Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Existence and relaxation results in special classes of deformations

by

M. A. Sychev

Preprint no.: 17 2000

Existence and relaxation results in special classes of deformations *

M.A.Sychev
Max Planck Institute for Mathematics in the Sciences
Inselstr. 22 - 26, D-04103 Leipzig
GERMANY †

Abstract

In this paper we deal with the existence and relaxation issues in variational problems from the mathematical theory of elasticity. We consider minimization of the energy functional in those classes of deformations which make the problem essentially scalar.

It turns out that in these cases the relaxation theorem holds for integrands that are bounded from below by a power function with power exceeding the dimension of the space of independent variables. The bound from below can be relaxed in the homogeneous case. The same bounds were used previously to rule out cavitation and other essential discontinuites in admissible deformations. In the homogeneous case we can also indicate a condition which is both necessary and sufficient for solvability of all boundary value minimization problems of the Dirichlet type.

Key words Existence and relaxation, Mathematical Theory of Elasticity, weak convergence, Young measures

*This research was partially supported by the grant N 97-01-00508 of the Russian Foundation for Basic Research

†on leave from Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

1 Introduction

In this paper we deal with the existence and relaxation issues in variational problems of mathematical theory of elasticity. First general existence theorems were proved by J.M. Ball in [**B1**], for later work see e.g. [**MQY**], [**GMS**]. The results of those papers assert that the minimization problem

$$J(u) = \int L(Du)dx \to \min,\ u\Big|_{\partial\Omega} = f \tag{1.1}$$

has a solution in a Sobolev class of mappings $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$ if the integrand L is polyconvex and has sufficiently fast growth at infinity. Here polyconvexity means that

$$L(Du) = \tilde{L}(Du, \mathrm{adj}Du, \det Du), \tag{1.2}$$

where $\tilde{L}$ is a convex function of its variables and adj$A$ is the adjugate matrix of A.

The integral functional is sequentially weakly lower semicontinuous (s.w.l.s.) in $W^{1,p}(\Omega; \mathbf{R}^n)$ if

$$L(Du) \geq \max\{|Du|^p, |\mathrm{adj}Du|^r\}, \tag{1.3}$$

where e.g. $p \geq n-1$, $r \geq p/(p-1)$. In this case each minimizing sequence converges weakly to a solution of the minimization problem. A remarkable fact is that the theorem even covers energy densities L which meets the basic requirement coming from elasticity, which is

$$L(Du) \to \infty \text{ as } \det Du \to +0 \text{ and } L(Du) = \infty \text{ if } \det Du \leq 0. \tag{1.4}$$

The proof relies on two observations: weak continuity of the functionals $Du \to \mathrm{adj}Du$ and $Du \to \det Du$ with respect to the sequential weak convergence in Sobolev spaces associated with the exponents in (1.3), cf. [**B1**], and lower semicontinuity of integral functionals $\xi \to \int_\Omega F(\xi(x))dx$ with convex integrands F with respect to weak convergence of sequences in L^1.

Until this time no other existence or relaxation results meeting the requirement (1.4) were available in spite of extensive work devoted to some model cases where L has power growth at infinity or satisfies some estimates from above and below with power functions having sufficiently close exponents, see e.g. [**BFM**], [**FM**] and papers mentioned therein. Even the case of energies generated by isotropic materials is not completely studied. Recall that in the latter case L depends only on the main invariants of the matrix Du^tDu. The polyconvexity requirement holds for a number of such materials (see [**B1**], [**C**]), however does not cover all of them. For recent discussions see e.g. [**B2**], [**B3**], [**MSSp**].

In the model case

$$A_1|Du|^p + B_1 \leq L(Du) \leq A_2|Du|^p + B_2,\ A_2 \geq A_1 > 0,\ p > 1 \tag{1.5}$$

a condition, which characterizes s.w.l.s. property, is well known. In 1952 C. Morrey showed that an integral functional with a continuous integrand L is lower semicontinuous with respect to weak* convergence of sequences in $W^{1,\infty}$ if and only if the integrand L is *quasiconvex*, i.e.

$$\int_\Omega L(A + D\phi(x))dx \geq L(A)\,\text{meas}\ \Omega$$

for each function $\phi \in W_0^{1,\infty}$, [**Mo**]. The fact that this requirement still characterizes s.w.l.s. in $W^{1,p}$ for integrands satisfying (1.5) was established in [**AF**] in case of dependence of L upon x and u also. For a simple proof in the homogeneous case see [**Ma**], see also [**Me**].

In case of general integrands L with p-growth the quasiconvexification

$$L^{qc}(x,u,A) := \frac{1}{\text{meas}\ \Omega}\inf_{\phi\in W_0^{1,\infty}} \int_\Omega L(x,u,A + D\phi(y))dy$$

presents an integral functional for which *the relaxation property holds, i.e. given $u \in W^{1,p}$ we can find a sequence $u_k \rightharpoonup u$ in $W^{1,p}$ such that $J(u_k) \to$* $J^{qc}(u)$. Note that $J^{qc} \leq J$ everywhere and that J^{qc} is a functional which is lower semicontinuous with respect to the weak convergence in $W^{1,p}$ since L^{qc} is a quasiconvex function, see [**AF**], [**D1**], [**S4**]. Therefore in this case we have the relaxation result in full generality, i.e. the functional J^{qc} presents the lower semicontinuous extension (in the weak topology of $W^{1,p}$) of the original functional J.

Recall that *quasiconvexity* implies *rank-one convexity*, where $L : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ is called rank-one convex if for every matrices $A, B \in \mathbf{R}^{m\times n}$ with rank$(A - B) = 1$ the function $t \to L(A + (B - A)t)$ is convex, cf. e.g. [**B1**], [**BM**]. Moreover the converse is false if $m \geq 3$, cf. [**Sv1**]. The case $m = 2$ is an open problem. In the case $m = 1$ quasiconvexity reduces to convexity.

Note that quasiconvexity by itself is not enough to assert s.w.l.s. of general integral functionals. In fact, an example in [**BM, §7**] shows that even the simplest quasiconvex integrand $|\mathrm{det}Du|$ is no longer weak lower semicontinuous with respect to the weak convergence in $W^{1,p}$ if $p < n$ (the latter holds if $p \geq n$). Using this example one can construct an integrand satisfying the requirements (1.2), (1.4), and the requirement $L(Du) \geq \alpha|Du|^p + \gamma$ with $\alpha > 0$, $p > 1$, which is not sequentially weak lower semicontinuous (see e.g. [**JS**]). However the fact that quasiconvexity or similar property fails to characterize s.w.l.s. in the latter case can still be explained by slow growth of integrands at infinity. In fact derivation of the total energy in the form (1.1) holds under assumptions of the continuum body model, cf. [**C**, Ch.2]. At the same time failure of assumptions (1.3) may result in occurrence of cavitation and other essential discontinuites (i.e. in this case the total energy can have a different form). The latter phenomena was discovered by Ball in [**B4**] and was studied in many subsequent papers, see e.g. [**JS**], [**MSp**], [**MSSp**] and papers mentioned therein. In particular Šveŕak [**Sv2**] and later Müller, Qi, & B.S.Yan [**MQY**] showed that the assumptions (1.3) or their weakened form $p \geq n-1$, $r \geq n/(n-1)$ prevent cavitation. Therefore one still may expect that the lower semicontinuity and the relaxation results hold for integrands satisfying (1.3), i.e. in the case when admissible deformations do not allow essential discontinuites.

In this paper we are able to give complete analysis in cases when the minimization problems are essentially scalar. We study minimization in the classes of deformations which include the following ones:

1. generalized anti-plane shear deformations

   $$u : (x_1, \dots, x_n) \to (x_1, \dots, x_{n-1}, h(x_1, \dots, x_n)),\ \ h_{x_n} \geq 0 \text{ a.e.}.$$

   A subclass consisting of deformations with $h_{x_n} = 1$ a.e. is known as a class of anti-plane shear deformations. In this case $\mathrm{det}Du = 1$ a.e. The slightly more general case which we consider here requires that we deal with integrands $L : \Omega \to \mathbf{R} \cup \{\infty\}$, see (1.4).

2. $u \in u_0 + V_c$ a.e., where given $c \in \mathbf{R}^n$ the set V_c is defined as

   $$V_c := \{g \in W^{1,p}(\Omega; \mathbf{R}^n) : g = c \otimes g_1\}.$$

In fact, we can extend the second class to the class of functions $u \in W^{1,p}(\Omega;\mathbf{R}^n)$ with $u \in u_0 + V_{c_l}$ a.e. in Ω_l, where the sets Ω_l, $l \in \mathbf{N}$, are Lipschitz, open and disjoint and meas $\{\Omega \setminus \cup_l \Omega_l\} = 0$.

In all cases the problem (1.1) can be rewritten as a scalar problem. Therefore we will assume further that the Carathéodory integrand

$$L : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$$

satisfies the following basic assumptions

(H1) for a.e. $x \in \Omega$ the integrand L is bounded in a neighborhood of each point $(x,u,v) \in \mathbf{R}^{2n+1}$ where its value is finite

(H2) $L(x,u,v) \geq \alpha|v|^p + \gamma,\ p > 1,\ \alpha > 0.$

To state the first result recall that a function $F : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ is called *convex at a point* v_0 if

$$\sum c_i F(v_i) \geq F(v_0) \tag{1.6}$$

for every $v_i \in \mathbf{R}^n$, $c_i \geq 0$, $i \in \{1,\ldots,q\}$, such that $\sum c_i = 1$, $\sum c_i v_i = v_0$. We say that F is *strictly convex* at v_0 if the inequality (1.6) is strict under the additional assumption $v_i \neq v_0$, $i = 1,\ldots,q$.

Recall that a function F is convex at a point v_0 if and only if its *subgradient*

$$\partial F(v_0) := \{f \in \mathbf{R}^n : F(v) - F(v_0) - \langle f, v - v_0\rangle \geq 0, \forall v \in \mathbf{R}^n\}$$

is nonempty, cf. [**S1**], [**Y1, §56**].

Theorem 1.1 *Let L satisfy the requirements (H1), (H2) with $p > n$. Then the function L^{**}, which is obtained by convexification of L with respect to v, is a Carathéodory integrand which satisfies (H1), (H2).*

Moreover, for each $u_0 \in W^{1,1}(\Omega)$ with $J^c(u_0) < \infty$, where J^c is the integral functional associated with the integrand L^{**}, there exists a sequence $u_k \in W^{1,1}(\Omega)$ such that $u_k\big|_{\partial\Omega} = u_0\big|_{\partial\Omega}$, $u_k \rightharpoonup u_0$ in $W^{1,1}(\Omega)$, and

$$J(u_k) \to J^c(u_0) \text{ as } k \to \infty.$$

The equality $J(u_0) = J^c(u_0)$ holds if and only if for a.e. $x \in \Omega$ the function $L(x,u_0(x),\cdot)$ is convex at the point $Du_0(x)$. In this case the convergences

$u_k \rightharpoonup u_0$ *in* $W^{1,1}$, $J(u_k) \to J(u_0)$ *imply the convergence* $Du_k - Du_0 \to 0$ *in* L^1 *(and also the convergence* $L(\cdot, u_k(\cdot), Du_k(\cdot)) \to L(\cdot, u_0(\cdot), Du_0(\cdot))$ *in* L^1*) if and only if for a.e.* $x \in \Omega$ *the function* $L(x, u_0(x), \cdot)$ *is strictly convex at the point* $Du_0(x)$.

