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by

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PRESCRIBING THE MASLOV FORM OF LAGRANGIAN GRAPHS

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Abstract

We formulate and apply a modified Lagrangian mean curvature flow to prescribe the Maslov form of a Lagrangian graph in flat cotangent bundles. We prove longtime existence results based on a new energy estimate. In addition we derive miscellaneous results both for the Lagrangian mean curvature flow and the modified flow. Examples and counterexamples are given.

1 Introduction

Let $(T^*L, \bar{g}, \bar{J}, \bar{\omega})$ be the cotangent bundle of a compact, orientable, flat, n-dimensional Riemannian manifold L equipped with the flat metric $\bar{g}$, the standard complex structure $\bar{J}$ and the standard symplectic structure $\bar{\omega}(V, W) := \bar{g}(\bar{J}V, W)$. A submanifold $L_0 \subset T^*L$ is called Lagrangian if $\omega := \bar{\omega}_{|L_0} = 0$. Assume L_0 is a Lagrangian submanifold which in addition is a graph over the zero section L in T^*L. We study the evolution of L_0 under the modified Lagrangian mean curvature flow

$$\frac{d}{dt}F_t = g^{ij}(d_i f - H_i)\nu_j, \tag{1}$$

where $F_t : L \to L_t := F_t(L) \subset T^*L$ is a smooth family of diffeomorphisms, $H = H_i dx^i$ is the induced mean curvature one-form on L, $df = d_i f dx^i$ is the differential of a fixed smooth function on L, $\nu_j = J(\frac{\partial F}{\partial x^j})$ and g^{ij} is the pull-back of the induced Riemannian metric.

By Bieberbach's theorem (e.g. see [9] and [30]) we know that this is the same as analyzing the modified mean curvature flow for Lagrangian graphs sitting in the standard Euclidean space $\mathbb{C}^n$ which can be written as graphs over a compact fundamental domain $L = T^n \subset \mathbb{R}^n \subset \mathbb{C}^n$ of a flat torus.

(1) is a coupled system of nonlinear parabolic equations and for compact initial data always admits a smooth solution on a maximal time interval $[0, T)$. By the Codazzi equation it follows that the condition to be Lagrangian is preserved. The property to be Lagrangian is an integrability condition. If one looks at

the induced evolution equations for the n different height functions u_k then the Lagrangian property together with the Codazzi equations imply the integrability of (1). This yields the Monge-Ampère type equation (65).

Remark: *The evolution (1) becomes stationary if the mean curvature form equals the differential of f. Hence the modified mean curvature flow is an attempt to prescribe the form H by df. We will later see that there are certain natural restrictions on the class of functions f for which (1) can have a longtime solution. The mean curvature form (see below for the definition) of Lagrangian graphs over the zero section in cotangent bundles is always exact.*

Our main theorem can be stated as follows

Theorem 1.1 *Let L be a compact flat n-dimensional Riemannian manifold of diameter δ. Then there exists a constant B depending only on n and δ such that for all smooth functions f with*

$$||f||_{C^3} \leq B$$

we can find a Lagrangian submanifold in T^*L (equipped with the standard symplectic structure and flat metric) with mean curvature form $H = df$ that can be represented as a graph over the zero section in T^*L. In particular the modified mean curvature flow with L_0 being the zero section in T^*L exists for all $t \in [0, \infty)$ and the flow smoothly converges in the C^∞-topology to a smooth limiting Lagrangian graph over L with $H = df$.

Remark: *From the proof of Theorem 1.1 it will become clear that one can take any f for which*

$$\max\{\max_L |D^2 f|^2, \max_L |D^3 f|^2\} \leq \frac{c}{c+1}\Big(\frac{c}{17}\Big)^2$$

with $c := \frac{1}{2n\delta^2}$.

If L_0 is a Lagrangian graph over the zero section in the cotangent bundle of a compact, orientable flat Riemannian manifold L, then by the Lagrangian condition the n different height functions u_k all stem from one potential u that can be globally defined on L. On the other hand all maps $F : L \to T^*L$ with $F(x) := (x, du(x))$ are Lagrangian graphs.

The mean curvature flow for hypersurfaces has been studied extensively by many authors (e.g. [1], [2], [4], [5], [6], [7], [10], [11], [12], [14], [15], [16], [17], [21], [24], [25], [28], [29]). The mean curvature flow in higher codimension is an extremely subtle problem. Almost nothing is known. One of the few results is the existence theory for a weak formulation of the flow (see [4]) and results on the curve shortening flow in $\mathbb{R}^3$ [3]. In the Lagrangian category we are able to prove the following stability result which is (as far as we know) the first stability result in higher codimension:

Theorem 1.2 *Let L be a compact flat n-dimensional Riemannian manifold of diameter δ and let $c := \frac{1}{2n\delta^2}$. If for a smooth function $u : L \to \mathbb{R}$*

$$\begin{aligned} F &: L \to T^*L \\ F(x) &:= (x, du(x)) \end{aligned}$$

is a Lagrangian graph over L such that $q := cS_2 + S_3$ with $S_k := |D^k u|$ satisfies

$$q < \frac{c}{50},$$

then this remains true under the mean curvature flow ($f = 0$) and (1) admits an immortal solution such that the Lagrangian submanifolds L_t converge in the C^∞-topology to the zero section L as $t \to \infty$.

For hypersurfaces in $\mathbb{R}^{n+1}$ that can be written as graphs over $\mathbb{R}^n$ it is well known that the property to be a graph remains true (see [13]). Unfortunately a corresponding result in higher codimension is not known and in general this might be wrong. However in the Lagrangian case we will prove:

Theorem 1.3 *Let $M := T^*L$ be the cotangent bundle of a compact, flat Riemannian manifold L equipped with its standard complex structure and flat metric. Further let L_t be a family of Lagrangian submanifolds in T^*L defined by closed 1-forms $u_k(x,t)dx^k$ evolving from L_0 by its mean curvature. If at $t = 0$ all eigenvalues λ of the Hessian $D_{ij}u$ satisfy $\lambda^2 \le 1 - \epsilon$ with a constant $0 < \epsilon < 1$, then this remains true on L_t.*

In the case $n = 1$, i.e. for periodic graphs over $\mathbb{R}$ we will prove that (1) always admits an immortal solution but that in most cases we do not get a convergence result as described in Theorem 1.1. The reason is that the flow exists as long as it stays a graph (see below for details) which is always true although the slope might increase exponentially fast. On the other hand convergence of H to df can only occur if $\text{osc}(f) < \pi$. One example of this phenomenon is given in Figure 1. In contrast to this case we do have convergence in Figure 2.

Now we want to explain our motivation for this paper. First note that by the Lagrangian condition the complex structure $\overline{J}$ maps normal vectors to tangent vectors. Hence the mean curvature vector field $\overrightarrow{H}$ gets mapped to a vector field tangent to L. Using the induced metric it is easy to see that this gives a one-form on L. This one-form will be called **mean curvature form** $H = H_i dx^i$. It is an easy consequence of the Codazzi equations (see below) that H is closed. Let us now briefly recall the construction of the **Maslov form** for Lagrangian immersions in $\mathbb{R}^{2n}$. First consider the Grassmannian $G(n, 2n)$ of n-dimensional oriented planes in $\mathbb{R}^{2n} = \mathbb{R}^n \oplus i\mathbb{R}^n$ and denote by LAG $\subset G(n, 2n)$ the subset consisting of all Lagrangian planes. Since the unitary transformations $U(n)$ act transitively on LAG and the isotropy group at $p = \mathbb{R}^n \subset \mathbb{R}^n \oplus i\mathbb{R}^n$ is given by $SO(n)$ one observes that LAG can be identified with $U(n)/SO(n)$. An immersion in $\mathbb{R}^{2n}$ is Lagrangian if and only if its Gauss map defines a section in LAG. Given a Lagrangian immersion L in $\mathbb{R}^{2n}$ we obtain the following map:

$$L \to \text{LAG} \to U(n)/SO(n) \to S^1$$

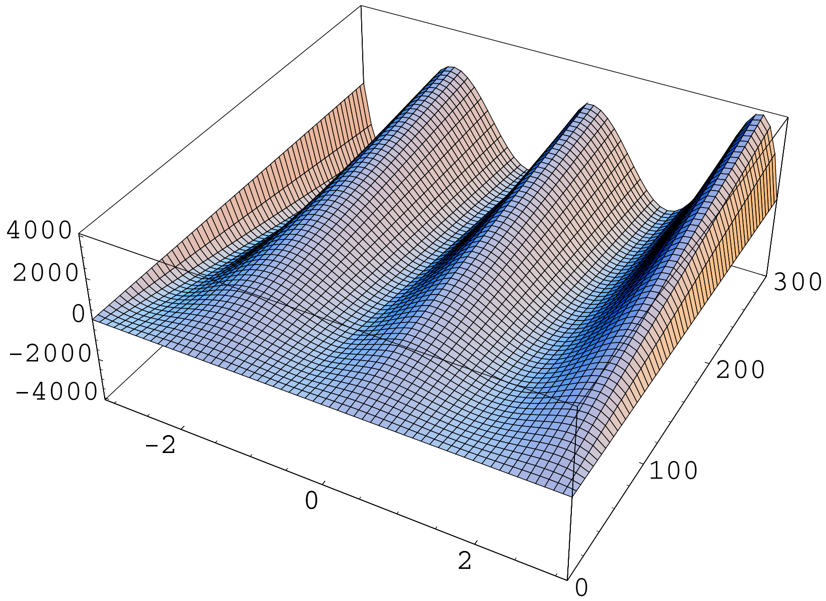

Figure 1: *A curve evolving under the modified mean curvature flow* $\frac{d}{dt}F = (\nabla^i f - H^i)\nu_i$. *Here* $f(x) = 10\sin(3x)$. *The oscillation of* f *is bigger than* π *and consequently the flow cannot converge to a smooth line with curvature form* $H = df$. *Nevertheless we have longtime existence and the curves which evolve from the real axis stay graphs over the initial curve for all times.*

