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On convergence to statistic equilibrium in wave equations with mixing

by

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Abstract

The wave equation with constant or variable coefficients in the whole space $\mathbb{R}^n$ with an arbitrary odd $n \geq 3$ is considered. The initial datum is a translation-invariant random function with zero expectation and finite mean density of the energy, which also fits the mixing condition of Ibragimov-Linnik-Rosenblatt type. We study the distribution μ_t of the random solution at the moment $t \in \mathbb{R}$. The main result is the convergence of μ_t to some Gaussian measure as $t \to \infty$. This is the central limit theorem for linear wave equations. For the case of constant coefficients the proof is based on a new analysis of Kirchhoff's and Herglotz-Petrovskii's integral representations of the solution and on S.N.Bernstein's "room-corridors" method. The case of variable coefficients is reduced to constant coefficients. For this purpose the scattering theory for infinite energy solutions is constructed. The relation to Gibbs measures is discussed. The investigation is inspired by the problems of the mathematical foundation of the statistical physics.

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1 Introduction

One of the central mathematical problems arising in mathematical foundation of statistical physics is the key role of the "canonical" equilibrium Gibbs measures (=statistics). The statistics have been introduced by J.C. Maxwell, L. Boltzmann and J.W. Gibbs in the theory of the gases. The measures are invariant with respect to corresponding dynamics and depend on a finite number of additive integrals (density and temperature mean energy density), [19]. The key role of the measures might be explained by a transition of "any" random solution to one of equilibrium Gibbs statistics. In other words, the distributions μ_t of a random solution converge to one of the Gibbs measure g in the long-time limit,

$$\mu_t \rightharpoondown g, \quad t \to \infty. \tag{1.1}$$

This convergence is known as a "thermodynamic behavior" of the dynamical system, [45]. For the first time the thermodynamic behavior is established by Ya.G. Sinai and others [16, 17, 18, 53] for an ideal gas in $\mathbb{R}^n$ with an arbitrary $n \geq 1$. Corresponding canonical Gibbs measures g depend on the mean density and on temperature (mean energy density) and satisfy mixing condition due to the Poisson distribution. The convergence (1.1) is proved under assumption that the initial measure μ_0 is absolutely continuous with respect to the Gibbs measure g. This means the mixing property for the Gibbs measure of the ideal gas. Even more strong mixing K-property is established in the papers cited above (and B-property, [6, 40]). The results are extended to the gas with a repulsive interaction concentrated in a finite region, [4], to one-dimensional hard rod systems, [1, 48], and to the infinite one-dimensional chains of harmonic oscillators, [37]. In [49] a convergence to statistic equilibrium is proved for the one-dimensional chains of harmonic oscillators for the initial states corresponding to two Gibbs measures with different temperatures at the ends of the chain. Let us note, however, that the limit measure in [49] is not a canonical Gibbs measure.

The equilibrium Gibbs measures also play the key role in the statistical physics of the continuous systems governed by partial differential equations. For instance, the Gibbs distribution for the quantized energy is postulated by M.Planck in his derivation of the black-body spectrum of the Maxwell field and by A. Einstein and P. Debye in the quantum theory of solid state. This suggests that the thermodynamic behavior might be an inherent property of a class of infinite dimensional Hamiltonian systems governed by partial differential equations like the coupled nonlinear Dirac-Maxwell system [5, 14].

Recently V. Jaksic and C.A. Pillet have established the mixing property for the Gibbs measures of the continuous nonlinear Hamiltonian system of the classical particle smoothly coupled to a wave field, [21]. However, for the nonlinear continuous systems with the local interaction the known problem of "ultraviolet divergence" arises immediately. The Gibbs measures are not well defined since the local energy formally is infinite a.s. This blocks further mathematical study, at least without a suitable renormalization. This situation suggests an investigation of the convergence to non-canonical equilibrium measures with finite mean local energy,

$$\mu_t \rightharpoondown \mu_\infty, \quad t \to \infty. \tag{1.2}$$

Main problem is to postulate an appropriate property for the initial measure μ_0 which could provide such a convergence. We make here one step in this direction for the case of linear wave equations and show that the mixing condition of Ibragimov-Linnik-Rosenblatt type for the measure μ_0 is sufficient. The mixing condition means the long-range correlations decay for the random initial field with the distribution μ_0. For instance, the mixing condition holds at the high temperatures for the canonical Gibbs measure in the discrete systems that corresponds to the absence of the phase transitions, [19].

Next question is the relation between the limit measures μ_∞ and the canonical Gibbs measures which are well defined for the linear equations. We prove that the Gibbs measures give a good approximation to the limit measures in scaling limit, when the radius of correlations of the initial data is small.

We consider the Cauchy problem for the hyperbolic wave equations in $\mathbb{R}^n$ with odd $n \ge 3$:

$$\left\{ \begin{array}{l} \ddot{u}(x,\, t) = \sum\limits_{j,k=1}^{n} \partial_j(a_{jk}(x)\partial_k u(x,\, t)) - a_0(x)\, u(x,\, t), \quad x \in \mathbb{R}^n, \\ u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x), \end{array} \right. \tag{1.3}$$

where $\partial_j \equiv \dfrac{\partial}{\partial x_j}$. We establish the convergence (1.2) for a wide class of initial measures μ_0 with a finite mean local energy and with the mixing condition. We do not assume that the initial measure μ_0 is absolutely continuous with respect to a limit measure μ_∞. The mixing property for the limit measures is established in [10] (we state this result in Section 2.5).

We prove that the limit measure μ_∞ is Gaussian though we do not assume that the initial measure μ_0 is Gaussian. This is the central limit theorem for the linear wave equations. Note that the corresponding Hamiltonian functional is formally a quadratic form. Respectively, the canonical Gibbs measures g_T, $T \ge 0$, are Gaussian (and have the infinite mean local energy, see Section 2.7). Hence, our result (1.2) is similar to the thermodynamic behavior (1.1). On the other hand, the limit equilibrium measure μ_∞ depends on a functional parameter and has a finite mean local energy. Hence μ_∞ is not a Gibbs measure. Therefore, the convergence (1.2) is different from the thermodynamic behavior (1.1). The existence of the infinite dimensional set of the equilibrium measures μ_∞ is related to the fact that the wave equation (1.3) is very degenerate and admits infinitely many additive first integrals.

We discuss the relation to the Gibbs measures for the case of constant coefficients, $a_{jk}(x) \equiv \delta_{ij}$. We consider the sequence of homogeneous and isotropic initial measures μ_0 such that r_0, the radius of correlations of the initial data, converges to zero, and for a fixed $T \ge 0$

$$\frac{1}{2} E_0(\nabla u_0(x) \cdot \nabla u_0(y) + v_0(x) v_0(y)) \to T\delta(x-y), \tag{1.4}$$

where E_0 denotes the mathematical expectation with respect to μ_0. Then correlation functions of corresponding limit measures μ_∞ converge to the correlation functions of the Gibbs measure g_T. For the Gaussian measures this means

$$\mu_\infty \sim g_T, \quad r_0 \ll 1. \tag{1.5}$$

We adjust these relations in Section 2.7. The local energy for Gibbs measures is infinite a.s. Roughly speaking, the canonical measures for the continuous systems are “improper” objects, like the Green function, which is also a solution with infinite local energy.

Let us describe our assumptions and tools. We assume that the coefficients of the equation (1.3) are constant outside a finite region, $a_{jk}(x) = \delta_{ij}$, $|x| \ge$const. Moreover, we assume that the nontrapping condition is satisfied, i.e. all rays go to infinity. This holds for the case of constant coefficients $a_{jk}(x) \equiv \delta_{ij}$ since all the rays are then the straight lines. Let us denote $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot,t), \dot{u}(\cdot,t))$, $Y_0 = (Y_0^0, Y_0^1) \equiv (u_0, v_0)$. Then (1.3) becomes

$$\dot{Y}(t) = \mathcal{F}(Y(t)), \quad t \in \mathbb{R}; \quad Y(0) = Y_0. \tag{1.6}$$

We assume that the initial datum $Y_0(x)$ is a random function with zero mean value in the functional phase space $\mathcal{H}$ of solutions with finite local energy. Let us denote by μ_t, $t \in \mathbb{R}$, the measure in $\mathcal{H}$, which is a distribution of the random solution $Y(t)$ to the Cauchy problem (1.6). We assume for the simplicity of exposition that the initial correlation functions are homogeneous, i.e. for $i, j = 0, 1$,

$$Q_0^{ij}(x,y) := E_0 Y_0^i(x) Y_0^j(y) = q_0^{ij}(x-y), \ \ x, y \in \mathbb{R}^n, \tag{1.7}$$

though a weaker condition $Q_0^{ij}(x,y) \sim q_0^{ij}(x-y)$, $|x-y| \to \infty$, is needed. Next we assume that the initial mean "energy" density is finite for $x \in \mathbb{R}^3$:

$$e_0 := E_0[|u_0(x)|^2 + |\nabla u_0(x)|^2 + |v_0(x)|^2] = q_0^{00}(0) - \Delta q_0^{00}(0) + q_0^{11}(0) < \infty. \tag{1.8}$$

Finally, we assume that the initial measure μ_0 fits a mixing condition. Roughly speaking, the random values

$$Y_0(x) \ \ \textit{and} \ \ Y_0(y) \ \ \textit{are asymptotically independent as} \ \ |x-y| \to \infty. \tag{1.9}$$

Our main result means the convergence (1.2) to an equilibrium measure μ_∞, which is a Gaussian measure in the phase space $\mathcal{H}$. A similar convergence holds for $t \to -\infty$ since our system is time-reversible. We give some optimal bounds for mixing coefficient of the initial measure μ_0. We construct generic examples of the initial measures satisfying all assumptions imposed. We get the explicit formulas (2.29) for the correlation functions of the limit measure μ_∞.

We prove the convergence (1.2) following the strategy stated in [23, 29, 41, 42]. At first, we prove (1.2) for the equations with constant coefficients $a_{jk}(x) \equiv \delta_{ij}$, in three steps.
I. The family of measures μ_t, $t \geq 0$, is compact in an appropriate Fréchet space.
II. The correlation functions converge,

$$Q_t^{ij}(x,y) \equiv E_0 Y^i(x,t) Y^j(y,t) \to Q_\infty^{ij}(x,y), \ \ t \to \infty. \tag{1.10}$$

III. The characteristic functionals converge as $t \to \infty$ for any Ψ from the dual space to $\mathcal{H}$,

$$\tilde{\mu}_t(\Psi) = \int e^{i<Y,\Psi>} \mu_t(dY) \to e^{-\frac{1}{2}<Q_\infty \Psi, \Psi>}, \tag{1.11}$$

where Q_∞ is the integral operator with the matrix-valued kernel $(Q_\infty^{ij}(x,y))_{i,j=0,1}$.

The compactness **I** follows from the Prokhorov criterion by the method [52]. Namely, we prove a uniform bound for the second moment functions of the measures μ_t, $t \geq 0$. Then the Prokhorov condition follows from the Sobolev embedding theorem by Chebyshev's inequality. We deduce the uniform bound from the explicit expression for the correlation functions $Q_t^{ij}(x,y)$. The expression follows from the Herglotz-Petrovskii formula for the wave equation with constant coefficients. In particular, for the case $n = 3$ and $u_0(x) \equiv 0$, the Kirchhoff formula holds:

$$u(x,t) = \frac{1}{4\pi t} \int\limits_{S_t(x)} v_0(z) dS(z), \tag{1.12}$$

where $dS(z)$ is the Lebesgue measure on the sphere $S_t(x) : \ |z-x| = t$. The convergence (1.10) also follows from the explicit expression for $Q_t^{ij}(x,y)$. The expression leads to an explicit formula for the limiting correlation functions $Q_\infty^{ij}(x,y)$.