Remark 1.2 The assumption (H1) does not guarantee a possibility to approximate u_0 by piece-wise functions u_k with $J(u_k) \to J(u_0)$. In fact, for each pair of convex functions θ_1, θ_2 with superlinear growth and such that $\lim_{|v|\to\infty} \theta_2(v)/\theta_1(v) = \infty$ one can construct an integrand $L : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ such that $\theta_1 \leq L \leq \theta_2$ and for some Sobolev function u_0 there is *no approximation in energy* by Lipschitz functions u_k with $u_k \to u_0$ in L^1, see [**S2**]. Therefore the standard approach through approximation in energy functions with finite energy by smooth or piece-wise affine ones (see e.g. [**ET**], [**MS**], [**Bu**]) can not be applied in this case.

Remark 1.3 The theorem implies that J^c is the lower semicontinuous envelope of J since the functionals having Carathéodory integrands with convex dependence on Du are automatically sequentially lower semicontinuous in the weak topology of $W^{1,1}$, see e.g. [**Ba**], [**D2**], and [**S2**]. Therefore the relaxed problem has a solution.

In case of homogeneous problems we can prove the relaxation result for L satisfying the requirement (H2) with $p > n - 1$. The latter inequality is a part of the set of conditions preventing occurrence of cavitation, see e.g. [**Sv 2**], [**MQY**]. Moreover in this case we can indicate a condition characterizing solvability of all minimization problems of the type (1.1).

Theorem 1.4 *Let* $L : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ *be a continuous function such that* $L(\cdot) \geq \alpha|\cdot|^p + \gamma$, *where* $\alpha > 0$, $p > n - 1$ *if* $n \geq 3$ *and* L *has at least superlinear growth at infinity if* $n = 2$. *Then:*

1) for each boundary datum f *with* $J^c(f) < \infty$ *the problem*

$$J^c(u) \to \min, \ u\Big|_{\partial\Omega} = f, \ u \in W^{1,1}(\Omega)$$

has a solution. Moreover for each such solution u_0 *there exists a sequence* $u_k \in u_0 + W_0^{1,1}(\Omega)$ *with* $J(u_k) \to J^c(u_0)$, $u_k \rightharpoonup u_0$ *in* $W^{1,1}(\Omega)$. *We have* $J(u_0) = J^c(u_0)$ *if and only if* L *is convex at* $Du_0(x)$ *for a.e.* $x \in \Omega$. *In* *this case the convergences $u_k \to u_0$ in L^1, $J(u_k) \to J(u_0)$ imply the convergence $Du_k \to Du_0$ in L^1 (and automatically the convergence $L(Du_k(\cdot)) \to L(Du_0(\cdot))$ in L^1) if and only if for a.e. $x \in \Omega$ the function L is strictly convex at the point $Du_0(x)$.*

2) Moreover for each admissible f the original problem

$$J(u) \to \min,\ u\Big|_{\partial\Omega} = f,\ u \in W^{1,1}(\Omega)$$

has a solution if and only if the following condition holds for each $v \in \mathbf{R}^n$ with $L^c(v) < \infty$

(C) either $\partial L(v) \neq \emptyset$ or there exist $v_1, \ldots, v_q$ such that $v \in \operatorname{int} \operatorname{co}\{v_1, \ldots, v_q\}$ and $\cap_{i=1}^q \partial L(v_i) \neq \emptyset$

Remark 1.5 Note that arguments similar to the ones we use in the proofs of Theorem 1.1 and Theorem 1.4 can be applied to show that the relaxation property holds on deformations of the types 1 and 2 for the rank-one convexification

$$L^{rc} := \sup\{H : H \leq L,\ H \text{ is rank-one convex in } Du\}$$

of L provided L has sufficiently fast growth at infinity. Therefore in those particular cases when L^{rc} gives a lower semicontinuous functional we obtain the relaxation result in full generality (at the deformations of the type discussed above).

We will include a detailed proof of this assertion in a forthcoming paper.

Remark 1.6 Note that the condition (C) from Theorem 1.4 characterizes solvability of all boundary value minimization problems under a number of different assumptions. In [**S3**] we showed that superlinear growth of L at infinity is enough to assert that it characterizes solvability in the class of boundary data satisfying so-called bounded slope condition, see [**Gi**]. Moreover an observation of Sverak [**Sv3**] allows us to conclude that a similar result holds if L meets the requirement of p-growth (1.5), see [**S3**] for more detailed information.

Note also that we will prove that given v the condition (C) characterizes solvability of the minimization problem with linear boundary data $f := l_v$.

This fact was established in [**Ce**], [**F**] for continuous integrands with superlinear growth at infinity and our arguments follow the lines of the proof from those papers.

All the results still hold if continuity of L is replaced by lower semicontinuity. To show this one can refine the arguments given here using constructions from [**S3**]. However, this is not the purpose of the paper and we leave the details to the interested reader.

Note that previously minimization of isotropic energies in the class of anti-plane shear deformations was studied in case of dependence of L only on the first invariant of the matrix Du^tDu, which is $|Du|^2$, cf. [**BP**], [**GT**], [**R**], [**SH**]. In this case the requirement $L = L(|Du|)$ does not contradict the assumptions (1.5) under which standard relaxation theorems hold and, consequently, the attainment question can be reduced to finding a solution of the relaxed problem along which values of the original and the relaxed integrands coincide. Our results show that the same scheme still can be applied in the case of general anti-plane shear problems, including the nonhomogeneous case.

Throughout the paper we use standard notation. For a subset U of $\mathbf{R}^n$ the sets $\mathrm{int}U$, $\mathrm{co}U$, and $\mathrm{extr}U$ are respectively the interior of U, the convex hull of U, and the set of extreme points of U (a point a belongs to $\mathrm{extr}U$ if it can not be represented as a convex combination of other points of U). The set $B(a,\epsilon)$ denotes the open ball of radius ϵ which is centered at the point $a \in \mathbf{R}^n$. $Q(a,\epsilon)$ is the open cube with side length ϵ and the center a. The function l_a is an affine function with the gradient equal to a everywhere.

We assume that $\Omega \subset \mathbf{R}^n$ is a *bounded Lipschitz domain* unless otherwise indicated. A function $u : \Omega \to \mathbf{R}^m$ is *piece-wise affine* if $u \in W^{1,\infty}(\Omega; \mathbf{R}^m)$ and there is a decomposition of Ω into a negligible set and an at most countable collection of the closures of Lipschitz domains on each of which the restriction of u is affine.

The weak and strong convergences will be denoted $\rightharpoonup$ and $\to$, respectively.

The paper is organized as follows. In §2 we recall some basic facts from Young measure theory, which presents some technical tools necessary in this paper. We view Young measures as measurable functions, see [**S4**], [**S5**], since it allows us to use some additional tools that are not easily available from the standard viewpoint of Young measures as elements of the duals of appropriate Banach spaces. These properties are especially convenient for studies of the behavior of integral functionals on weakly convergent sequences. In §3 we prove some auxiliary results. In §4 we prove Theorem 1.1 as a consequence of a more general Theorem 4.1. In §5 we prove Theorem 1.2.

2 Basic facts from Young measure theory

Recall the definition of Young measures.

Definition 2.1 *A family $(\nu_x)_{x\in\Omega}$ of probability measures $\nu_x \in C_0(\mathbf{R}^l)'$ is called a Young measure if there exists a sequence of measurable functions $z_k : \Omega \to \mathbf{R}^l$ such that for each $\Phi \in C_0(\mathbf{R}^l)$*

$$\Phi(z_k) \rightharpoonup^* \bar{\Phi} \text{ in } L^\infty(\Omega), \text{ where } \bar{\Phi}(x) = \langle \Phi; \nu_x \rangle$$

(here and later on $\langle \Phi; \nu \rangle$ denotes the action of the measure ν on Φ).

Recall that each sufficiently regular sequence of measurable functions contains a subsequence generating a Young measure. This result shows that Young measures presents reasonable extension of standard functions.

Theorem 2.2 *(**[Y1]**, **[Y2]**, **[T1]**, **[B5]**) Each sequence of measurable functions $\xi_k : \Omega \to \mathbf{R}^l$ contains a subsequence generating a Young measure $(\nu_x)_{x\in\Omega}$ provided it is bounded in $L^r(\Omega)$, $r > 0$. Moreover this subsequence converges in measure if and only if ν_x is a Dirac mass for a.a. $x \in \Omega$.*

A starting point of our approach is the characterization of Young measures as measurable functions given by Theorem 2.3.

Recall that the weak* convergence of elements of the set $M_c(\mathbf{R}^l)$, which is the set of all Radon measures supported in $\mathbf{R}^l$ with the total variation bounded by c, is equivalent to convergence in the metric

$$\rho(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i ||\Phi_i||_C} \Big| \langle \Phi_i; \mu \rangle - \langle \Phi_i; \nu \rangle \Big|,$$

where $\{\Phi_i\}$ is a dense sequence of elements of the space

$$C_0(\mathbf{R}^l) = \{\Phi \in C(\mathbf{R}^l) : \lim_{v \to \infty} |\Phi(v)| = 0\}.$$

The metric ρ characterizes Young measures.

Theorem 2.3 [S4]
Let $(\nu_x)_{x\in\Omega}$ be a family of probability measures. Then the following assertions are equivalent:
1) $(\nu_x)_{x\in\Omega}$ is a Young measure,
2) the function $\nu : \Omega \to (M_1, \rho)$ is measurable ,
3) the maps $x \to \langle \Phi; \nu_x \rangle$ are measurable for all $\Phi \in C_0(\mathbf{R}^l)$.

The idea of our approach is to use the characterization 2) of Young measures as measurable functions. Although these functions have more complex nature than the standard measurable functions with values in $\mathbf{R}^n$, they still have quite a broad spectrum of properties. In fact these properties allow us to prove all standard results of Young measure theory, cf. [**S4**], [**S5**].

The three basic properties of these functions are the following.

1. Note that the convergence $\langle \Phi; \nu^k_{(\cdot)} \rangle \rightharpoonup^* \langle \Phi; \nu_{(\cdot)} \rangle$ in L^∞ means convergence of the integrals $\int_{\tilde\Omega} \langle \Phi; \nu^k_x \rangle dx$ to the integral $\int_{\tilde\Omega} \langle \Phi; \nu_x \rangle dx$ for all measurable subsets $\tilde\Omega$ of Ω. On the other hand the functional

$$\Phi \to (1/\,\text{meas}\ \tilde\Omega) \int_{\tilde\Omega} \langle \Phi; \nu_x \rangle dx$$

is given by the action of a Radon measure which we denote $\mathrm{Av}(\nu_x)_{x\in\tilde\Omega}$, i.e.

$$\langle \Phi; \mathrm{Av}(\nu_x)_{x\in\tilde\Omega} \rangle := \frac{1}{\text{meas}\ \tilde\Omega} \int_{\tilde\Omega} \langle \Phi; \nu_x \rangle dx,\ \ \forall \Phi \in C_0(\mathbf{R}^l).$$

To compare actions of two families of measures $(\nu^1_x)_{x\in\tilde\Omega}$ and $(\nu^2_x)_{x\in\tilde\Omega}$ we have to compare the distance between the measures $\mathrm{Av}(\nu^1_x)_{x\in\tilde\Omega}$ and $\mathrm{Av}(\nu^2_x)_{x\in\tilde\Omega}$ in ρ-metric. The following proposition presents such estimates, see [**S4**], [**S5**] for proofs. Here we consider families of those Radon measures, which are elements of $M_c(\mathbf{R}^l)$. In this case the average Av is also an element of $M_c(\mathbf{R}^l)$.

