Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

Guofang Wang

Preprint no.: 24 2000

$\mathbb{S}^1$-INVARIANT HARMONIC MAPS FROM $\mathbb{S}^3$ TO $\mathbb{S}^2$

GUOFANG WANG

ABSTRACT. In this note, we show that a homotopy class of $[\mathbb{S}^3, \mathbb{S}^2]$ admits an $\mathbb{S}^1$-invariant harmonic map if and only if its Hopf invariant is $\pm k^2$ for some integer k, where $\mathbb{S}^3$ is the unit 3-sphere and $\mathbb{S}^2$ is a 2-sphere with an arbitrary metric.

1. INTRODUCTION

Let (M^m, g) and (N^n, h) be two compact Riemannian manifolds with dimension $m, n \geq 2$ respectively and $\mathcal{C}(M,N) = H^1 \cap C^0(M,N)$. A map $f \in \mathcal{C}(M,N)$ is harmonic if it is a critical point of the energy functional

$$E(f) = \frac{1}{2}\int_M e(f) dvol(M)$$

in $\mathcal{C}(M,N)$, where $e(f)$ is the energy density of f and $dvol(M)$ is the volume element of M. This means that, for any smooth family of maps $f_s \in \mathcal{C}(M,N)$ with $f_0 = f$,

$$\frac{d}{ds}_{|s=0} E(f_s) = 0.$$

In local coordinates (x^i) and (y^α) on M and N, the energy density is defined by

$$e(f)(x) = g^{ij}(x)\frac{\partial f^\alpha(x)}{\partial x^i}\frac{\partial f^\beta(x)}{\partial x^j}h_{\alpha\beta}(f(x)).$$

Due to a regularity result for Morrey [19], any harmonic maps in $\mathcal{C}(M,N)$ is smooth. In this note, we are interested in the existence

1991 *Mathematics Subject Classification.* 58E20.

Key words and phrases. Harmonic map, Hopf invariant.

of harmonic maps. The existence problem for harmonic maps can be formulated in the following way:

Does a given homotopy class of $[M, N]$ admit a harmonic representative?

When N has non-positive curvature, in a celebrated paper [11], Eells and Sampson showed that for any given homotopy class there is a harmonic representative. When the curvature of N is not non-positive, e.g. $N = \mathbb{S}^2$, the existence problem of harmonic maps becomes more delicate. There are a lot of existence and non-existence results, see [6, 7] and [27]. Roughly speaking, there are basically three kinds of existence results. One is when M has dimension two. In this case, the energy functional E is conformally invariant and various variational approaches were developed to study the existence and non-existence of harmonic maps. We refer to [12, 18, 21, 2, 15, 16, 25]. Another is reduction to a one dimensional problem, for example, the Smith construction, the Hopf construction of harmonic maps (cf. [23, 3, 8, 9, 20, 26]). In this case, the harmonic map equation is reduced to an ordinary differential equation. The third type results are obtained by direct constructions, for example, the twistor method ([10]) and the algebraic methods ([24] and [13]). We are not able to list all results here. For the interested reader, we refer to two surveys [6] and [7] and a very complete list of papers on harmonic maps [27].

Here we consider a simple case which is different from the three cases mentioned above. Let $\mathbb{S}^3$ be the unit 3-sphere with the standard metric g_0 and $\mathbb{S}^2$ the 2-sphere with an arbitrary metric h_0. It is well-known that $[\mathbb{S}^3, \mathbb{S}^2] = \pi_3(\mathbb{S}^2) = \mathbb{Z}$ and to each class corresponds an integer—its Hopf invariant. In [23], Smith gave examples of harmonic maps from $\mathbb{S}^3$ to $\mathbb{S}^2$ of Hopf invariant $\pm k^2$, for any integer k, by means of the Hopf fibration and posed the following problem (see also [7]):

Which classes of $[\mathbb{S}^3, \mathbb{S}^2]$ have harmonic representatives?

