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by

Alexander Lange

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Alexander Lange
alange@mis.mpg.de

MPI MIS *
D - 04103 Leipzig, Inselstr. 22-26

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Abstract

Along the calculation of a perturbative S-matrix, created by Epstein and Glaser and further developed by Scharf, an algebra for the involved distributions is extracted. But its members turn out to be more singular than the distributions one has to split (causally) within the original procedure. Moreover, an antipode is introduced and the properties for a Hopf algebra are checked. That "rather unexpected" kind of formulating perturbative QFT, only recently discovered by Kreimer for the BPHZ-approach, is straightforwardly implemented. EGS' causality, implying locality, is substituted by time-reflection symmetry. The latter, being a consequence of EGS' assumption anyway, is motivated, here, starting with unitarity. The achieved Hopf algebra establishes the (combinatorial) connection to BPHZ's procedure, where time-reflections correspond to counterterms.

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1 Introduction

Locality of perturbative QFT was proved by Epstein and Glaser in [7]. It was done by developing a new renormalization procedure, exemplified for ϕ^4-theory, different from the one commonly referred to as BPHZ [3, 8, 16]. The authors called their result to be equivalent to the latter approach, which was mentioned in the abstract already, and has been repeated at the end of section 7 (on p. 258) wrapped in some context. But indeed, apart from the analytical estimations concerning a single renormalization step in different so-called renormalization schemes, a proof which compares the combinatorial aspects (the so-called renormalization procedures), i.e. their iteration with the successive BPHZ-subtraction of counterterms, was not performed in any detail.

The pointed lack of language might have two reasons. Firstly, the two procedures do stand on different feet – expressed Feynman diagrammatically, EG is based on vertices and BPHZ on loops. And secondly, a common language itself has been missing. Starting to comment the latter, any choice for a mathematical formalism seams to be quite restricted now. Since Kreimer's work there is no doubt that a Hopf algebra describes BPHZ-renormalization (-combinatorics) most appropriately. And in the present paper it will be shown that such a statement somehow applies to EG's approach as well. The problems involved with the first discrepancy look better been investigated after having an adequate Hopf algebra (language) at hand and therefore a detailed discussion is addressed to forthcoming work. That means, at the end of the paper the correspondence of the two procedures will be provided on a formal level (only), leaving aside any graph or vertex structure. But anyway, EG's renormalization induction will be written in BPHZ fashion, even if the compensation of divergences by producing counterterms is, at the first look, not at all an obvious exercise of the Hopf algebra's antipode in the EG(S)-approach. Because, it is found out, that it serves as a realization of time-reflections there.

Additionally, the following chosen references would help arguing that EG's claim has not become unquestioned folklore in the physicists' community. The most convincing one belongs to Scharf (contributing the third letter), who is developing EG's method further, e.g. QED in [12]. And another is Kreimer's hope, expressed on p. 24 in [11], that "EG is reconciled with BPHZ" after having (re-)discovered, now, how locality is (mathematical) encoded in the BPHZ-procedure. By the way, he understands locality (referred to with the attribute perturbative here) as the firmed production of suitable counterterms for the Lagrangian to render the theory finite, cf. p. 3 in [9]. But EG did use locality as a synonym for microcausality. Therefore one might interpret the current result as a realization of the latter (algebraic) locality within a BPHZ-renormalization procedure.

The paper is organized as follows. It is started with an introduction to the method of EGS. The (formal) perturbative expansion of the S-matrix, consisting of the n-point functions, is defined. The assumed properties of causality for the S-matrix are reformulated on the level of the n-point functions. In section

3 the (causal) construction of EGS is presented. The treatment of the operator valued distributions, i.e. their reduction to scalar distributions, is explained. Translation invariance is claimed to hold for whole the exposition. Under those suppositions the heart of the EGS-construction, the distribution splitting, is described in section 4. The assumption underlying all renormalization prescriptions, in some way, is formulated. Therefore the degree of singularity is introduced, this is done via powercounting.

In section 5 the EGS-procedure, applied here, gets another foundation. Time-reflection symmetry, motivated by unitarity, is supposed to hold primary. The splitting is recognized to be subordinated to a proper definition for the multiplication of the involved distributions, what is done in section 6 by the direct product. Starting from distributions with singular point-support, representing the propagators as motivated in section 3, all the distributions produced by the EGS-construction are shown to form an algebra. That property still holds for the operators built by those distributions.

Furthermore, in section 7 an antipode is introduced, which makes the structure underlying the operator valued distributions a Hopf algebra. The antipode, here, corresponds to time-reflections whereas the antipode of Kreimer's Hopf algebra represents counterterms. That is emphasized in section 8, where the connection to BPHZ's approach is established identifying the common structure. Based on the new foundation, an additional solution for the EGS-induction is found, which enables one to write down a forest formula in the EGS-approach.

2 Preparation for the EGS-construction

The basic physical object of perturbative QFT is the S(cattering)-matrix, which encodes the interaction of the considered particles. It maps a finite set of incoming to a finite set of outgoing free particle-fields, or, when they are described by a Fock space, an initial Φ_i to a final state Φ_f. In the context of Feynman diagrams it is referred to those as external particles or lines, respectively. Intermediate states, however, will not be defined (neither on a Fock space), but one might think of those when it is talked about internal lines.

To make a brief comment on its physical importance, in scattering experiments the so-call cross-section is determined. The latter is proportional to the scattering-probability ρ and formally given by the square of the modulus of the dual bra-ket (in a physicist's notation),

$$\rho(\Phi_i, \Phi_f) = |\langle \Phi_f \,|\, \mathbb{S} \,|\, \Phi_i \rangle|^2. \tag{1}$$

However, the external particles are modeled by wave-packets (in practical calculations) satisfying the free equation of motion belonging to its sort, e.g. a spin-1/2 particle ψ, living in $(L^2(\mathbb{R}^3))^4$, by the Dirac equation. But therefore the S-matrix $\mathbb{S}$ can be described to be "piecewise" defined for a certain number of those free fields as an operator-valued distribution.

The EGS-method starts with an expansion of the S-matrix, which will shortly reveal itself being used as a infrared regularized power expansion in the coupling constant, e.g. the charge (divided by Planck's constant) e for QED,

$$\mathbb{S}(g) := \mathbb{I} + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots \int dx_n T_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n). \tag{2}$$

The T_n are (the) operators on the free particle-fields, the so-called *n-point functions*, are *totally symmetric* distributions. That means, the order of the arguments $x_k \in \mathbb{R}^{1+d}$ does not matter within a T_n, and one can therefore write

$$T_n(\{x_1, \ldots, x_n\}) := T_n(x_1, \ldots, x_n). \tag{3}$$

Those arguments correspond to *vertices* in Feynman diagrams.

The T_n-distributions are tested by an "adiabatic switching" function $g \in \mathcal{S}(\mathbb{R}^{1+d})$, cutting off the long-range interaction by living in the Schwartzian space of rapidly decreasing functions. The configuration space $\mathbb{R}^{1+d}$ is a $1+d$ dimensional Minkowski space with signature $(+, -, \ldots, -)$, where the first component plays the role of time.

Remark 1 Problems with infrared divergences do only appear when observable quantities are calculated from the T_n and the so-called adiabatic limit $g \to e$ has to be taken, i.e. choosing $g \in \mathcal{S}(\mathbb{R}^{1+d})$ with $g_e(0) = e$ and considering the limit $\epsilon \to 0$, for test functions $g(x) := g_e(\epsilon x)$, which would formally give

$$\rho(\Phi_i, \Phi_f) = \lim_{\varepsilon \to 0} |\langle \Phi_f \,|\, \mathbb{S}(g_\varepsilon) \,|\, \Phi_i \rangle|^2. \tag{4}$$

Convention 2 To have compatibility with [12], the coupling constant will appear inside the n-point functions, i.e. let $e := 1$ in the adiabatic limit, above, and keep the letter together with its meaning.

The bare physical interaction, plugged into T_1, is the starting point, i.e. $n = 1$, of the iterative procedure, explained here.

Example 3 For QED, $T_1 = j^\mu A_\mu = e : \bar{\psi}\gamma^\mu\psi : A_\mu$ is the expression which usually appears in the (formal action-) Lagrangian. The coupling constant e reappears (linearly), which will make the T_n indicating its power exponent n, downstairs.

The method is based on supposing,

Assumption 4 that an inverse of $\mathbb{S}$ exists, and that it can analogously be expressed,

$$\mathbb{S}^{-1}(g) = \mathbb{I} + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots \int dx_n \tilde{T}_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n). \tag{5}$$

The corresponding n-point functions in its expansion, also called *inverse*, are denoted by a tilde.

EGS do not make any primary assumptions for this tilde-operation. That means, they do not connect the T_n with the $\tilde{T}_n$ directly. But giving the inverse a representation, as it is done in section 5, this will turn out to be crucial for the algebraization of the explained EGS-method. If one supposes unitarity to hold, one can get the required connection in the shape of time-reflection symmetry. However, the latter result is reproducible by applying EGS' causality procedure only.

Notation 5 Let $\mathbb{N}_n := \{1, \ldots, n\}$ and let $|\cdot|$:=card abbreviate the cardinality of a set. The different arguments $x_k = (x_k^0, x_k^1, \ldots, x_k^d) \in \mathbb{R}^{1+d}$ of the considered distributions are abbreviated by its index $k \in \mathbb{N}$, the number of the belonging vertex, if only the combinatorial aspects are important, but not the coordinate of the point in space-time, e.g.,

$$T(\mathbb{N}_n) := T_{|\mathbb{N}_n|}(\mathbb{N}_n) :\equiv T_n(x_1, \ldots, x_n), \quad \text{where} \quad \mathbb{N}_n :\equiv (x_1, \ldots, x_n). \tag{6}$$

By the way, the set-braces indicate the total symmetry of the T_n additionally.

Right from this setup some formal conclusions are drawn, including two ways of expressing the n-point functions with tildes in terms of the ones without. The inversion of $\mathbb{I} - \mathbb{S}$ by a formal series, i.e. $\mathbb{S}^{-1} = \mathbb{I} + \sum_{k=1}^{\infty} (\mathbb{I} - \mathbb{S}(g))^k$, and multiplying out the k-powers leads to

$$\tilde{T}(\mathbb{N}_n) = \sum_{k=1}^{n} (-1)^k \sum_{\substack{X_i \neq \emptyset,\, 1 \leq i \leq k \\ X_1 \dot{\cup} \ldots \dot{\cup} X_k \in P_k(\mathbb{N}_n)}} T(X_1) \ldots T(X_k), \tag{7}$$

where $P_k(Z)$ denotes the set of partitions of a set Z into k disjointed subsets.

Another formulation, providing an inductive evaluation of the $\tilde{T}_n$'s, is obtained by multiplying out the identity

$$\begin{aligned}
\mathbb{I} &= \mathbb{S}(g)\mathbb{S}^{-1}(g) \\
&= \mathbb{I} + \sum_{n=1}^{\infty} \sum_{n_1+n_2=n} \Bigg(\frac{1}{n_1!\, n_2!} \int dx_1 \cdots \int dx_{n_1} \int dy_1 \cdots \int dy_{n_2} \\
&\qquad T_{n_1}(x_1, \ldots, x_{n_1}) \tilde{T}_{n_2}(y_1, \ldots, y_{n_2}) g(x_1) \ldots g(x_{n_1}) g(y_1) \ldots g(y_{n_2}) \Bigg),
\end{aligned}$$

which is equivalent to

$$0 = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{X \dot{\cup} Y \in P_2^0(\mathbb{N}_n)} \int dx_1 \cdots \int dx_n T(X) \tilde{T}(Y) g(x_1) \ldots g(x_n). \tag{8}$$

$P_2^0(Z)$ denotes the partitions of Z in two sets again. The zero indicates that empty sets are allowed, this is supposed to hold for other indices than two as well. And, let

$$T_0(\emptyset) := 1 =: \tilde{T}_0(\emptyset). \tag{9}$$

Having reached this point of the exposition, Scharf concludes on p. 162 in [12] two (sets of) formulas. Later on it will become clear, that those constitute the basis on which the EGS-construction rest upon (Hopf-) algebraically.