Lemma 2.4 *Let $\nu^1, \nu^2 : \Omega \to (M_c, \rho)$ be measurable functions.*

1. If $\rho(\mathrm{Av}(\nu^1_x)_{x\in\tilde\Omega}, \mathrm{Av}(\nu^2_x)_{x\in\tilde\Omega}) \leq \delta$ with $\tilde\Omega \subset \Omega$ such that $\mathrm{meas}\,(\Omega \setminus \tilde\Omega) \leq \delta\,\mathrm{meas}\,\Omega$, then $\rho(\mathrm{Av}(\nu^1_x)_{x\in\Omega}, \mathrm{Av}(\nu^2_x)_{x\in\Omega}) \leq (2c+1)\delta$.

2. If $\rho(\nu^1_x, \nu^2_x) \leq \delta$ for a.a. $x \in \tilde\Omega \subset \Omega$ with $\mathrm{meas}\,(\Omega \setminus \tilde\Omega) \leq \delta\,\mathrm{meas}\,\Omega$, then $\rho(\mathrm{Av}(\nu^1_x)_{x\in\Omega}, \mathrm{Av}(\nu^2_x)_{x\in\Omega}) \leq (2c+1)\delta$.

In particular given a measurable function $\nu : \Omega \to (M_1, \rho)$ we have

$$\rho(\mathrm{Av}(\nu_y)_{y\in B(x,\epsilon)}, \nu_x) \to 0 \text{ as } \epsilon \to 0, \text{ a.e. in } \Omega.$$

The other two properties of Young measures come from the general theory of measurable functions with values in a compact metric space.

2. The second property of such functions is the Lusin property.

Theorem 2.5 *Let $\Omega \subset \mathbf{R}^n$ be a measurable set, and let (K, d) be a compact metric space. The function $\xi : \Omega \to (K, d)$ is measurable in the usual Lebesque sense (i.e. preimages of closed sets are measurable sets) if and only if it has the Lusin property: for each $\epsilon > 0$ there exists a compact subset Ω_ϵ of Ω such that $\mathrm{meas}\,(\Omega \setminus \Omega_\epsilon) \leq \epsilon$ and the function $\xi\big|_{\Omega_\epsilon}$ is continuous.*

The proof of this theorem is almost identical to the proof in the case (K, d) equals $\mathbf{R}^n$ with the Eucledian metric.

3. The third property is a version of the theorem on measurable selections proved first in [**K-RN**] (for more sophisticated versions of such theorems see [**CV**]). This property shows how to construct a Young measure $(\nu_x)_{x\in\Omega}$ knowing that for a.a. $x \in \Omega$ all possible choices of measures ν_x are given by the sets $V(x)$.

Let Ω be a bounded measurable subset of $\mathbf{R}^n$ and let (K, d) be a compact metric space. A mapping $V : \Omega \to 2^K$ is a closed measurable multi-valued mapping if, for a.a. $x \in \Omega$, the set $V(x) \subset K$ is closed and if for each closed subset C of K the set $\{x \in \Omega : V(x) \cap C \neq \emptyset\}$ is measurable.

Theorem 2.6 *If $V : \Omega \to 2^K$ is a closed measurable multivalued mapping then there exists a measurable selection, i.e. a measurable map $\nu : \Omega \to (K, d)$ such that $\nu(x) \in V(x)$ for a.a. $x \in \Omega$.*

We need also a result on relation of the values of an integral functional along a sequence and the value it assumes on a Young measure generated by the sequence.

Theorem 2.7 *Let Ω be a bounded measurable subset of $\mathbf{R}^n$ and let $L(x,v) : \Omega \times \mathbf{R}^l \to \mathbf{R} \cup \{\infty\}$ be a Carathéodory integrand which is bounded from below. Suppose that a sequence of measurable functions ξ_i generates a Young measure $(\nu_x)_{x\in\Omega}$.*

Then
$$\liminf_{i\to\infty} \int_\Omega L(x,\xi_i(x))dx \geq \int_\Omega \langle L(x,\cdot);\nu_x\rangle dx.$$
Moreover, $\lim_{i\to\infty} \int_\Omega \int_{\mathbf{R}^l} L(x,\xi_i(x))dx \to \int_\Omega \int_{\mathbf{R}^l} \langle L(x,v);\nu_x\rangle dx$ if and only if the functions $L(\cdot,\xi_i(\cdot))$, $i \in \mathbf{N}$, are equi-integrable. In this case $L(\cdot,\xi_i(\cdot)) \rightharpoonup \langle L(\cdot,v);\nu_{(\cdot)}\rangle$ in L^1.

Proof

This can be found in [**Ba**], [**Kr**], [**S4**] in the case when L satisfies the requirements of the theorem and has finite values.

In the general case the result follows by approximation the integrand L by the integrands $L_k := \min\{L,k\}$. **QED**

3 Some auxiliary results

In this section we prove three auxiliary lemmas.

Lemma 3.1 *Let $L : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ be a continuous function. Let $v_1,\ldots,v_q$ be such points in $\mathbf{R}^n$ that $\sum_{i=1}^q c_i v_i = F$ for some $c_i \geq 0$ with $\sum_{i=1}^q c_i = 1$ and $L(v_i) < \infty$ for each $i \in \{1,\ldots,q\}$.*

Then, there exists a sequence of piece-wise affine functions $u_k \in W^{1,\infty}(\Omega)$ such that $u_k\big|_{\partial\Omega} = l_F$, $Du_k \in \cup_{i=1}^q B(v_i,1/k)$ a.e., and
$$\frac{\operatorname{meas}\{x\in\Omega : Du_k(x) \in B(v_i,1/k)\}}{\operatorname{meas}\Omega} \to c_i,\ i \in \{1,\ldots,q\}$$
(in this case we have also $J(u_k) \to \sum_{i=1}^q c_i L(v_i) \operatorname{meas}\Omega$).

If $F \in \operatorname{int}\operatorname{co}\{v_1,\ldots,v_q\}$ then there exists a piece-wise affine function $u \in l_F + W_0^{1,\infty}(\Omega)$ with the property $Du \in \{v_1,\ldots,v_q\}$ a.e.

Lemma 3.2 is a perturbation argument which will allow us to approximate those Sobolev functions which are a.e. differentiable in the classical sense by more regular ones.

Lemma 3.2 *Let $u_0 \in W^{1,1}(B(x_0, \epsilon))$ be a.e. differentiable in the classical sense in a subset of $B(x_0, \epsilon)$ of full measure (here we implicitly assume that u_0 is defined everywhere, i.e. u_0 is a fixed representative of its Sobolev class of equivalence) and let x_0 be a point of this set.*

Let $v_1, \ldots, v_q$ be extreme points of a compact convex set such that $Du(x_0) \in \operatorname{int} \operatorname{co}\{v_1, \ldots, v_q\}$. Define the function

$$w_s(\cdot) := u_0(x_0) + \langle Du_0(x_0), \cdot - x_0\rangle + \max_{1\le i\le q} \langle v_i - Du(x_0), \cdot - x_0\rangle - s. \quad (3.1)$$

There exists a sequence of sets $\Omega_i \subset B(x_0, \epsilon)$, $i \in \mathbf{N}$, and a sequence $s_i \to 0$ such that for every $i \in \mathbf{N}$ we have $x_0 \in \Omega_i$, $\operatorname{meas} \partial\Omega_i = 0$,

$$B(x_0, \delta s_i) \subset \Omega_i \subset B(x_0, s_i/\delta) \text{ with some } \delta > 0, \quad (3.2)$$

and if $u_i = w_{s_i}$ in Ω_i and $u_i = u_0$ in $B(x_0, \epsilon) \setminus \Omega_i$ then $u_i \in W^{1,1}(B(x_0, \epsilon))$ provided $i \in \mathbf{N}$ is sufficiently large.

In the following lemma we show that each probability measure with finite action on a continuous integrand $L : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ can be approximated in energy by convex combinations of Dirac masses.

Lemma 3.3 *Let $L : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ be a continuous function with superlinear growth, i.e. $L(v) \ge \theta(v)$, where $\theta(v)/|v| \to \infty$ as $|v| \to \infty$. Let also ν be a probability measure supported in $\mathbf{R}^n$ such that $\langle L; \nu\rangle < \infty$ and let A be the center of mass of ν.*

Then there exists a sequence of convex combinations of Dirac masses $\nu_j := \sum_i c_i^j \delta_{v_i^j}$ centered at A and such that $\nu_j \rightharpoonup^* \nu$, $\langle L; \nu_j\rangle \to \langle L; \nu\rangle$, and $\operatorname{dist}(\operatorname{supp} \nu_j, \operatorname{supp} \nu) \to 0$ *as $j \to \infty$.*

We will utilize the following version of the *Vitaly covering theorem.*

A family G of closed subsets of $\mathbf{R}^n$ is said to be a Vitaly cover of a bounded set A if for each $x \in A$ there exists a positive number $r(x) > 0$, a sequence of balls $B(x, \epsilon_k)$ with $\epsilon_k \to 0$, and a sequence $C_k \in G$ such that $x \in C_k$, $C_k \subset B(x, \epsilon_k)$, and $(\operatorname{meas} C_k / \operatorname{meas} B(x, \epsilon_k)) > r(x)$ for all $k \in \mathbf{N}$.

The version of the *Vitaly covering theorem* from [**Sa**,p.109] says that each Vitaly cover of A contains at most countable subfamily of disjoint sets C_k such that meas $(A \setminus \cup_k C_k) = 0$.

Proof of Lemma 3.1

Without loss of generality we can assume that $F = 0$.

Let $b_1, \ldots, b_q$ be extreme points of a compact subset in $\mathbf{R}^n$ with $0 \in \operatorname{int} \operatorname{co}\{b_1, \ldots, b_q\}$. Consider the function

$$w_s(\cdot) := \max_{v \in \{b_1, \ldots, b_q\}} \langle v, \cdot \rangle - s, \quad s > 0. \tag{3.3}$$

It is clear that $w_s(\cdot)$ is a Lipschitz function such that $Dw_s \in \{b_1, \ldots, b_q\}$ a.e. and $w_s(\cdot) = 0$ in ∂P_s, where P_s are polyhedrons with the property $P_s = sP_1$.