Recently, Eells and Ratto [8] obtained the second kind of existence result by considering the following special maps: For any two integers k, l, let $f_{k.l} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $f_{k,l}(e^{i\xi}, e^{i\eta}) = e^{i(k\xi + l\eta)}$, and consider

$$\phi_{k,l} : \mathbb{S}^3 \to \mathbb{S}^2$$

defined by

$$\phi(\sin s e^{i\xi}, \cos s e^{i\eta}) = (\sin \alpha(s) f_{k,l} e^{i\xi}, \cos \alpha(s) e^{i\eta}),$$

where $\alpha(s) : [0, \pi/2] \to [0, \pi]$ satisfies the boundary conditions $\alpha(0) = 0$ and $\alpha(\pi/2) = \pi$. This is the α-Hopf construction. They showed that such a map $\phi_{k,l}$ is a harmonic map for some α if and only if $k = \pm l$. It is easy to check that $\phi_{k,l}$ has the Hopf invariant kl. So if such a map is harmonic, then its Hopf invariant is $\pm k^2$. The method for obtaining this result involves an ordinary differential equation for α. Several years ago, W. Y. Ding [4] conjectured that a homotopy class of $[\mathbb{S}^3, \mathbb{S}^2]$ admits a harmonic representative if and only if its Hopf invariant is $\pm k^2$ for some integer k. A slightly stronger conjecture was proposed by Eells in [28] : Is every harmonic map from $\mathbb{S}^3 \to \mathbb{S}^2$ a harmonic morphism? We remark that a homotopy class of $[\mathbb{S}^3, \mathbb{S}^2]$ contains a harmonic morphism if and only if its Hopf invariant is $\pm k^2$ for some integer k. Thus, the latter implies the former.

Due to the result of [5], for any homotopy class of $[\mathbb{S}^3, \mathbb{S}^2]$, there is a metric g on $\mathbb{S}^3$ conformally equivalent to the standard metric such that this class admits a harmonic representative from $(\mathbb{S}^3, g)$ to $\mathbb{S}^2$. Thus, the non-existence of harmonic maps from $\mathbb{S}^3$ to $\mathbb{S}^2$ may be delicate.

In this note, we consider $\mathbb{S}^1$-invariant maps defined in section 2 below. The problem of the existence of $\mathbb{S}^1$-invariant harmonic maps can be reduced to the existence of critical points of a functional defined on the space of maps from a two dimensional orbifold to $\mathbb{S}^2$. We obtain

Theorem 1.1. *Any homopoty class of* $[\mathbb{S}^3, \mathbb{S}^2]$ *admits such* $\mathbb{S}^1$*-invariant harmonic representatives if and only is its Hopf invariant is* $\pm k^2$ *for some integer* k.

In fact, such a harmonic map is a harmonic morphism.

2. $\mathbb{S}^1$-invariant maps and the Hopf invariant

Let $\mathbb{S}^3 = \{(z, w) \in \mathbb{C} \times \mathbb{C} \,|\, |z|^2 + |w|^2 = 1\}$ be the unit three dimensional sphere in $\mathbb{R}^4$ and $\mathbb{S}^2$ the 2-sphere in $\mathbb{R}^3$. Given two relatively prime integers $k \geq l \geq 1$, define an isometric action of $\mathbb{S}^1$ on $\mathbb{S}^3$

$$T_{k,l} : \mathbb{S}^1 \to Iso(\mathbb{S}^3)$$

by

$$T^\theta_{k,l}(w, z) = (e^{ik\theta} w, e^{il\theta} z)$$

for any $\theta \in \mathbb{S}^1$ and $(w, z) \in \mathbb{S}^3$. A continuous map $f : \mathbb{S}^3 \to \mathbb{S}^2$ is $\mathbb{S}^1$-invariant under the action T if for any $\theta \in \mathbb{S}^1$ and $(z, w) \in \mathbb{S}^3$

$$f(T^\theta_{k,l}(w, z)) = f(w, z).$$

For a smooth map $f : \mathbb{S}^3 \to \mathbb{S}^2$, its Hopf invariant is defined by

$$H(f) = \int_{\mathbb{S}^3} dw \wedge w,$$

where w is a one-form on $\mathbb{S}^3$ so that $dw = f^*(\alpha)$ and α is a generator of $H^2_{DR}(\mathbb{S}^2)$ (cf. [1]). We also can define the Hopf invariant for a continuous map. If f is an $\mathbb{S}^1$-invariant map, then this Hopf invariant

$$H(f) = kln^2,$$

for some integer n. This can be seen as follows. Define $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ by

$$\pi(w, z) = (2\frac{\bar{w}^l}{|w|^{l-1}} \frac{z^k}{|z|^{k-1}}, |w|^2 - |z|^2).$$