$$p \mapsto T_pL \mapsto A_p \mapsto \det_{\mathbb{C}} A_p$$

where A_p is the element in $U(n)/SO(n)$ determined by the identification of LAG and $U(n)/SO(n)$. Writing $\det A_p$ as $e^{i\pi\gamma}$ with a multivalued function γ one obtains that $d\gamma$ is a well defined closed 1-form on L, the so called Maslov form. It has been shown by Morvan [18] that $d\gamma = \frac{1}{\pi}H$. This relation is very similar to the one which states that up to a constant factor 2π the Ricci form on a Kähler manifold equals its first Chern form. It is often useful to compare the Ricci flow $\frac{d}{dt}g_{ij} = -R_{ij}$ and the mean curvature flow. In [8] Cao has shown that a modified Ricci flow can be used to deform a given Kähler metric with first Chern class c_1 to any other Kähler metric with the same Chern class. Assuming that the Lagrangian immersion in $\mathbb{C}^n$ evolves according to $\frac{d}{dt}F = -\theta^k\nu_k$, the evolution equation for the mean curvature form (see (18)) is

$$\frac{d}{dt}H = dd^\dagger\theta.$$

If we now choose a closed 1-form m and set $\theta := H - m$, we obtain a flow that will become stationary if and only if $H = m$. Moreover since m is fixed we observe that

$$\frac{d}{dt}(H - m) = dd^\dagger(H - m)$$

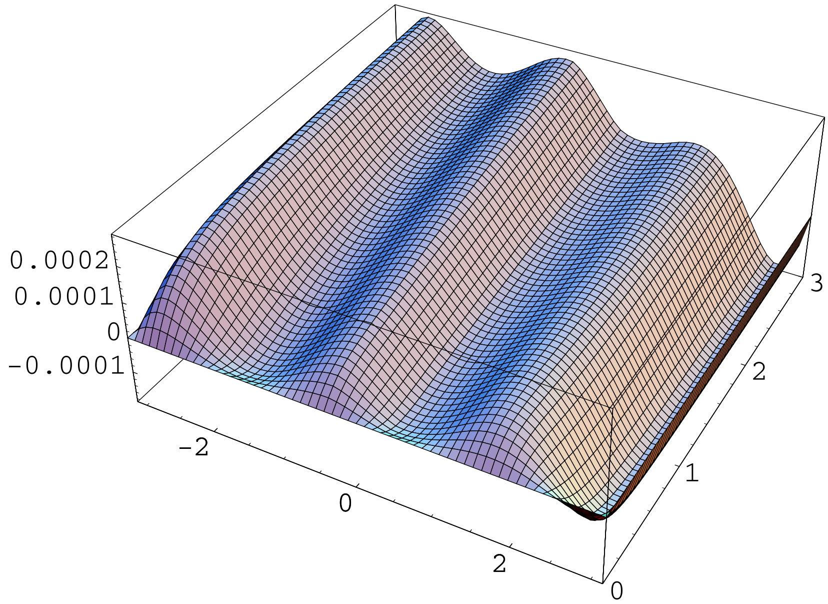

Figure 2: *A periodic graph over the real line in* $\mathbb{R}^2$ *evolving under the modified mean curvature flow* $\frac{d}{dt}F = (\nabla^i f - H^i)\nu_i$ *(as a model for the modified mean curvature flow in the cotangent bundle of* S^1 *equipped with the flat metric of a cylinder). The figure shows the space-time* $L_0 \times [0,T]$ *with* $L_0 = [-\pi, \pi]$ *and* $T = 3$. *The curves evolve exponentially fast from the zero section (*$t = 0$*, i.e. from the front line) to a curve with curvature form* $H = df$*, where* $f = \frac{1}{1000}\log(1 + \frac{1}{2}\sin(3x))$.

and this implies that $[H - m]$ does not change. The solution to the PDE

$$\frac{d}{dt}F_t = -(H^k - m^k)\nu_k, \tag{2}$$

where $m_i dx^i$ is a fixed closed 1-form on L will be called modified mean curvature flow. For $m = 0$ we obtain the Lagrangian mean curvature flow. For cotangent bundles we simply set $m = df$ since H is always exact in these cases. Having in mind the corresponding result for the Kähler-Ricci flow one is tempted to conjecture that in the case $[H - m] = 0$ one can use the modified LMCF to deform a given initial Lagrangian immersion with Maslov class m_1 to a new Lagrangian immersion with prescribed Maslov form m in the same cohomology class. That this in general is wrong, at least in the smooth setting, can be easily seen by considering the Whitney spheres. These can be constructed in the following way: Consider the map

$$\begin{aligned} \overline{f} &: \mathbb{R}^{n+1} \to \mathbb{C}^n \\ \overline{f}(x^1, \ldots, x^{n+1}) &:= \frac{1}{1 + (x^{n+1})^2}(x^1, \ldots, x^n, x^{n+1}x^1, \ldots, x^{n+1}x^n) \end{aligned}$$

and let

$$f := \overline{f}_{|S^n \subset \mathbb{R}^{n+1}}. \tag{3}$$

Then f defines a Lagrangian immersion of S^n into $\mathbb{C}^n$. This immersion is called **Whitney sphere** and will be denoted by W^n. Using the orthogonal projections

$$\begin{aligned}\pi_1(x^1,x^2,y^1,y^2) &:= (x^2,y^1,y^2)\\ \pi_3(x^1,x^2,y^1,y^2) &:= (x^1,x^2,y^2)\end{aligned}$$

one can illustrate the shape of W^2 as shown in Figure 3.

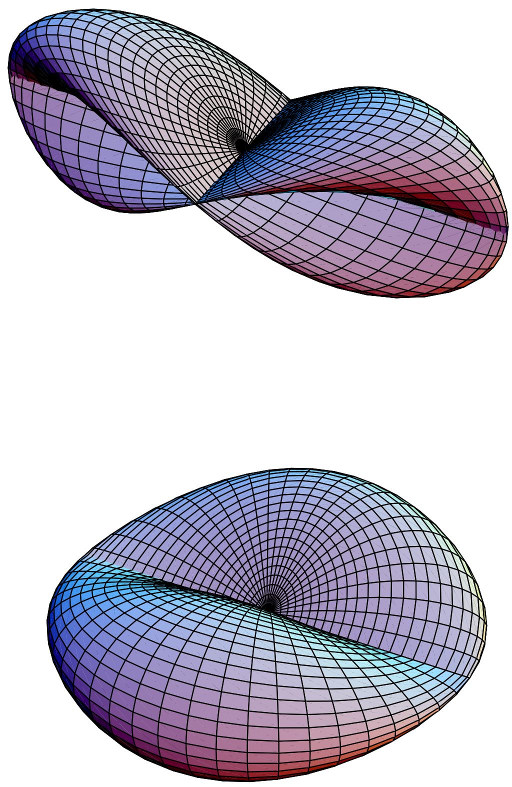

Figure 3: *The Whitney sphere*

Here the mean curvature form is exact and hence the Maslov class vanishes. But since there do not exist compact minimal submanifolds in $\mathbb{R}^{2n}$ we see that the modified LMCF cannot always have a longtime solution in time for which H tends to a fixed but arbitrary exact 1-form m.

The last example points out that there is a qualitative difference between the modified Lagrangian mean curvature flow and the modified Kähler Ricci flow.

The theory of Lagrangian immersions in a given Calabi-Yau manifold is very important. E.g. in the well-known paper by Strominger, Yau and Zaslow [26] it has been conjectured that the moduli space of special Lagrangian (these are minimal and Lagrangian) tori in a given Calabi-Yau manifold can be used to construct its mirror partner. While several geometers studied minimal Lagrangian immersions (e.g. see the article of [19] for an overview) the question which closed 1-forms m can be realized as the mean curvature form (or Maslov form) of a Lagrangian immersion is widely open. It should be noted that this problem is the Lagrangian analogue to the question arising in the theory of hypersurfaces with prescribed mean curvature. For minimal Lagrangian immersions of compact, orientable manifolds in hyperKähler manifold we recently deduced [23] that the second Betti number must be bigger than 1 and that they are Kähler submanifolds (w.r.t. one of the other complex structures) if the second Betti number is one.

The paper is organized as follows: In chapter 2 we introduce notations and recall basic geometric identities for Lagrangian submanifolds. In section 3 we will briefly discuss Lagrangian variations and then state the evolution equations for various geometric quantities. In additition we state a useful representation formula and show that the modified flow exists as long as the second fundamental form stays bounded. The energy estimate which is a key ingredient to prove our main theorem will be provided in section 3.1. In chapter 4 we control the condition to be a graph and the paper ends with chapter 5 where we finally prove longtime existence.

2 Geometric identities

Assume that $(y^\alpha)_{\alpha=1,\ldots,2n}$ are local coordinates for $\mathbb{R}^{2n}$, that doubled greek indices are summed from 1 to $2n$, that $\overline{J} = \overline{J}_\alpha{}^\beta dy^\alpha \otimes \frac{\partial}{\partial y^\beta}$ denotes the standard complex structure on $\mathbb{R}^{2n}$ and that $\overline{\omega} = \langle \overline{J}\cdot, \cdot\rangle$, where $\langle\cdot,\cdot\rangle = \overline{g}_{\alpha\beta} dy^\alpha \otimes dy^\beta$ is the standard inner product. Then

$$\overline{J}_\alpha{}^\beta \overline{J}_\gamma{}^\alpha = -\overline{\delta}_\gamma{}^\beta, \tag{4}$$

$$\overline{\omega}_{\alpha\beta} = \overline{J}_\alpha{}^\gamma \overline{g}_{\gamma\beta} = -\overline{J}_\beta{}^\gamma \overline{g}_{\gamma\alpha} = -\overline{\omega}_{\beta\alpha}. \tag{5}$$

Now assume that $F : L \to \mathbb{R}^{2n}$ is a Lagrangian immersion. If $(x^i)_{i=1,\ldots,n}$ are local coordinates for L, then we set

$$e_i := \frac{\partial F}{\partial x^i}\ ;\ \nu_i := \overline{J} e_i.$$

Note that by the Lagrangian condition ν_i is a normal vector for any $i = 1, \ldots, n$. The second fundamental form on L can then be defined as

$$h_{ijk} := -\langle \nu_i, \overline{\nabla}_{e_j} e_k\rangle$$

and the mean curvature 1-form $H_i dx^i$ is given by

$$H_i := g^{kl} h_{ikl},$$

where g^{ij} is the inverse of the induced Riemannian metric g_{ij} and we sum doubled latin indices from 1 to n. Hence the inward pointing mean curvature vector $\overrightarrow{H}$ can be written as

$$\overrightarrow{H} = -g^{ij} H_i \nu_j.$$

We also introduce

$$\begin{aligned} A_{ijkl} &:= h_{ijn} h^n{}_{kl}, \\ a_{kl} &:= A_i{}^i{}_{kl} = H^n h_{nkl}, \\ b_{kl} &:= A_k{}^i{}_{li} = h_k{}^{ij} h_{ijl}. \end{aligned}$$