Theorem 6 *The validity of the equations*

$$\sum_{X \dot{\cup} Y \in P_2^0(\mathbb{N}_n)} T(X) \tilde{T}(Y) \;=\; 0 \;=\; \sum_{X \dot{\cup} Y \in P_2^0(\mathbb{N}_n)} \tilde{T}(X) T(Y), \tag{10}$$

for all $1 \leq n \in \mathbb{N}$, *is equivalent to those of* $\mathbb{S}\mathbb{S}^{-1} = \mathbb{I} = \mathbb{S}^{-1}\mathbb{S}$, *respectively.*

Proof. Starting with $\mathbb{I} = \mathbb{S}^{-1}\mathbb{S}$, one immediately writes down the analog of (8) providing the right hand side of (10), which was left over to show. □

From (10) two possibilities for the announced induction are read off, keeping the equations' side,

$$-T(\mathbb{N}_n) - \sum_{X \dot{\cup} Y \in P_2(\mathbb{N}_n)} T(X)\tilde{T}(Y) = \tilde{T}(\mathbb{N}_n) = -T(\mathbb{N}_n) - \sum_{X \dot{\cup} Y \in P_2(\mathbb{N}_n)} \tilde{T}(X)T(Y). \quad (11)$$

From the one-point function T_1, which represents the interaction (Lagrangian) of the considered theory, EGS calculate all T_n iteratively, cleverly exploiting that

Assumption 7 the S-matrix is microcausal. That means, the latter is both, *causal*, i.e.

$$\operatorname{supp}(g_1) \lhd \operatorname{supp}(g_2) \quad \Longrightarrow \quad \mathbb{S}(g_1 + g_2) = \mathbb{S}(g_2)\mathbb{S}(g_1), \quad (12)$$

and *anticausal*, i.e.

$$\operatorname{supp}(g_1) \rhd \operatorname{supp}(g_2) \quad \Longrightarrow \quad \mathbb{S}(g_1 + g_2) = \mathbb{S}(g_1)\mathbb{S}(g_2), \quad (13)$$

for any $g_1, g_2 \in \mathcal{S}(\mathbb{R}^{1+d})$. The relations, $\lhd$ and $\rhd$, between the supports of the test functions g_1, g_2 denote the *causal* and the *anticausal order*, respectively. They are defined for $X, Y \subseteq \mathbb{R}^{1+d}$ by

$$\begin{array}{lll} X \lhd Y & :\Longleftrightarrow & X \cap (Y + c^<) = \emptyset \qquad \text{and} \\ X \rhd Y & :\Longleftrightarrow & X \cap (Y + c^>) = \emptyset, \qquad \text{with} \end{array} \quad (14)$$

elementwise addition and $c^<$ denoting the (time-like) *future cone* in $\mathbb{R}^{1+d}$, i.e., $c^< := \{(x^0, x^1, \ldots, x^d) \in \mathbb{R}^{1+d} \mid \sqrt{(x^1)^2 + \ldots (x^d)^2} \leq x^0\}$ and $c^> := -c^<$ defining the *past cone*.

Both these causal properties allow a special representation for the n-point functions, which actually will be exploited.

Proposition 8 *For causally ordered (finite) sets of space-time points, $X \lhd Y$ (and $Y \rhd X$), causality (anticausality, resp.) of $\mathbb{S}$ is equivalent to both,*

$$T(X \cup Y) = T(Y)T(X) \quad \textit{and} \quad \tilde{T}(X \cup Y) = \tilde{T}(X)\tilde{T}(Y). \quad (15)$$

Proof. Take the definitions (and swap the variables X, Y, resp.) and apply that
□

Lemma 9 *the left hand side (sets of) equations (15) are equivalent to (12) and the right hand side (sets of) equations (15) are equivalent to (12)'s inversion, i.e. $\mathbb{S}(g_1 + g_2)^{-1} = \mathbb{S}(g_1)^{-1}\mathbb{S}(g_2)^{-1}$, for $X \equiv (x_{k+1}, \ldots, x_n) \subseteq \operatorname{supp}(g_1)$ and $Y \equiv (x_1, \ldots, x_k) \subseteq \operatorname{supp}(g_2)$.*

Proof. The addenda in the expansions of $\mathbb{S}(g_1 + g_2)$ and $\mathbb{S}(g_2)\mathbb{S}(g_1)$, and their inverses respectively, just correspond via the proposed equations. Therefore compare

$$\mathbb{S}(g_1+g_2) = \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\int dx_1 \cdots \int dx_n T_n(x_1,\ldots,x_n)$$
$$\big(g_1(x_1)+g_2(x_1)\big)\ldots\big(g_1(x_n)+g_2(x_n)\big)\Big)$$
$$= \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=1}^{n} \binom{n}{k}}_{=2^n} \Big(\int dx_1 \cdots \int dx_n T_n(x_1,\ldots,x_n)$$
$$g_2(x_1)\ldots g_2(x_k) g_1(x_{k+1})\ldots g_1(x_n)\Big) \quad \text{and}$$

$$\mathbb{S}(g_2)\mathbb{S}(g_1) = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{k!\,(n-k)!}\Big(\int dx_1 \cdots \int dx_n T_k(x_1,\ldots,x_k)$$
$$T_{n-k}(x_{k+1},\ldots,x_n) g_2(x_1)\ldots g_2(x_k) g_1(x_{k+1})\ldots g_1(x_n)\Big),$$

and do the same with the constituents of the inverted equation, provided by Assumption 4. □

The section finishes with the remark that (algebraic) locality is fulfilled under the assumptions here. The statements follow straight from the definitions and Proposition 8.

Corollary 10 Microcausality implies local commutativity, i.e.

$$\mathrm{supp}(g_1) \bowtie \mathrm{supp}(g_2) \quad \Longrightarrow \quad \mathbb{S}(g_1+g_2) = \mathbb{S}(g_1)\mathbb{S}(g_2) = \mathbb{S}(g_2)\mathbb{S}(g_1),$$

for any $g_1, g_2 \in \mathcal{S}(\mathbb{R}^{1+d})$, whereby $(\cdot \bowtie \cdot) :\Leftrightarrow (\cdot \rhd \cdot) \wedge (\cdot \lhd \cdot)$ denotes the relation of being *space-like separated*. Local commutativity, also called *locality* simply, can equivalently be expressed by

$$\begin{aligned} T(X\cup Y) &= T(X)T(Y) = T(Y)T(X) \quad \text{and} \\ \tilde{T}(X\cup Y) &= \tilde{T}(X)\tilde{T}(Y) = \tilde{T}(Y)\tilde{T}(X), \quad \text{for} \quad X \bowtie Y. \end{aligned} \tag{16}$$

3 The causal induction and the splitting

The aim is to determine T_n from $T_1, \ldots, T_{n-1}$. Therefore the n-point function is decomposed in retarded and advanced functions. Those are the basic objects of EGS' method.

Definition 11 Let

$$T(\mathbb{N}_n) =: R(1,\ldots,n) - \overbrace{\sum_{X \dot\cup Y \in P_2(\mathbb{N}_{n-1})} T(Y\cup\{n\})\tilde{T}(X)}^{=:R'_n(1,\ldots,n)} \tag{17}$$

denote the *retarded* and

$$T(\mathbb{N}_n) =: A(1,\ldots,n) - \overbrace{\sum_{X\dot{\cup}Y\in P_2(\mathbb{N}_{n-1}))} \tilde{T}(X)T(Y\cup\{n\})}^{=:A'_n(1,\ldots,n)} \tag{18}$$

the *advanced* function, and call the primed versions *auxiliary* ones. The latter are built by $T_1,\ldots,T_{n-1}$, i.e. from the induction steps below. The difference of the two,

$$D := R - A = R' - A', \tag{19}$$

is the *causal* function, and it is referred to all of them as *EGS-functions*, abbreviated by J. In the next theorem it is expressed that the names are properly given.

The EGS-induction does go straight over $\mathbb{N}$, i.e. over the vertices. Therefore the functions are defined by including the new vertex x_n after this was done for "$x_1,\ldots,x_{n-1}$" inductively.

Remark 12 Due to the proposed (total) symmetry, the T_n do not depend on chosen iteration sequence. But having always appended the latest x_n at last position, the just defined EGS-functions do, i.e. they are not totally symmetric.

Or, in other words, the induction sensibly depends on the chosen order of vertices. Choosing a permutation of "$1,2,...,n$", this results in different EGS-functions.

Lemma 13 *For any decomposition* $X\cup Y = \mathbb{N}_{n-1}$*, with* $X\cap Y=\emptyset$*,* $X\neq\emptyset$*,*

$$\begin{aligned} Y\cup\{n\} \rhd X \quad &\Longrightarrow \quad R'_n(1,\ldots,n) = -T(Y\cup\{n\})\,T(X),\\ Y\cup\{n\} \lhd X \quad &\Longrightarrow \quad A'_n(1,\ldots,n) = -T(X)\,T(Y\cup\{n\}). \end{aligned} \tag{20}$$

Proof. One only needs to apply Proposition 8. However, the statements here are those of Theorem 1.1 in [12], pp. 165 and they are proved there. □

Denotation 14 The generalization of the time-like future (or past) cone to n points, built by the Cartesian product and with all the apexes at $x\in\mathbb{R}^{1+d}$, $C_n^<(x) := \times_{k=1}^n (c^< + \{x\})$ (and $C_n^>(x) := \times_{k=1}^n (c^> + \{x\})$, resp.) will be used.

Proposition 15 *D has a* causal support*,*

$$\operatorname{supp} D_n(1,\ldots,n) \subseteq C_n^<(x_n)\cup C_n^>(x_n), \tag{21}$$

if causality is supposed. That is concluded from

$$\begin{aligned} \operatorname{supp} R(1,\ldots,n) &\subseteq C_n^<(x_n),\\ \operatorname{supp} A(1,\ldots,n) &\subseteq C_n^>(x_n). \end{aligned} \tag{22}$$

Proof. For an indirect argument assume that the support properties (22) are not satisfied. Therefore suppose

$$\begin{aligned} &\emptyset\neq X_R\subseteq \operatorname{supp} R(1,\ldots,n),\ \text{i.e.,}\ X_R\not\subseteq C_n^<(x_n),\ \text{i.e.}\ \{n\}\rhd X_R,\\ &\emptyset\neq X_A\subseteq \operatorname{supp} A(1,\ldots,n),\ \text{i.e.,}\ X_A\not\subseteq C_n^>(x_n),\ \text{i.e.}\ \{n\}\lhd X_A. \end{aligned}$$

That just reproduces the prerequisites in Lemma 13, for appropriate Y_R, Y_A, i.e. $Y_R \cup \{n\} \rhd X_R$ and $Y_A \cup \{n\} \lhd X_A$. Write down the definitions and apply causality of T_n as well as the lemma,

$$\begin{aligned} R(1,\ldots,n) &= \overbrace{T_n(X_R \cup Y_R \cup \{n\})}^{T(Y_R\cup\{n\})\,T(X_R),\text{ by (15)}} + \overbrace{R'(1,\ldots,n)}^{-T(Y_R\cup\{n\})\,T(X_R),\text{ by (20)}} = 0, \\ A(1,\ldots,n) &= \underbrace{T_n(X_A \cup Y_A \cup \{n\})}_{T(X_A)\,T(Y_A\cup\{n\}),\text{ by (15)}} + \underbrace{A'(1,\ldots,n)}_{-T(X_A)\,T(Y_A\cup\{n\}),\text{ by (20)}} = 0. \end{aligned}$$

Hence, $X_R \not\subseteq \operatorname{supp} R(1,\ldots,n)$ and $X_A \not\subseteq \operatorname{supp} A(1,\ldots,n)$, what contradicts the assumption. □

The induction is based on assuming causality only up to the $(n-1)^{\text{th}}$ order.

Theorem 16 *Without supposing causality of T_n, but assuming causality for $T_1,\ldots,T_{n-1}$, equation (21) does hold for $n \geq 3$, i.e., D_n has a causal support.*

Proof. This is Theorem 1.4 in [12], proved on pp. 167 there. □

Induction 17 For $k \leq n-1$ one knows how to compute $D_n = R'_n - A'_n$ from the T_k, cf. Definition 11. By a so-called *causal splitting* $\theta^<$, or $\theta^>$, which will be defined below, one recovers $R_n = \theta^< D_n$, or $A_n = \theta^> D_n$ resp., from $D_n = R_n - A_n$ using its support property (22). And therefore,

$$T_n = \theta^< D_n - R'_n = \theta^> D_n - A'_n. \tag{23}$$

To "continue" the induction at the n^{th} order, the T_n obtained that way, still has to satisfy causality. *Proof.* Assume $X_R \cup X_A = \mathbb{N}_n$ and $X_R \lhd X_A$. Then either $n \in X_A$, i.e. $X_R \not\subseteq C_n^<(x_n)$, i.e $R_n = 0$, i.e. $T_n = -R'_n$, or $n \in X_R$, i.e. $X_A \not\subseteq C_n^>(x_n)$, i.e $A_n = 0$, i.e. $T_n = -A'_n$. And hence, Lemma 13 proves the claim in both of the cases, $T_n(\mathbb{N}_n) = T(X_A)\,T(X_R)$. □
Note that, due to the theorem, the induction only works, if

Assumption 18 D_n has a causal support for $n = 1, 2$ as well.

Example 19 For QED, that is shown to be fulfilled in [12].

Remark 20 Note that, in addition to the splitting, the induction only requires *addition* and *single products.*

Assumption 21 Having in mind to model a "Lagrangian theory", cf. Example 3, the general form of bare interaction is supposed to be

$$T_1(x) = \sum_q \prod_{s=1}^{\#} \, : \prod_{r=1}^{s^{\#}} \phi_{sr}(x) : \; t_1^q(x) \quad \in i^p\mathbb{R}, \qquad \text{for } x \in \mathbb{R}^{1+d}, \tag{24}$$

and ": · :" denoting the normal order.

t_1^q is an *essentially real* function (in $\mathcal{C}^\infty(\mathbb{R}^{1+d})$, say), i.e. $p \in \mathbb{N}$, thus either equal to its own real or imaginary part, which contains the coupling constant.

r is the running number for the occurrence of a particle ϕ_s, of some sort s. All together there are $\#$ different sorts of particles, and $s^\#$ denotes the number of a certain sort.

The product symbols, together with the variables, encode a certain order for the multiplication of the particle fields, which is non-commutative in general. However, to describe quantum particles one needs to introduce spin, i.e. a certain geometrical structure.