π is continuous, but is not smooth. It is easy to compute that $H(\pi) = kl$. In fact, π is the composition of the following two maps $\sigma : \mathbb{S}^3 \to \mathbb{S}^3$ and $v : \mathbb{S}^3 \to \mathbb{S}^2$ defined by

$$\sigma : (w, z) \to (\frac{w^l}{|w|^{l-1}}, \frac{z^k}{|z|^{k-1}})$$

and

$$v : (w, z) \to (-2\bar{w}z, |w|^2 - |z|^2).$$

Here v is the Hopf map of Hopf invariant 1. We have that $H(\pi) = \deg \sigma H(v)$ (see e.g. [14]) and $\deg \sigma = kl$. Let $u_f : Q \to \mathbb{S}^2$ be defined in the next section (see (3.2) below). Here Q is a topological sphere. Clearly $f = u_f \circ \pi$. From [14],

$$H(f) = H(\pi)(\deg u_f)^2 = kln^2,$$

for some integer n. Now we restate our result as follows

Theorem 2.1. *If f is a harmonic $\mathbb{S}^1$-map w.r.t. $T_{k,l}$, then $k = l$. In this case, the Hopf invariant of f, $H(f) = k^2n^2$ for some integer n.*

3. Reduction to a 2-dimensional case

Given two relatively prime integers $k \geq l \geq 1$, Let $T = T_{k,l}$ be the $\mathbb{S}^1$-action defined in previous section. Let

$$\mathcal{C}(\mathbb{S}^3, \mathbb{S}^2)^T = \{f \in H^1 \cap C^0(\mathbb{S}^3, \mathbb{S}^2) |\ f \text{ is an } \mathbb{S}^1\text{-map w.r.t.} T\}.$$

First, we have

Lemma 3.1. *Let $f \in \mathcal{C}(\mathbb{S}^3, \mathbb{S}^2)^T$. Then f is harmonic if and only if f is a critical point of $E_{|\mathcal{C}(\mathbb{S}^3,\mathbb{S}^2)^T}$.*

Proof: See V(1.4) in [9]. □

In fact, for such a harmonic map, we can characterize it as a critical point of another functional as follows.

Let $Q = \mathbb{S}^3/T_{k,l}$ be the quotient space. When $k > l$, Q is an orbifold with two conical singularities of angles $\frac{2\pi}{k}$ and $\frac{2\pi}{l}$ respectively. For the definition of *orbifold*, see [22].

Let $\rho : \mathbb{S}^3 \to Q$ be the projection map. Set $I = \{(\theta, \eta) | \theta \in [0, 2\pi), \eta \in [0, \pi]\}$ and define the map $\psi : I \to \mathbb{S}^3$ by $\psi(\theta, \eta) = (0, \theta, \eta/2)$ in the cylindrical coordinates on $\mathbb{S}^3$. Clearly, $\rho \circ \psi : I \to Q$ serves as a coordinate chart of Q and $\rho : (\mathbb{S}^3, g_0) \to (Q, h)$ is a Riemannian submersion (see [9]), where

$$h = \frac{k^2 \sin^2 \eta}{k^2 \sin^2(\eta/2) + l^2 \cos^2(\eta/2)} d\theta^2 + d\eta^2, \tag{3.1}$$

in the coordinates θ, η. Note that $Q = \mathbb{S}^2$ (up to homothety) if and only if $k = l$.

Using the metric h, we can define the Sobolev space $H^1(Q, \mathbb{S}^2)$. Let $\mathcal{C}(Q, \mathbb{S}^2) = H^1(Q, \mathbb{S}^2) \cap C^0(Q, \mathbb{S}^2)$. We identify $\mathcal{C}(\mathbb{S}^3, \mathbb{S}^2)^T$ with $\mathcal{C}(Q, \mathbb{S}^2)$ as follows. Let $f \in \mathcal{C}(\mathbb{S}^3, \mathbb{S}^2)$. We define a map $u_f : Q \to \mathbb{S}^2$ by

$$u_f(x) = f(\rho^{-1}(x)) \tag{3.2}$$

for $x \in Q$. Since f is an $\mathbb{S}^1$-map, u_f is well defined and continuous. It is easy to check that

$$E(f) = \int_Q |\nabla u_f|_h^2 \, W dvol(h), \tag{3.3}$$

where $W : Q \to \mathbb{R}$ is defined by

$$W(\eta, \theta) = (k^2 \sin^2(\eta/2) + l^2 \cos^2(\eta/2))^{\frac{1}{2}}.$$

We remark that $W(\eta)$ is the length of $\mathbb{S}^1$ orbit of the point $\psi(\theta, \eta)$. Since $1 \le l \le W(\eta, \theta) \le k$, by (3.3) $u_f \in H^1(Q, \mathbb{S}^2)$. Therefore $u_f \in \mathcal{C}(Q, \mathbb{S}^2)$. On the other hand, for any $u \in \mathcal{C}(Q, \mathbb{S}^2)$, by (3.3) it is also easy to check that $f_u = u \circ \rho \in \mathcal{C}(\mathbb{S}^3, \mathbb{S}^2)^T$. From Lemma 3.1, we now have the following

Lemma 3.2. *Let f be a smooth $\mathbb{S}^1$-map from $\mathbb{S}^3$ to $\mathbb{S}^2$. The f is harmonic if and only if u_f is a critical point of the following functional*

$$J(u) = \int_Q |\nabla u|_h^2 \, W dvol(h) \tag{3.4}$$

in the space $\mathcal{C}(Q, \mathbb{S}^2)$.