Let us denote the Christoffel symbols for the Levi-Civita connection $\overline{\nabla}$ on $\mathbb{R}^{2n}$ by $\overline{\Gamma}^\alpha_{\beta\gamma}$ and let us also set

$$\overline{\Gamma}_{\alpha\beta} := \overline{\Gamma}^\gamma_{\alpha\beta} \frac{\partial}{\partial y^\gamma}.$$

The Christoffel symbols for the induced Levi-Civita connection on L will be written without a bar, i.e. in the form Γ^i_{jk}. To distinguish objects on the ambient space from the corresponding induced objects on L we will often use a bar for the ambient objects. Then we obtain the following equations:

$$\begin{aligned} h_{ijk} &= h_{jik} = h_{jki}, & (6) \\ -h_{jk}{}^n \nu_n &= \frac{\partial}{\partial x^j} e_k - \Gamma^n_{jk} e_n + \overline{\Gamma}_{\alpha\beta} e^\alpha_j e^\beta_k, & (7) \\ h_{jk}{}^n e_n &= \frac{\partial}{\partial x^j} \nu_k - \Gamma^n_{jk} \nu_n + \overline{\Gamma}_{\alpha\beta} e^\alpha_j \nu^\beta_k, & (8) \\ h_{ijk} &= -\langle \frac{\partial}{\partial x^j} e_i + \overline{\Gamma}_{\alpha\beta} e^\alpha_i e^\beta_j, \nu_k \rangle, & (9) \\ \text{Gauss equation} &: R_{ijkl} = A_{ikjl} - A_{iljk}, & (10) \\ \text{Codazzi equation} &: \nabla_i h_{jkl} - \nabla_j h_{ikl} = 0, & (11) \\ \text{traced Codazzi equation} &: \nabla_k H_l - \nabla_l H_k = 0. & (12) \end{aligned}$$

We also have the following Simons type identity:

Lemma 2.1

$$\begin{aligned} \Delta h_{ijk} &= \nabla_i \nabla_j H_k \\ &+ a_i{}^s h_{sjk} - b_i{}^s h_{sjk} - b_j{}^s h_{ski} - b_k{}^s h_{sij} + 2 h_{in}{}^m h_{jm}{}^s h_{ks}{}^n. \quad (13) \end{aligned}$$

3 Variations of Lagrangian immersions

We mention that Oh [20] studied normal variations of an initial Lagrangian immersion L_0 in Kähler manifolds. Let $F_t : L \times (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$ be a smooth family of Lagrangian immersions. The deformation vector field (or velocity) at time $t = 0$ is given by $V := \frac{d}{dt}_{|t=0} F_t$. Since we assume that L_0 is Lagrangian we can write V in the form

$$V = -\theta^k \nu_k + \lambda^k e_k.$$

Tangential variations do not alter the shape of an immersion L and merely correspond to the diffeomorphism group acting on L but they are often useful to reparametrize a given immersion. Hence without loss of generality we can assume that $\lambda_k = 0$. The induced metric g on L_0 can be used to identify the vector field $\theta^k \frac{\partial}{\partial x^k}$ with a 1-form on L_0, namely $\theta_k dx^k = g_{kl}\theta^l dx^k$. In the sequel this form will be called deformation 1-form. One can then compute the evolution equations for various geometric objects:

Proposition 3.1

$$\begin{aligned}
\frac{d}{dt}_{|t=0} g_{ij} &= -2\theta^n h_{nij}, &(14)\\
\frac{d}{dt}_{|t=0} d\mu &= -\theta^i H_i d\mu, &(15)\\
\frac{d}{dt}_{|t=0} \omega_{ij} &= \nabla_i\theta_j - \nabla_j\theta_i, &(16)\\
\frac{d}{dt}_{|t=0} h_{ijk} &= \nabla_i\nabla_j\theta_k - \theta^n(A_{ijkn} + A_{kijn}), &(17)\\
\frac{d}{dt}_{|t=0} H_i &= \nabla_i d^\dagger\theta, &(18)
\end{aligned}$$

where $d^\dagger\theta$ is shorthand for $\nabla^i\theta_i$.

In view of equation (16) we see that the Lagrangian condition $\omega_{ij} = \overline{\omega}(e_i, e_j) = 0$ is preserved only if θ is closed. If θ is exact, then this will be called a Hamiltonian variation. In the case where $\theta = H$ we can compute

Proposition 3.2

$$\begin{aligned}
\frac{d}{dt} g_{ij} &= -2a_{ij}, &(19)\\
\frac{d}{dt} d\mu &= -|H_i|^2 d\mu, &(20)\\
\frac{d}{dt} h_{ijk} &= \nabla_i\nabla_j H_k - H^n(A_{ijkn} + A_{kijn}), &(21)\\
\frac{d}{dt} H &= dd^\dagger H. &(22)
\end{aligned}$$

To rewrite equation (21) we use equation (13) and see that

Proposition 3.3

$$\frac{d}{dt} h_{ijk} = \Delta h_{ijk} - R_i^{\ s} h_{sjk} - R_j^{\ s} h_{ski} - R_k^{\ s} h_{sij} - 2h_{in}^{\ \ m} h_{jm}^{\ \ s} h_{ks}^{\ \ n},$$

where $R_i^{\ s} := a_i^{\ s} - b_i^{\ s}$ is the induced Ricci curvature.

In addition

Proposition 3.4

$$\frac{d}{dt}H_i = \Delta H_i - R_i{}^s H_s. \tag{23}$$

The evolution equation for A_{ikjl} is given by

Proposition 3.5

$$\begin{aligned}\frac{d}{dt}A_{ikjl} &= \Delta A_{ikjl} - 2\nabla^m h^n{}_{ik}\nabla_m h_{njl} \\ &- R_i{}^s A_{skjl} - R_k{}^s A_{sijl} - R_j{}^s A_{slik} - R_l{}^s A_{sjik} \\ &+ 2b_{ns}h^n{}_{ik}h^s{}_{jl} - 2A_{itks}A_{jl}{}^{ts} - 2A_{jtls}A_{ik}{}^{ts}.\end{aligned}$$

The proof of these equations is a straightforward computation. We also have

Proposition 3.6

$$\begin{aligned}\frac{d}{dt}a_{ik} &= \Delta a_{ik} - 2\nabla_m h_{nik}\nabla^m H^n - R_i{}^s a_{sk} - R_k{}^s a_{si} \\ &+ 2a^{mn}(A_{ikmn} - A_{imnk}), \end{aligned} \tag{24}$$

$$\begin{aligned}\frac{d}{dt}b_{ij} &= \Delta b_{ij} - 2\nabla_i h_{klm}\nabla_j h^{klm} \\ &- R_i{}^s b_{sj} - R_j{}^s b_{si} + 2(A_{ilkm} - A_{imkl})(A_j{}^{lkm} - A_j{}^{mkl}).\end{aligned} \tag{25}$$

Let us also write $A = h_{ijk}dx^i \otimes dx^j \otimes dx^k$. Then $|A|^2 = |h_{ijk}|^2 = g^{ij}b_{ij}$ and $|H_i|^2 = g^{ij}a_{ij}$ imply

Proposition 3.7

$$\frac{d}{dt}|H_i|^2 = \Delta|H_i|^2 - 2|\nabla_i H_j|^2 + 2|a_{ij}|^2, \tag{26}$$

$$\frac{d}{dt}|h_{ijk}|^2 = \Delta|h_{ijk}|^2 - 2|\nabla_i h_{jkl}|^2 + 2|b_{ij}|^2 + 2|A_{ilkm} - A_{imkl}|^2. \tag{27}$$

From the evolution equation of the mean curvature form (18) under the modified mean curvature flow we can deduce the following important result for the 1-form $H - m$:

Lemma 3.8 (Representation formula) *Assume that $L_t = F_t(L)$ is compact, orientable and evolves under the modified MCF (2) and let Δ_t be the Laplace-Beltrami operator at time t w.r.t. the induced metrics $g_{ij}(t)$. If $(H-m)(0)$ is exact, then there exists a unique smooth family of functions α_t on L such that*

$$(H-m)(t) = d\alpha_t$$

$$\frac{d}{dt}\alpha_t = \Delta_t\alpha_t$$

$$\min_L \alpha_0 = 0.$$

Proof: Compare with Lemma 2.4 in [22]. □

The parabolic maximum principle implies

Corollary 3.9 *Assume that L is a compact, orientable Lagrangian immersion evolving by the modified MCF and that $[H] = [m]$. Let α_t be the functions defined as in Lemma 3.8. Then*

$$\begin{aligned} \min_L \alpha_0 \le \min_L \alpha_t &\le \max_L \alpha_t \le \max_L \alpha_0, \\ osc\alpha_t &\le osc\alpha_0. \end{aligned}$$

For hypersurfaces it is well known (see [16], Theorem 8.1) that the mean curvature flow for closed initial data has a smooth solution as long as the norm of the second fundamental form $|A|^2$ stays bounded. In other words a singularity can only form if the curvature blows up somewhere. We will see that this also holds in the Lagrangian setting.