We can decompose Ω into domains $\Omega_i := x_i + s_i P_1$, $i \in \mathbf{N}$, and a set N of null measure, i.e. $\Omega := \cup_{i \in \mathbf{N}} (x_i + s_i P_1) \cup N$. We define $u(x) := w_{s_i}(x - x_i)$ for $x \in x_i + s_i P_1$, $i \in \mathbf{N}$, $u = 0$ otherwise. Then $u \in W_0^{1,\infty}(\Omega)$, $Du \in \{b_1, \ldots, b_q\}$ a.e. in Ω. This proves the second part of the lemma.

We first prove the first part of the lemma in case $q = 2$, i.e. when $F = c_1 v_1 + c_2 v_2$.

Let $k \in \mathbf{N}$. We can take $b_1 = v_1$, $b_2 = v_2$ and assume that $b_i \in B(v_1, 1/k)$, $i \in \{3, \ldots, l\}$, are such points that $0 \in \operatorname{int} \operatorname{co}\{b_1, \ldots, b_l\}$, where $b_1, \ldots, b_l$ are extreme points of a compact convex set. By (3.3) we can find a piece-wise affine function $u_k \in W_0^{1,\infty}(\Omega)$ such that $Du_k \in \{b_1, \ldots, b_l\}$. It is clear that $Du_k \in B(v_1, 1/k) \cup B(v_2, 1/k)$ a.e. and

$$\frac{\text{meas}\,\{x \in \Omega : Du_k(x) \in B(v_i, 1/k)\}}{\text{meas}\ \Omega} \to c_i, \ \ i = 1, 2.$$

This proves the lemma in case $q = 2$.

We assume that the lemma is valid for $q \geq 2$. We will show that it also holds for $q + 1$. We define $\tilde{c}_1 := (c_1 + c_2)$, $\tilde{v} := (c_1 v_1 + c_2 v_2)/\tilde{c}$. Then we can apply the induction assumption to the case of convex combination $\sum_{i=3}^{q+1} c_i v_i + \tilde{c}\tilde{v}$ to find a sequence of piece-wise affine functions $u_k \in W_0^{1,\infty}(\Omega)$ such that $Du_k \in \cup_{i=3}^{q+1} B(v_i, 1/k) \cup B(\tilde{v}, 1/k)$ a.e. and

$$\frac{\text{meas}\,\{x \in \Omega : Du_k(x) \in B(v_i, 1/k)\}}{\text{meas}\ \Omega} \to c_i, \ \ i \in \{3, \ldots, q+1\},$$

$$\frac{\text{meas}\,\{x \in \Omega : Du_k(x) \in B(\tilde{v}, 1/k)\}}{\text{meas}\,\Omega} \to \tilde{c}.$$

Let $j \in \mathbf{N}$. We can apply the same construction as in the case $q = 2$ to perturb every function $u := u_k$, where $k \in \mathbf{N}$ is sufficiently large, in each open subset of $\tilde{\Omega} := \{x \in \Omega : Du(x) \in B(\tilde{v}, 1/k)\}$ where Du is constant in such a way that the perturbation ϕ_j has the property $D\phi_j \in \cup_{i=1}^{2} B(v_i, 1/j)$ a.e. in the set. This is possible since $B(\tilde{v}, 1/k) \subset \text{int}\,\text{co}\{b_1, \ldots, b_l\}$ for all sufficiently large k. Note that

$$\frac{\text{meas}\,\{x \in \tilde{\Omega} : D\phi_j \in B(v_i, 1/j)\}}{\text{meas}\,\tilde{\Omega}} \to c_i, \;\; i \in \{1, 2\}, \;\text{ as } j \to \infty.$$

Therefore we can select a subsequence u_k (not relabeled) and a sequence of their perturbations $\phi_{j(k)}$ with $j(k) \to \infty$ as $k \to \infty$ such that it meets all the requirements of the theorem. Then the claim of the theorem holds for $q+1$. Then it holds in the general case.

The proof is complete. **QED**

Proof of Lemma 3.2

Without loss of generality we can assume that u_0 is the standard representative of its Sobolev class, i.e. we have

$$u_0(x_0) = \lim_{\epsilon \to 0} \frac{\int_{B(x_0,\epsilon)} u_0(x)dx}{\text{meas}\; B(x_0, \epsilon)}$$

for all $x_0 \in \Omega$ where the limit exists (this holds a.e. in Ω). In fact it is not difficult to see that the standard representative is a.e. differentiable in the classical sense if there exists another representative with this property.

Let $\tilde{\Omega}$ be the set of those points of Ω where the function u_0 has the classical derivative. Then $\text{meas}\,(\Omega \setminus \tilde{\Omega}) = 0$. Let $x_0 \in \tilde{\Omega}$.

Consider the function

$$f_s(\cdot) = \max_{1 \le i \le q} \langle v_i - Du_0(x_0), \cdot \rangle - s$$

Note that $Df_s \in \{v_i - Du_0(x_0) : i = 1, \ldots, q\}$ a.e. and $f_s|_{\partial P_s} = 0$, where

$$P_s = \{x : \max_{1 \le i \le q} \langle v_i - Du_0(x_0), x \rangle \le s\}$$

is a compact set with Lipschitz boundary and nonempty interior. Moreover $P_s = sP_1$.

Note that for each $s > 0$ we have

$$f_s(\cdot) - f_s(0) = \max_{v\in\{v_1,\dots,v_q\}}\langle v - Du(x_0), \cdot\rangle \geq \delta|\cdot|,\ \ \delta > 0,$$

$$f(0) = -s,\ \ f(x) = s \text{ for } x \in P_{2s}.$$

Then for all sufficiently small $s > 0$ we have

$$f_s(\cdot - x_0) < u_0(\cdot) - u_0(x_0) - \langle Du_0(x_0), \cdot - x_0\rangle \text{ in } x_0 + P_{s/2},$$

$$f_s(\cdot - x_0) > u_0(\cdot) - u_0(x_0) - \langle Du_0(x_0), \cdot - x_0\rangle$$

in a neighborhood of $x_0 + \partial P_{2s}$ since the right-hand side of the inequalities is $o(|\cdot - x_0|)$.

We define Ω_s as the set of all those $x \in x_0 + P_{2s}$ where

$$f_s(\cdot - x_0) < u_0(x) - u_0(x_0) - \langle Du_0(x_0), \cdot - x_0\rangle. \tag{3.4}$$

The set $\Omega_s \subset\subset (x_0 + P_{2s})$ consists of an open set and a set of null measure Ω'_s. In fact if $y \in \Omega_s$ and $Du(y)$ exists in the classical sense, then (3.4) holds in a neighborhood of y. Therefore we can assume that the set Ω_s is open. To prove existence of a sequence $s_i \to 0$ such that meas $(\partial\Omega_{s_i}) = 0$ note that meas $\{\bar{\Omega}_{\delta_2} \setminus \Omega_{\delta_1}\} = 0$ if $\delta_2 < \delta_1$. Therefore $\delta \to$ meas Ω_δ is an increasing function with jumps at the points δ where meas $(\partial\Omega_\delta) > 0$. Since each monotone function has at most countably many jumps we deduce existence of a sequence $s_i \to 0$ for which meas $(\partial\Omega_{s_i}) = 0$, $i \in \mathbf{N}$.

Recall that a function $u \in L^1(\Omega)$ belongs to the class $W^{1,1}(\Omega)$ if and only if it has a representative $\tilde{u}$ (i.e. $\tilde{u} = u$ a.e. in Ω) such that $\tilde{u}$ is absolutely continuous on almost all lines parallel to the coordinate axes and the partial derivatives belong to the class $L^1(\Omega)$, see e.g. [**EG**, §4.9]. The standard representative of a Sobolev function always has this property. We will use this characterization to show that the function $u_i := \min\{w_{s_i}, u_0\}$ lies in the class $W^{1,1}(x_0 + P_{2s_i})$. In fact if for $(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n) \in \mathbf{R}^{n-1}$ the function

$$y \to \bar{u}_0(y) := u_0(y_1, \dots, y_{j-1}, y, y_{j+1}, y, y_n)$$

is absolutely continuous and if

$$\operatorname{meas}\{y : (y_1, \ldots, y_{j-1}, y, y_{j+1}, \ldots, y_n) \in (\Omega \setminus \tilde{\Omega}) \cup (\partial\Omega_{s_i} \cup \Omega'_{s_i})\} = 0,$$

then we can use openness of Ω_s to show that the function

$$\bar{u}_i(\cdot) := u_i(y_1, \ldots, y_{j-1}, \cdot, y_{j+1}, \ldots, y_n)$$

is also absolutely continuous and

$$||\dot{u}_i||_{L^1} \le 2\epsilon \max_{1\le i\le q} |v_i| + ||\dot{\bar{u}}_0||_{L^1}.$$

This proves that $u_i \in W^{1,1}(x_0 + P_{2s_i})$. Then $u_i \in W^{1,1}(B(x_0, \epsilon))$ if $u_i := u_0$ in $B(x_0, \epsilon) \setminus (x_0 + P_{2s_i})$ and if $i \in \mathbf{N}$ is sufficiently large.

The lemma is proved.

QED

Proof of Lemma 3.3

Let ν be a probability measure with finite action on L. Let A be the center of mass of ν. Without loss of generality we can assume that $A = 0$.

We will construct convex combinations of Dirac masses $\nu_j = \sum c_i^j \delta_{v_i^j}$, $j \in \mathbf{N}$, centered at 0 with the properties

$$\langle L; \nu_j\rangle \to \langle L; \nu\rangle,\ \nu_j \stackrel{*}{\rightharpoonup} \nu,\ \max_i \operatorname{dist}(v_i^j, \operatorname{supp}\nu) \to 0,\ \text{as } j \to \infty. \tag{3.5}$$

There exists a point $A' \in \mathbf{R}^n$ and $\epsilon > 0$ such that $|L| < M_1$ in $B(A', 2\epsilon)$ and $\nu(B(A', \epsilon)) = c_0 > 0$. For each integer $j \ge M_1$ consider the set

$$U_j = \{v \in \mathbf{R}^n : L(v) \le j\}.$$

We can decompose U_j into sets U_j^i, $i = 1, \ldots, l(j)$, $U_j^{i'}$, $i' = 1, \ldots, l'(j)$, with diameters minorizing $1/j$ in such a way that the oscillation of L in each element of the decomposition does not exceed $1/j$, and $U_j^{i'} \subset B(A', \epsilon)$, $i' = 1, \ldots, l'(j)$, $U_j^i \subset U_j \setminus B(A', \epsilon)$, $i = 1, \ldots, l(j)$.