Proof: First, Lemma 3.1 implies that a harmonic map $f \in \mathcal{C}(\mathbb{S}^3, \mathbb{S}^2)^T$ satisfies

$$\frac{d}{dt}_{|t=0} E(f_t) = 0, \tag{3.5}$$

for any smooth family of maps $f_t \in \mathcal{C}(\mathbb{S}^3, \mathbb{S}^2)^T$ with $f_0 = f$. From the discussion above, we know that any smooth family of $f_t \in \mathcal{C}(\mathbb{S}^3, \mathbb{S}^2)^T$ with $f_0 = f$ corresponds to a smooth family of maps $u_t \in \mathcal{C}(Q, \mathbb{S}^2)$ with $u_0 = u_f$. Therefore, in view of (3.3) we know that (3.5) is equivalent to

$$\frac{d}{dt}_{|t=0} J(u_t) = 0, \tag{3.6}$$

for any smooth family of maps $u_t \in \mathcal{C}(Q, \mathbb{S}^2)$ with $u_0 = u_f$. This proves the Lemma. □

Remark. When $k = l$, $Q = \mathbb{S}^2$ (up to homothety), u_f is smooth and W is a constant. In this case, f is a smooth harmonic map if and only if u_f is a harmonic map from $\mathbb{S}^2$ to $\mathbb{S}^2$. Hence, there are many harmonic maps from $\mathbb{S}^3$ to $\mathbb{S}^2$ of Hopf invariant $\pm k^2$. Moreover, in this case, all such harmonic maps are harmonic morphisms.

4. Proof of Theorem 2.1

Proof of Theorem 2.1. Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be a smooth harmonic $\mathbb{S}^1$-map and $u_0 = u_f$. To prove the Theorem, we first claim that there is a family of diffeomorphisms $\alpha_s : [0, \pi] \to [0, \pi]$ such that

(a) $\alpha_0 = \mathrm{id}$,

(b) $\alpha_s(0) = 0$ and $\alpha_s(\pi) = \pi$,

(c) $\alpha'_s(\eta) := \frac{\partial \alpha_s}{\partial \eta}(\eta) > 0$ in $[0, \pi]$ and $\alpha'_s(0) = \alpha'_s(\pi) = 1$,

(d) $\frac{W(\alpha_s)}{\sin \alpha_s}(\eta)\alpha'_s(\eta) = \frac{W(\eta)}{\sin \eta}$ in $(0, \pi)$,

(e) $\frac{d}{ds}_{|s=0}\alpha_s(\eta) = 1$.

If this is done, we define $\phi_s : I \to I$ by

$$\phi_s(\theta, \eta) = (\theta, \alpha_s(\eta)).$$

Let $u_s = u_0 \circ \phi_s : Q \to \mathbb{S}^2$. First, by (a), (b) and (c) it is easy to see that $u_s \in \mathcal{C}(Q, \mathbb{S}^2)$ with $u_0 = u_f$, i.e. u_s is a deformation of u_f. We then claim that $\{\phi_s\}$ is a family of conformal diffeomorphisms. Namely, for any s,

$$\phi_s^*(h) = |\alpha'_s|^2 h. \tag{4.1}$$

In fact, from (d) we have

$$\begin{aligned} \phi_s^*(h)(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}) &= h(\phi_{s*}\frac{\partial}{\partial \theta}, \phi_{s*}\frac{\partial}{\partial \theta}) \\ &= \frac{k^2 \sin^2 \alpha_s(\eta)}{W^2(\alpha_s(\eta))} = |\alpha'_s(\eta)|^2 \frac{k^2 \sin^2 \eta}{W^2(\eta)} \\ &= |\alpha'_s|^2 h(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}), \end{aligned}$$

$$\begin{aligned} \phi_s^*(h)(\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \eta}) &= h(\phi_{s*}\frac{\partial}{\partial \eta}, \phi_{s*}\frac{\partial}{\partial \eta}) \\ &= |\alpha'_s|^2 h(\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \eta}) \end{aligned}$$

and

$$\phi_s^*(h)(\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \theta}) = 0 = |\alpha'_s|^2 h(\frac{\partial}{\partial \eta}, \frac{\partial}{\partial \theta}).$$