If μ_0 is a Gaussian measure, the convergence (1.11) follows from (1.10). In the case when the measure μ_0 is non-Gaussian, the proof of the convergence (1.11) follows by a development of S.N. Bernstein's "room-corridors" method, The development is suggested by the structure of the Kirchhoff formula (1.12). Roughly speaking, we divide the sphere of the integration in (1.12), $S_t(x)$, into the "rooms" R_k, $1 \leq k \leq N$, separated by the "corridors" C_j of a fixed dimension $d > 0$. If the area $|R_k| \sim 1$, then $N \sim t^2$ and (1.12) becomes

$$u(x,\,t) \sim \frac{\sum_{k=1}^{N} \xi_k}{\sqrt{N}}, \tag{1.13}$$

where ξ_k is the integral over the "room" R_k. Mixing condition (1.9) implies that the random values ξ_k are almost independent for large d since the distance between the rooms is $\geq d$. Therefore, the random value $u(x,t)$ is asymptotically Gaussian by the Central Limit Theorem of Lindeberg. This is only an idea of the proof of (1.11). In Section 6 we adjust the choice of the rooms R_k. We verify the Lindeberg condition in Appendix A.

The key role is played by the fact that the Kirchhoff formula (4.6) itself contains CLT due to the integration over the sphere $|z - x| = t$ and to the first power of t in the denominator. The last is related closely to the energy conservation for the wave equation since the energy is a quadratic form. The Herglotz-Petrovskii formulae for the wave equation have the same geometrical structure in arbitrary odd dimension $n \geq 5$. The case of even dimension is much more complicate since then the integration is over the ball $|z - x| \leq t$. The same difficulty arises for the Klein-Gordon equation in any dimension.

All three steps **I-III** depend drastically on the mixing condition. Simple examples show that all the assertions can fail if the mixing condition breaks down: If we take $u_0(x) \equiv 0$ and $v_0(x) \equiv \pm 1$ with probability $p_\pm = 0.5$, then $u(x,\,t) \equiv \pm t$ a.s.

At last, we reduce the case of variable coefficients to the case of constant coefficients. For this purpose the scattering theory for the solutions of infinite energy is constructed (this strategy is similar to [4]). The key role in proving the scattering theory is played by B.R. Vainberg's estimates for the local energy decay, [51]. We state these estimates in the appropriate form in Appendix B.

We state main result in Section 2. Sections 3 - 6 concern the case of constant coefficients: the statement of the result (Section 3), the compactness of measures family **I** (Section 4), the convergence (1.10) (Section 5), and the convergence (1.11) (Section 6). In Section 7 we construct the scattering theory, and in Section 8 we prove the convergence (1.2). Appendices A, B concern the Lindeberg condition and B.R. Vainberg's estimates, respectively.

The mixing condition for initial measure provides the convergence (1.2) and plays the key role in our investigation. The convergence (1.2) is close to (1.1), hence, roughly speaking, the mixing condition substitutes the ergodic hypothesis. Moreover, the condition seems to be an appropriate postulate from the physical point of view.

The mixing condition has been introduced by R.L. Dobrushin and Yu.M. Suhov in the context of systems of infinitely many particles, [8]. Under this assumption, they have proved the convergence (1.2) for the free motion in infinite particle systems and in one-dimensional hard rod systems, [9]. The mixing and K-property are established for the equilibrium limit measures. The results were extended to the group of the Bogolubov transforms of C^*-algebras, [47]. All these systems are degenerate and admit infinitely many additive first integrals, similarly to the wave equation (1.3). Respectively, the limit equilibrium measure μ_∞ for the systems also depends on a functional parameter and in general is not a canonical Gibbs measure.

In [3] the result [49] is extended to the initial states of 1D chain of harmonic oscillators with the mixing condition. Our result is close to [3, 49] since the equation (1.3) describes a continuous n-dimensional family of harmonic oscillators. Some steps in our proof also reassemble [3] (usage of the Lindeberg CLT and the estimates of the 4-th order momentum to check the Lindeberg condition). However, the key idea of our proof is based on the structure of 3D Kirchhoff's formula (1.12) and is quite different from 1D case. The generalization of the results [3, 49] to the 3D wave equations is done in [12].

Our results seem to be a realization of the R.L. Dobrushin-Yu.M. Suhov's program for linear wave equations. The idea to introduce a mixing condition into the context of the wave equation is inspired by R.L.Dobrushin's talk at Moscow Mathematical Society, [7]. He considered the infinite-particle system and suggested that (1.1) follows by CLT, since for large t remote particles bring their almost independent contributions into a fixed region of the space. Let us note that the Kirchhoff formula (1.12) itself contains this very mechanism. The formula shows explicitly the dispersive mechanism of the wave propagation. Coupled to the mixing initial field, this dispersive mechanism provides the statistical stabilization.

The dispersive mechanism is studied now for a wide range of nonlinear wave problems, [39, 46, 50], [24]-[28], [30]-[33]. Similar dispersive mechanism has been exploited in [21] to prove the mixing property for the Gibbs measures of the Hamiltonian wave system with a smooth nonlinear nonlocal coupling. However, for the nonlinear wave equations with the local interaction the proof of a convergence to statistic equilibrium is still an open problem.

Main part of our results was established in [43] and is announced in [23, 29, 41, 42], however the whole exposition and proofs are published here for the first time. The extension of the results to the Klein-Gordon equation in any dimension $n \geq 2$ is given in [11, 23, 34, 35, 36].

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2 Main results

Let us describe our results more precisely.

2.1 Notations

We denote $D = C_0^\infty(\mathbb{R}^n)$. We assume that the following properties **E1**–**E3** of the equation (1.3) are satisfied:

E1 $a_{jk}(x) = \delta_{jk} + \hat{a}_{jk}(x)$, where $\hat{a}_{jk}(x) \in D$; also $a_0(x) \in D$.

E2 $a_0(x) \geq 0$, and the hyperbolicity condition holds: $\exists \alpha > 0$

$$H(x,\xi) \equiv \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \quad x,\xi \in \mathbb{R}^n. \tag{2.14}$$

E3 Non-trapping condition [51]: for $(x(0),\xi(0)) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\xi(0) \neq 0$,

$$|x(t)| \to \infty \quad \text{as} \quad \text{t} \to \infty, \tag{2.15}$$

where $(x(t), \xi(t))$ is a solution to the Hamiltonian system

$$\dot{x}(t) = H_\xi(x(t), \xi(t)), \quad \dot{\xi}(t) = -H_x(x(t), \xi(t)).$$

Example. **E1-E3** hold for the acoustic equation with constant coefficients

$$\ddot{u}(x, t) = \Delta u(x, t), \quad x \in \mathbb{R}^n. \tag{2.16}$$

For instance, **E3** follows because $\dot{\xi}(t) \equiv 0 \Rightarrow x(t) \equiv \xi(0)t + x(0)$.

We assume that the initial datum Y_0 belongs to the phase space $\mathcal{H}$.

Definition 2.1 *$\mathcal{H} \equiv H^1_{\rm loc}(\mathbb{R}^n) \oplus H^0_{\rm loc}(\mathbb{R}^n)$ is the Fréchet space of $Y(x) \equiv (u(x), v(x))$ with real valued functions $u(x)$, $v(x)$, which is endowed with local energy seminorms*

$$\|Y\|^2_R = \int\limits_{|x|<R} (|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2) dx < \infty, \quad \forall R > 0. \tag{2.17}$$

The following proposition is well-known [38].

Proposition 2.2 *i) For any $Y_0 \in \mathcal{H}$ there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H})$ to Cauchy problem (1.6).*
ii) For any $t \in \mathbb{R}$, the operator $U(t) : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{H}$

We introduce the appropriate Hilbert spaces of initial datum of infinite energy. Let δ be an arbitrary positive number.

Definition 2.3 *$\mathcal{H}_\delta$ is the Hilbert space of the functions $Y = (u, v) \in \mathcal{H}$ with the finite norm*

$$|\!|\!|Y|\!|\!|^2_\delta = \int e^{-2\delta|x|}(|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2)\, dx < \infty.$$

Denote by $H^s_{\rm loc}(\mathbb{R}^n)$, $s \in \mathbb{R}$, the local Sobolev spaces, i.e. the Fréchet spaces of distributions $u \in D'(\mathbb{R}^n)$ with finite seminorms

$$\|u\|_{s,R} \equiv \sup_{\|\psi\|_{-s}=1} | < u,\, \psi > |.$$

Here the sup is taken over all $\psi \in D$ such that $\psi(x) = 0$ for $|x| > R$, and

$$\|\psi\|^2_{-s} \equiv \int_{\mathbb{R}^n} (1 + |\xi|)^{-2s} |\tilde{\psi}(\xi)|^2 d\xi, \quad \tilde{\psi}(\xi) = F\psi(\xi) = \int e^{i\xi\cdot x}\psi(x)dx.$$

The brackets $< \cdot, \cdot >$ denote the scalar product in $L^2(\mathbb{R}^n)$ or in $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ or their different extensions.

Definition 2.4 *For $\varepsilon > 0$ let us denote*

$$\mathcal{H}^{-\varepsilon} \equiv H^{1-\varepsilon}_{\rm loc}(\mathbb{R}^n) \oplus H^{-\varepsilon}_{\rm loc}(\mathbb{R}^n).$$

Then $\mathcal{H} \subset \mathcal{H}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding $\mathcal{H} \subset \mathcal{H}^{-\varepsilon}$ is compact by the Sobolev theorem.

2.2 Random solution. Convergence to equilibrium

Now we assume that Y_0 in (1.6) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$. Here $\mathcal{B}(\mathcal{H})$ is the Borel σ-algebra of subsets in $\mathcal{H}$. We denote by $\mu_0(dY_0)$ a Borel probability measure in $\mathcal{H}$ which is the distribution of the random function Y_0. Then, due to Proposition 2.2, $Y(t) = U(t)Y_0$ is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$.

Definition 2.5 *μ_t is a Borel probability measure in $\mathcal{H}$ which is the distribution of $Y(t)$:*

$$\mu_t(B) = \mu_0(U(-t)B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbb{R}. \tag{2.18}$$

Our main goal is to derive the convergence of the measures μ_t as $t \to \infty$. We establish the weak convergence of the measures μ_t in the Fréchet spaces $\mathcal{H}^{-\varepsilon}$ with any $\varepsilon > 0$:

$$\mu_t \overset{\mathcal{H}^{-\varepsilon}}{\longrightarrow} \mu_\infty \quad \text{as} \quad \mathrm{t} \to \infty, \tag{2.19}$$

where μ_∞ is a Borel probability measure in the space $\mathcal{H}$. By definition this means the convergence

$$\int f(Y)\mu_t(dY) \to \int f(Y)\mu_\infty(dY) \quad \text{as} \quad \mathrm{t} \to \infty \tag{2.20}$$

for any bounded continuous functional $f(Y)$ in the space $\mathcal{H}^{-\varepsilon}$.

For the simplicity we assume that $Y_0 = (u_0, v_0)$ is a unit random function in the probability space $(\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_0)$ and denote by E_0 the corresponding mathematical expectation operator.

Definition 2.6 *The correlation functions of the measure μ_t are the distributions*

$$Q_t^{ij}(x, y) \equiv E_0 Y^i(x, t) Y^j(y, t), \quad i, j = 0, 1, \tag{2.21}$$

where $Y^i(x, t)$ are the components of $Y(t) = (Y^0(x, t), Y^1(x, t))$.

This means that for any $\varphi, \psi \in D$

$$< Q_t^{ij}(x, y), \varphi(x)\psi(y) >= E_0 < Y^i(x, t), \varphi(x) > < Y^j(y, t), \psi(y) > . \tag{2.22}$$

We will denote $\mathcal{D} = D \oplus D$, and

$$< Y, \Psi >=< Y^0, \Psi^0 > + < Y^1, \Psi^1 >$$

for $Y = (Y^0, Y^1) \in \mathcal{H}$, $\Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$. For a Borel probability measure μ in the space $\mathcal{H}$ we denote by $\tilde{\mu}$ the characteristic functional (Fourier transform)

$$\tilde{\mu}(\Psi) \equiv \int \exp(i < Y, \Psi >) \, \mu(dY), \quad \forall \Psi \in \mathcal{D}.$$

The measure μ is called Gaussian (with zero expectation) if its characteristic functional has the form

$$\tilde{\mu}(\Psi) = e^{-\frac{1}{2}<Q\Psi, \Psi>}, \quad \Psi \in \mathcal{D},$$

where Q is a linear operator $\mathcal{D} \to \mathcal{D}'$. μ is called translation-invariant if $\forall h \in \mathbb{R}^3$

$$\mu(\hat{h}B) = \mu(B), \quad \forall B \in \mathcal{B}(\mathcal{H}),$$

where $\hat{h}Y(x) = Y(x + h)$.