Theorem 3.10 *Assume that for $t \in [0, T)$ L_t is smooth family of Lagrangian immersions in a Kähler-Einstein manifold M^{2n} evolving by its mean curvature, that all ambient curvature quantities $\overline{\nabla}^r \overline{R}$ are bounded by a constant C_r and that $\lim_{t\to T} \max_{L_t} |A|^2$ is bounded. Then there exists an $\epsilon > 0$ such that the mean curvature flow admits a smooth solution on the extended time interval $[0, T+\epsilon)$.*

Proof: We will prove this only in the euclidean case. The proof for the general case is similar and repeatedly uses the equations of Gauß, Weingarten and Codazzi. Let us write any contraction of two tensors S and T by $S * T$ and let c_r be any constant depending only on r. We claim: For any $r \ge 0$ the evolution equation for $\nabla^r A$ can be expressed as

$$\frac{d}{dt}\nabla^r A = \Delta \nabla^r A + c_r \sum_{k+l+m=r} \nabla^k A * \nabla^l A * \nabla^m A. \tag{28}$$

To prove (28) we observe that

$$\frac{d}{dt}\nabla T = \nabla \frac{d}{dt} T + c \frac{d}{dt}\Gamma * T,$$

where T is an arbitrary tensor and Γ denotes the connection. Since

$$\frac{d}{dt}\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\nabla_i(\frac{d}{dt} g_{jl}) + \nabla_j(\frac{d}{dt} g_{il}) - \nabla_l(\frac{d}{dt} g_{ij})) = cA * \nabla A$$

we get

$$\frac{d}{dt}\nabla T = \nabla \frac{d}{dt} T + cA * \nabla A * T. \tag{29}$$

On the other hand

$$\nabla \Delta T = \Delta \nabla T + cA * A * \nabla T + cA * \nabla A * T \tag{30}$$

as can be seen by applying the rule for interchanging derivatives twice and by using the Gauß equations. Consequently

$$(\frac{d}{dt} - \Delta)\nabla T = \nabla((\frac{d}{dt} - \Delta)T) + cA * A * \nabla T + cA * \nabla A * T. \qquad (31)$$

Now equation (3.3) tells us that

$$\frac{d}{dt}A = \Delta A + c_0 A * A * A.$$

and using this together with equation (31) it follows by induction that (28) holds. Let again T be any tensor. Then

$$\begin{aligned} \frac{d}{dt}|T|^2 &= cA * A * T * T + 2\langle T, \frac{d}{dt}T\rangle \\ &= \Delta|T|^2 - 2|\nabla T|^2 + 2\langle T, (\frac{d}{dt} - \Delta)T\rangle + cA * A * T * T. \end{aligned}$$

Applying this to $T = \nabla^r A$ we obtain

$$\frac{d}{dt}|\nabla^r A|^2 = \Delta|\nabla^r A|^2 - 2|\nabla^{r+1}A|^2 + c_r \nabla^r A * \sum_{k+l+m=r} \nabla^k A * \nabla^l A * \nabla^m A. \qquad (32)$$

Now assume that $|\nabla^k A|^2$ is uniformly bounded on the time interval $[0,T)$ for $0 \leq k \leq r-1$. Then the evolution equation (32) and Schwarz' inequality show that there exists a constant c depending only on r and on the bounds for $|\nabla^k A|^2$ such that

$$\frac{d}{dt}|\nabla^r A|^2 \leq \Delta|\nabla^r A|^2 + c|\nabla^r A|^2,$$

provided $r \geq 1$. Consequently $|\nabla^r A|^2$ can grow at most exponentially on $[0,T)$, proving a uniform bound for $|\nabla^r A|^2$ also. Thus a bound for $|A|^2$ implies bounds for all higher derivatives $|\nabla^r A|^2$ as well. This proves the theorem. □

It is also well known that compact initial submanifolds in euclidean space can only have a smooth mean curvature flow evolution on a finite maximal time interval $[0,T)$. A point $p \in \mathbb{C}^n$ is called a blow-up point, if there exists a point $x \in L^n$ such that $\lim_{t\to T} F_t(x) = p$ and $\lim_{t\to T}|A|^2(x) = \infty$. The next Lemma estimates the blow up rate from below:

Lemma 3.11 *The function* $\max_{L_t}|A|^2$ *is Lipschitz continous and satisfies*

$$\max_{L_t}|A|^2 \geq \frac{1}{6(T-t)}. \qquad (33)$$

Proof: Recall that $|A|^2 = |h_{ijk}|^2$ and that by equation (27) we have

$$\frac{d}{dt}|h_{ijk}|^2 = \Delta|h_{ijk}|^2 - 2|\nabla_i h_{jkl}|^2 + 2|b_{ij}|^2 + 2|A_{ilkm} - A_{imkl}|^2.$$

Since $b_{ij} = h_{ikl}h^{kl}{}_j$ is a quadratic tensor and hence positive semidefinite and since $g^{ij}b_{ij} = |h_{ijk}|^2 = |A|^2$, we estimate

$$\frac{d}{dt}|h_{ijk}|^2 \le \Delta|h_{ijk}|^2 + 2|A|^4 + 2|A_{ilkm} - A_{imkl}|^2.$$

Now since

$$2|A_{ilkm} - A_{imkl}|^2 \le 4|A_{ilkm}|^2 = 4h_{iln}h^n{}_{km}h^{il}{}_s h^{skm} = 4|b_{ij}|^2 \le 4|A|^4$$

we conclude that

$$\frac{d}{dt}\max_{L_t}|A|^2 \le 6(\max_{L_t}|A|^2)^2$$

and integrating from t to T implies the result. □

Following a similar procedure as above we obtain:

Theorem 3.12 *Let M be a complete Kähler-Einstein manifold with uniform bounds on all curvature quantities. Further let L_t be a smooth family of Lagrangian immersions in M evolving by the modified mean curvature flow (2). Assume that for $t \in [0,T)$ $\lim_{t\to T}\max_{L_t}|A|^2$ is bounded. Then there exists an $\epsilon > 0$ such that the modified mean curvature flow admits a smooth solution on the extended time interval $[0, T+\epsilon)$. If max$|A|^2$ stays uniformly bounded on the maximal time interval $0 \le t < T$, then all higher covariant derivatives $|\nabla^k A|^2$ are uniformly bounded as well and the solution for the modified mean curvature flow exists for all $t \in [0,\infty)$.*

To obtain a Lagrangian immersion with presribed mean curvature form $H = m$ using the modified mean curvature flow, we will have to prove two things: First, in order to obtain longtime existence, we need uniform estimates for the norm of the second fundamental form $|A|^2 = h_{ijk}h^{ijk}$. In the next step we have to show that the immersions converge in the C^∞-topology to a smooth limiting Lagrangian immersion with $H = m$. To prove the latter we will need the results in the following section.

3.1 An energy estimate

The next Harnack inequality is Theorem 2.1. in [8].

Proposition 3.13 *Let L be a compact manifold of dimension n and let $g_{ij}(t)$ be a family of Riemannian metrics on L with the following properties:*

$$\begin{aligned}
&(a) \quad C_1 g_{ij}(0) \le g_{ij}(t) \le C_2 g_{ij}(0),\\
&(b) \quad |\frac{\partial g_{ij}}{\partial t}|(t) \le C_3 g_{ij}(0),\\
&(c) \quad R_{ij}(t) \ge -C_4 g_{ij}(0),
\end{aligned}$$

where C_1, C_2, C_3, C_4 are positive constants independent of t. If Δ_t denotes the Laplace-Beltrami operator w.r.t. $g_{ij}(t)$ and f is a positive solution for the "heat" equation

$$\frac{d}{dt}f = \Delta f,$$

on $L \times [0, \infty)$, then for any $\alpha > 1$ we have

$$\sup_L f(t_1) \le \inf_L f(t_2)(\frac{t_2}{t_1})^{\frac{n}{2}} exp(\frac{C_2^2\delta^2}{4(t_2 - t_1)} + (\frac{n\alpha C_4}{2(\alpha - 1)} + C_2C_3(n + A))(t_2 - t_1)),$$

where δ is the diameter of L measured by $g_{ij}(0)$, $A = \sup |\nabla^2 \log f|$ and $0 < t_1 < t_2 < \infty$.

Gauß equations imply that the Ricci curvatures are given by $R_{ij} = a_{ij} - b_{ij}$. The evolution equation for the metric $\frac{d}{dt}g_{ij} = -2(H^n - m^n)h_{nij}$ then implies that the conditions in Proposition 3.13 are satisfied if L_t is a family of Lagrangian immersions evolving by the modified mean curvature flow $\frac{d}{dt}F = (m^n - H^n)\nu_n$ such that the second fundamental forms satisfy $|A| \le C$ uniformly in t and such that all metrics $g_{ij}(x, t)$ are uniformly equivalent to $g_{ij}(x, 0)$. On the other hand the strong parabolic maximum principle implies that α chosen as in Theorem 3.8 is either identical zero or a positive solution of the heat equation. Similarly as in [8] we obtain

Lemma 3.14 *Let osc $\alpha(t) := \sup \alpha(t) - \inf \alpha(t)$ be the oscillation of α, where α is the solution of the heat equation given by Theorem 3.8. If L_t is a family of compact Lagrangian immersions in $\mathbb{R}^{2n}$ evolving by the modified mean curvature flow such that the norm of the second fundamental forms on L_t are uniformly bounded and all induced metrics $g_{ij}(t)$ are uniformly equivalent to $g_{ij}(0)$, then there exist constants C_5, C_6 independent of t such that*

$$osc\ \alpha(t) \le C_5 e^{-C_6 t}, \quad \forall t \in [0, \infty). \tag{34}$$

Define the following energy

$$E(t) := \frac{1}{2}\int k^2 d\mu_t,$$

where we set

$$k := \alpha - \frac{\int \alpha d\mu_t}{\int d\mu_t}.$$

Lemma 3.15 *Under the assumptions in Lemma 3.14 we have*

$$E(t) \le C_9 e^{-C_{10} t} \tag{35}$$

for positive constants C_9, C_{10}.