Let $c_{j,i} := \nu(U_j^i)$, $c_{j,i'} := \nu(U_j^{i'})$ and $c_j := \nu(\mathbf{R}^n \setminus U_j)$. Note that

$$c_j + \sum_{i'=1}^{l'} c_{j,i'} + \sum_{i=1}^{l} c_{j,i} = 1,\ \sum_{i'=1}^{l'} c_{j,i'} = c_0.$$

Let $A_j^i \in U_j^i$, $i = 1, \dots, l(j)$, $A_j^{i'} \in U_j^{i'}$, $i' = 1, \dots, l'(j)$. Consider the probability measure

$$\mu_j := \sum_{i'} \frac{c_{j,i'}}{1-c_j} \delta_{A_j^{i'}} + \sum_i \frac{c_{j,i}}{1-c_j} \delta_{A_j^i}.$$

Let z_j be the center of mass of μ_j. It easy to check that superlinear growth of L at infinity implies the convergence $z_j \to 0$, $j \to \infty$. Then the measure

$$\nu_j := \sum_{i'} \frac{c_{j,i'}}{1-c_j} \delta_{(A_j^{i'} - z_j \frac{1-c_j}{c_0})} + \sum_i \frac{c_{j,i}}{1-c_j} \delta_{A_j^i}.$$

is centered at 0. Moreover, by construction we have

$$\nu_j \overset{*}{\rightharpoonup} \nu,\ \langle L; \nu_j \rangle \to \langle L; \nu \rangle,\ \max_i \operatorname{dist}(v_i^j, \operatorname{supp} \nu) \to 0,\ \text{as } j \to \infty.$$

This way we establish existence of the measures ν_j with the properties (3.5). The proof is complete. **QED**

4 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We start with an auxiliary result which has certain interest by itself.

Theorem 4.1 *Let L satisfy the requirement (H1) and assume that $L(x, u, v) \geq \theta(v)$, where the function $\theta : \mathbf{R}^n \to \mathbf{R}$ has superlinear growth.*

Assume that the function $u_0 \in W^{1,1}(\Omega)$ is a.e. differentiable in the classical sense (here we consider a fixed representative of the Sobolev class) and assume that $(\nu_x)_{x \in \Omega}$ is a Young measure with the centers of mass at $Du_0(x)$ a.e. and with the finite action on L, i.e. $\int_\Omega \langle L(x, u_0(x), \cdot); \nu_x \rangle dx < \infty$. Then there is a sequence $u_k \in W^{1,1}(\Omega)$ such that Du_k generates the Young measure $(\nu_x)_{x \in \Omega}$ and

$$u_k \rightharpoonup u_0 \text{ in } W^{1,1}(\Omega),\ u_k\Big|_{\partial\Omega} = u_0\Big|_{\partial\Omega},$$
$$L(\cdot, u_k(\cdot), Du_k(\cdot)) \rightharpoonup \langle L(\cdot, u_0(\cdot), v); \nu_{(\cdot)} \rangle \text{ in } L^1. \tag{4.1}$$

In particular $J(u_k) \to \int_\Omega \langle L(x, u_0(x), \cdot); \nu_x \rangle dx$.

Proof of Theorem 4.1

For each $k \in \mathbf{N}$ there exists a compact subset Ω_k of the set $\operatorname{int}\Omega$ such that $\operatorname{meas}(\Omega \setminus \Omega_k) \leq 1/k$ and the functions

$$Du_0 : \Omega_k \to \mathbf{R}^n,$$

$$L : \Omega_k \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{\infty\},$$
$$\nu : \Omega_k \to (M_1(\mathbf{R}^n), \rho),$$
$$x \to \langle L(x, u_0(x), \cdot); \nu_x\rangle,\ x \in \Omega_k,$$

are continuous (the metric ρ was defined in §2). $\tilde{\Omega}$ denotes those subset of the union of the sets of Lebesgue points of Ω_k, $k \in \mathbf{N}$, where the requirement on boundedness of L holds, i.e. for each $x \in \tilde{\Omega}$ the integrand is bounded in a neighborhood of each point (x, u, v) where it takes finite value.

The probability measure obtained by exchanging the center of mass of ν by A will be denoted $\nu \diamond A$, e.g. $\delta_B \diamond A = \delta_A$.

By Lemma 3.3 given $\epsilon > 0$ and $x \in \tilde{\Omega}$ we can find a measure $\tilde{\nu}_x$, which is a finite convex combination of Dirac masses, such that it has the same center of mass as ν_x and

$$\rho(\nu_x, \tilde{\nu}_x) \leq \epsilon/2,\ |\langle L(x, u_0(x), \cdot); \nu_x\rangle - \langle L(x, u_0(x), \cdot); \tilde{\nu}_x\rangle| < \epsilon/2. \qquad (4.2)$$

Let $x_0 \in \tilde{\Omega}$ and assume that $v_1, \ldots, v_q$ are extreme points of a compact convex set with $Du_0(x_0) \in \operatorname{int}\operatorname{co}\{v_1, \ldots, v_q\}$. By Lemma 3.2 for each $\eta \in]0, \epsilon[$ we can find $s_i > 0$ and $\Omega_i \subset \Omega$, $i \in \mathbf{N}$, such that $s_i \to 0$ as $i \to \infty$ and if w_{s_i}, $i \in \mathbf{N}$, are the functions associated with the vectors

$$v_1^i := Du_0(x_0) + \eta(v_1 - Du_0(x_0)), \ldots, v_q^i := Du_0(x_0) + \eta(v_q - Du_0(x_0)),$$

then $u_i \in W^{1,1}(\Omega)$ with $u_i = w_{s_i}$ on Ω_i (consequently $Du_i \in \{v_1^i, \ldots, v_q^i\}$ a.e. in Ω_i) and $u_i = u_0$ otherwise, and

$$B(x_0, \delta s_i) \subset \Omega_i \subset B(x_0, s_i/\delta) \text{ with } \delta = \delta(\eta) > 0,\ i \in \mathbf{N}. \qquad (4.3)$$

Note that in this case

$$\limsup_{i\to\infty}(\operatorname{ess\,sup}_{x\in\Omega_i}\{|w_{s_i}(x) - u_0(x_0)| + |Dw_{s_i}(x) - Du_0(x_0)|\}) \leq \eta.$$

If $\eta > 0$ is sufficiently small then (4.2), (4.3) and the assumptions on L imply the following inequalites for all sufficiently large $i \in \mathbf{N}$

$$\rho(\nu_x, \tilde{\nu}_{x_0} \diamond Dw_{s_i}(x)) < \epsilon \text{ a.e. in } \Omega_i \text{ (cf. Lemma 2.4)}, \tag{4.4}$$

$$\int_{\Omega_i} |\langle L(x, w_{s_i}(x), \cdot); \tilde{\nu}_{x_0} \diamond Dw_{s_i}(x)\rangle - \langle L(x_0, u_0(x_0), \cdot); \tilde{\nu}_{x_0}\rangle| dx < \epsilon/2 \operatorname{meas} \Omega_i.$$

Note that the last inequality, (4.2), and the assumptions on $\tilde{\Omega}$ imply

$$\int_{\Omega_i} |\langle L(x, w_{s_i}(x), \cdot); \tilde{\nu}_{x_0} \diamond Dw_{s_i}(x)\rangle - \langle L(x, u_0(x), \cdot); \nu_{x_0}\rangle| dx \leq \epsilon \operatorname{meas} \Omega_i \tag{4.5}$$

if $i \in \mathbf{N}$ is sufficiently large.

Since $\tilde{\Omega}$ contains almost all points of Ω and u_0 is a.e. classically differentiable we can apply the Vitaly covering theorem and Lemma 3.2 to decompose Ω into at most countable collection of the sets $\bar{\Omega}_k$ with meas $(\partial\Omega_k) = 0$ and a set of null measure. Let x_k be the points x_0 associated with the sets Ω_k, $k \in \mathbf{N}$, and let w_{s_k} be the functions associated with x_k and Ω_k for which both (4.4) and (4.5) hold. Defining u_ϵ as w_{s_k} in Ω_k, $k \in \mathbf{N}$, we obtain a Sobolev function which is piece-wise affine in each Ω_k and coincides with u_0 on $\partial\Omega_k$.

Define a Young measure $(\nu_x^\epsilon)_{x\in\Omega}$ as $\tilde{\nu}_{x_k} \diamond Du_\epsilon(x)$ for $x \in \Omega_k$, $k \in \mathbf{N}$. By Lemma 3.1 we can find a sequence of functions $u_j^\epsilon \in u_\epsilon + W_0^{1,\infty}(\Omega)$, $j \in \mathbf{N}$, which are piece-wise affine in each set Ω_k and which satisfy $u_j^\epsilon = u_\epsilon$ in $\partial\Omega_k$,

$$L(\cdot, u_j^\epsilon(\cdot), Du_j^\epsilon(\cdot)) \rightharpoonup \langle L(\cdot, u_\epsilon(\cdot), v); \nu_{(\cdot)}^\epsilon\rangle \text{ in } L^1(\Omega_k), \tag{4.6}$$

$$(\delta_{Du_j^\epsilon(x)})_{x\in\Omega_k} \rightharpoonup^* (\nu_x^\epsilon)_{x\in\Omega_k}, \ j \to \infty, \ k \in \mathbf{N}, \tag{4.7}$$

where the latter means that the sequence Du_j^2 generates the measure $(\nu_x^\epsilon)_{x\in\Omega_k}$ in Ω_k.

By (4.5), (4.6) and since $\epsilon > 0$ can be taken arbitrary small we can find a sequence $u_j^{\epsilon_j} \in W^{1,1}(\Omega)$ with $\epsilon_j \to 0$ as $j \to \infty$ having the properties $u_j^{\epsilon_j} \rightharpoonup u_0$ in $W^{1,1}(\Omega)$, $u_j^{\epsilon_j}\Big|_{\partial\Omega} = u_0$, and

$$\int_\Omega \{L(x, u_j^{\epsilon_j}(x), Du_j^{\epsilon_j}(x)) - \langle L(x, u_0(x), \cdot); \nu_x\rangle\} dx \to 0.$$

Applying Lemma 2.4 together with (4.4), (4.7) we can select the sequence $u_j^{\epsilon_j}$ with one more property

$$(\delta_{Du_j^{\epsilon_j}(x)})_{x\in\Omega} \rightharpoonup^* (\nu_x)_{x\in\Omega}, \ j \to \infty.$$

This proves Theorem 4.1. **QED**

To prove Theorem 1.2 we will need one more auxiliary lemma which is an extension of a lemma in [**ET**,Ch.9] to the case of integrands with possibly infinite values.

Lemma 4.2 *Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and let $L : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ be a Carathéodory integrand which satisfy both (H1) and the inequality $L \geq \theta$ with a continuous convex function $\theta : \mathbf{R}^n \to \mathbf{R}$ having superlinear growth.*

Then the integrand L^c, which is obtained by convexification of L with respect to v, i.e.

$$L(x, u, v) = \inf\{\sum_{i=1}^{q} c_i L(x, u, v_i) : q \in \mathbf{N}, c_i \geq 0, \sum_{i=1}^{q} c_i = 1, \sum_{i=1}^{q} c_i v_i = v\},$$

is a Caratheodory integrand which satisfies (H1) and the inequality $L^c \geq \theta$.

Moreover for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$, $v \in \mathbf{R}^n$ there exist $c_i \geq 0$, $v_i \neq v$, $i \in \{1, \ldots, n+2\}$, such that $\sum_i c_i = 1$, $\sum_i c_i v_i = v$, and $L(x, u, v) = \sum_i c_i L(x, u, v_i)$.