2.3 Mixing condition

Let $O(r)$ denote the set of all pairs of open subsets $\mathcal{A},\ \mathcal{B}\subset \mathbb{R}^n$ with $\rho(\mathcal{A},\ \mathcal{B})\geq r$ and let $\sigma(\mathcal{A})$ be the σ-algebra of subsets of $\mathcal{H}$ generated by all linear functionals $Y\mapsto <Y,\Psi>$, where $\Psi\in\mathcal{D}$ with supp $\Psi\subset\mathcal{A}$. We define the Ibragimov-Linnik mixing coefficient [20]

$$\varphi(r)\equiv \sup_{(\mathcal{A},\mathcal{B})\in O(r)} \sup_{\substack{A\in\sigma(\mathcal{A}),\,B\in\sigma(\mathcal{B})\\ \mu_0(B)>0}} \frac{|\mu_0(A\cap B)-\mu_0(A)\mu_0(B)|}{\mu_0(B)}. \tag{2.23}$$

Definition 2.7 *The measure μ_0 fits strong uniform Ibragimov-Linnik mixing condition if*

$$\varphi(r)\to 0 \quad \text{as} \quad \mathrm{r}\to\infty. \tag{2.24}$$

Below, we will determine the rate of this decay.

2.4 Main theorem

We assume that the initial measure μ_0 satisfies the following properties **S0–S3**:

S0 The measure μ_0 has zero expectation value,

$$E_0 Y_0(x)\equiv 0,\ \ x\in\mathbb{R}^n. \tag{2.25}$$

S1 The correlation functions of the measure μ_0 are translation invariant, i.e. (1.7) holds.
S2 The initial mean "energy" density is finite, (1.8).
S3 The measure μ_0 satisfies the *strong uniform* Ibragimov-Linnik mixing condition, and

$$\int_0^\infty r^{n-2}\varphi^{1/2}(r)dr<\infty. \tag{2.26}$$

The integrals over the measure μ_0 in (2.25), (1.7) and (1.8) are defined similarly to (2.21), by the identities of type (2.22).

Remark 2.8 (1.8) implies that the measure μ_0 is concentrated in $\mathcal{H}_\delta$ for all $\delta>0$, since

$$\int |\!|\!|Y_0|\!|\!|_\delta^2\,\mu_0(dY_0)=e_0\int e^{-2\delta|x|}\,dx<\infty. \tag{2.27}$$

On the other hand, a translation-invariant probabilty measure μ_0 cannot be concentrated on the space of nonzero states with the finite *global* energy, i.e. $\mu_0(\{Y_0\in\mathcal{H}:\ \|Y_0\|_\infty<\infty, Y_0(x)\not\equiv 0\})=0$.

Let $\mathcal{E}(x)=-C_n|x|^{2-n}$ be the fundamental solution of the Laplacian, i.e. $\Delta\mathcal{E}(x)=\delta(x)$ for $x\in\mathbb{R}^n$. Define the matrix-valued function

$$Q_\infty(x,y)=\Big(Q_\infty^{ij}(x,y)\Big)_{i,j=0,1}=\Big(q_\infty^{ij}(x-y)\Big)_{i,j=0,1}, \tag{2.28}$$

where

$$\Big(q_\infty^{ij}\Big)_{i,j=0,1}=\frac{1}{2}\begin{pmatrix} q_0^{00}-\mathcal{E}*q_0^{11} & q_0^{01}-\ \ q_0^{10}\\ q_0^{10}-\ \ q_0^{01} & q_0^{11}-\Delta q_0^{00}\end{pmatrix}. \tag{2.29}$$

Note that q_0^{ij} are estimated by the mixing coefficient (cf.[20], ch.17, §2): $\forall\alpha\in\mathbb{Z}_+^n$, $|\alpha|\le 2-i-j$, $i,j=0,1$,

$$|\partial_z^\alpha q_0^{ij}(z)|\le 2e_0\varphi^{1/2}(|z|),\quad \forall z\in\mathbb{R}^n. \tag{2.30}$$

Hence, (2.26) implies the existence of the convolution $\mathcal{E}*q_0^{11}$ in (2.29).

Our main result is the following theorem.

Theorem 2.9 [23, 29] *Let $n\ge 3$ be odd, and let* **E1–E3**, **S0–S3** *hold. Then*
i) the convergence (2.19) holds for any $\varepsilon>0$.
ii) The limit measure μ_∞ is a Gaussian equilibrium measure on $\mathcal{H}$.
iii) The limit characteristic functional has the form

$$\tilde\mu_\infty(\psi)=\exp\,(-\frac{1}{2}<Q_\infty\Omega\Psi,\,\Omega\Psi>),\quad \Psi\in\mathcal{D}.$$

Here $\Omega:\mathcal{D}\to\mathcal{H}'_\delta$ is a linear continuous operator for sufficiently small $\delta>0$, and $Q_\infty:\mathcal{H}'_\delta\to\mathcal{H}_\delta$ is a linear continuous operator with the integral kernel $Q_\infty(x,y)$.
iv) For the case of constant coefficients, $a_{ij}(x)\equiv\delta_{ij}$, the measure μ_∞ is translation invariant, and $\Omega=I$.

Remark 2.10 i) The *uniform* Rosenblatt mixing condition [44] is sufficient together with a higher power $p>2$ in the bound (1.8).
ii) The mixing condition **S3** also can be weakened by introducing the σ-algebras $\sigma_{i\alpha}$ generated by the functionals $Y\mapsto<D^\alpha Y^i,\Psi>$, $|\alpha|\le 1$. The rate of decay of the corresponding mixing coefficients $\varphi_{i\alpha,j\beta}$ can be specified (see [12]). For the sake of simplicity, we do not go into further details.

2.5 Ergodicity and mixing for equilibrium measures

(2.19) implies a convergence of the random solution $Y(t)=U(t)Y_0$ to the stationary process corresponding to the limit equilibrium measure μ_∞, [29]. In [10] the mixing is established for the stationary random solution: $\forall f,g\in L^2(\mathcal{H},\mu_\infty)$,

$$\lim_{t\to\infty}\int f(U(t)Y)g(Y)\mu_\infty(dY)=\int f(Y)\mu_\infty(dY)\int g(Y)\mu_\infty(dY). \tag{2.31}$$

In particular, the group $U(t)$ is ergodic with respect to the measure μ_∞:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(U(t)Y)dt=\int f(Y)\mu_\infty(dY)\ \ (\mathrm{mod}\ \mu_\infty). \tag{2.32}$$

2.6 Examples

2.6.1 Gaussian measures

We construct Gaussian initial measures μ_0 satisfying **S0** – **S3**. Let μ_0 be a Gaussian measure in the space $\mathcal{H}$ with the characteristic functional

$$\tilde\mu_0(\Psi)\equiv E_0\exp(i<Y,\Psi>)=\exp(-\frac{1}{2}<Q_0\Psi,\Psi>),\quad \Psi\in\mathcal{D}. \tag{2.33}$$

Here Q_0 is a correlation operator with a matrix integral kernel $(Q_0^{ij}(x,y))_{i,j=0,1}$. Let

$$Q_0^{ij}(x,y) \equiv q_0^{ij}(x-y), \tag{2.34}$$

for any i, j, where the function $q_0^{ij} \in C^2(\mathbb{R}^n)$ has a compact support. Then **S0**, **S1** and **S2** are satisfied; **S3** holds with $\varphi(r) \equiv 0$ for $r \geq r_0$ if $q_0^{ij}(z) \equiv 0$ for $|z| \geq r_0$. For a given matrix function $\left(q_0^{ij}(z)\right)$ such a Gaussian measure μ_0 exists in the space $\mathcal{H}$ iff the corresponding Fourier transform is a nonnegative matrix-valued measure: $\left(\tilde{q}_0^{ij}(\xi)\right) \geq 0$, $\xi \in \mathbb{R}^n$, [15]. For example, all these conditions hold if $\tilde{q}_{\pm}^{ij}(\xi) = \delta^{ij} f(\xi_1) \cdot \ldots \cdot f(\xi_n)$, with

$$f(z) = \left(\frac{1-\cos(r_0 z)}{z^2}\right)^2, \quad z \in \mathbb{R}.$$

2.6.2 Non-Gaussian measures

Let us choose some bounded odd nonconstant functions f^0, $f^1 \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$ with bounded derivatives. Let us define $\hat{\mu}_0$ as the distribution of the random function $(f^0(Y(x)), f^1(Y(x)))$, where $Y(x)$ is a random function with the Gaussian distribution μ_0 from the previous example. Then **S0-S3** hold for $\hat{\mu}_0$ with mixing coefficients $\hat{\varphi}(r) \equiv 0$ for $r \geq r_0$. The measure $\hat{\mu}_0$ is not Gaussian since the functions f^0, f^1 are bounded and nonconstant.

2.7 Comparison to Gibbs measures

We duscuss a relation of our result to Gibbs measures and to thermodynamic behavior. Let us restrict our consideration to the case of the wave equation (1.3) with constant coefficients. Then Gibbs measures g_T with the absolute temperature $T > 0$ are defined formally by

$$g_T(du_0, dv_0) = \frac{1}{Z} e^{-\beta \int (|\nabla u_0(x)|^2 + |v_0(x)|^2) dx} \prod_x du_0(x) dv_0(x), \quad \beta = T^{-1}. \tag{2.35}$$

More precisely, for $T \geq 0$, g_T is a Gaussian measure with the correlation functions

$$g_T^{00}(x-y) = -T\mathcal{E}(x-y), \quad g_T^{11}(x-y) = T\delta(x-y), \quad g_T^{01}(x-y) = g_T^{10}(x-y) = 0. \tag{2.36}$$

Such a measure exists in a weighted Sobolev space of negative degree, which is a space of tempered distributions. This follows by the Minlos theorem [15].

g_T is an equilibriun measure for the wave equation which formally follows from (2.29). It is translation invariant, so **S1** holds. The condition **S2** fails since the mean "energy" density $g_T^{00}(0) - \Delta g_T^{00}(0) + g_T^{11}(0)$ is infinite; this means the "ultraviolet divergence". Mixing condition in the form **S3** does not hold. However it holds in a more concise form of Remark 2.10 ii) since $(\nabla Y_0(x), Y_1(x))$ is δ-correlated by (2.36). The convergence of type (1.1) holds for the initial measures μ_0 absolutely continuous with respect to the Gibbs measure g_T, and the limit measure coincides with g_T. This mixing property (and even K-property) can be proved by known methods for Gaussian processes [6], and we do not touch it here.

Let us consider a fixed initial mesures μ_0 satisfying all assumptions **S0-S3**. Let, moreover,

$$q_0^{00}(z) = q_0^{11}(z) = 0, \quad |z| > r_0 \tag{2.37}$$

with a finite $r_0 > 0$, as in Examples above. We may as well assume the symmetry (or isotropy):

$$q_0^{01}(z) = q_0^{01}(-z), \quad z \in \mathbb{R}^n. \tag{2.38}$$

Then also $q_0^{01}(z) = q_0^{10}(z)$, and (2.29) implies

$$q_\infty^{01}(z) = q_\infty^{10}(z) = 0 \tag{2.39}$$

which correspons to (2.36). At last, let us define the "mean temperature"

$$\overline{T} := \frac{1}{2}\int (q_0^{11}(z) - \Delta q_0^{00}(z))dz. \tag{2.40}$$

Then $\overline{T} \geq 0$ by Bohner theorem. Let us denote by $Y_0(x) = (Y_0^0(x), Y_0^1(x))$ a random function with the distribution μ_0. Denote by $\mu_{0,\varepsilon}$, $\varepsilon > 0$, the distribution of the random function $Y_{0,\varepsilon}(x) = (\varepsilon^{-1/2}Y_0^0(\varepsilon^{-1}x), \varepsilon^{-3/2}Y_0^1(\varepsilon^{-1}x))$. Then corresponding correlation functions are

$$q_{0,\varepsilon}^{ij}(z) = \varepsilon^{-1-i-j} q_0^{ij}(\varepsilon^{-1}z). \tag{2.41}$$

Therefore, (2.37) implies

$$q_{0,\varepsilon}^{00}(z) = q_{0,\varepsilon}^{11}(z) = 0, \;\; |z| > \varepsilon r_0. \tag{2.42}$$

On the other hand, the corresponding mean temperature does not depend on ε:

$$\overline{T}_\varepsilon = \frac{1}{2}\int (q_{0,\varepsilon}^{11}(z) - \Delta q_{0,\varepsilon}^{00}(z))dz = \overline{T}, \;\; \varepsilon > 0. \tag{2.43}$$

Next lemma means that the corresponding limit equilibrium measures $\mu_{\infty,\varepsilon}$ converge to the Gibbs measure $g_{\overline{T}}$ when $\varepsilon \to 0$.