Proof: $\frac{d}{dt}d\mu = -\langle H, H-m\rangle d\mu = -\langle H, d\alpha\rangle d\mu$ and $\int k d\mu = 0$ give

$$\begin{aligned}
\frac{d}{dt}E &= \int k(\Delta k + \frac{1}{\int d\mu}\int \alpha\langle H, d\alpha\rangle d\mu - \frac{1}{(\int d\mu)^2}\int \alpha d\mu \int \langle H, d\alpha\rangle d\mu) \\
&- \frac{1}{2}\int k^2\langle H, d\alpha\rangle d\mu \\
&= -\int |\nabla k|^2 d\mu - \frac{1}{2}\int k^2 \langle H, d\alpha\rangle d\mu.
\end{aligned}$$

Now $H = H - m + m = d\alpha + m$ and $d\alpha = dk$ give

$$\frac{d}{dt}E = -\int |\nabla k|^2 d\mu - \frac{1}{2}\int k^2(|\nabla k|^2 + \langle m, dk\rangle)d\mu.$$

Using Schwarz' inequality we conclude that for any $\epsilon > 0$ we obtain

$$\frac{d}{dt}E \leq -\int |\nabla k|^2 d\mu + \frac{1}{4\epsilon}\int k^2|\nabla k|^2 d\mu + \frac{\epsilon}{4}\int k^2|m|^2 d\mu.$$

Let us now assume that L_t is a family of compact Lagrangian immersions in $\mathbb{R}^{2n}$ that evolve by the modified mean curvature flow and which have uniformly bounded second fundamental form and uniformly equivalent metrics $g_{ij}(t)$. Then conditions $(a), (b)$ and (c) of Proposition 3.13 are all satisfied. Consequently there exists a constant C_7 independent of t such that $|m|^2 \leq C_7$. Since osc $k =$ osc α we conclude with Lemma 3.14

$$\frac{d}{dt}E \leq (\frac{C_5^2}{4\epsilon}e^{-2C_6 t} - 1)\int |\nabla k|^2 d\mu + \frac{\epsilon C_7}{2}E.$$

Since $\int k d\mu = 0$ and all metrics are uniformly equivalent we obtain from Poincaré's inequality that there exists a constant $C_8 > 0$ independent of t such that

$$\int |\nabla k|^2 d\mu \geq C_8 \int k^2 d\mu.$$

Choose $\epsilon := \frac{C_8}{C_7}$ and let t_0 be such that $\frac{C_5^2}{4\epsilon}e^{-2C_6 t} < \frac{1}{2}, \forall t > t_0$. Consequently

$$\begin{aligned}
\frac{d}{dt}E &\leq -\frac{1}{2}\int |\nabla k|^2 d\mu + \frac{C_8}{2}E, \quad \forall t > t_0 \\
&\leq -C_8 E + \frac{C_8}{2}E = -\frac{C_8}{2}E, \quad \forall t > t_0.
\end{aligned}$$

This implies that there exist positive constants C_9, C_{10} such that

$$E(t) \leq C_9 e^{-C_{10}t}.$$

□

4 Lagrangian graphs over Lagrangian immersions

Let L be a compact smooth manifold of dimension n and assume that $G : L \to (\mathbb{C}^n, \overline{J})$ is a smooth Lagrangian immersion. For $i, j, k = 1, \ldots, n$ let us define

$$\begin{aligned} t_i &:= \frac{\partial G}{\partial x^i} \\ n_i &:= \overline{J}(t_i) \\ \sigma_{ij} &:= \langle t_i, t_j \rangle, \quad \sigma^{ij} = (\sigma_{ij})^{-1} \\ \tau_{kij} &:= -\langle \overline{\nabla}_{t_i} t_j, n_k \rangle \\ \tau_k &:= \sigma^{ij} \tau_{kij}, \end{aligned}$$

where we have used the summation convention. Here x^i are local coordinates for L and $\overline{J}$ is the usual complex structure on $\mathbb{C}^n$. Now assume that $u_i dx^i$ is a smooth 1-form on L and define a new map

$$\begin{aligned} F &: L \to L_u := F(L) \subset \mathbb{C}^n \\ F(x) &:= G(x) + u^k(x) n_k(x), \end{aligned}$$

i.e. L_u is a graph over $L_0 = G(L)$. We introduce the symmetric tensor

$$T_{ij} := \sigma_{ij} + \tau_{ijk} u^k.$$

As usual let $e_i := \frac{\partial F}{\partial x^i}, \nu_i := \overline{J}(e_i)$ and let g_{ij}, h_{kij} denote the induced metric and second fundamental form on $L_u = F(L)$. Then we get

$$\begin{aligned} e_i &= T_i{}^l t_l + D_i u^l n_l, \\ \nu_j &= -D_j u^k t_k + T_j{}^k n_k, \\ g_{ij} &= T_i{}^l T_{jl} + D_i u^l D_j u_l, \end{aligned} \tag{36}$$

where here and in the following a raised index will always be raised with respect to the metric σ^{ij} and D will always denote the Levi-Civita connection w.r.t. σ_{ij}. The only exception will be g^{ij}, denoting the inverse of g_{ij}. If T_{ij} is positive definite, then F is an immersion. F is Lagrangian if and only if $\langle e_i, \nu_j \rangle = 0, \ \forall i, j$. A short computation shows that this is true if and only if the 1-form $s_i dx^i$ with

$$s_i := u_i + \frac{1}{2} \tau_{ikl} u^k u^l \tag{37}$$

is closed. The Gauß-Weingarten-Codazzi equations for L_0 are

$$D_i t_j = -\tau_{ij}{}^k n_k, \tag{38}$$

$$D_i n_j = \tau_{ij}{}^k t_k, \tag{39}$$

$$D_i \tau_{jkl} = D_j \tau_{ikl}. \tag{40}$$

We have

$$D_i s_j = T_j{}^l D_i u_l + \frac{1}{2} D_i \tau_{jkl} u^k u^l. \tag{41}$$

In view of (40) we obtain that $F(L)$ is Lagrangian iff

$$D_i s_j - D_j s_i = T_j{}^l D_i u_l - T_i{}^l D_j u_l = 0. \tag{42}$$

Throughout the rest of this paper we will assume that (42) is valid. Now assume that we have a family F of Lagrangian graphs over L_0 such that

$$\frac{d}{dt}F = -g^{mn}\theta_m \nu_n$$

with a family of closed 1-forms θ on L (Recall that θ must be closed to preserve the Lagrangian condition). Then

$$\frac{d}{dt}F = -g^{mn}\theta_n (T_m{}^l n_l - D_m u^l t_l). \tag{43}$$

On the other hand using (39)

$$\begin{aligned} \frac{d}{dt}F &= \frac{d}{dt}(G + u^k n_k) \\ &= T_i{}^l \frac{dx^i}{dt} t_l + \frac{du^k}{dt} n_k. \end{aligned} \tag{44}$$

Combining (43) and (44) we obtain

$$-g^{mn}\theta_n T_m{}^l = \frac{du^l}{dt}, \tag{45}$$

$$g^{mn}\theta_n D_m u^l = T_i{}^l \frac{dx^i}{dt}. \tag{46}$$

Next we compute

$$\frac{du^l}{dt} = \frac{\partial u^l}{\partial t} + D_i u^l \frac{dx^i}{dt} \tag{47}$$

and

$$\frac{\partial}{\partial t} s_k = T_{kl} \frac{\partial u^l}{\partial t}. \tag{48}$$

But then

$$\begin{aligned} \frac{\partial}{\partial t} s_k &= T_{kl} \frac{\partial u^l}{\partial t} \\ &= T_{kl} \left(\frac{du^l}{dt} - D_i u^l \frac{dx^i}{dt}\right) \\ &= -g^{mn}\theta_n T_{kl} T_m{}^l - T_{kl} D_i u^l \frac{dx^i}{dt}\ , \quad \text{with (45)} \\ &= -g^{mn}\theta_n T_{kl} T_m{}^l - T_{il} D_k u^l \frac{dx^i}{dt}\ , \quad \text{with (42)} \\ &= -\theta_k\ , \quad \text{with (36) and (46)}. \end{aligned} \tag{49}$$

Our idea is to use the last equation together with the integrability condition (42) to rewrite the evolution equation $\frac{d}{dt}F = -g^{mn}\theta_n \nu_m$ in terms of smooth 1-forms $s_i dx^i$ on L equipped with the fixed background metric σ_{ij}. Observe that for Lagrangian graphs over flat Lagrangian immersions the expressions simplify enormously. Here we have that the two 1-forms $s_i dx^i$ and $u_i dx^i$ coincide. Moreover $\tau_{ijk} = 0$, $T_{ij} = \sigma_{ij}$. In the flat case the equations for the induced metric and second fundamental form become

$$g_{ij} = \sigma_{ij} + D_i u^k D_j u_k, \tag{50}$$

$$h_{kij} = -D_i D_j u_k, \tag{51}$$

and the 1-form $u_i dx^i$ is closed (Hence $D_i u_j = D_j u_i$). From now on we will only consider the flat case and note that in the general case the previous constructions and equations can be used to deduce the same as stated in Theorem 1.1 with the exception that the constant B now depends also on $\max_{L_0} |A|^2$ and that it is crucial to make sure that the tensor T_{ij} is invertible.

If $x_0 \in L$ is a fixed point and $w = w^k \frac{\partial}{\partial x^k}$ an arbitrary vector in $T_{x_0} L_0$, then the fact that L_0 is a flat torus implies that we can extend w uniquely to a parallel vector field on L_0. We define the function

$$f^{(w)} := u^k w_k. \tag{52}$$

We will also denote the second fundamental form w.r.t. w by $A^{(w)}$, i.e.

$$A^{(w)}{}_{ij} := h_{kij} w^k. \tag{53}$$

If we choose an orthonormal basis $E_1, \dots, E_n$ such that $D_i u^k$ becomes diagonal, i.e. $D_i u^k = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then we see from (50) that $g_{ij} = \operatorname{diag}(1 + \lambda_1^2, \dots, 1 + \lambda_n^2)$ and $g^{ij} = \operatorname{diag}(\frac{1}{1+\lambda_1^2}, \dots, \frac{1}{1+\lambda_n^2})$. This proves that g^{ij} and $D_i u^k$ commute, i.e. we have

$$g^{ij} D_j u^k = g^{kj} D_j u^i. \tag{54}$$

Using (51) we also obtain

$$D_i f^{(w)} = D_i u^k w_k, \tag{55}$$

$$D_i D_j f^{(w)} = -h_{kij} w^k = -A^{(w)}{}_{ij} \tag{56}$$

$$|D^2 f^{(w)}|^2 = |A^{(w)}|^2. \tag{57}$$

If L is a Lagrangian graph over L_0, then we obtain two metrics on L_0 namely σ_{ij} and the pullback metric of L to L_0 which is given by g_{ij}. Consequently we can measure the length of a vector $w \in T_{x_0} L$ w.r.t. σ_{ij} and g_{ij}. To destinguish them we label expressions with σ resp. g, e.g.

$$\begin{aligned} |w|_\sigma^2 &:= \sigma_{ij} w^i w^j, \\ |w|_g^2 &:= g_{ij} w^i w^j. \end{aligned}$$

We compute

$$\begin{aligned} |Df^{(w)}|_\sigma^2 &= \sigma_{ij} \nabla^i u_k w^k \nabla^j u_l w^l \\ &= \sigma_{ik} \nabla^i u_j w^k \nabla^j u_l w^l \\ &= \delta^i{}_k (g_{il} - \sigma_{il}) w^k w^l \\ &= |w|_g^2 - |w|_\sigma^2. \end{aligned} \tag{58}$$

Moreover if w is an eigenvector of $D_i u^k$ with respect to an eigenvalue λ, then

$$|Df^{(w)}|^2_\sigma = \lambda^2 |w|^2_\sigma. \tag{59}$$

Now a family of Lagrangian immersions evolving in $\mathbb{R}^{2n}$ will maintain the property to be a graph over L_0 if and only if all eigenvalues of the quadratic matrix $D_i u^k D_j u_k$ stay bounded. By (59) this is equivalent to require

$$|Df^{(w)}|^2_\sigma < \infty$$

for all $w \in TL_0$.