Proof

The facts that $L^c \geq \theta$ and L^c satisfies (H1) are obvious. To prove the lemma it is enough to show that if Ω' is a compact subset of Ω such that the restriction of L to $\Omega' \times \mathbf{R} \times \mathbf{R}^n$ is continuous, then the restriction of L^c to the same set is continuous.

Note that if $F : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ is a continuous function with superlinear growth then by the Carathéodory theorem, cf. [**ET**], we have

$$F^c(v_0) = \inf\{\sum_{i=1}^{n+2} c_i F(v_i) : c_i \geq 0, \sum_{i=1}^{n+2} c_i = 1, \sum_{i=1}^{n+2} c_i v_i = v_0\}.$$

Moreover there exist c_i^0, v_i^0, $i \in \{1, \ldots, n+2\}$ at which the infimum on the right-hand side is attained. To show this we take a minimizing sequence c_i^k, v_i^k, $i \in \{1, \ldots, n+2\}$. We can find a subsequence (not relabeled) such that for each $i \in \{1, \ldots, n+2\}$ we have $v_i^k \to v_i^0$, $c_i^k \to c_i^0$ or $v_i^k \to \infty$, $c_i^k \to c_i^0 = 0$.

Note that if $v_i^k \to \infty$ or $F(v_i^0) = \infty$ then $c_i^0 = 0$. Assuming that $c_i^0 F(v_i^0) = 0$ in those two cases we infer that

$$\sum_{i=1}^{n+2} c_i^0 F(v_i^0) \leq \liminf_{n\to\infty} \sum_{i=1}^{n+2} c_i^k F(v_i^k).$$

These arguments also show that $L^c : \Omega' \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ is a lower semicontinuous function. Moreover, because of continuity of the restriction of L, we infer upper semicontinuity of L^c at points where its value is finite. In case $L^c(x,u,v) = \infty$ and $(x_i, u_i, v_i) \to (x,u,v)$ we have $L^c(x_i,u_i,v_i) \to \infty$, $i \to \infty$, since otherwise we can use the above arguments to show that $L^c(x,u,v) < \infty$, which is a contradiction. Therefore $L^c : \Omega' \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ is both lower and upper semicontinuous. Then it is continuous.

This way we establish that L^c is a Carathéodory integrand. **QED**

Proof of Theorem 1.1

Note first that by Lemma 4.2 the integrand L^c is Carathéodory and satisfies both the requirements (H1) and (H2). It is clear also that the subset of Ω where $L(x, u_0(x), Du_0(x)) = L^c(x, u_0(x), Du_0(x))$ is measurable. Let $\tilde{\Omega}$ be its complement.

We will show that there exist measurable functions $c_i : \tilde{\Omega} \to [0,1]$, $v_i : \tilde{\Omega} \to \mathbf{R}^n$, $i \in \{1,\dots,n+2\}$, such that

$$v_i(\cdot) \neq Du_0(\cdot),\ i \in \{1,\dots,n+2\},\ \sum_{i=1}^{n+2} c_i(\cdot) = 1,\ \sum_{i=1}^{n+2} c_i(\cdot) v_i(\cdot) = Du_0(\cdot),$$

$$\sum_{i=1}^{n+2} c_i(\cdot) L(\cdot, u_0(\cdot), v_i(\cdot)) = L^c(\cdot, u_0(\cdot), Du_0(\cdot)) \text{ a.e. in } \tilde{\Omega}. \qquad (4.8)$$

Then the family of probability measures $(\nu_x)_{x\in\Omega}$, where $\nu(x) = \sum_{i=1}^{n+2} c_i(x) \delta_{v_i(x)}$ in $\tilde{\Omega}$ and $\nu(x) = \delta_{Du_0(x)}$ in $\Omega \setminus \tilde{\Omega}$, has measurable actions on elements of $C_0(\mathbf{R}^n)$. By Theorem 2.3 $(\nu_x)_{x\in\Omega}$ is a Young measure.

Applying Theorem 4.1 to this case we can assert existence of a sequence $u_k \in W^{1,1}(\Omega)$ with the properties

$$u_k \rightharpoonup u_0 \text{ in } W^{1,1}(\Omega),\ u_k\Big|_{\partial\Omega} = u_0\Big|_{\partial\Omega},\ L(\cdot, u_k(\cdot), Du_k(\cdot)) \rightharpoonup \langle L(\cdot, u_0(\cdot), v); \nu_{(\cdot)}\rangle \text{ in } L^1,$$

where in view of (4.8) the identity $\langle L(x, u_0(x), \cdot); \nu_x\rangle = L^c(x, u_0(x), Du_0(x))$ holds for a.e. $x \in \Omega$.

This proves the theorem.

To establish (4.8) we will construct a sequence of compact subsets Ω_j of $\tilde{\Omega}$ such that $\Omega_{j_1} \cap \Omega_{j_2} = \emptyset$ if $j_1 \neq j_2$, meas $(\Omega \setminus \cup_{j=1}^{\infty} \Omega_j) = 0$, and for each $j \in \mathbf{N}$ the functions

$$L : \Omega_j \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{\infty\},$$

$$u_0 : \Omega_j \to \mathbf{R},\ Du_0 : \Omega_j \to \mathbf{R}^n$$

are continuous.

It is enough to establish (4.8) in case $\tilde{\Omega} = \Omega_j$. We will construct by induction a sequence $\tilde{\Omega}_k$ of pairwisely disjoint compact subsets of $\tilde{\Omega}$ such that (4.8) holds in each set $\tilde{\Omega}_k$ and meas $\{\tilde{\Omega} \setminus \cup_k \tilde{\Omega}_k\} = 0$.

Assume that the sets $\tilde{\Omega}_j$, $j \in \{1, \ldots, k-1\}$, and the functions $v_i : \cup_{j=1}^{k-1} \tilde{\Omega}_j \to \mathbf{R}^n$, $c_i : \cup_{j=1}^{k-1} \tilde{\Omega}_j \to [0,1]$ are already defined. To define $\tilde{\Omega}_k$ consider the set Ω'_k consisting of those $x \in \tilde{\Omega}$ where we can find $c_i \geq 0$ and $v_i \in \mathbf{R}^n$, $i \in \{1, \ldots, n+2\}$, such that

$$1/k \leq |L(x, u_0(x), v_i)| \leq k, 1/k \leq |v_i| \leq k, \sum_{i=1}^{n+2} c_i = 1, \sum_{i=1}^{n+2} c_i v_i = Du_0(x),$$

$$\sum_{i=1}^{n+2} c_i L(x, u_0(x), v_i) = L^c(x, u_0(x), Du_0(x)). \tag{4.9}$$

This way we define a multivalued mapping

$$x \in \Omega'_k \to \{c_1, \ldots, c_{n+2}, v_1, \ldots, v_{n+2} : (4.9) \text{ holds}\}.$$

The set Ω'_k is closed (in $\tilde{\Omega}$) and the multivalued mapping is upper semicontinuous in Ω'_k since validity of (4.9) for $x_m \in \Omega'_k$ with c_i^m, v_i^m, $i = 1, \ldots, n+2$, and the convergences $c_i^m \to c_i$, $v_i^m \to v_i$, $x_m \to x$ imply validity of (4.9) for x with c_i, v_i, $i = 1, \ldots, n+2$. Hence $V : \Omega'_k \to 2^{\mathbf{R}^{(n+1)(n+2)}}$ is a measurable multifunction, which is defined in a closed subset Ω'_k of $\tilde{\Omega}$. By Theorem 2.6 we can find a measurable selection $x \to (c_1, \ldots, c_{n+2}, v_1, \ldots, v_{n+2})$ of this function. Then we have

$$\sum_{i=1}^{n+2} c_i(x) L(x, u_0(x), v_i(x)) = L^c(x, u_0(x), Du_0(x))$$

everywhere in Ω'_k. We define $\tilde{\Omega}_k$ as a compact subset of the set $\Omega'_k \setminus \cup_{i=1}^{k-1}\tilde{\Omega}_i$ such that meas $\tilde{\Omega}_k \geq$ meas $(\Omega'_k \setminus \cup_{i=1}^{k-1}\tilde{\Omega}_i) - 1/k$.

We have meas $(\tilde{\Omega} \setminus \cup_{k=1}^{\infty}\tilde{\Omega}_k) = 0$ since, by Lemma 4.2, for each $x \in \tilde{\Omega}$ there exists $c_i \geq 0$, $v_i \neq Du_0(x)$, $i \in \{1, \dots, n+2\}$, with the properties

$$\sum_{i=1}^{n+2} c_i = 1,\ \sum_{i=1}^{n+2} c_i v_i = Du_0(x),\ \sum_{i=1}^{n+2} c_i L(x, u_0(x), v_i) = L^c(x, u_0(x), Du_0(x)).$$

This completes proof of the first part of the theorem.

To prove the second part we note first that $J(u_0) = J^c(u_0)$ if and only if

$$L(x, u_0(x), Du_0(x)) = L^c(x, u_0(x), Du_0(x)) \text{ a.e.,}$$

where the latter holds if and only if for a.e. $x \in \Omega$ the function $L(x, u_0(x), \cdot)$ is convex at the point $Du_0(x)$, cf. [**S6, §3**]

Assume now that for a.e. $x \in \Omega$ the function $L(x, u_0(x), \cdot)$ is strictly convex at $Du_0(x)$ and $u_k \rightharpoonup u_0$ in $W^{1,p}(\Omega)$, $J(u_k) \to J(u_0)$. We show that $Du_k \to Du_0$ in L^1 by contradiction. If the latter does not hold then we can assume that Du_k generates a nontrivial Young measure $(\nu_x)_{x\in\Omega}$, i.e. ν_x is not a Dirac mass for a set of x of positive measure, see Theorem 2.2. By Theorem 2.7

$$J(u_0) = \lim_{k\to\infty} J(u_k) \geq \int \langle L(x, u_0(x), \cdot); \nu_x\rangle dx, \tag{4.10}$$

where for a.e. $x \in \Omega$ the center of mass of ν_x is $Du_0(x)$. By strict convexity at $Du_0(x)$ we have

$$\langle L(x, u_0(x), \cdot); \nu_x\rangle \geq L(x, u_0(x), Du_0(x)) \text{ a.e.,}$$

moreover the inequality is strict at the points where the measure ν_x is nontrivial, i.e. ν_x is not a Dirac mass, see [**S6, §3**]. This observation and (4.10) imply the inequality $\liminf_{k\to\infty} J(u_k) > J(u_0)$, which is a contradiction. The contradiction shows that $\nu_{(\cdot)} = \delta_{Du_0(\cdot)}$ a.e. in Ω. Then Theorem 2.2 implies the convergence $Du_k \to Du_0$ in L^1. Since $J(u_k) \to J(u_0)$ we also have by Theorem 2.7 that

$$L(\cdot, u_k(\cdot), Du_k(\cdot)) \to L(\cdot, u_0(\cdot), Du_0(\cdot)) \text{ in } L^1.$$

We prove the last assertion of the theorem again by contradiction, i.e. we assume that it is no longer true that $L(x, u_0(x), \cdot)$ is strictly convex at $Du_0(x)$ for a.a. $x \in \Omega$. Then we can apply the above arguments to construct a nontrivial Young measure in the form of convex combinations of at most $n+2$ Dirac masses, i.e. $\nu_{(\cdot)} = \sum_{i=1}^{n+2} c_i \delta_{v_i(\cdot)}$ a.e., with the centers of mass at $Du_0(\cdot)$ and with the property $\sum c_i L(\cdot, u_0(\cdot), v_i) = L(\cdot, u_0(\cdot), Du_0(\cdot))$ a.e. in Ω (see (4.9)).