It is assumed that one can, with the help of Wick's theorem, cf. p. 184 in [12], write the EGS-functions J_n at the n^{th} order in the form of (24), i.e.

$$\begin{gathered} J_n(\mathbb{N}_n) = \sum_q \prod_{s=1}^{\#} : \prod_{r=1}^{s^\#} \phi_{sr}(\xi_{sr}) : \; j_n^q(\mathbb{N}_n), \quad \text{with an} \\ \text{injective} \quad \xi : \mathbb{N}_\# \times \mathbb{N}_n \to \mathbb{N}_n, \quad \sum_{s=1}^{\#} s^\# = n, \end{gathered} \tag{25}$$

for different (running) variables, and where the j_n^q represent the scalar parts of J_n. The map ξ assigns coordinates to the particle fields. That is, using the Assumption 24 below, just the content of EG's Theorem 0 in [7], pp. 229.

Therefore one can do analysis with the numerical interaction of the particles only, i.e. with the *scalar* functions j, i.e. *distributions*, which come into the game via Wick's theorem, because of the propagators representing the (anti-) commutator relations.

To consider singularities will be necessary. Not that much, as it is emphasized on p. 177 in [12], because of the splitting only, which can not be realized by a multiplication with a $\mathcal{C}^\infty$-function cutting out the required support, but by a step-function, living in $\mathcal{D}'$. But also, due to the fact that the propagators, i.e. essentially j_2, are singular for two equal arguments in general. Together with the next Assumption 24, supposing translation invariance, those propagators can therefore be represented by distributions with singular point-support. And usually products of distributions require renormalization.

Example 22 QED satisfies that form, cf. p. 161 and p. 170 in [12].

Notation 23 For the analytical part, where the geometrical context does not play any role, operators are represented by distributions, written in small letters. In the operator-context, they are written in capital letters, and the order of the composition gets important.

Anyway, the following restriction makes the presentation more transparent.

Assumption 24 Let $\mathbb{S}$, and thus all the T_n, be *translation invariant.*

Firstly, this implies that the t_1^q are (essential real) constants.

Remark 25 In [4], for instance, it is investigated how translation invariance can be dropped, what thus enables one to work on a curved background. The therefore necessary equivalent for EG's Theorem 0 is also given.

The assumption is exploited here by using the (resulting) invariance of the $j = t, d, r, a, r', a'$, at every order n, while performing the space-time translation

$$(x_1, \ldots, x_n) \mapsto (x_1 - x_n, \ldots, x_{n-1} - x_n, 0), \quad \text{i.e.}$$

$$\underbrace{j_n(x_1 - x_n, \ldots, x_{n-1} - x_n, 0)}_{= (j_n * \varepsilon_{x_n})(x_1, \ldots, x_n)} = j_n(x_1, \ldots, x_n). \tag{26}$$

Convention 26 Hence, at every (single) occurrence one can substitute j_n by the convolution $j_n * \varepsilon_{x_n}$. One will need this substitution for the definition of the splitting, to sent the singular point-support of the propagators j_2 to $0 \in \times_{k=1}^{n} \mathbb{R}^{1+d}$. Therefore it is thought to be performed, even if not indicated explicitly.

4 Realization of the splitting on distributions

The step-functions which, being multiplied, perform the splitting on the occurring distributions are introduced. They are defined as distributions as well.

Definition 27 For testfunctions $f \in \mathcal{S}(\times_{k=1}^{n} \mathbb{R}^{1+d})$

$$\begin{aligned} \theta_n : f &\mapsto \int_{(0,\infty)\times\mathbb{R}^d} dx_1 \cdots \int_{(0,\infty)\times\mathbb{R}^d} dx_{n-1} \int_{\mathbb{R}^{1+d}} dx_n\, f(x_1, \ldots, x_n) \quad \text{and} \\ \tilde{\theta}_n : f &\mapsto \int_{(-\infty,0)\times\mathbb{R}^d} dx_1 \cdots \int_{(-\infty,0)\times\mathbb{R}^d} dx_{n-1} \int_{\mathbb{R}^{1+d}} dx_n\, f(x_1, \ldots, x_n) \end{aligned} \tag{27}$$

denote distributions which define step functions on $\times_{k=1}^{n} \mathbb{R}^{1+d}$ in an a.e.-sense, i.e.

$$\theta_n(x_1, \ldots, x_n) = \begin{cases} 1, \\ 0, \end{cases} \quad \tilde{\theta}_n(x_1, \ldots, x_n) = \begin{cases} 0, \\ 1, \end{cases} \quad \forall k \leq n-1 \begin{cases} 0 < x_k^0, \\ x_k^0 < 0. \end{cases} \tag{28}$$

Corollary 28 Obviously,

$$\theta + \tilde{\theta} = 1 : f \mapsto \int_{\mathbb{R}^{1+d}} dx_1 \int_{\mathbb{R}^{1+d}} dx_n\, f(x_1, \ldots, x_n) \quad \text{and} \tag{29}$$

$$\theta(f) = \tilde{\theta}(x_1, \ldots, x_n)\big(f(x_1^{\perp}, \ldots, x_n^{\perp})\big), \quad \tilde{\theta}(f) = \theta(x_1, \ldots, x_n)\big(f(x_1^{\perp}, \ldots, x_n^{\perp})\big), \tag{30}$$

where $x^{\perp} = (-x^0, \ldots, x^d)$, for $x = (x^0, \ldots, x^d) \in \mathbb{R}^{1+d}$.

To define products of distributions, especially for the splitting in this section, one needs to recall a well known statement.

Lemma 29 *A distribution $\tau \in \mathcal{D}'(\mathbb{R}^{(1+d)n})$ with point-support, e.g.* $\operatorname{supp}(\tau) = \{0\}$*, is a tempered distribution, $\tau \in \mathcal{S}'(\mathbb{R}^{(1+d)n})$, i.e., it is of finite order,*

$$\exists \pi \geq 0,\ \exists c > 0 \quad s.t. \quad |\tau(f)| \leq c \sup_{\substack{|\alpha| \leq \pi \\ x \in \mathbb{R}^{(1+d)n}}} (1 + \|x\|)^{\pi} |f^{(\alpha)}(x)|,\ \forall f \in \mathcal{S}(\mathbb{R}^{(1+d)n}). \tag{31}$$

Call the infimum ω, of those π, (fractal) order *of τ. Thus, if $\delta^{(\alpha)}$ denotes the (distributive) derivative of the delta distribution w.r.t. the multi-index α, it can be represented by*

$$\tau = \sum_{|\alpha| \leq \omega} c_\alpha \delta^{(\alpha)}, \qquad \textit{where} \quad c_\alpha = \tau(x^\alpha h), \tag{32}$$

for some $h \in \mathcal{D}$, with $h(x) = 1$, $\forall x : \|x\| < 1$.

Proof. Cf. section 3, 4 of § 8 in [15], pp. 111, even if the (fractal) order has to be in $\mathbb{N}$. But that bit of generalization, here, only changes the "$\mathcal{O}(1/k)$" on p. 113 into "$\mathcal{O}(k^m/k^{[m]+1})$", what does not do any harm to the required behavior for $k \to \infty$, there. □

Note, (31) is the characterization of $\tau \in \mathcal{S}'(\times_{k=1}^n \mathbb{R}^{(1+d)})$, which is due to Schwartz. It also holds for a τ restricted to any regular subset. The lemma is used here in the following way.

Proposition 30 *A distribution $\tau \in \mathcal{S}'(\mathbb{R}^{(1+d)n})$ with singular support only at the origin, i.e.* sing supp$(\tau) \subseteq \{0\} \subset \mathbb{R}^{(1+d)n}$, *is represented by*

$$\begin{aligned} \tau &= \tau|_{\mathbb{R}^{(1+d)n}\setminus\{0\}} + \sum_{|\alpha| \leq \omega} c_\alpha \delta^{(\alpha)}, \\ \textit{with} \quad \tau|_{\mathbb{R}^{(1+d)n}\setminus \mathrm{H}} &:= \begin{cases} \tau, & \textit{on } \mathbb{R}^{(1+d)n} \setminus \mathrm{H}, \\ 0, & \textit{elsewhere}, \end{cases} \quad \textit{being a density.} \end{aligned} \tag{33}$$

Proof. Apply Lemma 29 on the difference $\tau - \tau|_{\mathbb{R}^{(1+d)n}\setminus\{0\}} \in \mathcal{S}'(\mathbb{R}^{(1+d)n})$, which has a point-support at the origin, in the worst case. □

The (fractal) order ω corresponds to the *scaling degree* sd, defined in [4], on pp. 21. It is just equivalent to the so-called *degree of singularity (divergence)* w.r.t. the origin in $\mathbb{R}^{(1+d)n}$,

$$\omega_p := \mathrm{sd} - \#\text{ of (multi-)dimensions}, \tag{34}$$

where, as usual, "multi" refers to the multiple arguments.

Remark 31 A physicist might call (34) *power counting*, even if it is not specified for a certain theory, i.e. graph structure. Things are presenting more generally here. Note that the EGS-procedure covers nonrenormalizable theories as well.

Proposition 32 *If renormalization is required for a $\tau \in \mathcal{S}'(\mathbb{R}^{(1+d)n})$, i.e. if $\omega(\tau) \geq 0$, and* sing supp$(\tau) = \{0\}$,

$$\omega(\tau) = \omega_p(\tau), \qquad \textit{if} \quad \mathrm{sd}(\tau) \geq \text{dimensions} \equiv (1+d)n, \ \textit{here.} \tag{35}$$

Proof. Changing variables, i.e. introducing a scaling parameter, $\lambda > 0$, in (31), this leads to

$$\left|\tau\big(f(\lambda^{-1}\cdot)\big)\right| \leq c \sup_{\substack{|\alpha| \leq \pi \\ x \in \mathbb{R}^{(1+d)n}}} (1 + \|x\|)^\pi \left|\lambda^{-|\alpha|} f^{(\alpha)}(\lambda^{-1}x)\right|.$$

Thus, ω is just the infimum for all the π satisfying

$$\lambda^{\pi+(1+d)n}\big|\lambda^{-(1+d)n}\,\tau\big(f(\lambda^{-1}\cdot)\big)\big| \overset{\lambda\to 0}{\longrightarrow} 0, \quad \text{i.e.,} \quad \mathrm{sd} = \omega + (1+d)n.$$

□

Corollary 33 Hence, in (33) one can substitute $\omega := \omega_p$.

Definition 34 One says, the usual *renormalization assumption* is made, if

$$\omega_p(\tau) = \omega_p(\tau|_{\mathbb{R}^{(1+d)n}\setminus \mathrm{H}}), \text{ i.e. } \ \omega(\tau) = \omega_p(\tau|_{\mathbb{R}^{(1+d)n}\setminus \mathrm{H}}), \quad \text{for } \mathrm{H} = \text{sing supp}(\tau). \tag{36}$$

Definition 35 To satisfy the required support properties (22) one realizes the causal splitting (23) as a multiplication by θ and $\tilde{\theta}$, resp.,

$$\begin{aligned}(\theta^{<}d)(x) &:= \theta(x)\,d(x), \quad \text{to ensure that} \quad \text{supp}\,(\theta^{<}d) \subseteq C^{<},\\ (\theta^{>}d)(x) &:= \tilde{\theta}(x)\,d(x), \quad \text{to ensure that} \quad \text{supp}\,(\theta^{>}d) \subseteq C^{>},\end{aligned} \tag{37}$$

which still has to be defined. But therefore the singular support of the d_n could only be the cone's apex, i.e.,

$$\text{sing supp}\,(d_n) \subseteq \{0\}. \tag{38}$$

This fact dictates the form of the multiplication. Demanding the splittings to result in the tempered distributions, i.e. in $\mathcal{S}'(\times_{k=1}^{n}\mathbb{R}^{1+d})$, and demanding their densities to agree with the pointwise multiplication on the non-singular (supported) points, i.e.

$$\begin{aligned}&\text{supp}\,\big(\theta_n(x)\,d_n(x) - \theta_n(x)\,d_n(x)|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\{0\}}\big) \subseteq \{0\},\\ &\text{supp}\,\big(\tilde{\theta}_n(x)\,d_n(x) - \tilde{\theta}_n(x)\,d_n(x)|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\{0\}}\big) \subseteq \{0\},\end{aligned}$$

according to Lemma 30, one is forced to set

$$\begin{aligned}\theta_n(x)\,d_n(x) &:= \theta_n(x)\,d_n(x)|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\{0\}} + \sum_{|\alpha|\le\omega} c_\alpha\,\delta^{(\alpha)}(x),\\ \tilde{\theta}_n(x)\,d_n(x) &:= \tilde{\theta}_n(x)\,d_n(x)|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\{0\}} + \sum_{|\alpha|\le\tilde{\omega}} \tilde{c}_\alpha\,\delta^{(\alpha)}(x),\end{aligned} \tag{39}$$

for $x \in \times_{k=1}^{n}\mathbb{R}^{1+d}$. Due to Assumption 21, only the scalar part d of the causal function D is considered. Therefore the notation is chosen to be the same as, if the splitting were applied to (whole the) operator.

Remark 36 Try to imagine (39) applied on testfunctions, together with the representation for the constants, in (33), or Fourier transformed into momentum space, cf. [12], pp. 177, where this is written out. That certainly reminds one of the T-operation in the BPHZ-procedure, which performs a Taylor expansion (around zero in space and momentum, resp.) and picks out the terms up to order ω, cf. [5], on p. 102.