It is desirable to derive uniform bounds for the eigenvalues of $D_j u_i$.

Proof of Theorem 1.3: We need the evolution equation for $|Df^{(w)}|^2_\sigma$. Since $m = df = 0$ we compute

$$\begin{aligned}
\frac{\partial}{\partial t}|Df^{(w)}|^2 &= 2D^k f^{(w)}(\widetilde{\Delta} D_k f^{(w)} + D_k g^{ij} D_i D_j f^{(w)}) \\
&= \widetilde{\Delta}|Df^{(w)}|^2 - 2g^{ij} D_i D_k f^{(w)} D_j D^k f^{(w)} \\
&- 4g^{kn} g^{lm} D_i D_m u_s D_n u^s D^i f^{(w)} D_k D_l f^{(w)}
\end{aligned}$$

which directly followed from

$$\frac{\partial}{\partial t} f^{(w)} = \widetilde{\Delta} f^{(w)},$$

where $\widetilde{\Delta} = g^{ij} D_i D_j$ is the normalized Laplacian. At a fixed point (x_0, t_0) let w be an eigenvector of $D_l u^k$ with respect to an eigenvalue λ. Choose normal coordinates $(x^i)_{i=1,\ldots,n}$ around x_0 such that $D_i u^k$ becomes diagonal. After rotating the coordinate system if necessary we can assume that $w = \frac{\partial}{\partial x^1}$. Then

$$\begin{aligned}
&- 4g^{kn} g^{lm} D_i D_m u_s D_n u^s D^i f^{(w)} D_k D_l f^{(w)} \\
&= -4 \sum_{k,l=1}^{n} \frac{\lambda \lambda_k}{(1+\lambda_l^2)(1+\lambda_k^2)} (D_k D_l f^{(w)})^2 \\
&\leq 4 \max_k \left|\frac{\lambda \lambda_k}{1+\lambda_k^2}\right| \sum_{k,l=1}^{n} \frac{1}{1+\lambda_l^2} (D_k D_l f^{(w)})^2 \\
&\leq 2 \sum_{k,l=1}^{n} \frac{1}{1+\lambda_l^2} (D_k D_l f^{(w)})^2 = 2g^{ij} D_i D_k f^{(w)} D_j D^k f^{(w)}
\end{aligned}$$

provided all eigenvalues λ, λ_k satisfy $|\lambda| \leq 1$. Consequently we must have

$$\frac{\partial}{\partial t}|Df^{(w)}|^2 \leq \widetilde{\Delta}|Df^{(w)}|^2$$

as long as all eigenvalues λ satisfy $\lambda^2 < 1$. Then the parabolic maximum principle proves that $|Df^{(w)}|^2$ remains uniformly bounded in t and consequently by (59) λ^2 stays bounded by the initial bound $1 - \epsilon$. □

5 Longtime existence

The evolution equation for the closed 1-forms $s = u = u_k dx^k$ (49) with $\theta = H - df = d\alpha$ implies that the cohomology class of $u_k dx^k$ is fixed for all t. Consequently there exists a smooth family of functions β such that

$$\begin{aligned} u &= [u] + d\beta, \\ \frac{\partial}{\partial t}\beta &= \alpha, \end{aligned} \tag{60}$$

where as usual we abbreviated the harmonic part of u by $[u]$. We define the normalized function

$$v := \beta - \frac{\int_L \beta d\mu_0}{\int_L d\mu_0}, \tag{61}$$

where $d\mu_0$ denotes the induced initial volume element on L. Let us assume that all third order derivatives of v are uniformly bounded. Since $\int v d\mu_0 = 0$ we see that this implies uniform bounds on all lower order derivatives and then also uniform bounds on all derivatives of $u_k dx^k$ up to second order. Then as in [8] we conclude

Proposition 5.1 *Let $u = s$ be the solution of (49) with $\theta = d\alpha = H - df$ on the maximal time interval $0 \le t < T$. Further let v be the normalization of u as defined in (61). Assume that the second fundamental forms of the Lagrangian graphs corresponding to the generating 1-forms u are uniformly bounded in t. Then the C^∞-norm of v is uniformly bounded for all $t \in [0, T)$ and consequently $T = \infty$. Moreover there exists a time sequence $t_n \to \infty$ such that $v(x, t_n)$ converges in the C^∞-topology to a smooth function $v_\infty(x)$ on L as $n \to \infty$.*

Hence we see that the solution for the modified mean curvature flow in flat cotangent bundles exists for $t \in [0, \infty)$ if $|A|^2$ is uniformly bounded and that under this assumption a subsequence of the normalized function v converges in the C^∞-topology to a smooth limit function v_∞. We will now use Lemma 3.14 together with our energy estimate 3.15 to investigate this in more detail.

Theorem 5.2 *Make the same assumptions as in Proposition 5.1. Then the normalized function v converges in the C^∞-topology to a smooth limit function v_∞ such that the Maslov form $M_i dx^i$ of the Lagrangian immersion in $\mathbb{R}^{2n}$ induced by v_∞ is given by $\frac{1}{\pi} df$.*

Proof: We want to control the L^1-norm of v. For $t > s$ we compute

$$\begin{aligned} \int_L |v(x,t) - v(x,s)| d\mu_0 &= \int_L |\int_s^t \frac{\partial v}{\partial \tau}(x,\tau) d\tau| d\mu_0 \\ &\le \int_L \int_s^t |\frac{\partial v}{\partial \tau}(x,\tau)| d\tau d\mu_0 \\ &= \int_s^t \int_L |\frac{\partial \beta}{\partial \tau} - \frac{\int \frac{\partial \beta}{\partial \tau} d\mu_0}{\int d\mu_0}| d\mu_0 d\tau. \end{aligned}$$

By equation (60) we have $\frac{\partial\beta}{\partial\tau}(x,\tau) = \alpha(x,\tau)$. Therefore

$$\frac{\partial\beta}{\partial\tau} - \frac{\int \frac{\partial\beta}{\partial\tau} d\mu_0}{\int d\mu_0} = \alpha - \frac{\int \alpha d\mu_0}{\int d\mu_0}$$

and then

$$\begin{aligned}\int_L |v(x,t) - v(x,s)| d\mu_0 &\leq \int_s^t \int_L |\alpha - \frac{\int \alpha d\mu_0}{\int d\mu_0}| d\mu_0 d\tau \\ &\leq \int_s^t \int_L |k| + |\frac{\int \alpha d\mu_t}{\int d\mu_t} - \frac{\int \alpha d\mu_0}{\int d\mu_0}| d\mu_0 d\tau.\end{aligned}$$

Since for all τ, t

$$\inf_{x\in L} \alpha(x,\tau) \leq \frac{\int \alpha(x,\tau) d\mu_t}{\int d\mu_t} \leq \sup_{x\in L} \alpha(x,\tau),$$

we must have

$$|\frac{\int \alpha d\mu_t}{\int d\mu_t} - \frac{\int \alpha d\mu_0}{\int d\mu_0}| \leq \text{osc } \alpha.$$

From Hölder's inequality we obtain

$$\int_L |v(x,t) - v(x,s)| d\mu_0 \quad \leq \quad (\int d\mu_0)^{\frac{1}{2}} \int_s^\infty (\int_L k^2 d\mu_0)^{\frac{1}{2}} d\tau + \int d\mu_0 \int_s^\infty \text{osc } \alpha d\tau.$$

Since all metrics are uniformly equivalent and L is compact we can find a positive constant C_{11} such that

$$\int k^2 d\mu_0 \leq C_{11} \int k^2 d\mu_t = 2C_{11}E.$$

Then equations (34) and (35) imply the existence of two positive constants C_{12}, C_{13} such that

$$\int_L |v(x,t) - v(x,s)| d\mu_0 \leq C_{12} \int_s^\infty e^{-C_{13}\tau} d\tau.$$

This shows that v is a Cauchy sequence in L^1. Since we have already shown in Proposition 5.1 that one can extract a subsequence that converges in the C^∞-topology to a smooth limit function v_∞ it follows by standard arguments (e.g. see [8]) that v converges to v_∞ in C^∞. Using the estimate for the oscillation of α and the fact that $d\alpha = H - df$ we see that α converges to a constant and $H - df$ tends to zero. Since the Maslov form $M_i dx^i$ is given by $M_i = \frac{1}{\pi} H_i$ we obtain the theorem. □

Now we introduce these quantities

$$\begin{aligned} S_2 &:= D_l u_k D^l u^k = |D_l u_k|^2, \\ S_3 &:= D_l D_m u_k D^l D^m u^k = |D_l D_m u_k|^2. \end{aligned}$$

From equation (49) we obtain

$$\frac{\partial}{\partial t}du = -d\alpha,$$

where α is the function defined as in Theorem 3.8.