By Theorem 4.1 there is a sequence $u_k \rightharpoonup u_0$ in $W^{1,p}(\Omega)$ generating the Young measure $(\nu_x)_{x\in\Omega}$ such that $u_k\Big|_{\partial\Omega} = u_0\Big|_{\partial\Omega}$ and

$$J(u_k) \to \int_\Omega \langle L(x, u_0(x), \cdot); \nu_x \rangle dx = J(u_0).$$

By Theorem 2.2 the sequence Du_k does not converge in measure.

This proves the last assertion of the theorem. **QED**

5 Proof of Theorem 1.4

To prove Theorem 1.4 we will need two auxiliary lemmata.

Recall that a function $u \in W^{1,p}(\Omega)$ is called monotone if for a.a. $x \in \Omega$ and each $\epsilon > 0$ such that $Q(x, \epsilon) \subset \Omega$ (recall that $Q(x, \epsilon)$ is the open cube with the side length ϵ and the center at x) we have that for a.a. $\epsilon' \in]0, \epsilon[$

$$\operatorname{ess}\inf_{\partial Q(x,\epsilon')} u \le \operatorname{ess}\inf_{Q(x_0,\epsilon')} u, \ \operatorname{ess}\sup_{Q(x,\epsilon')} u \le \operatorname{ess}\sup_{\partial Q(x,\epsilon')} u. \tag{5.1}$$

Note that this definition has sense for Sobolev functions since for a.a. $\epsilon' \in]0, \epsilon[$ the trace $u\Big|_{\partial Q(x,\epsilon')}$ is defined.

Lemma 5.1 *Assume that the minimization problem*

$$J(u) = \int_\Omega L(Du(x))dx \to \min, \ u \in W^{1,1}(\Omega), \ u\Big|_{\partial\Omega} = f,$$

where $L : \mathbf{R}^n \to \mathbf{R}$ is a convex integrand such that $L \ge \alpha|\cdot|^p + \gamma$ with $\alpha > 0$ and $p > n-1$ in case $n \ge 3$ and L has superlinear growth at infinity in case $n = 2$, has at least one admissible function. Then there is a monotone solution u_0.

Another important observation is

Lemma 5.2 *Assume that $u \in W^{1,p}(\Omega)$ with $p > n-1$ if $n \geq 3$ and with $p = 1$ if $n = 2$. If u is a monotone function then it is a.e. differentiable in the classical sense.*

Proof
See (**[Ri, Ch.VI, §4]**). There the result is stated under additional assumption of continuity of the function u. However the same arguments can be applied in the general case.

Proof of Lemma 5.1
Superlinear growth of L at infinity allows us to find a minimizing sequence u_k which converges weakly in $W^{1,1}(\Omega)$ to u_0. Convexity of L implies lower semicontinuity of the integral functional with respect to this convergence. Therefore

$$J(u_0) \leq \liminf_{k\to\infty} J(u_k),$$

i.e. u_0 is a solution of the minimization problem.

We have $L \geq \theta$, where θ is a strictly convex function with superlinear growth at infinity. By the above arguments for each $\mu > 0$ the problem

$$J_\mu(u) := \int_\Omega \{L(Du) + \mu\theta(Du)\}dx \to \min,\ u\Big|_{\partial\Omega} = f$$

has a solution u_μ.

First we prove that each function u_μ, $\mu > 0$, is monotone. We take $x_0 \in \Omega$ and $Q(x_0, \epsilon) \subset \Omega$. For a.a. $\delta \in]0, \epsilon[$ the trace $u_\mu\Big|_{\partial Q(x_0,\delta)}$ is well defined and is a continuous function. Fix such a δ. Let

$$M_\delta := \operatorname{ess} \sup_{\partial Q(x_0,\epsilon')} u_\mu,\ m_\delta := \operatorname{ess} \inf_{\partial Q(x_0,\epsilon')} u_\mu.$$

We define $u_\delta(x) = u_\mu(x)$ if $x \in Q(x_0, \delta)$ and $u_\mu(x) \in [m_\delta; M_\delta]$, $u_\delta(x) = M_\delta$ if $x \in Q(x_0, \delta)$ and $u_\mu > M_\delta$, $u_\delta(x) = m_\delta$ if $x \in Q(x_0, \delta)$ and $u_\mu(x) < m_\delta$. Then $u_\delta \in W^{1,p}(Q(x_0, \delta))$ and

$$u_\delta\Big|_{\partial Q(x_0,\delta)} = u_\mu\Big|_{\partial Q(x_0,\delta)}.$$

The later property also implies that $u_\delta \in W^{1,p}(\Omega)$ if we assume $u_\delta = u_\mu$ in $\Omega \setminus Q(x_0, \delta)$.

Let $L_\mu := L + \mu\theta$ and let $g \in \partial L_\mu(0)$. Then we have

$$\int_\Omega \langle g, Du_\delta(x) - Du_\mu(x)\rangle dx = 0$$

since both the functions u_δ and u_μ coincide at the boundary of Ω. Then

$$0 \geq J(u_\mu) - J(u_\delta) = \int_\Omega \{L_\mu(Du_\mu) - L_\mu(Du_\delta) - \langle g, Du_\mu - Du_\delta\rangle\} dx. \quad (5.2)$$

Define

$$\Omega_\delta := \{x \in Q(x_0, \delta) : u_\delta \neq u_\mu\}.$$

The expression under the integral in the right-hand side of (5.2) vanishes in the set $\Omega \setminus \Omega_\delta$. Then the integral over Ω in the right-hand side of (5.2) is equal to the integral over Ω_δ. If meas $\Omega_\delta > 0$ then $J(u_\mu) > J(u_\delta)$ since $Du_\delta = 0$ in Ω_δ, $Du_\mu \neq 0$ in a subset of Ω_δ with positive measure, and $L_\mu(v) - L_\mu(0) - \langle g, v\rangle > 0$ for $v \neq 0$. This contradiction with (5.2) proves that u_μ is a monotone function.

We also have

$$J_\mu(u_\mu) \to \inf\{J(u) : u \in W^{1,1}(\Omega), u\Big|_{\partial\Omega} = f\}, \ \mu \to 0. \quad (5.3)$$

We can find a subsequence $\mu_k \to 0$, $k \to \infty$, such that $u_{\mu_k} \rightharpoonup \bar{u}$ in $W^{1,p}(\Omega)$. Then

$$J(\bar{u}) \leq \liminf_{k\to\infty} J_{\mu_k}(u_{\mu_k})$$

and, in view of (5.3), $\bar{u}$ is a solution of the original problem.

Since each function u_{μ_k}, $k \in \mathbf{N}$, is monotone we can infer that $\bar{u}$ is also a monotone function. We prove this only in case $n \geq 3$, however the remaining case can be treated analogously.

Let x_0 be fixed and let $Q(x_0, \epsilon) \subset \Omega$. Then for a.a. $\epsilon' \in]0, \epsilon[$ we have

$$\liminf_{k\to\infty} ||Du_k||_{L^p(\partial Q(x_0,\epsilon'))} < \infty.$$

Then for each such ϵ' and an appropriate subsequence u_k (not relabeled) we have

$$||u_k - u_0||_{L^\infty(\partial Q(x_0,\epsilon'))} \to 0.$$

Since (5.1) holds for each u_k and $u_k \to u_0$ in L^p we infer that (5.1) holds also for the limit function u_0. **QED**

Proof of Theorem 1.4

By Lemma 4.2 the integrand L^c is a convex continuous function with values in $\mathbf{R} \cup \{\infty\}$. Moreover $L^c \geq \alpha|\cdot|^p + \gamma$ with $\alpha > 0$ and $p > n-1$ if $n \geq 3$ and $L^c \geq \theta$, where $\theta : \mathbf{R}^n \to \mathbf{R}$ is a convex function with superlinear growth, if $n = 2$.

By Lemma 5.1 we can find a monotone solution u_0 of the relaxed problem. By Lemma 5.2 this solution is a.e. differentiable in the classical sense. Applying Lemma 4.2 and Theorem 4.1 we can find a sequence $u_k \rightharpoonup u_0$ in $W^{1,1}(\Omega)$ such that $u_k\big|_{\partial\Omega} = f$ and $J(u_k) \to J^c(u_0)$.

The proof of the remaining part of the first assertion of the theorem follows lines of the proof of the analogous assertions of Theorem 1.1. This proves the first part of the theorem.

To complete the proof we have to show that the condition (C) is both necessary and sufficient for solvability of all boundary value minimization problems with nonempty set of admissible functions. We first show that this condition characterizes solvability of problems with linear boundary data v_0, i.e. when $f = l_{v_0}$ and $L^c(v_0) < \infty$.

Without loss of generality we can assume that $L^c(v_0) = 0$ and $L^c \geq 0$ everywhere. In fact we can replace the integrand L by the integrand $L(\cdot) - L(v_0) - \langle g, \cdot - v_0\rangle$ with $g \in \partial L^c(v_0)$ since the functional

$$u \to \int_\Omega \langle g, Du(x)\rangle dx$$

assumes the same value at all functions $u \in W^{1,1}(\Omega)$ with $u\big|_{\partial\Omega} = f$.

If $L^c(v_0) = L(v_0)$ then the function l_{v_0} is a solution of (1.1). Assume that $L(v_0) > L^c(v_0)$. Define $V := \{v \in \mathbf{R}^n : L(v) = 0\}$. We have $V \neq \emptyset$ since $L \geq 0$ and, by Lemma 4.2, we can find $c_i \geq 0$, $v_i \in \mathbf{R}^n$, $i \in \{1, \ldots, n+2\}$, such that

$$\sum_{i=1}^{n+2} c_i = 1, \quad \sum_{i=1}^{n+2} c_i L(v_i) = L^c(v_0) = 0.$$

If $v_0 \in \operatorname{int} \operatorname{co} V$ then by Lemma 3.1 we can find a function $u \in W^{1,1}(\Omega)$ with $u\big|_{\partial\Omega} = l_{v_0}$ and $Du \in \operatorname{extr} V$ a.e. Then u is a solution. The converse also holds, i.e. solvability of the problem implies $v_0 \in \operatorname{int}\operatorname{co}V$. To show this we can again apply Lemma 3.1 to find a sequence $u_k \in W^{1,1}(\Omega)$ such that $u_k\Big|_{\partial\Omega} = l_{v_0}$, $J(u_k) \to 0$. Therefore if a solution u_0 exists then $Du_0 \in V$ a.e. The requirements $Du_0 \in V$ a.e., $u_0\Big|_{\partial\Omega} = l_{v_0}$ imply that $v_0 \in \operatorname{int}\operatorname{co}V$. Otherwise we can apply the Hanh-Banach theorem to find a vector $y \in \mathbf{R}^n$ $(y \neq 0)$ such that

$$\frac{\partial u_0}{\partial y} = \langle Du_0, y\rangle \geq \langle v_0, y\rangle \text{ a.e. in } \Omega.$$

Then, since $u_0 = l_{v_0}$ on $\partial\Omega$, we obtain that $u_0 = l_{v_0}$ a.e. in Ω. Then $Du_0 = v_0$ a.e. in Ω, which is a contradiction with the assumption $v_0 \notin V$.