So far, only conclusions were drawn, but no real definitions were made. Actually, one would have to set the singular order ω and the constants c_α, which in accordance with the first statement (29) in Corollary 28 should satisfy,

$$\omega = \tilde{\omega} \quad \text{and} \quad c_\alpha = -\tilde{c}_\alpha, \quad \text{to ensure that} \quad d(x) = \theta(x)\, d(x) + \tilde{\theta}(x)\, d(x). \tag{40}$$

But, for the constants, that has to be done by extra physical assumptions, called *normalization conditions*. Those can be reasoned on the ground of symmetries (Ward identities). This will not be discussed here, cf. section 3.6 in [12], pp. 206, but also section 3.7, pp. 213, and section 3.13, pp. 258. However, to have a *unique* splitting the constants are assumed to be fixed.

Remark 37 By choosing the c_α adequately Lorentz invariance of the theory can be assured. How this is to be done is explained in section 4.5 of [12], pp. 282, in general terms, not for QED only.

The (fractal) order, instead, can be determined from d, just by *power counting*. Therefore, as usual, the following *renormalization assumption* has to be made,

$$\begin{aligned} \omega\Big(\theta(x)\, d(x)\Big) &= \omega\Big(\theta(x)\, d(x)|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\{0\}}\Big) \\ \text{i.e.,} \quad &= \omega(d) \qquad \big(\, e.g., \;= \omega_p(d)\,\big). \end{aligned} \tag{41}$$

Applying the splitting a couple of times, this does not change the degree of singularity in (39), i.e. the splitting is idempotent.

Corollary 38 In multiplicative terms, $\theta^2 d = \theta d$ and $\tilde{\theta}^2 d = \tilde{\theta} d$.

This finishes Definition 35.

Remark 39 According to the fact that supp (d) is in a cone (in the time-like one), and that the neighborhood around the singular point zero is of importance only, Scharf chooses the concept of the quasiasymptotics, cf. [14]. One says, that $d_n(x) \in \mathcal{S}'(\times_{k=1}^{n}\mathbb{R}^{1+d})$ has a *quasiasymptotics* $q(x) \in \mathcal{S}'(\times_{k=1}^{n}\mathbb{R}^{1+d})$ at $0 = x \in \times_{k=1}^{n}\mathbb{R}^{1+d}$ w.r.t. a positive continuous function ρ, if

$$\lim_{\epsilon\to 0} \epsilon^{(1+d)n}\rho(\epsilon) d_n(\epsilon x) = q(x) \quad \text{in} \;\; \mathcal{S}'(\times_{k=1}^{n}\mathbb{R}^{1+d}). \tag{42}$$

One exploits the *power-counting function* ρ of the quasiasymptotics, which is regular varying, i.e.

$$\lim_{\epsilon\to 0} \frac{\rho(r\epsilon)}{\rho(\epsilon)} = r^{\omega_q}, \quad \forall r > 0, \tag{43}$$

where ω_q defines the *singular order* of the quasiasymptotics. Thus, one is equipped with an order, $\omega := [\omega_q]$. The integer part is in accordance with the remark in the "Proof" of Proposition 30.

Example 40 In spite of the presentation of the Schwinger model in [2], by (9),

$$\hat{d}_2^{\mu\nu}(k) = \Big(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\frac{4m^2 \text{sgn}(k^0)\Theta(k^2 - 4m^2)}{2\pi k^2\sqrt{1 - 4m^2/k^2}},$$

there is not any discrepancy to obtain, between simple powercounting and using the quasiasymptotics. Applying the powercounting formula (34) for the singular order (in a symmetric case), noting that there is only a single argument, according to (26),

$$\omega_p = \begin{pmatrix} \text{power of the reciprocal} & - & \text{dimensions of} \\ \text{momentum, } k \to \infty, \text{ in } d_2^{\mu\nu} & & \text{space-time} \end{pmatrix} = 2 - 2 = 0,$$

this rather verifies the agreement with the singular order of the quasiasymptotics, $\omega_q = 0$, there. And the discontinuity, $\mathrm{sgn}(k^0)$, does not matter either, not even that it is in the time-like cone.

Remark 41 According to the Assumption 24 an inductively obtained T_n has to be translation invariant as well. But this invariance is not destroyed by the splitting. And, due to the fact that the splitting is unique, total symmetry in the arguments is provided, cf. Remark 12. Hence, indeed, the Induction 17 works properly with the concretely defined splitting.

Example 42 The method immediately leads to a general expression for the retarded and advanced function, and Wick's theorem transforms the non-commutative product into the form (25),

$$\begin{aligned} R_2'(x_2,x_1) &= -T_1(x_1)T_1(x_2) = A_2'(x_1,x_2) \\ &= e : \bar\psi_a(x_1)\gamma^\mu_{ab}\psi_b(x_1) : A_\mu(x_1)\, e : \bar\psi_c(x_2)\gamma^\nu_{cd}\psi_d(x_2) : A_\nu(x_2) \\ = ie^2\gamma^\mu_{ab}\gamma^\nu_{cd}\Big\{ &+ \Big(: \bar\psi_a(x_1)\psi_b(x_1)\bar\psi_c(x_2)\psi_d(x_2) : g_{\mu\nu} D_0^{(+)}(x_1,x_2)\Big) \\ &- \Big(: \psi_b(x_1)\bar\psi_c(x_2) :: A_\mu(x_1)A_\nu(x_2) : S^{(-)}_{da}(x_2,x_1)\Big) \\ &+ \Big(: \bar\psi_a(x_1)\psi_d(x_2) : iS^{(+)}_{bc}(x_1,x_2) g_{\mu\nu} D_0^{(+)}(x_1,x_2)\Big) \\ &\cdots \Big\}, \qquad \text{where the index 0 refers to } m=0, \text{ and} \\ S^{(\pm)}(x,y) &:= \frac{i}{(2\pi)^3}\int d^3p \frac{\not p \pm m}{2E} e^{\mp ip(x-y)} =: (i\not\partial + m) D^\pm(x,y), \end{aligned} \tag{44}$$

where m is the mass and E the energy. Every parenthesized addend corresponds to a possible Feynman graph, cf. [12], pp. 185 for the full expression.

That, practically, corresponds to defining a graph in the EGS-approach. And, while calculating a certain graph a lot of products of T's within the induction procedure disappear finally, in the adiabatic limit, cf. [1].

Remark 43 Note, that applying Wick's theorem implements the (anti-)commutation relations, i.e. the quantum, into the field theory. For the foregoing Example 42, i.e. for QED, one equips the EGS-scalar-functions by the following propagators, where all of them have a singularity at the origin

$$\begin{aligned} &\{\psi_a^{(-)}(x), \bar\psi_b^{(+)}(y)\} = -iS^{(+)}_{ab}(x,y), \quad \{\bar\psi_a^{(-)}(x), \psi_b^{(+)}(y)\} = -iS^{(-)}_{ba}(y,x), \\ &[A_\mu^{(-)}(x), A_\nu^{(+)}(y)] = ig_{\mu\nu} D_0^{(+)}(x,y), \quad \text{where the (parenthesized) sign} \end{aligned} \tag{45}$$

indicates the usual restriction on the corresponding part of the spectrum. Proper distributions (already) appear at the second order, especially for loops, when the propagators have to be multiplied, as in the third addend of (44), above.

5 Time-reflection symmetry instead of causality

Some consequences of causality are formulated, which will be crucial for the exposition. But it is started with some *heuristics*, whereby it is supposed, that

Assumption 44 the S-matrix, is *unitary*, i.e., $\mathbb{S}^{-1} = \mathbb{S}^{+}$, or $\tilde{T}_n = T_n^+$, applied on $\phi_i := \Phi|_{\text{i particles}}$ and $\phi_f := \Phi|_{\text{f particles}}$, $i + f = n$ arbitrarily chosen particles,

$$\langle \phi_f \,|\, \tilde{T}_n \,|\, \phi_i \rangle = \langle \phi_f \,|\, T_n^+ \,|\, \phi_i \rangle := \langle T_n \phi_f \,|\, \phi_i \rangle, \tag{46}$$

having defined the formal adjoined T_n^+ by the right hand side equation, as usual.

Remark 45 For some particle sectors EGS can conclude (physical) unitarity directly from the causal procedure, cf. section 4.7 in [12].

Definition 46 To exploit the assumption it will be useful to denote *time-reflection*, with $\perp\phi$ on particle fields ϕ, i.e. satisfying

$$\langle (\perp\phi_f)(\,\cdot^{\perp}) \,|\, \bullet \,|\, (\perp\phi_i)(\,\cdot^{\perp}) \rangle = \langle \phi_i \,|\, \bullet \,|\, \phi_f \rangle, \tag{47}$$

where " $\cdot^{\perp}$" denotes the *componentwise time-reflection* $x^{\perp} = (-x^0, \ldots, x^d)$ on the belonging configuration space elements $x = (x^0, \ldots, x^d) \in \mathbb{R}^{1+d}$. Using $x^{\perp\perp} = x$,

$$\begin{aligned}
&\langle \phi_f(x_{n-f}, \ldots, x_n) \,|\, \tilde{T}_n(x_1, \ldots, x_n) \,|\, \phi_i(x_1, \ldots, x_i) \rangle \\
&\quad = \langle T_n(x_1, \ldots, x_n) \phi_f(x_{n-f}, \ldots, x_n) \,|\, \phi_i(x_1, \ldots, x_i) \rangle \\
&\quad = \overline{\langle \phi_i(x_1, \ldots, x_i) \,|\, T_n(x_1, \ldots, x_n) \phi_f(x_{n-f}, \ldots, x_n) \rangle} \\
&\quad = \overline{\langle (\perp\phi_f)(x_1^{\perp}, \ldots, x_i^{\perp}) \,|\, T_n(x_1, \ldots, x_n) \,|\, (\perp\phi_i)(x_{n-f}^{\perp}, \ldots, x_n^{\perp}) \rangle} \\
&\quad = \overline{\langle \phi_f(x_{n-f}, \ldots, x_n) \,|\, \perp^{-1} T_n(x_1^{\perp}, \ldots, x_n^{\perp}) \perp \,|\, \phi_i(x_1, \ldots, x_i) \rangle}.
\end{aligned}$$

To go on here, one needs the scalar instances to be *essentially real*, i.e. either being equal to its own real or to its own imaginary part. Therefore one would have

$$\tilde{T}_n(x_1, \ldots, x_n) = \chi_{ifn} \perp^{-1} T_n(x_1^{\perp}, \ldots, x_n^{\perp}) \perp, \quad \text{with} \quad \chi_{ifn} = +1, -1, \tag{48}$$

depending on whether having to complex-conjugate ± 1 or $\pm i$, resp. However, the external field are modeled essentially real. And one actually assumes to consider (Lagrangian) theories, where the interaction T_1 is an essentially real scalar, cf. Assumption 21. This implies, that all the T_n and $\tilde{T}_n$ are essentially real. Therefore note, on one hand, that starting with a truly real interaction T_1 $(= -\tilde{T}_1)$, the n-point functions T_n and $\tilde{T}_n$, obtained with the causal procedure, are real as well. Products, sums, and causal splitting do not do any harm to reality. And on the other hand,

Lemma 47 *consider the EGS-theory where the starting interaction is multiplied by an arbitrary $z \in \mathbb{C}$, i.e. substituted by $T_1 \mapsto zT_1$. This implies the following change of the other n-point functions,*

$$T_n \mapsto z^n \, T_n \qquad \textit{and} \qquad \tilde{T}_n \mapsto z^n \, \tilde{T}_n. \tag{49}$$

Proof. Plug the z simply into the coupling g, against Convention 2, and pull it out again, at the n^{th} order being in the n^{th} power, and the job is done. □

Thus, one can thus chose $\chi_{ifn} := 1$, w.l.o.g., and

Corollary 48 it was shown so far, that unitarity implies *time-reflection symmetry*, i.e. $\mathbb{S}^{-1} = \mathbb{S}^{+} = (\perp^{-1}\mathbb{S}\perp)(\,\cdot^{\perp})$,

$$\tilde{T}_n(x_1, \ldots, x_n) = \perp^{-1} T_n(x_1^{\perp}, \ldots, x_n^{\perp})\perp =: (\perp^{-1} T_n \perp)(x_1^{\perp}, \ldots, x_n^{\perp}), \tag{50}$$

and therefore yielding the identity, when the tilde is applied twice,

$$T_n = \sim^2 \circ\, T_n = \perp^{-2} T_n \perp^2. \tag{51}$$

One can conclude, $\perp^2 = \pm 1$, cf. [12], (4.4.24) on p. 277. Applying Lemma 9 the forgoing characterization can be extended on products of (inverse) n-point functions T'_n, T''_m,

$$\sim \circ\, (T'_n T''_m) = \tilde{T}''_m \tilde{T}'_n \;=\; \perp^{-1} T''_m(\,\cdot^{\perp})\perp \perp^{-1} T'_n(\,\cdot^{\perp})\perp = (\perp^{-1} T''_m T'_n \perp)(\,\cdot^{\perp}). \tag{52}$$

Example 49 Considering a graph in QED for instance, with l internal lines $(= n - f - i$, above), then for a double time-reflection it holds, cf. [12] and [1],

$$\perp^2 = \begin{cases} +\mathrm{id} & \text{if } l \text{ is even,} \\ -\mathrm{id} & \text{else.} \end{cases} \tag{53}$$

This obviously encodes information about the general structure of the interaction.