Lemma 2.11 *The limit correlation functions converge in the sence of distributions,*

$$q_{\infty,\varepsilon}^{ij}(z) \to g_{\overline{T}}^{ij}(z) \quad \text{as} \quad \text{ve} \to 0. \tag{2.44}$$

Proof. The relation (2.41) implies that

$$\int (|q_{0,\varepsilon}^{11}(z)| + |\Delta q_{0,\varepsilon}^{00}(z)|)dz = \text{const} < \infty, \;\; \varepsilon > 0.$$

Hence, (2.29), (2.42), and (2.43) imply that

$$q_{\infty,\varepsilon}^{11}(z) = q_{0,\varepsilon}^{11}(z) - \Delta q_{0,\varepsilon}^{00}(z) \to \overline{T}\delta(z), \;\; \varepsilon \to 0. \tag{2.45}$$

Therefore, again by (2.29) and (2.42),

$$q_{\infty,\varepsilon}^{00} = \frac{1}{2}\mathcal{E} * (\Delta q_{0,\varepsilon}^{00} - q_{0,\varepsilon}^{11}) = -\mathcal{E} * q_{\infty,\varepsilon}^{11} \to -\overline{T}\mathcal{E}, \;\; \varepsilon \to 0. \tag{2.46}$$

$\square$

Remark $\overline{T}$ does not depend on q_0^{00} due to (2.37). However, the convergence (2.44) holds with a weaker assumption $\Delta q_0^{00}(z) = 0$, $|z| > r_0$, and then $\overline{T}$ depends on q_0^{00} too. We do not touch here this generalization since it involves a weaker mixing condition of the Remark 2.10 ii).

3 Equations with constant coefficients

We consider the Cauchy problem (1.3) with the constant coefficients, i.e.

$$\left\{ \begin{array}{l} \ddot{u}(x,\,t) = \Delta u(x,\,t), \quad x \in \mathbb{R}^n, \\ u|_{t=0} = u_0(x), \quad \dot{u}|_{t=0} = v_0(x). \end{array} \right. \tag{3.1}$$

We rewrite (3.1) in the form

$$\dot{Y}(t) = \mathcal{F}_0(Y(t)), \;\; t \in \mathbb{R}, \quad Y(0) = Y_0. \tag{3.2}$$

For $Y \equiv (u, v) \in \mathcal{H}$ and $R > 0$, let us denote

$$\mathcal{E}_R(Y) = \int_{|x|<R} [|\nabla u(x)|^2 + |v(x)|^2]\, dx. \tag{3.3}$$

Denote by $U_0(t)$, $t \in \mathbb{R}$, the dynamical group of the Cauchy problem (3.2), $Y(t) = U_0(t)Y_0$.

The following proposition is well-known [38]:

Proposition 3.1 *For any $Y_0 \in \mathcal{H}$ and $R > 0$,*

$$\mathcal{E}_R(U_0(t)Y_0) \le \mathcal{E}_{R+t}(Y_0), \quad t \ge 0. \tag{3.4}$$

Our main result for the problem (3.2) is the following theorem:

Theorem 3.2 *Let $n \ge 3$ be odd, the conditions* **S0–S3** *hold, and $\mu_t(B) = \mu_0(U_0(-t)B)$, $B \in \mathcal{B}(\mathcal{H})$, $t \in \mathbb{R}$. Then the conclusions of Theorem 2.9 hold with $\Omega = I$.*

This theorem follows from next two propositions, applying the method of [52].

Proposition 3.3 *The family of the measures μ_t, $t \ge 0$, is compact in the space $\mathcal{H}^{-\varepsilon}$,* $\forall \varepsilon > 0$.

Proposition 3.4 *For any $\Psi \in \mathcal{D}$,*

$$\tilde{\mu}_t(\Psi) \equiv \int \exp(i < Y, \Psi >)\mu_t(dY) \to \exp\,(-\frac{1}{2} < Q_\infty \Psi,\, \Psi >), \quad t \to \infty. \tag{3.5}$$

4 Compactness

Proposition 3.3 follows from the estimate (4.2) below, using the Prokhorov criterion [52, Lemma 3.1] by the method of [52]. Let us denote

$$e_t \equiv \int [|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2]\, \mu_t(dY). \tag{4.1}$$

Lemma 4.1 *Let* **S0–S3** *hold. Then*

$$\overline{e} \equiv \sup_{t \ge 0} e_t < \infty. \tag{4.2}$$

Proof. The inequality (4.2) follows immediately from the following relations:

$$\varepsilon_t \equiv \int [|v(x)|^2 + |\nabla u(x)|^2]\, \mu_t(dY) = \varepsilon_0, \tag{4.3}$$

$$\int |u(x)|^2 \mu_t(dY) \leq d < \infty. \tag{4.4}$$

Proof of (4.3). Applying mathematical expectation operator E_0 to both sides of (3.4) we get, by **S1**,

$$\varepsilon_t |B_R| \leq \varepsilon_0 |B_{R+t}|, \quad t \geq 0. \tag{4.5}$$

Here B_R is the ball $|x| < R$ in $\mathbb{R}^n$, and $|B_R|$ is its volume. Tending $R \to \infty$, we get $\varepsilon_t \leq \varepsilon_0$ for $t \geq 0$. By the same reasons we have $\varepsilon_0 \leq \varepsilon_t$, hence $\varepsilon_t \equiv \varepsilon_0$. □

Proof of (4.4) This estimate follows from the mixing condition **S3** and from **S1-S2**. For simplicity, we derive it in the case when $n = 3$ and $u_0(x) \equiv 0$, $u_0(x) \in C(\mathbb{R}^3)$ a.s. The proof for general case is similar. For the case considered the solution $u(x, t)$ to (3.1) is given by the Kirchhoff formula:

$$u(x, t) = \frac{1}{4\pi t} \int_{S_t(x)} v_0(y)\, dS(y), \quad x \in \mathbb{R}^3, \tag{4.6}$$

where $S_t(x) \equiv \{z \in \mathbb{R}^3 : |z - x| = t\}$. Hence

$$d_t \equiv \int |u(x)|^2 \mu_t(dY) = E_0 |u(x, t)|^2 = \frac{1}{(4\pi t)^2} \int_{S_t(x) \times S_t(x)} q_0^{11}(z - p)\, dS(z)\, dS(p). \tag{4.7}$$

Additionally, let

$$\varphi(r) = 0 \;\text{ for }\; r \geq r_0 \;\text{ with some }\; r_0 > 0, \tag{4.8}$$

as in examples of Section 2.5. Then $q_0^{11}(z - p) = 0$ for $|z - p| \geq r_0$ by (2.30), and (4.7) implies

$$d_t \leq \frac{C}{t^2} \int\limits_{\substack{S_t(x) \times S_t(x) \\ |z-p| \leq r_0}} dS(z)\, dS(p) \leq \frac{C_1 r_0^2}{t^2} \cdot t^2 = d < \infty, \;\; t \geq 0. \tag{4.9}$$

Hence, (4.4) follows with the additional assumption (4.8). We remove it by the following known lemma on spherical integral identity, [22].

Lemma 4.2 *Let $f(r) \in L^1_{\rm loc}[0, \infty)$. Then for any $r_0 > 0$ and $p \in S_t(x)$ the following identity holds:*

$$\int\limits_{\{z \in S_t(x) :\, |z-p| \geq r_0\}} f(|z - p|) dS(z) = 2\pi \int_{r_0}^{2t} r f(r) dr. \tag{4.10}$$

The lemma with $f(r) = \phi^{1/2}(r)$, $r_0 = 0$ and the inequalities (2.26), (2.30) imply

$$E_0 |u(x,t)|^2 \leq \frac{2e_0}{(4\pi t)^2} \int\limits_{S_t(x) \times S_t(x)} \varphi^{1/2}(|z - p|)\, dS(z) dS(p) \leq C < \infty. \tag{4.11}$$

Then (4.4) follows without assumption (4.8). □

Corollary 4.3 *Estimate (4.2) implies, similarly to (2.27),*

$$\sup_{t\geq 0}\int \|Y\|_\delta^2\,\mu_t(dY)\leq C_\delta\overline{e}<\infty,\quad \forall\delta>0. \tag{4.12}$$

Therefore, the measures μ_t, $t\geq 0$, are concentrated in $\mathcal{H}_\delta$ for any $\delta>0$, and the characteristic functionals $\tilde{\mu}_t$, are equicontinuous in the space $\mathcal{H}'_\delta$:

$$\begin{aligned}|\tilde{\mu}_t(\Psi_1)-\tilde{\mu}_t(\Psi_2)| &\leq \int|\exp(i<Y,\Psi_1-\Psi_2>)-1|\mu_t(dY)\leq\int|<Y,\Psi_1-\Psi_2>|\mu_t(dY)\\ &\leq \int\|Y\|_\delta\cdot\|\Psi_1-\Psi_2\|'_\delta\mu_t(dY)\leq C(\delta,\overline{e})\|\Psi_1-\Psi_2\|'_\delta,\ t\geq 0 \end{aligned}\tag{4.13}$$

for $\Psi_1,\Psi_2\in\mathcal{D}$, where $\|\cdot\|'_\delta$ denotes the norm in the Hilbert space $\mathcal{H}'_\delta$.

Remark Assumption **S3** is "necessary" for (4.4). Indeed, if we take $Y_0^0(x)\equiv 0$ and $Y_0^1(x)\equiv\pm 1$ with probability $p_\pm=0.5$, then **S0** – **S2** hold but **S3** fails, as well as (4.4), because $u(x,\,t)\equiv\pm t$ a.s.

5 Convergence of correlation functions

We prove the convergence on the correlation functions of the measures μ_t. We will need this convergence in the next section. Moreover, it also implies the convergence of characteristic functionals (3.5) in the case of Gaussian measures μ_0. Denote by $Q_t=(Q_t^{ij}(x,y))$, $i,j=0,1$, the correlation matrix of the measures μ_t:

$$Q_t^{ij}(x,y)\equiv\int Y^i(x)Y^j(y)\mu_t(dY)=q_t^{ij}(x-y),\quad x,y\in\mathbb{R}^n, \tag{5.1}$$

where $Y=(Y^0,Y^1)$ and the integral is defined similarly to (1.7).

Lemma 5.1 *The following convergence holds: $\forall i,j=0,1$,*

$$q_t^{ij}(z)\to q_\infty^{ij}(z),\quad t\to\infty,\quad \forall z\in\mathbb{R}^n. \tag{5.2}$$

Proof. We prove Lemma 5.1 for the case $u_0(x)\equiv 0$ a.s. and $(i,j)=(0,0)$. For the general case the proof is similar. We show that

$$q_t^{00}(z)\to-\frac{1}{2}\mathcal{E}*q_0^{11}(z),\ \ t\to\infty,\ \ \forall z\in\mathbb{R}^n. \tag{5.3}$$

For the solution to the Cauchy problem (3.1) we have

$$\tilde{u}(\xi,t)=\frac{\sin t|\xi|}{|\xi|}\,\tilde{v}_0(\xi).$$

Hence, denoting $\tilde{q}_t^{ij}=Fq_t^{ij}$, we have

$$\tilde{q}_t^{00}(\xi)=\frac{\sin^2 t|\xi|}{|\xi|^2}\,\tilde{q}_0^{11}(\xi)=\frac{1}{2|\xi|^2}\,\tilde{q}_0^{11}(\xi)-\frac{\cos 2t|\xi|}{2|\xi|^2}\,\tilde{q}_0^{11}(\xi). \tag{5.4}$$

To prove (5.3) we show that the oscillatory term in (5.4) is absolutely continuous and summable.

