee A\times P),\\
&\quad Q_5 = -\frac{7}{128}\,(A\vee A\vee A\vee A\vee P),
\end{aligned} \tag{51}$$

$$\underset{-2}{B} = -Y(3,1,(\dot{P})), \tag{52}$$

$$\underset{-1}{B} = X(1,1,(Q_1)) + Y(3,2,(Q_2)), \tag{53}$$

$$Q_1 = -\frac{1}{4}\,(\dot{A} \times P) - \frac{1}{4}\,(A \times \dot{P}), \tag{54}$$

$$Q_2 = \frac{3}{4}\,(A \vee \dot{P}) + \frac{1}{4}\,(\dot{A} \vee P), \tag{55}$$

$$\begin{aligned}
\underset{0}{B} &= Y(1,1,(Q_1)) + X(1,2,(Q_2)) + Y(3,3,(Q_3)),\\
Q_1 &= \frac{1}{2}\,\dddot{P} + \frac{1}{8}\,(A \times A \times \dot{P}) + \frac{1}{8}\,(A \times \dot{A} \times P) - \frac{9}{40}\,(A\rfloor\dot{A} \vee P) +\\
&\quad - \frac{3}{8}\,(A\rfloor A \vee \dot{P}) - \frac{3}{20}\,(\dot{A}\rfloor A \vee P),\\
Q_2 &= -\frac{1}{32}\,(A \times A \vee \dot{P}) - \frac{1}{96}\,(A \times \dot{A} \vee P) + \frac{5}{32}\,(A \vee \dot{A} \times P) +\\
&\quad + \frac{5}{32}\,(A \vee A \times \dot{P}) + \frac{1}{16}\,(\dot{A} \vee A \times P),\\
Q_3 &= -\frac{5}{8}\,(A \vee A \vee \dot{P}) - \frac{5}{12}\,(A \vee \dot{A} \vee P),
\end{aligned} \tag{56}$$

$$\begin{aligned}
\underset{1}{B} &= X(-1,1,(Q_1)) + Y(1,2,(Q_2)) + X(1,3,(Q_3)) + Y(3,4,(Q_4)),\\
Q_1 &= \frac{1}{8}\,(A \times \dddot{P}) - \frac{1}{8}\,(\dot{A} \times \ddot{P}) - \frac{3}{16}\,(\ddot{A} \times \dot{P}) - \frac{1}{16}\,(\dddot{A} \times P) - \frac{3}{160}\,(\dot{A} \times A\rfloor A \vee P) +\\
&\quad - \frac{9}{80}\,(A \times A\rfloor A \vee \dot{P}) - \frac{9}{160}\,(A \times \dot{A}\rfloor A \vee P) - \frac{3}{40}\,(A \times A\rfloor\dot{A} \vee P) +\\
&\quad + \frac{3}{160}\,(\dot{A}\rfloor A \vee A \times P) + \frac{3}{160}\,(A\rfloor A \vee \dot{A} \times P) + \frac{3}{160}\,(A\rfloor\dot{A} \vee A \times P) +\\
&\quad + \frac{3}{160}\,(A\rfloor A \vee A \times \dot{P}) + \frac{1}{32}\,(A \times A \times \dot{A} \times P) + \frac{1}{32}\,(A \times A \times A \times \dot{P}),\\
Q_2 &= -\frac{1}{16}\,(\dddot{A} \vee P) - \frac{5}{16}\,(\ddot{A} \vee \dot{P}) - \frac{5}{8}\,(A \vee \dddot{P}) - \frac{5}{8}\,(\dot{A} \vee \ddot{P}) +\\
&\quad + \frac{22}{63}\,(A\rfloor\dot{A} \vee A \vee P) + \frac{2}{21}\,(\dot{A}\rfloor A \vee A \vee P) + \frac{1}{3}\,(A\rfloor A \vee A \vee \dot{P}) +\\
&\quad + \frac{9}{160}\,(\dot{A} \vee A\rfloor A \vee P) + \frac{39}{160}\,(A \vee A\rfloor A \vee \dot{P}) + \frac{27}{160}\,(A \vee A\rfloor\dot{A} \vee P) +\\
&\quad + \frac{21}{160}\,(A \vee \dot{A}\rfloor A \vee P) - \frac{1}{16}\,(A \times A \vee \dot{A} \times P) - \frac{1}{32}\,(A \times \dot{A} \vee A \times P) +\\
&\quad - \frac{1}{96}\,(\dot{A} \times A \vee A \times P) - \frac{1}{16}\,(A \vee A \times \dot{A} \times P) - \frac{1}{16}\,(A \times A \vee A \times \dot{P}) +\\
&\quad + \frac{1}{96}\,(A \times A \times A \vee \dot{P}) + \frac{1}{288}\,(A \times A \times \dot{A} \vee P),\\
Q_3 &= \frac{1}{576}\,(\dot{A} \times A \vee A \vee P) + \frac{5}{288}\,(A \vee A \times A \vee \dot{P}) + \frac{5}{864}\,(A \vee A \times \dot{A} \vee P) +\\
&\quad + \frac{11}{576}\,(A \times A \vee A \vee \dot{P}) + \frac{13}{864}\,(A \times A \vee \dot{A} \vee P) +\\
&\quad - \frac{13}{144}\,(A \vee \dot{A} \vee A \times P) - \frac{11}{96}\,(A \vee A \vee A \times \dot{P}) - \frac{11}{96}\,(A \vee A \vee \dot{A} \times P),\\
Q_4 &= \frac{35}{64}\,(A \vee A \vee A \vee \dot{P}) + \frac{35}{64}\,(A \vee A \vee \dot{A} \vee P).
\end{aligned} \tag{57}$$

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