Leaving aside unitarity, the time-reflection symmetry can be obtained by only applying the causal induction.

Proposition 50 *If only microcausality is supposed, equation (50) holds.*

Proof. This is done on pp. 280 in [12], in general terms, not only for QED. □

Assumption 51 Drop both the assumptions, unitarity and causality, and only suppose to hold time-reflection symmetry, (50) and (52), from now on.

This has to be completed by an extension of the Definition 35 for the (causal) splitting, (37) and (39),

$$\theta^{<}(T'_n) = \theta\, T'_n \qquad \text{and} \qquad \theta^{>}(T'_n) = \tilde{\theta}\, T'_n, \tag{54}$$

represented as a multiplication of operator T_n, here, with θ and $\tilde{\theta}$ (in the space of distributions).

This will astonishingly be enough in the end, being applied as follows.

Proposition 52 *The multiplication of θ and $\tilde{\theta}$ with the inverse n-point function $\tilde{T}_n$ provides the connection to T_n,*

$$\theta \tilde{T}_n = (\perp^{-1} \tilde{\theta} T_n \perp)(\,\cdot^{\perp}) \qquad \textit{and} \qquad \tilde{\theta} \tilde{T}_n = (\perp^{-1} \theta T_n \perp)(\,\cdot^{\perp}). \tag{55}$$

Proof. Exploit Corollary 28,

$$\begin{aligned}(\theta\, \tilde{T}_n)(x_1, \ldots, x_n) &= \theta(x_1, \ldots, x_n)\, (\perp^{-1} T_n \perp)(x_1^{\perp}, \ldots, x_n^{\perp}) \\ &= \tilde{\theta}(x_1^{\perp}, \ldots, x_n^{\perp}) \perp^{-1} T_n(x_1^{\perp}, \ldots, x_n^{\perp})\perp \\ &= (\perp^{-1} \tilde{\theta}\, T_n \perp)(x_1^{\perp}, \ldots, x_n^{\perp}),\end{aligned}$$

and analog for $\tilde{\theta}$. □

6 Towards an algebra of distributions

In accordance with Assumption 21 the product for the involved distributions being obtained by EGS' construction, cf. Induction 17, will (actually) be defined. That does supplementary give a proper meaning to any formal usage before.

Definition 53 The (operator) product $T'_n T''_m$ is represented by the *direct product*, i.e. the composition

$$(t'_n t''_m)(f) = t'_n\big(t''_m(f)\big), \quad \text{for } f \in \mathcal{S}(\times_{k=1}^{m} \mathbb{R}^{1+d}), \tag{56}$$

of the distributions $t'_n \in \mathcal{S}'(\times_{k=1}^{n} \mathbb{R}^{1+d})$ and $t''_m \in \mathcal{S}'(\times_{k=1}^{m} \mathbb{R}^{1+d})$, which is well-defined in $\mathcal{S}'(\times_{k=1}^{n+m} \mathbb{R}^{1+d})$, cf. § 8.5 in [15]. This product is *associative* and *commutative*. But the latter property does by no means imply, that the operator product is commutative.

Example 54 For delta-distributions the direct product yields,

$$(\delta^{(\alpha)}(x)\delta^{(\beta)}(y))(f(x,y)) = \delta^{(\alpha)}(f^{\beta}(x,0))) = f^{\alpha+\beta}(0,0) = \delta^{\alpha+\beta}(x,y)f(x,y),$$
$$\text{i.e.,} \qquad \delta^{(\alpha)}\delta^{(\beta)} = \delta^{\alpha+\beta}, \quad \text{with adequate multi-indices } \alpha, \beta. \tag{57}$$

It is started to investigate products of distributions, whose form is given in Proposition 32.

Lemma 55 *The degree of singularity of the direct product of the distributions is just the sum of the factors' degree of singularity,*

$$\omega_p(t'\,t'') = \omega_p(t') + \omega_p(t''). \tag{58}$$

Proof. A rather implicit version of it is done by citing the additivity of the scaling degree, $\mathrm{sd}(t'\,t'') = \mathrm{sd}(t') + \mathrm{sd}(t'')$, cf. Lemma 5.1(b) in [4], p. 22, and recalling the additivity of the dimension, $(1+d)(n+m) = (1+d)n + (1+d)m$, remembering the definition in Proposition 32.

The explicit estimations for the proof are written down in the case, that t' and t'' have its singular support at the origin, i.e. $\operatorname{sing\,supp}(t') = \{0\}$ and $\operatorname{sing\,supp}(t'') = \{0\}$. Then, using the characterization (31) of Schwartz, $0 \leq \omega_p(t') = \inf \omega'$, s.t.

$$|t'(t''(f))| \leq c' \sup_{\substack{|\alpha'|\leq\omega' \\ x'\in\mathbb{R}^{(1+d)n}}} (1+\|x'\|)^{\alpha'} |(t''^{(\alpha')}(f))(x')|, \quad \text{and further,}$$

$0 \leq \omega_p(t'') = \inf \omega''$, s.t.

$$|t'(t''(f))| \leq c'c'' \sup_{\substack{|\alpha'|\leq\omega',\, x'\in\mathbb{R}^{(1+d)n} \\ |\alpha''|\leq\omega'',\, x''\in\mathbb{R}^{(1+d)m}}} \overbrace{(1+\|x'\|)^{\alpha'}(1+\|x''\|)^{\alpha''}}^{\leq 2^{\alpha'+\alpha''}\left(1+\sqrt{\|x'\|+\|x''\|}\right)^{\alpha'+\alpha''}} |f^{(\alpha'+\alpha'')}(x',x'')|,$$

for any $f \in \mathcal{S}(\times_{k=1}^{n+m} \mathbb{R}^{(1+d)})$. This results in $\omega(t'\,t'') = \omega(t') + \omega(t'')$. □

Lemma 56 *The direct product* $t't'' \in \mathcal{S}'(\times_{k=1}^{n+m}\mathbb{R}^{1+d})$ *of* $t' \in \mathcal{S}'(\mathbb{R}^{(1+d)n})$ *and* $t'' \in \mathcal{S}'(\mathbb{R}^{(1+d)m})$, *with* $\operatorname{sing\,supp}(t') = \{0\}$ *and* $\operatorname{sing\,supp}(t'') = \{0\}$, *has the following form,*

$$(t't'')(x,y) = t'(x)\,t''(y)|_{\times_{k=1}^{n+m}\mathbb{R}^{1+d}\setminus\mathrm{H}} + \sum_{|\alpha|\le\omega_p(t')+\omega_p(t'')} c_\alpha(t't'')\,\delta^{(\alpha)}(x,y) \tag{59}$$

$$\begin{aligned} &+ \sum_{|\alpha|\le\omega_p(t')} c_\alpha(t')\,\delta^{(\alpha)}(x)\,\big(t''(y)|_{\times_{k=1}^{m}\mathbb{R}^{1+d}\setminus\{0\}}\big) \\ &+ \sum_{|\alpha|\le\omega_p(t'')} c_\alpha(t'')\,\big(t'(x)|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\{0\}}\big)\,\delta^{(\alpha)}(y), \end{aligned} \tag{60}$$

$$c_\alpha(t't'') := \sum_{\substack{\alpha'+\alpha''=\alpha\\ |\alpha'|\le\omega(t')\\ |\alpha''|\le\omega(t'')}} c_{\alpha'}(t')c_{\alpha''}(t''), \tag{61}$$

denoted as being applied on testfunctions $f : \times_{k=1}^{n+m}\mathbb{R}^{1+d} \ni (x,y) \mapsto f(x,y) \in \mathcal{S}$ *and with* $\mathrm{H} = \{0\}\times\mathbb{R}^{(1+d)m} + \mathbb{R}^{(1+d)n}\times\{0\}$.

Proof. Use t' and t'' in the form written out in (33), and straightforwardly do the multiplication by applying Example 54. □

Those products have to be built by the EGS-procedure, e.g. the auxiliary EGS-distributions r', a', cf. Definition 11. But they do not have a singular point support any more.

From the foregoing Lemma 56 one concludes the general *form* of products here, called *special*, obtained by multiplying distributions, starting from those with singular point support. The latter are characterized in Proposition 32, and they do enter the EGS-induction with Wick's theorem at the second order cf. Example 42 and Remark 43, actually, when loops do appear in the Feynman graphs. Apply Convention 26 now.

Corollary 57 The distribution $t'_n \in \mathcal{S}'(\times_{k=1}^{n}\mathbb{R}^{1+d})$, in the case of n arguments, is supposed to be such a product of special form. Then

$$t'_n = t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\mathrm{H}_n^{1+d}} + \sum_{\substack{X\dot\cup Y=\mathbb{N}_n\\ X,Y\setminus\{n\}\neq\emptyset}} \sum_{|\alpha|\le\omega_p(t'_n)} t'^{\alpha}_X\,\delta^{(\alpha)} + \sum_{|\alpha|\le\omega_p(t'_n)} c_\alpha(t'_n)\delta^{(\alpha)} \tag{62}$$

with t'^{α}_X depending on the multi-index α and being a function with $|X|$ arguments, characterized by the special combination X of arguments, where the latter are, according to Notation 5, abbreviated by X as well. Note, that t'^{α}_X is build by the same factors, i.e. functions with a lower number of arguments, as $t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\mathrm{H}_n^{1+d}}$ is made of,

$$t'^{\alpha}_X \quad \text{is a factor of} \quad t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus\mathrm{H}_n^{1+d}}, \quad \text{where} \quad t'^{\alpha}_X(\mathrm{H}_n^{1+d}) = 0. \tag{63}$$

Analog to the constants c_α in Lemma 56, t'^{α}_X is specified for $t'_n = t''_l\,t'''_m$, with $n = l+m$, and a reasonable usage of notation,

$$t'^{\alpha}_X(X) := \sum_{\substack{X''\dot\cup X'''=X\\ |X''|\le l\\ |X'''|\le m}} \sum_{\substack{\alpha''+\alpha'''=\alpha\\ |\alpha''|\le\omega(t'')\\ |\alpha'''|\le\omega(t''')}} t''^{\alpha''}_{X''}(X'')\,t'''^{\alpha'''}_{X'''}(X'''). \tag{64}$$

According to the Convention 26,

$$\text{sing supp}(t'_n) \subseteq \mathrm{H}_\mathrm{n}^{1+\mathrm{d}} := \{(x_1, \ldots, x_{n-1}, 0) \in \times_{k=1}^{n} \mathbb{R}^{1+d} \mid \exists k < n, x_k = 0\}. \tag{65}$$

Even if the singular support is not any more the origin, what is exploited by the causal splitting in the original EGS-induction, the degree of singularity is still determined by $t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus \mathrm{H}_n^{1+d}}$ alone.

Lemma 58 *The renormalization assumption (41) does survive the operation of direct multiplication of distributions in special form, i.e.*

$$\omega_p(t'_n) = \omega_p(t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus \mathrm{H}_n^{1+d}}), \tag{66}$$

if that property holds for any factor constituting t'_n.

Proof. Therefore one only has to show the property to hold for a single product $t'_n = t''_l\, t'''_m$, say. Apply (58) of Lemma 55 twice, i.e. for t'_n and $(t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus \mathrm{H}_n^{1+d}})$,

$$\begin{aligned} \omega_p(t'_n) &= \omega_p(t''_l) + \omega_p(t'''_m) \\ &= \omega_p(t''_l|_{\times_{k=1}^{l}\mathbb{R}^{1+d}\setminus \mathrm{H}_n^{1+d}}) + \omega_p(t'''_m|_{\times_{k=1}^{m}\mathbb{R}^{1+d}\setminus \mathrm{H}_n^{1+d}}) = \omega_p(t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus \mathrm{H}_n^{1+d}}). \end{aligned}$$

□

The special form (62) is stable under addition, the analog to Lemma 55 is given.

Corollary 59 For $t'_n, t''_n \in \mathcal{S}'(\times_{k=1}^{n}\mathbb{R}^{1+d})$ in special form (62), the degree of singularity after addition can be expressed by

$$\omega_p(t'_n + t''_n) = \max\{\omega_p(t'_n), \omega_p(t''_n)\}, \tag{67}$$

where the functions $t_X^{'\alpha}$ and the constants c_α do simply add, i.e.

$$(t' + t'')_X^\alpha = t_X^{'\alpha} + t_X^{''\alpha}, \qquad c_\alpha(t'_n + t''_n) = c_\alpha(t'_n) + c_\alpha(t''_n). \tag{68}$$

The splitting also does not change the special form.

Corollary 60 Multiplying t'_n, in special form, with θ and $\tilde\theta$, then in accordance with (66), this yields

$$\begin{aligned} \theta_n t'_n &= \theta_n t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus \mathrm{H}_n^{1+d}} + \sum_{\substack{X \dot\cup Y = \mathbb{N}_n \\ X, Y\setminus\{n\} \neq \emptyset}} \sum_{|\alpha| \le \omega_p(t'_n)} \theta_{|X|}\, t_X^{'\alpha}\, \delta^{(\alpha)} + \sum_{|\alpha| \le \omega_p(t'_n)} c_\alpha(t'_n)\delta^{(\alpha)}, \\ \tilde\theta_n t'_n &= \tilde\theta_n t'_n|_{\times_{k=1}^{n}\mathbb{R}^{1+d}\setminus \mathrm{H}_n^{1+d}} + \sum_{\substack{X \dot\cup Y = \mathbb{N}_n \\ X, Y\setminus\{n\} \neq \emptyset}} \sum_{|\alpha| \le \omega_p(t'_n)} \tilde\theta_{|X|}\, t_X^{'\alpha}\, \delta^{(\alpha)} + \sum_{|\alpha| \le \omega_p(t'_n)} \tilde c_\alpha(t'_n)\delta^{(\alpha)}. \end{aligned} \tag{69}$$

Applying the renormalization assumption (41), $\tilde\omega_p(t'_n) = \omega_p(t'_n)$ is already implemented. The other part of (40) still requires

$$\tilde c_\alpha(t'_n) = -c_\alpha(t'_n), \tag{70}$$

where, again, this constants have been chosen in accordance with adequate normalization conditions based on some extra physical ground.