Lemma 5.2 *$\tilde q_0^{11}(\xi)\in C(\mathbb{R}^n)$, and*

$$\int |\xi|^{-2}|\tilde q_0^{11}(\xi)|\,d\xi<\infty. \tag{5.5}$$

Proof. The inequalities (2.30) and (2.26) imply that $q_0^{11}(\xi)\in L^1(\mathbb{R}^n)$, hence $\tilde q_0^{11}(\xi)\in C(\mathbb{R}^n)$. Moreover, $\tilde q_0^{11}(\xi)\geq 0$. Therefore,

$$\int |\xi|^{-2}|\tilde q_0^{11}(\xi)|\,d\xi=\int |\xi|^{-2}\tilde q_0^{11}(\xi)\,d\xi=C\int |x|^{2-n}q_0^{11}(x)\,dx. \tag{5.6}$$

Then (2.30) and (2.26) imply

$$\int |\xi|^{-2}|\tilde q_0^{11}(\xi)|\,d\xi\leq C_1\int\limits_0^{+\infty} h\phi^{1/2}(h)\,dh<\infty.$$

□

By the Lebesgue - Riemann theorem, this Lemma implies the following corollary.

Corollary 5.3 *For any $z\in\mathbb{R}^n$*

$$\int e^{-iz\xi}\frac{\cos t|\xi|}{|\xi|^2}\tilde q_0^{ij}(\xi)\,d\xi\to 0\quad\text{as}\quad \mathrm{t}\to\infty. \tag{5.7}$$

Now the convergence (5.3) follows from (5.4) and (5.7). □

6 S.N.Bernstein's rooms-corridors method

We prove Proposition 3.4. For the simplicity of exposition we consider the case $n=3$ and assume additionally (4.8). Moreover, we assume μ_0-almost sure

$$u_0(x)\equiv 0,\quad v_0\in C(\mathbb{R}^n)\ \text{ and }\ |v_0(x)|\leq b<\infty,\quad x\in\mathbb{R}^3. \tag{6.1}$$

General case can be considered similarly. Under these assumptions the Kirchhoff representation (4.6) holds, and we establish (3.5) by a variant of S.N. Bernstein's "rooms-corridors" method. We prove (3.5) only for $\Psi=(\phi(x),0)$, $\phi(x)\in D$; the proof for general case follows similarly. Moreover, we will consider only $\Psi_*=(\delta(x),0)$. Then formally

$$<Q_\infty\Psi_*,\Psi_*>=q_\infty^{00}(0). \tag{6.2}$$

Therefore, (3.5) is formally equivalent to the following proposition.

Lemma 6.1 *Let (6.1) hold. Then*

$$E_0\exp\{iu(0,t)\}\to\exp\{-\frac{1}{2}q_\infty^{00}(0)\},\quad t\to\infty. \tag{6.3}$$

Remark 6.2 The case $\phi(x)\in D$ can be reduced to $\phi(x)=\delta(x)$ as follows. The operator $U_0(t)$ commutes with the translations in $\mathbb{R}^3$. Therefore, denoting $\Psi=(\phi,0)$ and $\Psi_*=(\delta(x),0)$, we have

$$<U_0(t)Y_0,\Psi>=<U_0(t)Y_0*\check\phi,\Psi_*>$$

where $\check\phi(x)\equiv\phi(-x)$. It remains to notice that the distribution μ_0^* of $Y_0*\check\phi$ satisfies the same assumptions **S0-S3** as μ_0.

Proof of Lemma 6.1 (6.1) implies the Kirchhoff representation (4.6) for $u(0,t)$. For large $t>0$ we divide the sphere $S_t(0)$ by hyperplanes orthogonal to the axis Ox_3 into parts which we call the "rooms" R_k^t ($k=1,...,N\equiv N_t$), separated by the "corridors" C_k^t ($k=1,...,N-1$). We choose $2t>(N-1)r_0$ and $N,d>0$ such that

$$\left\{\begin{array}{l} R_k^t=\{x\in S_t(0):\ x_3\in[a_k,a_k+d]\},\\ C_k^t=\{x\in S_t(0):\ x_3\in[a_k+d,a_{k+1}]\},\end{array}\right. \tag{6.4}$$

$$a_1=-t,\quad a_{k+1}=a_k+d+r_0,\quad a_N+d=t. \tag{6.5}$$

We introduce the notations

$$I(R_k^t)=\int\limits_{R_k^t} v_0(y)\,dS(y),\quad I(C_j^t)=\int\limits_{C_j^t} v_0(y)\,dS(y).$$

Let us note that (6.4) and (6.5) imply that

$$C_k^t=\{x\in S_t(0):\ x_3\in[a_{k+1}-r_0,a_{k+1}]\}. \tag{6.6}$$

Then the "width" of the corridors C_k^t is r_0, and the distance between the different rooms R_k^t is greater than or equal to r_0. Hence, it follows from (4.8) that the random variables $I(R_k^t)$, $k=1,...,N_t$, are independent. (4.6) implies

$$u(0,t)=\frac{1}{4\pi t}\left(\sum_{k=1}^N I(R_k^t)+\sum_{j=1}^{N-1} I(C_j^t)\right). \tag{6.7}$$

By the triangle inequality

$$|E_0e^{iu(0,t)}-e^{-\frac12 q_\infty^{00}(0)}|\le|E_0e^{iu(0,t)}-E_0e^{i\sum I(R_k^t)}|+$$

$$+|E_0e^{i\sum I(R_k^t)}-e^{-\frac12\sum E_0(I(R_k^t))^2}|+|e^{-\frac12\sum E_0(I(R_k^t))^2}-e^{-\frac12 q_\infty^{00}(0)}|\equiv I_1+I_2+I_3. \tag{6.8}$$

We show successively that I_1, I_2, I_3 tend to zero as $t\to\infty$.

Step 1 We estimate I_1. Using the inequality

$$|\exp(i\xi)-1|\le|\xi|,\quad \forall\xi\in\mathbb{R}, \tag{6.9}$$

we obtain

$$I_1=|E_0e^{i\sum I(R_k^t)}(e^{i\sum I(C_j^t)}-1)|\le\sum E_0|I(C_j^t)|\le\sum(E_0|I(C_j^t)|^2)^{1/2}. \tag{6.10}$$

We have

$$E_0|I(C_j^t)|^2=\frac{1}{(4\pi t)^2}\int_{C_j^t\times C_j^t} q_0^{11}(y_1-y_2)\,dS(y_1)\,dS(y_2). \tag{6.11}$$

(4.8) and (2.30) imply that $q_0^{11}(z)=0$, for $|z|\ge r_0$. Hence we may restrict the domain of integration in (6.11) to

$$\{(y_1,y_2)\in C_j^t\times C_j^t\subset(\mathbb{R}^3)^2:\ |y_1-y_2|\le r_0\}.$$

Then (6.11) implies that

$$E_0|I(C_j^t)|^2\le\frac{C(r_0)}{t}. \tag{6.12}$$

We choose $N = N_t \to \infty$ such that

$$\frac{N_t}{\sqrt{t}} \to 0, \quad t \to \infty. \tag{6.13}$$

Hence we conclude from (6.10) and (6.12) that

$$I_1 \le N_t (C(r_0)/t)^{1/2} \to 0, \quad t \to \infty. \tag{6.14}$$

Step 2 Now we estimate I_3. The inequality $|e^{-\xi} - e^{-\eta}| \le |\xi - \eta|$, $\forall \xi, \eta \ge 0$, implies that

$$I_3 \le \frac{1}{2} \Big| \sum_{k=1}^{N_t} E_0 (I(R_k^t))^2 - q_\infty^{00}(0) \Big| \to 0, \quad \text{as} \quad t \to \infty. \tag{6.15}$$

Indeed, the random variables $I(R_k^t)$, $k = 1, ..., N_t$, are independent. Therefore,

$$\sum_{k=1}^{N} E_0 (I(R_k^t))^2 = E_0 (\sum_{k=1}^{N} I(R_k^t))^2. \tag{6.16}$$

On the other side, (6.7) implies

$$\begin{aligned} E_0 (\sum_{k=1}^{N} I(R_k^t))^2 &= E_0 \Big(u(0,t) - \sum_{j=1}^{N-1} I(C_j^t) \Big)^2 \\ &= E_0 u^2(0,t) - 2 E_0 u(0,t) \sum I(C_j^t) + E_0 (\sum I(C_j^t))^2. \end{aligned} \tag{6.17}$$

(5.2) implies the convergence of the first term,

$$E_0 u^2(0,t) \to q_\infty^{00}(0), \quad t \to \infty. \tag{6.18}$$

Two last terms converge to zero by (6.12) and (6.13). Finally, (6.15) - (6.18) imply $I_3 \to 0$, as $t \to \infty$.

Step 3 It remains to prove that $I_2 \to 0$ as $t \to \infty$. Since $I(R_k^t)$ are independent,

$$E_0 e^{i \sum_k I(R_k^t)} = \prod_k E_0 e^{i I(R_k^t)}.$$

It suffices to verify the Lindeberg condition:

$$\forall \varepsilon > 0, \quad \frac{1}{\sigma_t} \sum_{k=1}^{N_t} E_{|I(R_k^t)| > \varepsilon \sqrt{\sigma_t}} (I(R_k^t))^2 \to 0 \quad \text{as} \quad \mathrm{t} \to \infty, \tag{6.19}$$

where $E_{|f(Y)|>\delta} \equiv \int\limits_{|f(Y)|>\delta} \mu_0(dY)$, and $\sigma_t \equiv \sum\limits_{k=1}^{N_t} E_0 (I(R_k^t))^2$. Then Lindeberg's Central Limit Theorem implies that

$$I_2 = \Big| \prod_k E_0 e^{i I(R_k^t)} - e^{-\frac{1}{2} \sum_k E_0 (I(R_k^t))^2} \Big| \to 0 \quad \text{as} \quad t \to \infty, \tag{6.20}$$

and (6.16) - (6.18) imply that

$$\sigma_t \to q_\infty^{00}(0), \quad t \to \infty.$$

We may assume that $q_\infty^{00}(0) \neq 0$ (otherwise (6.3) follows immediately). Hence it suffices to verify that for any $\varepsilon > 0$

$$\sum_{k=1}^{N_t} E_{|I(R_k^t)|>\varepsilon}(I(R_k^t))^2 \to 0 \quad \text{as} \quad \mathrm{t} \to \infty.$$

Chebyshev inequality implies

$$\sum_{k=1}^{N_t} E_{|I(R_k^t)|>\varepsilon}(I(R_k^t))^2 \le \sum_{k=1}^{N_t} \frac{E_0|I(R_k^t)|^4}{\varepsilon^2}.$$

Therefore, the proof of (6.19) reduces to verifying that

$$\sum_{k=1}^{N_t} E_0|I(R_k^t)|^4 \to 0 \quad \text{as} \quad \mathrm{t} \to \infty. \tag{6.21}$$

We prove this in Appendix A. □

Corollary 6.3 *Estimate (4.13) implies that the convergence (3.5) holds for all $\Psi \in \mathcal{H}'_\delta$ with any $\delta > 0$. For instance, $Q_\infty : \mathcal{H}'_\delta \to \mathcal{H}_\delta$ is a linear continuous operator for any $\delta > 0$.*

7 Variable coefficients. Scattering theory

We reduce the proof of Theorem 2.9 to Theorem 3.2 by means of a special variant of scattering theory. This is the scattering theory for solutions of infinite energy. This is necessary, because $\mu_0\{Y_0 \in \mathcal{H} : \ Y_0 \neq 0, \ \ \mathcal{E}_\infty(Y_0) < \infty\} = 0$ for a homogeneous probability measure μ_0.