Due to the conformal invariance of the energy functional, $\int_Q |\nabla u_s|_h^2 \, dvol(h)$ is independent of s. Furthermore, we have

$$\begin{aligned} J(u_s) &= \int_Q |\nabla u_s|^2 \; W dvol(h) \\ &= \int_Q |\nabla u_f|^2 \; W \circ \phi_s^{-1} \; dvol(h). \end{aligned} \tag{4.2}$$

By Lemma 3.2, (4.2), (a) and (e), we have

$$\begin{aligned} 0 = \frac{d}{ds}_{|s=0} J(u_s) &= \int_Q |\nabla u_f|^2 \frac{d}{dt}_{|s=0} (W \circ \phi^{-1}) \; dvol(h) \\ &= -\int_Q |\nabla u_f|^2 W^{-1}(k^2 - l^2)(\sin \alpha_s(\eta) \frac{d}{\alpha_s} \, dt)_{|s=0} \; dvol(h) \\ &= -\int_Q |\nabla u_f|^2 W^{-1}(k^2 - l^2) \sin \eta \; dvol(h). \end{aligned}$$

It follows that $k = l$, which proves Theorem 2.1.

Hence, to prove Theorem 2.1, we only need to show the existence of such α_s. Set

$$F(\eta) = \int_{\pi/2}^{\eta} \frac{W(t)}{\sin t} dt.$$

Recall that $W(t) = (k^2 \sin^2(\eta/2) + l^2 \cos^2(\eta/2))^{\frac{1}{2}}$. Clearly, $F'(\eta) = W(\eta)/\sin \eta > 0$ in $[0, \pi]$, $\lim_{\eta \to 0} F(\eta) = -\infty$ and $\lim_{\eta \to \pi} F(\eta) = +\infty$. In fact, we have the following asymptotic behavior of F.

$$F(\eta) - l \log \eta = O(1) \qquad \text{as } \eta \to 0$$

and

$$F(\eta) + k \log(\pi - \eta) = O(1) \qquad \text{as } \eta \to \pi.$$

Define $\alpha_s : [0, \pi] \to [0, \pi]$ by

$$\alpha_s(\eta) = F^{-1}(F(\eta) + s).$$

By definition, we have

$$\alpha'(\eta) = \frac{W(\eta)}{\sin \eta} \frac{\sin \alpha(\eta)}{W(\alpha_s(\eta))} \sin \alpha_s(\eta).$$

Now it is easy to check that α_s satisfies conditions (a)-(e). This completes the proof of the Theorem. □

Remark. Our result is motivated by the Pohozaev identity for semilinear equations and the Kazdan-Warner condition [KW] in the problem of prescribing Gauss curvature.

We can also generalize the Theorem as follows. Let $E(a,b)$ be the ellipsoid defined by

$$E(a,b) = \{(w,z) \in \mathbb{C} \times \mathbb{C} \,|\, \frac{|w|^2}{a^2} + \frac{|z|^2}{b^2} = 1\}.$$

Given two relatively prime integers $k \geq l \geq 1$, we define an $\mathbb{S}^1$-action $T_{k,l}$ on $E(a,b)$ by

$$T^{\theta}_{k,l}(w,z) = (we^{ik\theta}, ze^{il\theta}).$$

Clearly, we can also define $\mathbb{S}^1$-map from $E(a,b)$ to $\mathbb{S}^2$ as before. By the same argument, we have

Theorem 4.1. *There is a harmonic $\mathbb{S}^1$-map from $E(a,b)$ to $\mathbb{S}^2$ if and only if*

$$\frac{b}{a} = \frac{l}{k}.$$

The author would like to thank Professor W. Y. Ding for sharing his ideas with him. The last version of this note was written while the author was visiting the Max-Planck Institute for Mathematics in the Sciences, Leipzig. He would like to thank Professor J. Jost and the institute for hospitality. He also would like to thank the referee for his or her critical reading and Professor R. Gulliver for his carefully correcting the context.

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Institute of Mathematics, Academic Sinica, Beijing, China

E-mail address: gwang@math03.math.ac.cn

Current address: Max-Planck Institute for Mathematics in the Sciences, Leipzig 04103, Germany

E-mail address: gwang@mis.mpg.de