Remark 61 The relation (61) between those constants does not restrict the normalization conditions. However, there are new constants appearing at every order, coming from the propagators which are introduced by applying Wick's theorem, reproducing the form (25).

One recognizes the splitting to be idempotent again, cf. Corollary 38, and furthermore,

Corollary 62 for $t'_n \in \times_{k=1}^n \mathbb{R}^{1+d}$ and $t'_m \in \times_{k=1}^m \mathbb{R}^{1+d}$ in special form (62),

$$\theta_{n+m} t'_n t''_m = \theta_n t'_n \, \theta_m t''_m. \tag{71}$$

But the analog equation, for $\tilde{\theta}$, does not hold. According to (70), there is the following discrepancy between (61) and the version with the tilde,

$$\tilde{c}_\alpha(t't'') = -c_\alpha(t't'') = -\sum_{\substack{\alpha'+\alpha''=\alpha \\ |\alpha'|\le\omega(t') \\ |\alpha''|\le\omega(t'')}} c_{\alpha'}(t')c_{\alpha''}(t'') = -\sum_{\substack{\alpha'+\alpha''=\alpha \\ |\alpha'|\le\omega(t') \\ |\alpha''|\le\omega(t'')}} \tilde{c}_{\alpha'}(t')\tilde{c}_{\alpha''}(t''). \tag{72}$$

Hence,

$$\tilde{\theta}_{n+m} t'_n t''_m = \tilde{\theta}_n t'_n \, \tilde{\theta}_m t''_m + \sum_{|\alpha|\le\omega_p(t'_n t'_m)} 2c(t'_n t''_m)\delta^{(\alpha)}. \tag{73}$$

Remark 63 One observes that the degree of singularity does not depend on any splitting which has been performed to build t'_n. The the degree of singularity is obtained additively from the distributions regular part, i.e. where the origin is excluded from the domain. In other words, θ and $\tilde{\theta}$ do not multiply with $\delta^{(\alpha)}$ within the inductive procedure. One does only multiply the latter ones, the deltas.

Proposition 64 *The (direct) multiplication of the distribution in special form (62) and the addition satisfy the following distributivity law,*

$$t'_n t'_m + t''_n t'''_m = (t'_n + t''_n) t'''_m, \tag{74}$$

for $t'_n, t''_n \in \times_{k=1}^n \mathbb{R}^{1+d}$ *and* $t'''_m \in \times_{k=1}^m \mathbb{R}^{1+d}$.

Proof. Using the special form (64) and applying Lemma 55 and Corollary 59, the problem is easily reduced to the associativity of numbers and functions, which do multiply pointwise. To check the constants c_α, use (61),

$$c_\alpha(t'_n t'''_m) + c_\alpha(t''_n t'''_m) = \sum_{\substack{\beta' + \alpha''' = \alpha \\ |\beta'| \le \omega(t'_n) \\ |\alpha'''| \le \omega(t'''_m)}} c_{\beta'}(t'_n)c_{\alpha'''}(t'''_m) + \sum_{\substack{\beta'' + \alpha''' = \alpha \\ |\beta''| \le \omega(t''_n) \\ |\alpha'''| \le \omega(t'''_m)}} c_{\beta''}(t''_n)c_{\alpha'''}(t'''_m)$$

$$= \sum_{\substack{\beta + \alpha''' = \alpha \\ |\beta| \le \max\{\omega(t'_n), \omega(t''_n)\} \\ |\alpha'''| \le \omega(t'''_m)}} c_\beta(t'_n + t''_n)c_{\alpha'''}(t'''_m) \; = \; c_\alpha\big((t'_n + t''_n)t'''_m\big),$$

and to check the functions t'^{α}_X, use (64),

$$
\begin{aligned}
&(t't''')^{\alpha}_X + (t''t''')^{\alpha}_X \\
&\quad = \sum_{\substack{Y\dot\cup X'''=X\\ |Y|\le n\\ |X'''|\le m}} \sum_{\substack{\beta'+\alpha'''=\alpha\\ |\beta'|\le\omega(t'_n)\\ |\alpha'''|\le\omega(t'''_m)}} t'^{\beta'}_Y\, t'''^{\alpha'''}_{X'''} + \sum_{\substack{Y\dot\cup X'''=X\\ |Y|\le n\\ |X'''|\le m}} \sum_{\substack{\beta''+\alpha'''=\alpha\\ |\beta''|\le\omega(t''_n)\\ |\alpha'''|\le\omega(t'''_m)}} t''^{\beta''}_Y\, t'''^{\alpha'''}_{X'''} \\
&\quad = \sum_{\substack{Y\dot\cup X'''=X\\ |Y|\le n\\ |X'''|\le m}} \sum_{\substack{\beta+\alpha'''=\alpha\\ |\beta|\le\max\{\omega(t'_n),\omega(t''_n)\}\\ |\alpha'''|\le\omega(t'''_m)}} (t'^{\beta'}_Y + t''^{\beta''}_Y)\, t'''^{\alpha'''}_{X'''} = \big((t'+t'')t'''\big)^{\alpha}_X,
\end{aligned}
$$

having dropped the arguments, for a better reading. □

Induction 65 The general form of all the distributions which have to be calculated within the EGS-procedure, i.e. by addition, multiplication and splitting, cf. Definition 11, Induction 17, is represented by the special form (62). And it is shown so far, that those distributions constitute an (commutative) *algebra.*

Remark 66 This algebra is not a differential algebra, therefore the no-go theorem of Schwartz, cf. [13], is not violated.

7 Identification of a Hopf algebra

There is an algebra to extract, which describes the operations with the n-point functions T_n of the previous sections.

Definition 67 Let $\mathcal{T} := \{X \mapsto T_{|X|}(X) \,|\, X \in \mathcal{F}(\mathbb{N})\}$ be the set of all mapping of symbols $T_{|X|}$ on finite subsets of $\mathbb{N}$, i.e. $\mathcal{F}(\mathbb{N}) := \{X \,|\, X \subset \mathbb{N} \text{ finite}\}$, representing the physical operator-valued distributions, possessing only a finite number of arguments. Caution, the index at the T's might be suppressed sometimes.

The multiplication is the composition of the operator-valued distributions, which in general is non-commutative. The concrete form is encoded by the (geometry of the) considered theory, i.e. by T_1 finally, cf. Example 3. But remember Assumption 21. The foregoing section does provide the realization of the product in the space of distributions, one has to consider. Through whole this section and without having it mentioned any further one applies that those distributions, of special form, constitute an algebra.

Definition 68 Based on the operator-composition, one inductively defines the space of formal products $\Pi\mathcal{T}$ of elements in $\mathcal{T}$,

$$\Pi\mathcal{T} \ni m(T'_{|X|} \otimes T''_{|Y|})(X\dot\cup Y) := T'(X)T''(Y),$$

firstly with elements $T', T'' \in \mathcal{T}$, and secondly with elements T', T'' in both, $\mathcal{T}$ and $\Pi\mathcal{T}$. The second ones are, due to their construction, indexed by the cardinality of its set of arguments as well, which does simply add.

Expressed differently, it was just defined a multiplication m on $\Pi\mathcal{T}$, i.e.

$$m : \Pi\mathcal{T} \otimes \Pi\mathcal{T} \to \Pi\mathcal{T}, \quad T'_{|X|} \otimes T''_{|Y|} \mapsto T'_{|X|}T''_{|Y|} := m(T'_{|X|}, T''_{|Y|}).$$

Remark 69 Consider the multiplication as being abstractly given now, enriched with some geometry. And note again, thinking in the abstraction of Assumption 21, i.e. on the level of distributions, and remembering Definition 53, i.e. the multiplication to correspond with the direct product of distributions, the arguments of the factors have to be disjoint. This setting is reasonable from a physical point of view as well, because it excludes tadpoles.

Lemma 70 *m is associative, and $E := T_0(\emptyset)$ serves to be a unit.*

Proof. Associativity can be concluded from the corresponding property of the operator-composition, i.e. by dropping brackets between the T's,

$$\begin{aligned} m\big(m(T^{(1)}_{|X|} \otimes T^{(2)}_{|Y|}) \otimes T^{(3)}_{|Z|}\big)(X \cup Y \cup Z) &= T^{(1)}(Z)T^{(2)}(Y)T^{(3)}(X) \\ &= m\big(T^{(1)}_{|X|} \otimes m(T^{(2)}_{|Y|} \otimes T^{(3)}_{|Z|})\big)(X \cup Y \cup Z), \end{aligned}$$

for $T^{(*)} \in \Pi\mathcal{T}$ and nonempty (and disjoint) $X, Y, Z \in \mathcal{F}(\mathbb{N})$. If only one of X, Y, Z were empty, there would not be any associativity to show.

Use the setting, $T_0(\emptyset) = 1$, of (9) in the more general way introduced below[1], cf. (75),

$$m\big(T(X) \otimes E\big) = T(X) = m\big(E \otimes T(X)\big), \quad \forall X \in \mathcal{F}(\mathbb{N}),$$

this proves E to be a unit. □

Using associativity, Lemma 9 and the left hand side equations of (15) can naturally be extended to $\Pi\mathcal{T}$. An arbitrary element $T' = \prod_i T_{|X_i|} \in \Pi\mathcal{T}$ of noncommutative factors $T_{|X_i|} \in \mathcal{T}$ corresponds to $\mathbb{S}(\sum_i g_i) = \prod_i \mathbb{S}(g_i)$ with $X_i \subseteq \mathrm{supp}(g_i)$ for all i (in the required finite set). The index at T', i.e. the number of its arguments, is $\sum_i |X_i|$. This will be denoted by $T' \in \Pi\mathcal{T}_{\sum_i |X_i|}$.

Definition 71 Let $\Pi\mathcal{T}_n := \{T_n \in \Pi\mathcal{T}\}$ denote the subset of $\Pi\mathcal{T}$ with elements having the same number $n \in \mathbb{N}$ of arguments, this introduces a $\mathbb{N}$-graduation on $\Pi\mathcal{T}$. Define the following addition on it,

$$(T'_n + T''_n)(X) := T'_n(X) + T''_n(X).$$

With this Abelian addition on the projections of the Cartesian product $\times_{n=0}^{\infty}\Pi\mathcal{T}_n$, one has just defined an Abelian addition on the direct sum

$$\mathcal{H} := \oplus_{n=0}^{\infty}\Pi\mathcal{T}_n,$$

denoted by $\oplus$. Equipping $\mathcal{H}$ with the usual (two sided) multiplication of complex scalars, which identifies

$$\mathbb{C} \otimes \mathcal{H} \cong \mathcal{H} \cong \mathcal{H} \otimes \mathbb{C} \tag{75}$$

by the isomorphisms $1 \otimes T \mapsto T$ and $T \otimes 1 \mapsto T$, that defines $\mathcal{H}$ to be a vector space on $\mathbb{C}$.

Remark 72 Even if not mentioned any further, the definitions (and proofs), formulated for elements of an arbitrary $\Pi\mathcal{T}_n$ only, are (can be, resp.) understood as naturally extended to the whole vector space $\mathcal{H}$ by linear continuation.

[1]There is no circle, logic-freaks simply do the unit afterwards.

Definition 73 Use the associativity of the multiplication by letting

$$T^{(4)}_{|W|}\left(T^{(1)}_{|X|}\oplus T^{(2)}_{|X|}\right)T^{(3)}_{|Z|} := T^{(4)}_{|W|}T^{(1)}_{|X|}T^{(3)}_{|Z|}\oplus T^{(4)}_{|W|}T^{(2)}_{|X|}T^{(3)}_{|Z|},$$

for $T^{(*)}\in\Pi\mathcal{T}$ and $W,X,Z\in\mathcal{F}(\mathbb{N})$. This ensures the distributivity of $\mathcal{H}$, cf. Proposition 64.

All the presented structure is summarized in,

Proposition 74 *$(\mathcal{H},m,E)$ is a unital algebra, with the unity map $\eta:\mathbb{C}\to\mathcal{H}$, $z\mapsto\eta(z):=zE$.*

Proof. To get the unity map η, apply its defining equation,

$$m(\eta\otimes\mathrm{id})=\mathrm{id}=m(\mathrm{id}\otimes\eta), \tag{76}$$

respectively on $z\otimes E$ and $E\otimes z$, for any $z\in\mathbb{C}$, and use (75). □

Remark 75 At this stage Corollary 62 can be restated in the operator notation, i.e. in capital letters (and caring about the order of those) and with a tiny little change. For $T'_n, T''_m\in\mathcal{H}$ the splitting on its product yields the following (regularized) multiplication rules

$$\theta\, T'_nT''_m \;=\; \theta T'_n\,\theta T''_m, \tag{77}$$

$$\tilde\theta\, T'_nT''_m \;=\; \tilde\theta T'_n\,\tilde\theta T''_m + \Bigg(\sum_{|\alpha|\le\omega_p(T'_nT''_m)} c_\alpha\,\delta^{(\alpha)}\Bigg)E, \tag{78}$$

with c_α representing the constants which are fixed by normalization conditions. Recalling the splitting of Induction 17 in the current language,

$$T_n=\theta(R'_n-A'_n)-R'_n=-\theta A'_n-\tilde\theta R'_n, \tag{79}$$

one observes the occurrence of renormalizing delta-polynomials to be necessary, cf. Remark 36.