If $\lambda_1, \ldots, \lambda_n$ are the n eigenvalues of $D_l u^k$ then we set

$$\gamma_k := -\text{arctan}\lambda_k = -\arg(1 + i\lambda_k).$$

Define

$$\begin{aligned} a &:= \mathbf{Im}(\det_{\mathbb{C}}(\delta^k{}_l + iD^k u_l)), \\ b &:= \mathbf{Re}(\det_{\mathbb{C}}(\delta^k{}_l + iD^k u_l)). \end{aligned}$$

Differentiating $\arctan(\frac{a}{b})$ and using equations (50) and (51) we conclude that the Lagrangian angle γ, i.e. the potential for H, is given by

$$\gamma = \gamma_1 + \cdots + \gamma_n = -\arctan(\frac{a}{b}), \tag{62}$$

The evolution equation (49) can be viewed as a parabolic equation of Monge-Ampère type in the following way. a and b are the imaginary and real parts of $\det_{\mathbb{C}}(\delta^k{}_l + iD^k u_l)$ and $a^2 + b^2 = \det_{\mathbb{R}}(\sigma^{kj} g_{jl})$. Thus

$$\frac{\det_{\mathbb{C}}(\sigma^k{}_l + iD^k u_l)}{\sqrt{\det_{\mathbb{R}}(\sigma^{kj} g_{jl})}} = \frac{b + ia}{\sqrt{a^2 + b^2}} = e^{-i\gamma}.$$

Consequently

$$\gamma = i \log(\frac{\det_{\mathbb{C}}(\sigma^k{}_l + iD^k u_l)}{\sqrt{\det_{\mathbb{R}}(\sigma^{kj} g_{jl})}}). \tag{63}$$

Moreover $d\alpha = d\gamma - df$.

Remark: *It is actually not a restriction to assume that $u_k dx^k$ is exact. To see this observe that the harmonic 1-forms on L are exactly those forms of the form $c_i dx^i$, where c_i are constants. Using the Hodge decomposition theorem we find that $u_k dx^k$ can be decomposed into $\widetilde{u}_k dx^k + c_k dx^k$, where $\widetilde{u}_k dx^k$ is exact on L. By letting*

$$\widetilde{u}(x, t) := u(x, t) - c_i x^i$$

we obtain

$$\begin{aligned} \widetilde{u}_k &= u_k - c_k, \\ \widetilde{u}_{kl} &= u_{kl}, \\ \frac{\partial}{\partial t}\widetilde{u} &= \frac{\partial}{\partial t}u. \end{aligned}$$

Consequently $\widetilde{u}$ is also a solution of (49). Choosing the right constants c_k we can assume that $\widetilde{u}_k dx^k$ is exact. This transformation corresponds to an isometry (a translation) since the height function u_k gets transformed into the new height function $\widetilde{u}_k = u_k + c_k$.

In view of the last remark we will now assume w.l.o.g. that $u_k dx^k = du$ is exact (on the torus L_0). Integrating equation (49) then implies that we can find a smooth family of functions $\phi(t)$ depending only on t such that

$$\frac{\partial}{\partial t} u = \arctan\left(\frac{a}{b}\right) + f + \phi. \tag{64}$$

On the other hand if $\phi(t)$ is a smooth family of functions for which (64) admits a solution u, then the Lagrangian graphs generated by du solve the modified mean curvature flow. We can therefore choose $\phi = 0$. The modified mean curvature flow then boils down to solve the nonlinear parabolic PDE:

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) &= \arctan\left(\frac{a}{b}\right) + f, \\ u(x,0) &= u_0(x). \end{aligned} \tag{65}$$

where $u_0(x)$ denotes the initial generating function. Corollary 3.9 implies

Lemma 5.3 *If u is a smooth family of functions solving (65), then*

$$\begin{aligned} \min_L \frac{\partial}{\partial t} u_{|0} \leq \min_L \frac{\partial}{\partial t} u_{|t} &\leq \max_L \frac{\partial}{\partial t} u_{|t} \leq \max_L \frac{\partial}{\partial t} u_{|0}, \\ osc \frac{\partial}{\partial t} u_{|t} &\leq osc \frac{\partial}{\partial t} u_{|0}. \end{aligned}$$

We obtain the following evolution equations

$$\frac{\partial}{\partial t} u_k = \widetilde{\Delta} u_k + f_k.$$

$$\begin{aligned} \frac{\partial}{\partial t} S_2 &= \widetilde{\Delta} S_2 - 4 g^{is} g^{jt} u^{kl} u_t{}^r u_{srl} u_{ijk} \\ &- 2 g^{ij} u^{kl}{}_j u_{kli} + 2 u^{kl} f_{kl}, \end{aligned} \tag{66}$$

$$\begin{aligned} \frac{\partial}{\partial t} S_3 &= \widetilde{\Delta} S_3 - 2 g^{is} T_s{}^{jkp} T_{ijkp} \\ &+ 18 g^{jn} g^{km} g^{st} u_n{}^r u_m{}^q u_{sr}{}^l u_{pkl} u_{tq}{}^a u^p{}_{ja} \\ &- 4 g^{is} g^{jt} u_t{}^{rp} u_{sr}{}^l u_{ij}{}^k u_{pkl} \\ &+ 4 g^{is} g^{jn} g^{tb} u_{ab} u_t{}^r u_n{}^{ap} u_{srl} u_{ijk} u_p{}^{kl} \\ &+ 12 g^{is} g^{jn} g^{tb} u^a{}_n u_t{}^r u_{bap} u_{srl} u_{ijk} u^{pkl} \\ &+ 2 u^{pkl} f_{pkl}, \end{aligned} \tag{67}$$

where for simplicity we have set $f_{kl} = D_k D_l f, u_{kl} = D_k u_l, u_{klj} = D_k D_l u_j$ etc. and

$$T_{sjkp} := u_{sjkp} - 3 \sigma_{ij} g^{it} u_{tr} u_s{}^{rl} u_{pkl}$$

We can estimate

$$18 g^{jn} g^{km} g^{st} u_n{}^r u_m{}^q u_{sr}{}^l u_{pkl} u_{tq}{}^a u^p{}_{ja} \leq 18 S_2 S_3^2$$

and similarly

$$\begin{aligned}
-4g^{is}g^{jt}u_t{}^{rp}u_{sr}{}^{l}u_{ij}{}^{k}u_{pkl} &\leq 4S_3^2,\\
4g^{is}g^{jn}g^{tb}u_{ab}u_t{}^{r}u_n{}^{ap}u_{srl}u_{ijk}u_p{}^{kl} &\leq 4S_2S_3^2,\\
12g^{is}g^{jn}g^{tb}u^a{}_n u_t{}^{r}u_{bap}u_{srl}u_{ijk}u^{pkl} &\leq 12S_2S_3^2,\\
-2g^{ij}u_i{}^{kl}u_{jkl} &\leq -2\frac{S_3}{1+S_2}.
\end{aligned}$$

Using Schwarz' inequality and the previous estimates together with the evolution equations for S_2 and S_3 we get the following inequality for $q := cS_2 + S_3$ with a constant c to be determined later

$$\frac{\partial}{\partial t}q \leq \widetilde{\Delta}q + 4qS_3 - 2c\frac{S_3}{1+S_2} + 34S_2S_3^2 + c\epsilon q + \frac{c+1}{c\epsilon}M_3, \tag{68}$$

where M_3 is defined as

$$M_3 := \max\{\sup_L |D_kD_pf|^2, \sup_L |D_kD_qD_pf|^2\}.$$

and $\epsilon > 0$ is arbitrary. We will need the next Lemma.

Lemma 5.4 *Let η be a closed smooth 1-form on a compact orientable connected Riemannian manifold (L, σ_{ij}) of diameter δ. If $[\eta]$ denotes the harmonic part of η then for any $x_0 \in L$ we obtain the inequality*

$$|\eta|(x_0) \leq \max_L |[\eta]| + \delta \max_L |D\eta|. \tag{69}$$

Proof: $D|\eta|$ exists whenever $|\eta| > 0$. If $|\eta|(x_0) = 0$ we are done. Otherwise we destinguish two cases. If $|\eta|(x) \neq 0$, $\forall x \in L$, then we use the decomposition theorem to split η into a harmonic part and an exact part, i.e. $\eta = [\eta] + d\beta$ with a smooth function β. Let x be a point where β assumes its minimum and choose a smooth curve γ of length smaller than δ connecting x_0 and x. Integration gives

$$\begin{aligned}
|\eta|(x_0) &= |\eta|(x) + \int_\gamma d|\eta|\\
&\leq |[\eta]|(x) + \delta \max_L |D|\eta||\\
&\leq \max_L |[\eta]| + \delta \max_L |D\eta|,
\end{aligned}$$

where we used that

$$|D|\eta||^2 = \frac{1}{|\eta|^2}D_j\eta_i\eta^j D_k\eta^i\eta^k \leq D_j\eta_i D^j\eta^i = |D\eta|^2.$$

If $|\eta|(x) = 0$ somewhere on L we can find a smooth curve γ of length smaller than δ connecting x_0 and a point $x \in L$ with $|\eta|(x) = 0$ in such a way that $|\eta|(y) \neq 0$ for all points $y \neq x$ on γ. Integrating gives

$$\begin{aligned}
|\eta|(x_0) &= |\eta|(x) + \int_\gamma d|\eta|\\
&\leq \delta \max_L |D|\eta||\\
&\leq \max_L |[\eta]| + \delta \max_L |D\eta|.
\end{aligned}$$

□

Theorem 5.5 *Let L be a compact flat n-dimensional Riemannian manifold of diameter δ and let $c := \frac{1}{2n\delta^2}$. Assume that $u_k dx^k$ is a smooth family of 1-forms generating Lagrangian graphs $F(x) = (x, u_1(x), \ldots, u_n(x)), x \in L$ in the cotangent bundle (with induced flat metric) of L that evolve according to the modified mean curvature flow $\frac{d}{dt}F = g^{ij}(d_i f - H_i)\nu_j$, where df is the differential of a fixed smooth function f on L with*

$$M_3 \le \frac{c}{c+1}(\frac{c}{17})^2,$$

and M_3 is defined as

$$M_3 := \max\{\sup_L |D_k D_p f|^2, \sup_L |D_k D_q D_p f|^2\}.$$

If $q := cS_2 + S_3$ satisfies

$$q < \frac{c}{50}, \text{ at } t = 0,$$

then

$$q < \frac{c}{50}, \quad \forall t.$$

In particular the modified mean curvature flow exists for all $t \in [0, \infty)$ and the smooth 1-form $du = u_k dx^k$ converges in the C^∞-topology to a smooth limit 1-form du_∞ such that the mean curvature form $H_i dx^i$ of the Lagrangian graph generated by du_∞ is given by the 1-form df.