Let us recall that $U_0(t)$ ($U(t)$) is the dynamical group of the Cauchy problem (3.2) (problem (1.6)). Now we will formulate the scattering theory for solutions of infinite energy.

Theorem 7.1 *Let* **E1**-**E3** *hold, and let $n \ge 3$ be odd. Then there exist $\delta, \gamma > 0$ and linear continuous operators $\theta, r(t) : \ \mathcal{H}_\delta \to \mathcal{H}$ such that for $Y_0 \in \mathcal{H}_\delta$*

$$U(t)Y_0 = \theta U_0(t)Y_0 + r(t)Y_0, \ \ t \ge 0, \tag{7.1}$$

and for any $R > 0$ there exists a constant $C = C(R, \delta, \gamma)$, such that for $Y_0 \in \mathcal{H}_\delta$

$$\|r(t)Y_0\|_R \le C \ e^{-\gamma t} |||Y_0|||_\delta, \ \ t \ge 0. \tag{7.2}$$

We deduce this Theorem at the end of this section by duality from a refined finite energy scattering theory. For $t \in \mathbb{R}$ we introduce the operators $U'(t)$, $U_0'(t)$ in the Hilbert space $\mathcal{H}'_\delta$, which are conjugate to operators $U(t)$, $U_0(t)$ in $\mathcal{H}_\delta$. For example,

$$< Y, U'(t)\Psi > = < U(t)Y, \Psi >, \ \ \Psi \in \mathcal{D}, \ \ Y \in \mathcal{H}_\delta, \ \ t > 0. \tag{7.3}$$

Lemma 7.2 *The following bound holds:*

$$|||U_0'(t)\Psi|||'_\delta \le Ce^{\delta t} |||\Psi|||'_\delta, \quad \forall \Psi \in \mathcal{H}'_\delta, \quad t \ge 0. \tag{7.4}$$

This Lemma follows by duality from the next one.

Lemma 7.3 *Let* **E1**–**E2** *hold. Then* $\forall\delta > 0$ *the operator* $U_0(t)$ *is continuous in* $\mathcal{H}_\delta$*, and there exists a constant* $C = C(\delta) > 0$ *such that for* $Y_0 \in \mathcal{H}_\delta$

$$|\!|\!| U_0(t)Y_0 |\!|\!|_\delta \le Ce^{\delta t} |\!|\!| Y_0 |\!|\!|_\delta, \;\; t \ge 0. \tag{7.5}$$

Proof. It suffices to consider $Y_0 \in \mathcal{D}$. Then $U_0(t)Y_0 \in \mathcal{D}$. Denote

$$E_\delta(Y) \equiv \int e^{-2\delta|x|}[|\nabla u(x)|^2 + |v(x)|^2]\, dx, \;\; Y = (u,v) \in \mathcal{H}_\delta.$$

Then

$$\dot{E}_\delta(U_0(t)Y_0) = 2\int e^{-2\delta|x|}[\nabla u(x,t)\nabla \dot{u}(x,t) + \dot{u}(x,t)\ddot{u}(x,t)]\, dx.$$

Substituting $\ddot{u}(x,t) = \triangle u(x,t)$ and integrating by parts, we obtain

$$\dot{E}_\delta(U_0(t)Y_0) = -2\int \nabla e^{-2\delta|x|} \cdot \nabla u(x,t)\dot{u}(x,t)\, dx.$$

Then $\dot{E}_\delta(U_0(t)Y_0) \le 2\delta E_\delta(U_0(t)Y_0)$ by Yang's inequality. Therefore, Gronwall's inequality implies

$$E_\delta(U_0(t)Y_0) \le e^{2\delta t}E_\delta(Y_0), \;\; t \ge 0.$$

In other words,

$$\int e^{-2\delta|x|}[|\dot{u}(x,t)|^2 + |\nabla u(x,t)|^2]\, dx \le e^{2\delta t}E_\delta(Y_0), \;\; t \ge 0. \tag{7.6}$$

It remains to estimate

$$\| u(\cdot,t)\|_\delta^2 \equiv \int \exp{(-2\delta|x|)}|u(x,t)|^2\, dx.$$

We have:

$$\| u(\cdot,t)\|_\delta \le \| u^0(x)\|_\delta + \| \int_0^t \dot{u}(\cdot,\tau)\, d\tau \|_\delta \le \| u^0(x)\|_\delta + \int_0^t \| \dot{u}(\cdot,\tau)\|_\delta\, d\tau.$$

Using (7.6), we get

$$\| u(\cdot,t)\|_\delta \le Ce^{\delta t} |\!|\!| Y_0 |\!|\!|_\delta.$$

Therefore, (7.6) implies (7.5). $\square$

Next lemma means that the action of the group $U'(t)$ coincides with the action of $U(t)$ up to the order of the components.

Lemma 7.4 *For* $\Psi = (\phi,\psi) \in \mathcal{D}$

$$U'(t)\Psi = (\dot{\psi}(\cdot,t), \psi(\cdot,t)), \tag{7.7}$$

where $\psi(x,t)$ *is the solution of Eq.(1.3) with the initial data* $(u_0,v_0) = (\psi,\phi)$.

Proof. Differentiating (7.3) with $Y,\Psi \in \mathcal{D}$, we obtain

$$< Y, \dot{U}'(t)\Psi > = < \dot{U}(t)Y, \Psi > . \tag{7.8}$$

The group $U(t)$ has the generator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix},$$

where $Au = \sum \partial_j(a_{jk}(x)\partial_k u) - a_0(x)u$. Therefore, it follows from (7.8) that the generator of $U'(t)$ is the adjoint operator

$$\mathcal{A}' = \begin{pmatrix} 0 & A \\ 1 & 0 \end{pmatrix}. \tag{7.9}$$

In other words, (7.7) holds, where $\ddot{\psi} = A\psi$. □

Next we involve B.R. Vainberg's estimates of the local energy decay which play the key role in the proof of Theorem 7.1. We use the Sobolev space $\mathcal{H}_R = H^1(B_R) \oplus L^2(B_R)$ with the norm $\|\cdot\|_R$. Let us recall that $H_0^{-1}(B_R)$ is a completion of $D_R = \{\psi \in D : \operatorname{supp}\psi \subset B_R\}$ in the Hilbert norm of the Sobolev space $H^{-1}(\mathbb{R}^n)$. The following lemma is well known, [13].

Lemma 7.5 *$H_0^{-1}(B_R)$ is the dual to the Hilbert space $H^1(B_R)$ with respect to the scalar product $<\cdot,\cdot>$.*

Corollary 7.6 *The dual space to the Hilbert space $\mathcal{H}_R$ with respect to the scalar product $<\cdot,\cdot>$ is*

$$\mathcal{H}'_R = H_0^{-1}(B_R) \oplus L^2(B_R). \tag{7.10}$$

Definition 7.7 *$\mathcal{H}'$ denotes the space $H_0^{-1}(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ endowed with the following convergence: a sequence Ψ_n converges to Ψ in $\mathcal{H}'$ iff $\exists R > 0$ s.t. all $\Psi_n \in \mathcal{H}'_R$, and Ψ_n converge to Ψ in the norm of $\mathcal{H}'_R$.*

Remark Below, we consider the continuity of the maps from $\mathcal{H}'$ or into $\mathcal{H}'$ only in the sense of the sequential continuity.

Denote $\mathcal{D}_R = D_R \oplus D_R$. B.R. Vainberg's results [51] imply the following lemma which we prove in Appendix B.

Lemma 7.8 *Let* **E1**–**E3** *hold, and let $n \geq 3$ be odd. Then $\forall R_0 > 0$ there exist constants $\kappa, C(R, R_0) > 0$ and $T = T(R, R_0) > 0$ such that for $\Psi \in \mathcal{D}_R$*

$$\|U'(t)\Psi\|_{L^2(B_{R_0})\oplus H^1(B_{R_0})} \leq C(R, R_0)e^{-\kappa t}\|\Psi\|'_R, \quad t \geq T > 0, \tag{7.11}$$

where $\|\cdot\|'_R$ denotes the norm of the space $\mathcal{H}'_R$.

Now we can state the refined dual scattering theory for the solutions of finite energy.

Proposition 7.9 *Let* **E1**–**E3** *hold, and $n \geq 3$ is odd. Then there exist $\delta, \gamma > 0$ and linear continuous operators $\Omega, \rho(t) : \mathcal{H}' \to \mathcal{H}'_\delta$ such that*

$$U'(t)\Psi = U'_0(t)\Omega\Psi + \rho(t)\Psi, \ t \geq 0, \tag{7.12}$$

and for any $R > 0$ there exists a constant $C_R = C(R, \delta, \gamma)$ such that for $\Psi \in \mathcal{D}_R$

$$\|\rho(t)\Psi\|'_\delta \leq C_R e^{-\gamma t}\|\Psi\|'_R, \quad t \geq 0. \tag{7.13}$$

Proof. We apply Cook's method. Let $\Psi \in \mathcal{D}_R$. Let us define $\Omega\Psi$ formally by

$$\Omega\Psi = \lim_{t\to+\infty} U'_0(-t)U'(t)\Psi = U'_0(-T)U'(T)\Psi + \int_T^{+\infty} \frac{d}{d\tau}U'_0(-\tau)U'(\tau)\Psi \, d\tau \tag{7.14}$$

with an appropriate $T > 0$. It suffices to prove the convergence of the integral in the norm of the space $\mathcal{H}'_\delta$. We have

$$\frac{d}{dt}U'_0(t)\Psi = \mathcal{A}'_0 U'_0(t)\Psi, \quad \frac{d}{dt}U'(t)\Psi = \mathcal{A}'U'(t)\Psi, \ t \in \mathbb{R}$$

where $\mathcal{A}'_0$ and $\mathcal{A}'$ are the generators to the groups $U'_0(t)$, $U'(t)$, respectively. Similarly to (7.9), we have

$$\mathcal{A}'_0 = \begin{pmatrix} 0 & \Delta \\ 1 & 0 \end{pmatrix}. \tag{7.15}$$

Therefore,

$$\frac{d}{dt}U'_0(-t)U'(t)\Psi = U'_0(-t)(\mathcal{A}' - \mathcal{A}'_0)U'(t)\Psi. \tag{7.16}$$

(7.9) and (7.15) imply

$$\mathcal{A}' - \mathcal{A}'_0 = \begin{pmatrix} 0 & A - \Delta \\ 0 & 0 \end{pmatrix}. \tag{7.17}$$

Let us note, that $A - \Delta = \sum \partial_j \hat{a}_{jk}(x)\partial_k - a_0(x)$, where $\hat{a}_{jk}(x)$, $a_0(x) \in C_0^\infty(B_{R_0})$ with some $R_0 > 0$, according to **E1**. Therefore, (7.16), (7.17) and (7.5), (7.7) imply that for $Y \in \mathcal{H}_\delta$

$$\begin{aligned} |< Y, \frac{d}{dt}U'_0(-t)U'(t)\Psi >| &= |< U_0(-t)Y, (\mathcal{A}' - \mathcal{A}'_0)U'(t)\Psi >| \\ &\le Ce^{\delta t}|||Y|||_\delta \cdot ||\psi(\cdot,t)||_{H^1(B_{R_0})}, \qquad t \ge 0. \end{aligned} \tag{7.18}$$

Applying (7.11), we get for $t \ge T = T(R, R_0)$,

$$|||\frac{d}{dt}U'_0(-t)U'(t)\Psi|||'_\delta \le C(R, R_0)e^{\delta t}e^{-\kappa t}||\Psi||'_R = C(R)e^{-\beta t}||\Psi||'_R, \tag{7.19}$$

where $\beta = \kappa - \delta$. Let us choose $\delta > 0$ sufficiently small: $\delta < \kappa$. Then we have $\beta > 0$, and (7.19) implies

$$\int_T^{+\infty} |||\frac{d}{dt}U'_0(-t)U'(t)\Psi \, dt|||'_\delta \le C_1(R)||\Psi||'_R < \infty.$$

Therefore, the existence of the limit in (7.14) follows, and the operator $\Omega : \mathcal{H}' \to \mathcal{H}'_\delta$ is continuous. Moreover, (7.14) and (7.19) imply

$$|||(U'_0(-t)U'(t) - \Omega)\Psi|||'_\delta \le C_2(R)e^{-\beta t}||\Psi||'_R, \ t \ge T. \tag{7.20}$$