To discover the promised Hopf algebra, finally, some notations have to be introduced, and some coalgebraic as well as some bialgebraic properties have to be revealed, at first.

Definition 76 Let $\triangle:\mathcal{H}\to\mathcal{H}\otimes\mathcal{H}$ be a linear map, given on $\mathcal{T}$, by

$$\triangle\, T(Z) := \sum_{X\dot\cup Y\in P^0_2(Z)} T(X)\otimes T(Y), \qquad \text{for any } Z\in\mathcal{F}(\mathbb{N}), \tag{80}$$

and for any $T'=\prod_i T_{|Z_i|}\in\Pi\mathcal{T}_{\sum_i|Z_i|}$ defined by $\triangle\, T':=\prod_i\triangle T_{|Z_i|}$.

Let $\varepsilon:\mathcal{H}\to\mathbb{C}$ be the linear map, given on $\mathcal{T}$, by

$$\varepsilon\big(T(X)\big) := \begin{cases}1, & \text{if } X=\emptyset,\\ 0, & \text{else,}\end{cases} \qquad \text{for any } X\in\mathcal{F}(\mathbb{N}), \tag{81}$$

i.e., $\varepsilon(zE):=z$, for $z\in\mathbb{C}$, and therefore being well-defined on $\mathcal{H}$.

Example 77 Note that, $\triangle(zT' \oplus T'') = z\triangle T' \oplus \triangle T''$, for $z \in \mathbb{C}$, according to Remark 72.

Let $\sigma : T' \otimes T'' \mapsto T'' \otimes T'$ denote the *flip operator*. And one immediately gets from the definition that

Corollary 78 $\triangle$ is cocommutative, i.e. $\sigma \circ \triangle = \triangle$.

Proposition 79 *The vector space $\mathcal{H}$, together with the coproduct $\triangle$ and counit ε, constitutes a cocommutative coalgebra, i.e. $\triangle$ is coassociative,*

$$(\triangle \otimes id) \circ \triangle = (id \otimes \triangle) \circ \triangle, \tag{82}$$

and the following counit properly holds,

$$(\varepsilon \otimes id) \circ \triangle = id = (id \otimes \varepsilon) \circ \triangle. \tag{83}$$

Proof. One checks, having $T'_{|V|} = \prod_i T_{|V_i|} \in \Pi\mathcal{T}_n$ and $V := \cup_i V_i$,

$$\begin{aligned}
(\mathrm{id} \otimes \triangle) \circ \triangle T'(V) &= (\mathrm{id} \otimes \triangle) \circ \Big(\prod_i \triangle T(V_i)\Big) \\
&= \prod_i \Big(\sum_{X \dot\cup W \in P_2^0(V_i)} T(X) \otimes \triangle T(W) \Big) \\
&= \prod_i \Big(\sum_{X \dot\cup W \in P_2^0(V_i)} T(X) \otimes \sum_{Y \dot\cup Z \in P_2^0(W)} T(Y) \otimes T(Z) \Big) \\
&= \prod_i \Big(\sum_{X \dot\cup Y \dot\cup Z \in P_3^0(V_i)} T(X) \otimes T(Y) \otimes T(Z) \Big) \\
&= \quad \dots \quad = (\triangle \otimes \mathrm{id}) \circ \triangle T'(V), \qquad \text{and}
\end{aligned}$$

$$\begin{aligned}
(\mathrm{id} \otimes \varepsilon) \circ \triangle T'(V) &= \prod_i \Big(\sum_{X \dot\cup Y \in P_2^0(V_i)} T(X) \otimes \varepsilon\big(T(Y)\big) \Big) = \prod_i \big(T(V_i) \otimes 1\big) \\
&= \quad T'(V) \\
&= \quad \dots \quad = (\varepsilon \otimes \mathrm{id}) \circ \triangle T'(V), \qquad \text{for any } V \in \mathcal{F}(\mathbb{N}).
\end{aligned}$$

□

Proposition 80 *$\mathcal{H}$, already proved to be an algebra and a coalgebra, is also a bialgebra, i.e. further it holds, for $T', T'' \in \mathcal{H}$,*

$$\begin{aligned}
\triangle(T'T'') &= \triangle(T')\triangle(T''), \qquad & \triangle(E) &= E \otimes E, \\
\varepsilon(T'T'') &= \varepsilon(T')\varepsilon(T''), \qquad & \varepsilon(E) &= 1.
\end{aligned}$$

Proof. Use $E = T(\emptyset)$, then the second equation in the second line was given by the definition already, and the equation above follows from the first one. The latter (i.e. the 1st one in the 1st line) is obtained by the following calculation, i.e., for $T'_{|U|} = \prod_i T_{|U_i|}$, $T''_{|X|} = \prod_i T_{|X_i|} \in \Pi\mathcal{T}_n$, and disjoint $U_i, X_j \in \mathcal{F}(\mathbb{N})$,

$$\begin{aligned}
\big(\triangle(T'_{|U|})\triangle(T''_{|X|})\big)(U \cup X) &= \Big(\prod_i \triangle(T_{|U_i|})\Big)(U)\Big(\prod_j \triangle(T_{|X_j|})\Big)(X) \\
&= \prod_{i,j}\Big(\sum_{V\dot\cup W\in P_2^0(U_i)} T_{|V|}\otimes T_{|W|}\Big)(U_i)\Big(\sum_{Y\dot\cup Z\in P_2^0(X_j)} T_{|Y|}\otimes T_{|Z|}\Big)(X_j) \\
&= \prod_{i,j}\Big(\sum_{V\dot\cup W\in P_2^0(U_i)}\ \sum_{Y\dot\cup Z\in P_2^0(X_j)} (T_{|V|}T_{|Y|})(V\cup Y)\otimes(T_{|W|}T_{|Z|})(W\cup Z)\Big) \\
&= \prod_{i,j}\Big(\sum_{(V\cup Y)\dot\cup(W\cup Z)\in P_2^0(U_i\cup X_j)} (T'_{|V|}T''_{|Y|})(V\cup Y)\otimes(T'_{|W|}T''_{|Z|})(W\cup Z)\Big) \\
&= \prod_{i,j}\triangle(T_{|U_i|}T_{|X_j|})(U_i\cup X_j) = \triangle(T'_{|U|}T''_{|X|})(U\cup X).
\end{aligned}$$

The equation, left over to prove, reduces to $\varepsilon(T'T'') = 0 = \varepsilon(T')\varepsilon(T'')$, for $T' \neq E \neq T''$. $\square$

Definition 81 Let $S : \mathcal{H} \to \mathcal{H}$ be a linear map, given by

$$S(T) := \tilde{T}, \qquad \text{for } T \in \mathcal{T}, \tag{84}$$

and defined for $T'_{|Z|} = \prod_i T_{|X_i|} \in \Pi\mathcal{T}_n$ by an extension of (10) and (52), i.e.

$$\prod_i\Big(\sum_{X\dot\cup Y\in P_2^0(Z_i)} T(X)\ S(T)(Y)\Big) = 0 = \prod_i\Big(\sum_{X\dot\cup Y\in P_2^0(Z_i)} S(T)(X)\ T(Y)\Big), \tag{85}$$

using the defining equation (84) already. Hence, all the brackets have to vanish. Thus, $S(T')$ can be expressed iteratively, by either

$$\begin{aligned}
&\overset{\curvearrowleft}{\prod_i}\underbrace{\Big(-T(Z_i) - \sum_{X\dot\cup Y\in P_2(Z_i)} T(X)\ S(T)(Y)\Big)}_{=S(T)(Z_i)} =: S(T')(Z) \qquad \text{or} \\
&S(T')(Z) := \overset{\curvearrowleft}{\prod_i}\underbrace{\Big(-T(Z_i) - \sum_{X\dot\cup Y\in P_2(Z_i)} S(T)(X)\ T(Y)\Big)}_{=S(T)(Z_i)},
\end{aligned} \tag{86}$$

having applied the *inverse order of the product*, denoted by a "$\curvearrowleft$", upstairs. Remember (11). One can recognize the two definitions to be equivalent. Therefore watch the expressions at the braces, they follow (85).

The inverted multiplication-order ensures, that

Proposition 82 *S is an anti-homomorphism, with $S(E) = E$.*

Proof. In accordance with Lemma 70 and setting (9), $S(E) = S(T(\emptyset)) = E$. It is still necessary to show, that the property of being an anti-homomorphism is fulfilled for $T \in \mathcal{T}$. But this follows from Lemma 9 by recognizing

$$S(T(Y)T(X)) = S(T(X \cup Y)) = \tilde{T}(X \cup Y) = \tilde{T}(X)\tilde{T}(Y) = S(T(X))S(T(Y)),$$

with the help of (15), and using the notation there. □

Another property, which, due to cocommutativity, would follow after the next theorem in the formal context (of Hopf algebras) anyway, is

Proposition 83 $S^2 = id$.

Proof. But it offers the opportunity to emphasize, that it is only based on $S^2 T = T$ in $\mathcal{T}$, i.e. (51), $\sim^2 \circ T = T$, cf. Corollary 48. Use this fact after having applied S on both sides of the equations (86). On the way, remember the fulfilled properties, anti-homomorphism and linearity. Take advantage of (85) (analogously, i.e. at the braces), the two equations (86) turn into

$$\prod_i T(Z_i) = T'(Z) = S^2(T')(Z) \quad \text{and} \quad S^2(T')(Z) = T'(Z) = \prod_i T(Z_i).$$

□

Theorem 84 *The bialgebra $\mathcal{H}$ is a Hopf algebra, and S defines the antipode, i.e.*

$$m \circ (id \otimes S) \circ \triangle = \eta \circ \varepsilon = m \circ (S \otimes id) \circ \triangle. \tag{87}$$

Proof. First of all, check that $\eta \circ \varepsilon : \Pi\mathcal{T}_n \to \Pi\mathcal{T}_n$ is determined by

$$(\eta \circ \varepsilon)\, T'(X) = \begin{cases} E, & \text{if } X = \emptyset \\ 0, & \text{else.} \end{cases} \tag{88}$$

Hence, there are two cases to consider. The first one, $X = \emptyset$, reproduces (87),

$$(\mathrm{id} \otimes S)\triangle(E) = E \otimes S(E) = E \otimes E = S(E) \otimes E = (S \otimes \mathrm{id})\triangle(E).$$

The other case requires

$$m \circ (\mathrm{id} \otimes S) \circ \triangle T'(Z) = 0 = m \circ (S \otimes \mathrm{id}) \circ \triangle T'(Z) \tag{89}$$

to be fulfilled, for $T'_{|Z|} = \prod_i T_{|X_i|} \in \Pi\mathcal{T}_n$ and $Z \in \mathcal{F}(\mathbb{N})$, as always. But those,

$$m \circ \prod_i \Bigg(\sum_{X \dot{\cup} Y \in P_2^0(Z_i)} T(X) \otimes S(T)(Y) \Bigg) = 0 \qquad \text{and}$$

$$0 = m \circ \prod_i \Bigg(\sum_{X \dot{\cup} Y \in P_2^0(Z_i)} S(T)(X) \otimes T(Y) \Bigg),$$

are just the defining equations (85) of S, when the multiplication m has been applied to each factor and to each addend, afterwards. □

8 The solution and its connection with BPHZ

The following decomposition of the coalgebraic expressions of the left and right hand side of (89) illuminates again what EGS' procedure is based on.

Definition 85 Writing out the left and the right hand side of the antipode condition, surprisingly, one can identify analog coalgebraic expressions for the advanced and retarded functions, being underlined,

$$
\begin{aligned}
&(\mathrm{id}\otimes S)\circ\triangle T_n(\mathbb{N}_n)\\
&= \overbrace{T_n(\mathbb{N}_n)\otimes E + \underbrace{\sum_{\substack{Y\neq\emptyset\\ X\dot\cup Y\in P_2^0(\mathbb{N}_{n-1})}} T_{|X|+1}(X\cup\{n\})\otimes ST_{|Y|}(Y)}_{=:\underline{R}'_n(\mathbb{N}_n)}}^{=:\underline{R}_n(\mathbb{N}_n)} \\
&\quad + \overbrace{\sum_{\substack{X\neq\emptyset\\ X\dot\cup Y\in P_2^0(\mathbb{N}_{n-1})}} T_{|X|}(X)\otimes ST_{|Y|+1}(Y\cup\{n\})}^{=:\underline{R}''_n(\mathbb{N}_n)} + E\otimes ST_n(\mathbb{N}_n),
\end{aligned} \tag{90}
$$

and for the flipped version,

$$
\begin{aligned}
&(S\otimes \mathrm{id})\circ\triangle T_n(\mathbb{N}_n)\\
&= \overbrace{E\otimes T_n(\mathbb{N}_n) + \underbrace{\sum_{\substack{Y\neq\emptyset\\ X\dot\cup Y\in P_2^0(\mathbb{N}_{n-1})}} ST_{|Y|}(Y)\otimes T_{|X|+1}(X\cup\{n\})}_{=:\underline{A}'_n(\mathbb{N}_n)}}^{=:\underline{A}_n(\mathbb{N}_n)} \\
&\quad + \overbrace{\sum_{\substack{X\neq\emptyset\\ X\dot\cup Y\in P_2^0(\mathbb{N}_{n-1})}} ST_{|Y|+1}(Y\cup\{n\})\otimes T_{|X|}(X)}^{=:\underline{A}''_n(\mathbb{N}_n)} + ST_n(\mathbb{N}_n)\otimes E,
\end{aligned} \tag{91}
$$

where $\underline{D}_n := \underline{R}'_n - \underline{A}'_n$ is the analog of the causal function. Call those EGS-functions *generalized.*

Remark 86 Applying the multiplication one recovers the EGS-functions.

$$J_n^* = m\circ \underline{J}_n^*, \qquad \text{and hence} \quad S(J_n^*) = m\circ\sigma\circ\underline{S}(\underline{J}_n^*), \tag{92}$$

where $\underline{S} := S\otimes S$ and "$*$" serves as a joker for a certain number (including zero) of primes.