Proof: Let us assume that $(x_0, t_0) \in L_0 \times [0, t_0]$ with $t_0 > 0$ is a point where $q(x_0, t_0) = \frac{c}{50} = \max_{L_0 \times [0,t_0]} q$. Let $h_{(1)}, \ldots, h_{(n)}$ be a set of n harmonic and even parallel 1-forms on L_0 such that

$$\sum_{w=1}^{n} h_{(w)i} h_{(w)j} = \sigma_{ij} \tag{70}$$

(Here we assume that (x^i) are normal coordinates). For each $w = 1, \ldots, n$ define the function

$$f_{(w)} := u^i h_{(w)i}.$$

We apply Lemma (5.4) to $f_{(w)}$ and sum over w to obtain

$$\begin{aligned} S_2 &= \sum_{w=1}^{n} D^i u^j h_{(w)j} D_i u^l h_{(w)l} = \sum_{w=1}^{n} |Df_{(w)}|^2 \\ &\le \delta^2 \sum_{w=1}^{n} \max_L |D^2 f_{(w)}|^2 \le n\delta^2 \max_L \sum_{w=1}^{n} |D^2 f_{(w)}|^2 \\ &\le n\delta^2 \max_L S_3. \end{aligned}$$

Consequently

$$q(x_0) = cS_2(x_0) + S_3(x_0) \leq cn\delta^2 \max_L S_3 + S_3(x_0) \leq cn\delta^2 q(x_0) + S_3(x_0) \quad (71)$$

from which we conclude that at (x_0, t_0)

$$S_3(x_0) \geq (1 - cn\delta^2)q \geq \frac{q}{2}.$$

Then inequality (68) implies with $\epsilon = \frac{1}{2}$

$$\begin{aligned} 0 \leq \frac{\partial}{\partial t} q &\leq 4qS_3 - 2c\frac{S_3}{1+S_2} + 34S_2S_3^2 + c\epsilon q + \frac{c+1}{c\epsilon}M_3^2 \\ &\leq 4q^2 - 2c^2(1-cn\delta^2)\frac{q}{c+q} + \frac{34}{c}q^3 + \frac{c}{2}q + 2\frac{c+1}{c}M_3^2 \\ &= c^2(\frac{1}{25^2} - \frac{1}{51} + \frac{17}{50^2 \cdot 25} + \frac{1}{100}) + 2\frac{c+1}{c}M_3^2 \\ &\leq c^2(\frac{1}{25^2} - \frac{1}{51} + \frac{17}{50^2 \cdot 25} + \frac{1}{100} + \frac{2}{17^2}) \\ &< 0. \end{aligned}$$

This contradiction proves that $q < \frac{c}{50}$ for all t. But this proves that $|A|^2$ is uniformly bounded for all t and that the Lagrangian submanifolds stay graphs over the zero section. Then the theorem follows from Theorem 5.2. □

Proof of Theorem 1.1: This is now an easy consequence of Theorem 5.5. Assume f is chosen such that

$$\max\{\sup_L |D_k D_p f|^2, \sup_L |D_k D_q D_p f|^2\} \leq \frac{c}{c+1}(\frac{c}{17})^2.$$

Choose the zero section in T^*L as the initial Lagrangian graph. Since the 1-forms $u_k dx^k$ evolve by an exact form it follows that $u_k dx^k$ stays exact. □

Proof of Theorem 1.2: Again we use Theorem 5.5. The evolution equation for u^k in the case $f = 0$ is given by

$$\frac{\partial}{\partial t} u^k = \widetilde{\Delta} u^k.$$

Applying the Harnack inequality 3.13 we can derive the estimate

$$\operatorname{osc}(u^k) \leq ce^{-\lambda t},$$

for two positive constants λ, C. This proves that u^k converges to a constant and therefore $du_\infty = 0$. □

Let us end this article with a last observation. In the special case $n = 1$ the metric can be written as $g = 1 + (u'')^2$. Let $\lambda := u''$. Then $S_2 = \lambda^2$ and the evolution equation (66) reduces to

$$\frac{\partial}{\partial t}\lambda^2 = \widetilde{\Delta}\lambda^2 - 2(3\lambda^2 + 1)\frac{(\lambda')^2}{(1+\lambda^2)^2} + 2\lambda f'' \leq \widetilde{\Delta}\lambda^2 + 2\lambda f''.$$

Since f'' is fixed in t and therefore bounded we can conclude that λ^2 can grow at most exponentially in t, i.e. there exist constants k, l depending only on $\max f''$ such that $\lambda^2 \leq ke^{lt}$. In particular λ^2 is bounded on any finite time interval. This proves that a periodic curve over $\mathbb{R}$ evolving by the modified mean curvature flow must stay a graph over $\mathbb{R}$ and it also proves longtime existence for the flow. Although the modified flow for periodic graphs over $\mathbb{R}$ always admits a longtime solution it does not necessarily converge to a smooth limiting graph with $H = df$. An example is illustrated in Figure 1. One can see that the height increases exponentially fast. On the other hand the flow cannot converge since the Lagrangian angle for a periodic graph over the real axis in $\mathbb{R}^2$ (this serves as a model for the cotangent bundle of S^1) must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This is not the case for the given function f.

References

[1] Abresch, U.; Langer, J.: The normalized curve shortening flow and homothetic solutions. *J. Differential Geom.* **23** *(1986), no. 2, 175–196.*

[2] Altschuler, Steven J.: Singularities of the curve shrinking flow for space curves. *J. Differential Geom.* **34** *(1991), no. 2, 491–514.*

[3] Altschuler, Steven J.: Shortening space curves. *Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), 45–51, Proc. Sympos. Pure Math.,* **54***, Part 3, Amer. Math. Soc., Providence, RI, 1993.*

[4] Ambrosio, Luigi; Soner, Halil Mete: Level set approach to mean curvature flow in arbitrary codimension. *J. Differential Geom.* **43** *(1996), no. 4, 693–737.*

[5] Andrews, B.: Harnack inequalities for evolving hypersurfaces, *Math. Zeitschrift,* **217***, 1994, 179–197.*

[6] Angenent, Sigurd: On the formation of singularities in the curve shortening flow. *J. Differential Geom.* **33** *(1991), no. 3, 601–633.*

[7] Brakke, Kenneth A.: The motion of a surface by its mean curvature. *Mathematical Notes,* **20***. Princeton University Press, Princeton, N.J., 1978. i+252 pp. ISBN: 0-691-08204-9.*

[8] Cao, H.-D.: Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.* **81** *(1985), 359–372.*

[9] Charlap, L.S.: Bieberbach groups and flat manifolds. *Springer Verlag, Universitext (1986).*

[10] Chen, Yun Gang; Giga, Yoshikazu; Goto, Shun'ichi: Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.* **33** *(1991), no. 3, 749–786.*

[11] Chopp, David L.: Computation of self-similar solutions for mean curvature flow. *Experiment. Math.* **3** *(1994), no. 1, 1–15.*

[12] Ecker, Klaus: Interior estimates and longtime solutions for mean curvature flow of noncompact spacelike hypersurfaces in Minkowski space. *J. Differential Geom.* **46** *(1997), no. 3, 481–498.*

[13] Ecker, Klaus and Huisken, Gerhard: Mean curvature evolution of entire graphs *Ann. of Math.* **130** *(1989), 453–471*

[14] Gage, Michael E.: An isoperimetric inequality with applications to curve shortening. *Duke Math. J.* **50** *(1983), no. 4, 1225–1229.*

[15] Grayson, Matthew A.: The shape of a figure-eight under the curve shortening flow. *Invent. Math.* **96** *(1989), no. 1, 177–180.*

[16] Huisken, Gerhard: Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.* **20** *(1984), no. 1, 237–266.*

[17] Ilmanen, Tom: Generalized flow of sets by mean curvature on a manifold. *Indiana Univ. Math. J.* **41** *(1992), no. 3, 671–705.*

[18] Morvan, J.-M.: On Lagrangian immersions. *Symplectic geometry (Toulouse, 1981), 167–171, Res. Notes in Math.,* **80**, *Pitman, Boston, Mass.-London, 1983.*

[19] Morvan, Jean-Marie: Minimal Lagrangian submanifolds. A survey. *Geometry and topology of submanifolds, III (Leeds, 1990), 206–226, World Sci. Publishing, River Edge, NJ, 1991.*

[20] Oh, Yong-Geun: Volume minimization of Lagrangian submanifolds under Hamiltonian deformations. *Math. Z.* **212** *(1993), no. 2, 175–192.*

[21] Smoczyk, Knut: Starshaped hypersurfaces and the mean curvature flow. *Manuscripta Math.* **95** *(1998), no. 2, 225–236.*

[22] Smoczyk, Knut: Harnack inequality for the Lagrangian mean curvature flow. *Calc. of Var.* **8**, *247–258 (1999)*

[23] Smoczyk, Knut: Nonexistence of minimal Lagrangian spheres in hyperKähler manifolds. *Calc. of Var.* **10**, *41–48 (2000)*

[24] Soner, H. M.; Souganidis, P. E.: Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature. *Comm. Partial Differential Equations* **18** *(1993), no. 5-6, 859–894.*

[25] Stahl, Axel: Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition. *Calc. Var. Partial Differential Equations* **4** *(1996), no. 4, 385–407.*

[26] Strominger, A.; Yau, S.-T.; Zaslow, E.: Mirror Symmetry is T-duality, *Nuclear Phys.* **B 479** *(1996), 243–259.*

[27] Tachibana, M.; Okumura, S.-I.: On the almost complex structure of tangent bundles of Riemannian spaces. *Tohoku Math. Jour.,* **14** *(1962), 156–161.*

[28] Visintin, Augusto: Nucleation and mean curvature flow. *Comm. Partial Differential Equations* **23** *(1998), no. 1-2, 17–53.*

[29] White, Brian: Partial regularity of mean-convex hypersurfaces flowing by mean curvature. *Internat. Math. Res. Notices 1994, no. 4,* **186** *ff., approx. 8 pp. (electronic).*

[30] Wolf, J.A.: Spaces of constant curvature. *McGraw-Hill, Inc. (1967).*