This way we establish that the condition (C) is both necessary and sufficient to resolve all minimization problems with affine boundary data. It remains to show that (C) implies solvability of all nonlinear boundary data problems.

Assume f is such that the problem

$$J^c(u) \to \min,\ u \in W^{1,1}(\Omega),\ u\Big|_{\partial\Omega} = f$$

has an admissible function. Then it also has a monotone solution u_0, cf. Lemma 5.1. By Lemma 5.2 u_0 is a.e. classically differentiable. Then we can apply Lemma 3.2 at each point x_0, where $Du_0(x_0)$ exists in the classical sense, to find a sequence of open sets Ω_{i,x_0}, which meets the requirements of the Vitaly covering theorem at the point x_0 (see §3), and the perturbations w_{i,x_0} of u_0 in Ω_{i,x_0}, $i \in \mathbf{N}$, such that

$$Dw_{i,x_0} \in \{L = L^c\} \text{ a.e. in } \Omega_{i,x_0}, \int_{\Omega_{i,x_0}} L^c(Du_0(x))dx = \int_{\Omega_{i,x_0}} L^c(Dw_{i,x_0}(x))dx. \tag{5.4}$$

Applying the Vitaly covering theorem we can decompose Ω into the sets $\Omega_{i,x(i)}$, $i \in \mathbf{N}$, and a set of null measure in such a way that for each set $\Omega_{i,x(i)}$, $i \in \mathbf{N}$, there is a perturbation $w_{i,x(i)}$ such that (5.4) holds. Therefore if $\bar{u} = w_{i,x(i)}$ in $\Omega_{i,x(i)}$, $i \in \mathbf{N}$, then $\bar{u} \in W^{1,1}(\Omega)$, $\bar{u}\Big|_{\partial\Omega} = u_0\Big|_{\partial\Omega}$, $J(\bar{u}) = J(u_0)$, and $D\bar{u} \in \{L = L^c\}$ a.e. Then $\bar{u}$ is a solution of the original problem. **QED**

Acknowledgements I thank Jan Kristensen for helpful remarks.

References

[AF] *Acerbi E., Fusco N*. Semicontinuity problems in the calculus of variations. Arch. Rat. Mech. Anal. 86 (1984), pp. 125-145.

[Ba] *Balder E.J*. A general approach to lower semicontinuity and lower closure in optimal control theory. SIAM J.Control and Optimization, vol.22 (1984), pp. 570-598.

[B1] *Ball J.M.* Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rat. Mech. Anal. 63 (1978), pp. 337-403.

[B2] *Ball J.M.* Review of "Nonlinear problems of elasticity" by Stuart Antman. Bull. AMS 2 (1996), pp.269-276.

[B3] *Ball J.M.* The calculus of variations and materials science. Q. Appl. Math. 56 (1998), pp. 719-740.

[B4] *Ball J.M.* Discontinuous equilibrium solutions and cavitation in nonlinear Elasticity. Phil. Trans. Roy. Soc. London. 306A (1982), pp.557-612.

[B5] *Ball J.M.* A version of the fundamental theorem for Young measures, in: PDE's and Continuum Models of Phase Transitions, M.Rascle, D.Serre, M.Slemrod, eds., Lecture Notes in Physics 344. Springer-Verlag (1989), pp. 207-215.

[BL] *Berliocchi H., Lasry J.M.* Intégrandes normales et mesures paramétrées en calcul des variations. Bull. Soc. Math. France 101 (1973), pp. 129-184.

[BM] *Ball J.M., Murat F*. $W^{1,p}$-quasiconvexity and variational problems for multiple integrals. J. Funct. Anal. 58 (1984), 225-253.

[BP] *Bauman P., Phillips D.* A nonconvex variational problem related to change of phase. Appl.Math.Optim. 21(1990), pp.113-138.

[Bu] *Buttazzo G.* Lower semicontinuity, relaxation and integral representation in the calculus of variations. Pitman Research Notes in Math. Series 207, 1989.

[BFM] *Bouchitte G., Fonseca I., Maly J.* The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent. Proc. Royal Soc. Edinb. Sect A 128 (1998), pp. 463-479.

[C] *Ciarlet P.* Mathematical Elasticity, Vol.I: Three-Dimensional Elasticity. North-Holland, 1988.

[Ce] *Cellina A.* On minima of functionals of gradient: necessary and sufficient conditions. Nonlinear Analysis TMA, 20 (1993), pp.337-347.

[CV] *Castaing C., Valadier M.* Convex analysis and measurable multifunctions. Lecture Notes in Mathematics, vol. 580. Springer-Verlag, Berlin-New York, 1977.

[D1] *Dacorogna B.* Weak continuity and weak lower semicontinuity of non-linear problems. Lecture Notes in Math. 922, Springer-Verlag (1982).

[D2] *Dacorogna B.* Direct methods in the Calculus of Variations. Springer-Verlag (1989).

[EG] *Evans L.C., Gariepy R.F.*. Measure theory and fine properties of functions. 1992, CRC Press.

[ET] *Ekeland I., Temam R.* Convex analysis and variational problems. North-Holland, 1976.

[F] *Friesecke G*. A necessary and sufficient condition for nonattainment and formation of microstructure almost everywhere in scalar variational problems. Proc.Roy.Soc.Edinburgh. Sect.A 124 (1994), pp.437-471.

[FM] *Fonseca I., Maly J.* Relaxation of multiple integrals in Sobolev spaces below the growth exponent for the energy density. Ann. Inst. Henri Poincaré. 14 (1997), no. 3, pp. 309-338.

[Gi] *Giusti E.* Minimal Surfaces and Functions of Bounded Variation. Birkhäuser, Boston, 1984.

[GMS] *Giaquinta M., Modica G., Souček J.* A weak approach to finite elasticity. Calc. Var. Partial Differential Equations 2 (1994), no. 1, pp. 65-100.

[GT] *Gurtin M.E., Temam R.* On the antiplane shear problem in elasticity. J. of Elasticity 11 (1981),pp.197-206.

[JS] *James R.D., Spector S.J.* Remarks on $W^{1,p}$-quasiconvexity, interpenetration of matter and function spaces for elasticity. Ann. Inst. H. Poincaré Anal. Non.Linéaire, 9 (1992), pp. 263-280.

[KR-N] *Kuratowski K., Ryll-Nardzewski*. A general theorem of selectors. Bull.Acad.Polon.Sci., vol. XIII (1966), N 6, pp. 397-403.

[Kr] *Kristensen J*. Finite functionals and Young measures generated by gradients of Sobolev functions. Ph.D.Thesis, Technical University of Denmark, Kyngby, August 1994.

[Ma] *Marcellini P.* Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals. Manuscripta Math. 51 (1985), pp. 1-28.

[Me] *Meyers N.G.* Quasiconvexity and lower semicontinuity of multiple variational integrals of any order. Trans. Amer. Math. Soc. 119 (1965), pp.125-149.

[Mo] *Morrey C.* Multiple integrals in the Calculus of Variations. Springer-Verlag, 1966.

[MQY] *Müller S., Qi T., Yan B.S.* On a new class of elastic deformations not allowing for cavitation. Ann. Inst. Henri Poincaré, Analyse Nonlinéaire, 11 (1994), pp. 217-243.

[MS] *Marcellini P., Sbordone C.* Semicontinuity problems in the calculus of variations. Nonlinear Anal., vol.4 (1980), N 2, pp.241-257.

[MSp] *Müller S., Spector S.J.* An existence theory for nonlinear elasticity that allows for cavitation. Arch. Rational Mech. Anal., 131 (1995), pp.1-66.

[MSSp] *Müller S., Sivaloganathan J., Spector S.J.* An isoperimetric estimate and $W^{1,p}$-quasiconvexity in nonlinear elasticity. Calc. Var. Partial Differential Equations 8 (1999), no. 2, pp.159-176.

[R] *Raymond J.-P.* An anti-plane shear problem. J. Elasticity 33 (1993), pp.213-231.

[Ri] *Rickman S.* Quasiregular mappings. Berlin, Springer-Verlag 1993.

[S1] *Sychev M.* Necessary and sufficient conditions in theorems of lower semicontinuity and convergence with a functional. Russ. Acad. Sci. Sb. Math. 186 (1995), pp. 847-878.

[S2] *Sychev M.* A criterion for continuity of an integral functional on a sequence of functions. Siberian Math.J. 36 (1995), N 1, pp. 146-156.

[S3] *Sychev M.* Characterization of scalar homogeneous variational problems solvable for all boundary data. Proc. Edinburgh Royal Society, Sect. A (to appear).

[S4] *Sychev M.* A new approach to Young measure theory, relaxation and convergence in energy. Ann. Inst. H. Poincaré Anal. Non.Linéaire 16 (1999), N 6, pp. 773-812.

[S5] *Sychev M.* Young measures as measurable functions. (in preparation)

[S6] *Sychev M.* Young measure approach to characterization of behavior of integral functionals on weakly convergent sequences by means of their integrands. Ann. Inst. H. Poincaré Anal. Non.Linéaire 15 (1998), N 6, pp. 755-783.

[S7] *Sychev M.* Characterization of homogeneous gradient Young measures in case of arbitrary integrands. Report No. 98-216, May 1998, Carnegie Mellon University.

[Sa] *Saks S.* Theory of the integral. Hafner, New York, 1937.

[SH] *Swart P.J., Holmes P.J.* Energy minimization and the formation of microstructure in dynamic anti-plane shear. Arch. Rational Mech.Anal. 121 (1992), pp. 37-85.

[Sv1] *Šverák V.* Rank-one convexity does not imply quasiconvexity. Proc. Royal Soc. Edinburgh, 120A (1992), pp.185-189.

[Sv2] *Šverák V.* Regularity properties of deformations with finite energy. Arch. Rat. Mech. Anal. 100 (1988), pp. 105-127.

[Sv3] *Šverák V.* Oral communication, 1998.

[T1] *Tartar L.* Compensated compactness and applications to partial differential equations, in Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol.IV, Pitman Research Notes in Mathematics 39 (1979), pp. 136-212.

[T2] *Tartar L.* The compensated compactness method applied to systems of conservations laws, in Systems of Nonlinear Partial Differential Equations, J.M.Ball ed., NATO ASI Series, Vol. CIII, Reidel, 1982.

[Y1] *Young L.C.* Lectures on the Calculus of Variations and Optimal Control Theory. Saunders, 1969 (reprinted by Chelsea 1980).

[Y2] *Young L.C.* Generalized curves and the existence of an attained absolute minimum in the Calculus of Variations. Comptes Rendus de la Societe des Sciences et des Lettres de Varsovie, classe III, 30 (1937), pp. 212-234.