Having put that at the beginning, a couple consequences can be stated.

Corollary 87 From the decompositions (90) and (91) one reads off,

$$\underline{A}_n^* = \sigma\circ\underline{R}_n^*. \tag{93}$$

Therefore the following (redundantly listed) properties hold,

$$\begin{array}{lllllll} \underline{S}\,\underline{R}'_n & = & \underline{A}''_n & = & \sigma\circ\underline{R}''_n & = & \sigma\circ\underline{S}\,\underline{A}'_n, \\ \underline{S}\,\underline{R}''_n & = & \underline{A}'_n & = & \sigma\circ\underline{R}'_n & = & \sigma\circ\underline{S}\,\underline{A}''_n, \end{array} \tag{94}$$

from which one can read off $\underline{S}^2 = \underline{\mathrm{id}} := \mathrm{id}\otimes\mathrm{id}$ and $\sigma\circ\underline{S} = \underline{S}\circ\sigma$. Applying the multiplication m, one recovers $S^2 = \mathrm{id}$ and

$$\begin{array}{ll} S(R'_n) = R''_n, & A''_n = S(A'_n), \\ S(R''_n) = R'_n, & A'_n = S(A''_n). \end{array} \tag{95}$$

And again, straight from the definitions,

$$\underline{D}_n = \underline{R}_n - \underline{A}_n + E\otimes T_n - T_n\otimes E, \quad \text{i.e.} \quad D_n = m\circ\underline{D}_n = R_n - A_n, \tag{96}$$

and one, indeed, reproduces EGS' definition for the causal function with the underlined version.

Rewriting the Hopf algebra property (89) this results in

$$R_n + S(R_n) = 0 = A_n + S(A_n), \tag{97}$$

i.e. for non-empty primed EGS-functions J^*_n,

$$0 = T_n + J^*_n + S(J^*_n + T_n). \tag{98}$$

The EGS-induction obviously solves the latter equation (98) for T_n. Before this is alternatively done here, the time-reflection has to be formulated algebraically. Therefore the characterization of time-reflection symmetry in Corollary 48 is applied.

Lemma 88 *Let J^*_n denote all EGS-functions again, then*

$$(\perp^{-1} J^*_n \perp)(\,\cdot^{\perp}) = m\circ\underline{S}\,\underline{J}^*_n = S(m\circ\sigma\circ\underline{J}^*_n). \tag{99}$$

Proof. Check this addendwise, with the help of (52), applying the inverted sequence of equations. Note, that

$$\perp^{-1}(m\circ\underline{J}^*_n)\perp = m\circ(\underline{\perp}^{-1}\underline{J}^*_n\,\underline{\perp}), \qquad \text{where} \quad \underline{\perp} := \perp\otimes\perp. \tag{100}$$

And then, apply (50) on each factor of the tensorproduct. Using (51), this leads to

$$(\underline{\perp}^{-1}\underline{J}^*_n\,\underline{\perp})(\,\cdot^{\perp}) = \underline{S}\,\underline{J}^*_n, \tag{101}$$

and thus the left hand side equation is shown. And, the right hand side equation is just the right hand side of (92). □

Remark 89 Hence, in the EGS-approach, the antipode of the Hopf algebra corresponds to time-reflections. Therefore, obviously, the Hopf algebra of EGS takes care of the proper time direction during the renormalization procedure. The latter is of course hidden by the proper construction of the multiplication for the occurring distributions.

And one will need the following argument.

Lemma 90 *For $F_n, G_n \in \mathcal{H}$, with n arguments and singular support in H_n^{1+d},*

$$0 = F_n + (\perp^{-1} G_n \perp)(\,\cdot^{\perp}) \qquad \text{implies} \tag{102}$$

$$\theta F_n = 0 = \tilde{\theta} G_n \quad \text{and} \quad \tilde{\theta} F_n = 0 = \theta G_n \qquad \text{on} \times_{k=1}^{n} \mathbb{R} \setminus \mathrm{H}_n^{1+d}. \tag{103}$$

Proof. Multiply the premise (102) with θ and afterwards with $\tilde{\theta}$,

$$0 = \theta F_n + (\perp^{-1} \tilde{\theta} G_n \perp)(\,\cdot^{\perp}) \quad \text{and} \quad 0 = \tilde{\theta} F_n + (\perp^{-1} \theta G_n \perp)(\,\cdot^{\perp}),$$

to obtain the two conclusions, by considering the disjoint domains. □

Proposition 91 *The n-point function T_n can inductively be obtained by the following two solutions of (98) on $\times_{k=1}^{n} \mathbb{R}^{1+d} \setminus H_n^{1+d}$,*

$$T_n = -\tilde{\theta} R'_n - \theta A'_n, \qquad (= \;\; \theta D_n - R'_n = \;\; \tilde{\theta} D_n - A'_n\,) \tag{104}$$

$$T_n = -\theta R'_n - \tilde{\theta} A'_n. \qquad (= -\theta D_n - A'_n = -\tilde{\theta} D_n - R'_n\,) \tag{105}$$

Proof. Write down the condition for the antipode (98) with the help of Lemma 88 and with (94) and (95), i.e., for $J_n^* := R'_n$,

$$0 = T_n + R'_n + (\perp^{-1}(A'_n + T_n)\perp)(\,\cdot^{\perp}). \tag{106}$$

Applying the foregoing Lemma 90 now, this leads to

$$\theta T_n + \theta R'_n = 0 = \tilde{\theta} A'_n + \tilde{\theta} T_n \quad \text{and} \quad \tilde{\theta} T_n + \tilde{\theta} R'_n = 0 = \theta A'_n + \theta T_n. \tag{107}$$

The left hand side equations yield (105), after addition, and the right hand side equations do yield (104).

Check, that another choice, $J_n^* := A'_n, R''_n, A''_n$, does produce the same solutions, but no further ones. □

The right hand side equations, straightforwardly obtained by the definition of D_n, are written down to compare the result with the EGS-induction. The first equation just agrees with EGS' result, but the second one constitutes an extra solution.

Corollary 92 Subtracting the equation (104) from (105), this implies that the causal function D_n vanishes on $\times_{k=1}^{n} \mathbb{R}^{1+d} \setminus H_n^{1+d}$.

Example 93 Thus, $D_2(x, 0) = 0$, for all $x \notin \{0\} = \mathrm{H}_2^{1+d} \subset \mathbb{R}^{1+d}$. Note, D_2 is not the Jordan-Pauli function defined in section 2.3 of [12].

Theorem 94 *Supposing the renormalization assumption (36), the n-point function T_n is (inductively) determined on whole the $\times_{k=1}^{n} \mathbb{R}^{1+d}$ by*

$$T_n = -\frac{1}{2}(R'_n + A'_n), \tag{108}$$

applying the direct multiplication for all the involved distributions.

Proof. Adding the two equations (104) and (105) of the last Proposition 91, one obtains $-(R'_n + A'_n)/2$, for $\times_{k=1}^{n} \mathbb{R}^{1+d} \setminus H_n^{1+d}$. This result does not contain any multiplication with θ or $\tilde{\theta}$. Therefore it can straightforwardly be extended to $\times_{k=1}^{n} \mathbb{R}^{1+d}$ by applying the direct multiplication under the renormalization assumption, as explained in section 6, for all the occurring distributions. □

With the help of Kreimer's achievements, one can establish a connection to BPHZ's approach, where the renormalization procedure is organized by so-called counterterms. Kreimer has shown in [10] that the latter are representable by the antipode of a Hopf algebra.

Identification 95 The n-point function T_n is assumed to correspond to the full n-vertex graph $\Gamma_n := \mathbb{N}_n$, which is the sum of all graphs of lower order in the considered theory. To ensure generality, the forest $\mathcal{F}(\Gamma_n)$, i.e. the set of all subgraphs $\gamma \subseteq \Gamma_n$, is taken to be the power set of all the n vertices,

$$\mathcal{F}(\Gamma_n) := \{\gamma \,|\, \gamma \subseteq \mathbb{N}_n\}, \quad \text{with the reduced graph} \quad \Gamma_n/\gamma := \Gamma_n \setminus \gamma. \tag{109}$$

However, on p. 16 in [6], vertices, internal lines I, and loops $L = I - n + 1$, each of them is considered as a natural grading.

According to strictly proceeding with distributions and applying the renormalization assumption, one chooses the trivial renormalization map, $R = \text{id}$. Hence, cf. [10], the characterization for the antipode yields exactly zero,

$$0 = m[(S \otimes \text{id})\triangle[\Gamma_n]] = \overline{R}(\Gamma_n) + C(\Gamma_n), \tag{110}$$

and, cf. p. 29 in [6], one obtains the correct value $R(\Gamma_n)$ for Γ_n by

$$R(\Gamma_n) = U(\Gamma_n) + C(\Gamma_n), \tag{111}$$

where $U(\Gamma_n)$ is the so-called unrenormalized value of the graph Γ_n. Continuing with [6], especially with their notation, which agrees with the one in Collins' textbook [5], the $\overline{R}$ operation of BPH is represented by

$$\overline{R}(\Gamma_n) = U(\Gamma_n) + \sum_{\emptyset \neq \gamma \subsetneq \Gamma_n} C(\gamma) U(\Gamma_n \setminus \gamma). \tag{112}$$

$C(\gamma)$ denotes the *counterterm* belonging to γ, which is represented by the antipode S in Kreimer's Hopf algebra.

Corollary 96 Using the above identifications, the recursive version (111) of BPH(Z)'s forest formula yields a version of (BPH)Z's explicit one,

$$R(\Gamma_n) = -\sum_{\emptyset \neq \gamma \subsetneq \Gamma_n} C(\gamma) U(\Gamma_n \setminus \gamma). \tag{113}$$

Writing down the solution (108) for EGS, again, only in terms of the n-point functions,

$$T_n(\mathbb{N}_n) = -\frac{1}{2} \sum_{\substack{\gamma \neq \emptyset \\ \gamma \dot{\cup} \gamma' \in P_0^2(\mathbb{N}_{n-1})}} T(\gamma' \cup \{n\})\, ST(\gamma) - \frac{1}{2} \sum_{\substack{\gamma \neq \emptyset \\ \gamma \dot{\cup} \gamma' \in P_0^2(\mathbb{N}_{n-1})}} ST(\gamma)\, T(\gamma' \cup \{n\}), \tag{114}$$

this turns out to be a straightforward (non-commutative) generalization with the obvious structural identifications,

$$T_n(\mathbb{N}_n) = R(\Gamma_n), \quad T_{|\gamma|+1}(\gamma' \cup \{n\}) = U(\gamma' \cup \{n\}), \quad ST_{|\gamma|}(\gamma) = C(\gamma). \tag{115}$$

Therefore, the time-reflections in the EGS-approach, with their special implementation by Corollary 48, are the substitutions for the counterterms in the BPHZ-approach.

9 Conclusions

Locality (the algebraic one, provided by microcausality) is implemented in the EGS-method of perturbative QFT. Their induction is found to be writable in the form of BPHZ's forest formulas, and hence, the EGS-procedure reappears in the framework of BPHZ's renormalization. This is (formally) done for a general graph with the finest subgraph structure, using the vertices for the grading (and one will, therefore, still have to take a closer look at this).

However, that implies (perturbative) locality, which affirms the production of the required counterterms in the Lagrangian, using the perturbative renormalization method. Therefore causality is substituted by time-reflection symmetry, which realizes an inversion of the S-matrix. This is different from the original treatment, but turns out to be crucial for the algebraization. The belonging time-reflections correspond to the counterterms in a BPHZ-setting. Both instances are mathematically modeled by the antipode of a Hopf algebra.

Constructing a Hopf algebra for the EGS-procedure, this requires a different analysis compared to the original version. The structure of singularities, one has to take care of, is more complicated. Hence, it is not enough to consider distributions with a singular point support only.

Using the EGS-model, the physical role of the counterterms (of the corresponding BPHZ-formulation) is cleared up. They simply arrange the correct time-direction through whole the perturbation procedure. One might conjecture, that also generally, the counterterms allow the implementation of a time ordering, rather than, that they poorly solve the renormalization problem, what naturally appears when distributions are multiplied.

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