Let us choose, further, $2\delta < \kappa$. Then $\delta < \beta = \kappa - \delta$ and $\gamma = \beta - \delta = \kappa - 2\delta > 0$. Finally, (7.20) and (7.4) imply

$$\begin{aligned} |||\rho(t)\Psi|||'_\delta &= |||(U'(t) - U'_0(t)\Omega)\Psi|||'_\delta = |||U'_0(t)(U'_0(-t)U'(t) - \Omega)\Psi|||'_\delta \\ &\le C_3(R)e^{\delta t}|||(U'_0(-t)U'(t) - \Omega)\Psi|||'_\delta \le C_4(R)e^{-\gamma t}||\Psi||'_R, \quad t \ge T. \end{aligned}$$

□

Proof of Theorem 7.1. (7.12) implies that for $Y_0 \in \mathcal{H}$ and $\Psi \in \mathcal{H}'_R$ with any $R > 0$

$$< U(t)Y_0, \Psi > = < U_0(t)Y_0, \Omega\Psi > + < Y_0, \rho(t)\Psi >, \ t \ge 0 \tag{7.21}$$

By Proposition 7.9, the operators $\Omega_R \equiv \Omega|_{\mathcal{H}'_R}$ and $\rho_R(t) \equiv \rho(t)|_{\mathcal{H}'_R}$ are continuous $\mathcal{H}'_R \to \mathcal{H}'_\delta$. Therefore, the duality (7.10) implies the existence of the adjoint continuous operators $\theta_R = \Omega'_R : \mathcal{H}_\delta \to \mathcal{H}_R$ and $r_R(t) = \rho'_R(t) : \mathcal{H}_\delta \to \mathcal{H}_R$ for any $R > 0$. It remains to define $(\theta Y_0)|_{B_R} = \theta_R Y_0$ and $(r(t)Y_0)|_{B_R} = r_R(t)Y_0$ for any $R > 0$. □

8 Convergence to equilibrium

We deduce Theorem 2.9 from two following Lemmas 8.1 and 8.2 (cf. Propositions 3.3, 3.4).

Lemma 8.1 *The family of the measures* $\{\mu_t\}$, $t \in \mathbb{R}$ *is compact in the space* $\mathcal{H}^{-\varepsilon}$, $\forall \varepsilon > 0$.

Lemma 8.2 *For every* $\Psi \in \mathcal{D}$,

$$\tilde{\mu}_t(\Psi) \equiv \int \exp(i < Y, \Psi >)\mu_t(dY) \to \exp(-\frac{1}{2} < Q_\infty \Psi, \Psi >), \quad t \to \infty. \tag{8.1}$$

Proof of Lemma 8.1. Lemma 8.1 follows similarly to Lemma 3.3 from the estimates

$$\sup_{t \geq 0} E_0 \|U(t)Y_0\|_R^2 < \infty, \ \forall R > 0. \tag{8.2}$$

Theorem 7.1 implies that

$$E_0\|U(t)Y_0\|_R^2 \leq 2E_0\|\theta U_0(t)Y_0\|_R^2 + 2E_0\|r(t)Y_0\|_R^2 \leq C_1(R)|\!|\!|U_0(t)Y_0|\!|\!|_\delta^2 + C_2(R)e^{-2\gamma|t|}|\!|\!|Y_0|\!|\!|_\delta^2.$$

Then from (4.2) we obtain (8.2) (cf. (2.27)):

$$E_0\|U(t)Y_0\|_R^2 \leq C_1(R)\overline{e}\int e^{-2\delta|x|}\,dx + C_2(R)e^{-2\gamma|t|}e_0\int e^{-2\delta|x|}\,dx \leq C(R,\delta) < \infty, \quad t \geq 0.$$

□

Proof of Lemma 8.2. Let $\Psi \in \mathcal{D}_R$. Then Theorem 7.1 implies that

$$\tilde{\mu}_t(\Psi) \equiv E_0 e^{i<U(t)Y_0,\Psi>} = E_0 e^{i<\theta U_0(t)Y_0 + r(t)Y_0,\Psi>} = E_0 e^{i<\theta U_0(t)Y_0,\Psi>} + \nu(t), \tag{8.3}$$

where

$$\nu(t) = E_0\left[e^{i<\theta U_0(t)Y_0,\Psi>}(e^{i<r(t)Y_0,\Psi>} - 1)\right].$$

Note that $\nu(t)$ vanishes as $t \to \infty$. Indeed, (6.9) and Theorem 7.1 imply

$$|\nu(t)| \leq E_0\ |< r(t)Y_0, \Psi >| \leq C(R)\|\Psi\|_R' E_0\|r(t)Y_0\|_R \to 0, \quad t \to \infty. \tag{8.4}$$

At last, Corollary 6.3 implies that

$$E_0 e^{i<\theta U_0(t)Y_0,\Psi>} = E_0 e^{i<U_0(t)Y_0,\Omega\Psi>} \to \exp\{-\frac{1}{2} < Q_\infty(x,y)\Omega\Psi(x), \Omega\Psi(y) >\}, \quad t \to \infty. \tag{8.5}$$

(8.3)–(8.5) imply (8.1). □

9 Appendix A. Lindeberg condition

We prove (6.21). (4.6) implies that

$$E_0|I(R_k^t)|^4 \leq \frac{\text{const}}{t^4}\int_{(R_k^t)^4} E_0\Big(v_0(y_1)v_0(y_2)v_0(y_3)v_0(y_4)\Big)dS(y_1)dS(y_2)dS(y_3)dS(y_4). \tag{9.1}$$

Let us consider the domain of the integration in (9.1).

Lemma 9.1 *The integrand in (9.1) does not vanish only for the following configurations of the points y_1, y_2, y_3, y_4:*

$$\begin{array}{ll} \mathbf{A} & \exists y_i, \text{ such that } \quad \rho(y_i,y_k)\ge r_0, \quad \forall k\ne i, \\ \mathbf{B} & \rho(y_a,y_i)\le r_0,\ \rho(y_j,y_b)\le r_0,\ \rho\Big((y_a,y_i),(y_j,y_b)\Big)\ge r_0, \\ \mathbf{C} & \rho(y_i,y_j)\le 3r_0, \quad \forall i,j. \end{array}$$

Proof. We introduce the quantity $M(\mathbf{y}) = \max|y_i - y_j| = |y_a - y_b|$ for $\mathbf{y} = (y_1, y_2, y_3, y_4) \in (\mathbb{R}^3)^4$. Only the two cases are possible: I) $M(\mathbf{y}) \le 3r_0$, and **C** holds, II) $M(\mathbf{y}) > 3r_0$.

It remains to analyze the case II. We divide the segment $[y_a, y_b]$ into three equal segments greater than r_0: $[y_a, y_b] = \triangle_1 \cup \triangle_2 \cup \triangle_3$. We denote by y_p, y_q the points with $p, q \ne a, b$ and by $\bar{y}_p$, $\bar{y}_q$ the orthogonal projections of y_p, y_q onto $[y_a, y_b]$. Then the case II splits into the following two cases:
II' $\bar{y}_p, \bar{y}_q \in \triangle_2 \cup \triangle_3$ or $\bar{y}_p, \bar{y}_q \in \triangle_1 \cup \triangle_2$,
II" $\bar{y}_p \in \triangle_1$, $\bar{y}_q \in \triangle_3$ or $\bar{y}_p \in \triangle_3$, $\bar{y}_q \in \triangle_1$.

In the case II' we have $\rho(y_a, y_k) \ge r_0$, $\forall k \ne a$, or $\rho(y_b, y_k) \ge r_0$, $\forall k \ne b$. Therefore, **A** holds.

The case II" splits further into three cases:
a) $\rho(y_a, y_p) \ge r_0$. Hence $\rho(y_p, y_q) \ge r_0$. So, $\rho(y_p, y_k) \ge r_0$, $\forall k \ne p$. Therefore, **A** holds
b) $\rho(y_b, y_q) \ge r_0$. Then $\rho(y_q, y_k) \ge r_0$, $\forall k \ne q$, and **A** holds
c) $\rho(y_a, y_p) \le r_0$ and $\rho(y_b, y_q) \le r_0$. Since $\rho(y_p, y_q) \ge r_0$, then **B** holds. □

Further we denote the sets $\mathcal{A} = \{\mathbf{y} = (y_1, y_2, y_3, y_4) \in (R_k^t)^4 : \mathbf{A} \text{ holds}\}$, $\mathcal{B} = \{\mathbf{y} \in (R_k^t)^4 : \mathbf{B} \text{ holds}\}$, $\mathcal{C} = \{\mathbf{y} \in (R_k^t)^4 : \mathbf{C} \text{ holds}\}$. Then (9.1) and Lemma 9.1 imply that

$$E_0|I(R_k^t)|^4 \le \frac{C}{t^4}\left(\int\limits_{\mathcal{A}} + \int\limits_{\mathcal{B}} + \int\limits_{\mathcal{C}}\right) E_0\Big(v_0(y_1)v_0(y_2)v_0(y_3)v_0(y_4)\Big)dS(y_1)dS(y_2)dS(y_3)dS(y_4). \quad (9.2)$$

(4.8), (2.25) imply that for $\mathbf{y} = (y_1, y_2, y_3, y_4) \in \mathcal{A}$

$$E_0\Big(v_0(y_1)v_0(y_2)v_0(y_3)v_0(y_4)\Big) = E_0 v_0(y_i) \cdot E_0\Big(v_0(y_{j1})v_0(y_{j2})v_0(y_{j3})\Big) = 0.$$

Therefore, the integral over $\mathcal{A}$ in (9.2) vanishes. Let us estimate the volume of $\mathcal{B}$ and $\mathcal{C}$.
I. The volume $|\mathcal{B}|$ of $\mathcal{B}$ is less or equal to the volume of $\{\mathbf{y} = (y_1, y_2, y_3, y_4) \in (R_k^t)^4 : |y_a - y_i| \le r_0,\ |y_j - y_b| \le r_0\}$. Therefore,

$$\begin{aligned} |\mathcal{B}| &\le |\{(y_a, y_i) \in (R_k^t)^2 :\ |y_a - y_i| \le r_0\}| \cdot |\{(y_j, y_b) \in (R_k^t)^2 :\ |y_j - y_b| \le r_0\}| \\ &= C(r_0)|R_k^t|^2. \end{aligned} \quad (9.3)$$

II. Obviously, $|\mathcal{C}| \le C(r_0)|R_k^t|$.

Finally, the integrand in (9.2) is bounded by b^4 with b from (6.1). Therefore, (9.2) - (9.3) imply that

$$E_0|I(R_k^t)|^4 \le \frac{C(r_0)b^4}{t^4}\left(|R_k^t|^2 + |R_k^t|\right).$$

Note that $|R_k^t| \le C\dfrac{t^2}{N_t}$, and $N_t \to \infty$. Therefore,

$$\sum_{k=1}^{N_t} E_0|I(R_k^t)|^4 \le N_t \cdot \frac{C(r_0)b^4}{t^4}\left(\frac{t^4}{N_t^2} + \frac{t^2}{N_t}\right) \to 0, \quad t \to \infty,$$

which completes the proof of (9.1). □

10 Appendix B. B.R. Vainberg's estimates

We prove Lemma 7.8. Theorem 4 of Chapter X from [51] implies the following lemma.

Lemma 10.1 *Let* **E1**–**E3** *hold, and let $n \geq 3$ be odd. Then $\forall R, R_0 > 0$ there exist constants $\kappa, C(R, R_0) > 0$ and $T = T(R, R_0) > 0$ such that*

$$\|\frac{\partial^k}{\partial t^k} U(t) Y_0\|_R \leq C(R, R_0) e^{-\kappa t} \|Y_0\|_{R_0}, \quad t \geq T > 0, \quad k = 0, 1, \tag{10.1}$$

for any $Y_0 \in \mathcal{H}$ with $\operatorname{supp} Y_0 \subset B_R$.

Therefore, we get by duality the following lemma.

Lemma 10.2 *Let* **E1**–**E3** *are fulfilled, and $n \geq 3$ is odd. Then $\forall R, R_0 > 0$ there exist constants $\kappa, C(R, R_0) > 0$ and $T = T(R, R_0) > 0$ such that*

$$\|\frac{\partial^k}{\partial t^k} U'(t) \Psi\|'_{H^{-1}(B_{R_0}) \oplus L^2(B_{R_0})} \leq C(R, R_0) e^{-\kappa t} \|\Psi\|'_R, \quad t \geq T > 0, \quad k = 0, 1, \tag{10.2}$$

for any $\Psi \in \mathcal{H}'_R$.

Let us we deduce (7.11) from the bound (10.2). (7.7) implies the representation $U'(t)\Psi = (\dot{\psi}(\cdot, t), \psi(\cdot, t))$ where $\psi(x, t)$ is a solution to $\ddot{\psi} = A\psi$. Then (10.2) with $k = 0$ implies

$$\|\psi(\cdot, t)\|_{L^2(B_{R_0})} \leq C(R, R_0) e^{-\kappa t} \|\Psi\|'_R, \quad t \geq T > 0. \tag{10.3}$$

Similarly, (10.2) with $k = 1$ implies that

$$\|\dot{\psi}(\cdot, t)\|_{L^2(B_{R_0})} \leq C(R, R_0) e^{-\kappa t} \|\Psi\|'_R, \quad t \geq T > 0, \tag{10.4}$$

and also, by virtue of $\ddot{\psi} = A\psi$, we get

$$\|A\psi(\cdot, t)\|_{H^{-1}(B_{R_0})} \leq C(R, R_0) e^{-\kappa t} \|\Psi\|'_R, \quad t \geq T > 0. \tag{10.5}$$

Note that (10.4) implies a part of the bound (7.11). It remains to obtain the estimate for $\|\psi(\cdot, t)\|_{H^1(B_{R_0})}$. We deduce it from the interior Schauder estimates [51, Th.VI.5] for the elliptic operator A:

$$\|\psi(\cdot, t)\|_{H^1(B_{R_0})} \leq C \left(\|A\psi(\cdot, t)\|_{H^{-1}(B_{R_0+1})} + \|\psi(\cdot, t)\|_{L^2(B_{R_0+1})} \right). \tag{10.6}$$

We use (10.3) and (10.5) with $R_0 + 1$ instead of R_0 in the right hand side of (10.6) and get

$$\|\psi(\cdot, t)\|_{H^1(B_{R_0})} \leq C_1(R, R_0) e^{-\kappa t} \|\Psi\|'_R, \quad t \geq T_1 > 0. \tag{10.7}$$

Finally, (10.4), (10.7) imply (7.11). □

References

[1] M. Aizenman, S. Goldstein, J.L. Lebowitz, Ergodic properties of an infinite one dimensional hard rod systems, *Comm. Math. Phys.*, **39** (1975), no. 4, 289-301.

[2] C. Boldrigini, R.L. Dobrushin, Yu.M. Sukhov, One-dimensional hard rod caricature of hydrodynamics, *Journ. Stat. Phys.* **31** (1983), 577-615.

[3] C. Boldrigini, A. Pellegrinotti, L. Triolo, Convergence to stationary states for infinite harmonic systems, *Journ. Stat. Phys.*, **30** (1983), 123-155.

[4] D.D. Botvich, V.A. Malyshev, Unitary equivalence of temperature dynamics for ideal and locally perturbed ferni-gas *Comm. Math. Phys.*, **91** (1983), no. 4, 301-312.

[5] N. Bournaveas, Local existence for the Maxwell-Dirac equations in three space dimensions, *Comm. Partial Diff. Equs* **21** (1996), no.5-6, 693-720.

[6] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, Ergodic Theory, Springer, New York-Berlin, 1982.

[7] R.L. Dobrushin, Problem of mathematical foundation of statistical mechanics, *Uspekhi Matematicheskikh Nauk* **32** (1977), 164-165 (in Russian).

[8] R.L. Dobrushin, Yu.M. Suhov, On the problem of the mathematical foundation of the Gibbs postulate in classical statistical mechanics, p. 325-340 in: Mathematical Problems in Theoretical Physics, Lecture Notes in Physics, v. 80, Springer, Berlin, 1978.

[9] R.L. Dobrushin, Yu.M. Sukhov, Time asymptotics for some degenerate models of evolution of systems with an infinite number of particles, *J. Sov. Math.* **16** (1981), 1277-1340.

[10] T.V. Dudnikova, A.I. Komech, Ergodic properties of hyperbolic equations with mixing, *Theory Probab. Appl.* **41** (1996), no.3, 436-448.

[11] T.V. Dudnikova, A.I. Komech, E.A. Kopylova, On convergence to statistical equilibrium for the Klein-Gordon equation with mixing, in preparation.

[12] T.V. Dudnikova, A.I. Komech, H. Spohn, On convergence to statistic equilibrium in two-temperature problem for wave equations with mixing, preprint no.26, Max-Planck-Institute for Mathematics in the Sciences, Leipzig, 2000.

[13] G.I. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations, Translations of Mathematical Monographs, Vol.52, American Mathematical Society, Providence, R.I., 1981.

[14] M.J. Esteban, V. Georgiev, E. Sere, Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations, *Calc. Var. Partial Differ. Equ.* **4** (1996), no.3, 265-281.

[15] I.I. Gikhman, A.V. Skorokhod, The Theory of Stochastic Processes, Vol.I, Springer, Berlin, 1974.

[16] S. Goldstein, Space-time ergodic properties of systems of infinitely many independent particles, *Comm. Math. Phys.*, **39** (1975), no. 4, 303-327.

[17] S. Goldstein, J.L. Lebowitz, Ergodic properties of an infinite systems of particles moving independently in a periodic field, *Comm. Math. Phys.*, **37** (1974), no. 1, 1-18.

[18] S. Goldstein, J.L. Lebowitz, M. Aizenman, Ergodic properties of infinite systems, p. 69-96 in: Lecture Notes in Physics, V.38, Springer, Berlin, 1975.

[19] H.O. Georgii, Gibbs Measures and Phase Transitions, Walter de Gruyter, Berlin, 1974.

[20] I.A. Ibragimov, Yu.V. Linnik, Independent and Stationary Sequences of Random Variables, Ed. by J. F. C. Kingman, Wolters-Noordhoff, Groningen, 1971.

[21] V. Jaksic, C.-A.Pillet, Ergodic properties of classical dissipative systems. I. *Acta Math.* **181** (1998), no.2, 245-282.

[22] F. John, Plane Waves and Spherical Means applied to Partial Differential Equations, Interscience Publishers, New York-London, 1955.

[23] A.I. Komech, Stabilization of statistics in wave and Klein-Gordon equations with mixing. Scattering theory for solutions of infinite energy, *Rend. Sem. Mat. Fis. Milano* **65** (1995), 9–22.

[24] A.I. Komech, On stabilization of string-nonlinear oscillator interaction, *J. Math. Anal. Appl.* **196** (1995), 384-409.

[25] A.I. Komech, On the stabilization of string-oscillators interaction, *Russian Journal of Math.Phys.* **3** (1995), 227-248.

[26] A.I. Komech, On transitions to stationary states in Hamiltonian nonlinear wave equations, *Phys. Letters A* **241** (1998), 311-320.

[27] A.I. Komech, On transitions to stationary states in one-dimensional nonlinear wave equations, *Arch. Rat. Mech. Anal.* **149** (1999), no. 3, 213-228.

[28] A.I. Komech, M. Kunze, H. Spohn, Effective dynamics for a mechanical particle coupled to a wave field, *Comm. Math. Phys.* **203** (1999), 1-19.

[29] A.I. Komech, N.E. Ratanov, Stabilization of space-time stochastic solutions of a wave equation, Statistics and control of stochastic processes. Vol. 2, Pap. Steklov Semin., Moscow/USSR 1985-86, Transl. Ser. Math. Eng., 171-187 (1989).

[30] A.I. Komech, H. Spohn, Soliton-like asymptotics for a classical particle interacting with a scalar wave field, *Nonlin. Analysis* **33** (1998), no.1, 13-24.

[31] A.I. Komech, H. Spohn, Long-Time Asymptotics for the Coupled Maxwell-Lorentz Equations *Comm. Partial Diff. Equs* **25** (2000), no.3/4, 558-580.

[32] A.I. Komech, H. Spohn, M. Kunze, Long-time asymptotics for a classical particle interacting with a scalar wave field, *Comm. Partial Diff. Equs.* **22** (1997), no.1/2, 307-335.

[33] A.I. Komech, B.R. Vainberg, On asymptotic stability of stationary solutions to nonlinear wave and Klein-Gordon equations, *Arch. Rat. Mech. Anal.* **134** (1996), 227-248.

[34] E.A. Kopylova, On stabilization of statistical solutions of the Klein-Gordon equation, *Russian Mathematical Surveys* **40** (1985), no.5, 240–241.

[35] E.A. Kopylova, Stabilization of moment functions of the statistical solution of the wave equation, *Moscow Univ. Math. Bull.* **40** (1985), no. 6, 65–68.

[36] E.A. Kopylova, Stabilization of statistical solutions of the Klein-Gordon equation, *Moscow University Mathematics Bulletin* **41** (1986), no.2, 72–75.

[37] O.E. Lanford III, J.L. Lebowitz, Time evolution and ergodic properties of harmonic systems, p.144-177 in: Dyn. Syst., Theor. Appl. (Battelle Seattle 1974 Renc.), Lect. Notes Phys., Vol.38, Springer, 1975.

[38] J.L. Lions, E. Magenes, Problèmes aux Limites non Homogènes et Applications, v.1, Dunod, Paris, 1968.

[39] C.S. Morawetz, W.A. Strauss, Decay and scattering of solutions of a nonlinear relativistic wave equation, *Comm. Pure Appl. Math.* **25** (1972), 1-31.

[40] D. Ornstein, Ergodic Theory, Randomness, and Dynamical Systems, Yale University Press, New Haven and London, 1974.

[41] N.E. Ratanov, Stabilization of statistical solutions of second-order hyperbolic equations, *Russian Mathematical Surveys* **39** (1984), no.1, 179–180.

[42] N.E. Ratanov, Asymptotic normality of statistical solutions of the wave equation, *Mosc. Univ. Math. Bull.* **40** (1985), no.4, 77-79.

[43] N.E. Ratanov, Stabilization of statistical solutions of second-order hyperbolic equations, PhD Thesis, Moscow State University, 1984.

[44] M.A. Rosenblatt, A central limit theorem and a strong mixing condition, *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), no.1, 43-47.

[45] D. Ruelle, Statistical Mechanics. Rigorous Results, Singapore: World Scientific, London: Imperial College Press, 1999.

[46] I. Segal, Dispersion for nonlinear relativistic wave equations, *Ann. Sci. Ecole Norm. Sup.* **1** (1968), 459-497.

[47] A.G. Shuhov, Yu.M. Suhov, Ergodic properties of groups of the Bogoliubov transformations of CAR C^*-algebras, *Ann. Phys.* **175** (1987), 231-266.

[48] Ya. Sinai, Ergodic properties of a gas of one-dimensional hard globuls with an infinite number of degrees of freedom, *Funct. Anal. Appl.* **6** (1972), 35-43.

[49] H. Spohn, J.L. Lebowitz, Stationary non equilibrium states of infinite harmonic systems, *Comm. Math. Phys.* **54** (1977), 97-120.

[50] W.A. Strauss, Decay and asymptotics for $\Box u = F(u)$, *J. Funct. Anal.* **2** (1968), 409-457.

[51] B.R. Vainberg, Asymptotic Methods in Equations of Mathematical Physics, Gordon and Breach, New York, 1989.

[52] M.I. Vishik, A.V. Fursikov, Mathematical Problems of Statistical Hydromechanics, Kluwer Academic Publishers, Dordrecht, 1988.

[53] K.L. Volkovisskii, Ya. Sinai, Ergodic properties of an ideal gas with an infinite number of degrees of freedom, *Functional Anal. Appl.* **5** (1971), 185-187.