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On convergence to statistic equilibrium in two-temperature problem for wave equations with mixing

by

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On Convergence to Statistic Equilibrium in Two-Temperature Problem for Wave Equation with Mixing

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Abstract

The wave equation in the whole space $\mathbb{R}^3$ is considered. The initial datum is a random function with finite mean density of the energy which also fits the mixing condition of Ibragimov-Linnik-Rosenblatt type. The random function converges to different space-homogeneous processes as $x_3 \to \pm\infty$, with the distributions $\mu_\pm$. We study the distribution μ_t of the random solution at the moments $t \in \mathbb{R}$. The main result is the convergence of μ_t to an equilibrium Gaussian translation-invariant measure as $t \to \infty$. The application to the case of the Gibbs measures $\mu_\pm = g_\pm$ with two different temperatures $T_\pm$ is given. Limiting mean energy current density *formally* is $-\infty \cdot (0, 0, T_+ - T_-)$ for the Gibbs measures, and it is finite $-C(0, 0, T_+ - T_-)$ with $C > 0$ for the convolution with a nontrivial test function.

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1 Introduction

We consider the wave equation in $\mathbb{R}^3$ with the initial conditions

$$\left\{\begin{array}{l} \ddot{u}(x,t)=\Delta u(x,t), \quad x\in\mathbb{R}^3, \\ \\ u\Big|_{t=0}=u_0(x), \quad \dot{u}\Big|_{t=0}=v_0(x). \end{array}\right. \tag{1.1}$$

Let us denote $Y(t)=(Y^0(t),Y^1(t))\equiv(u(\cdot,t),\dot{u}(\cdot,t))$, $Y_0=(Y_0^0,Y_0^1)\equiv(u_0,v_0)$. Then (1.1) becomes

$$\dot{Y}(t)=\mathcal{F}(Y(t)), \quad t\in\mathbb{R}, \quad Y(0)=Y_0. \tag{1.2}$$

We assume that the initial datum Y_0 is a random function in the phase space $\mathcal{H}$ of the solutions with a finite local energy. Denote by $\mu_t(dY)$, $t\in\mathbb{R}$, the measure in $\mathcal{H}$, which is a distribution of the random solution $Y(t)$ to the Cauchy problem (1.2). We assume that the initial correlation functions $Q_0^{ij}(x,y)\equiv EY_0^i(x)Y_0^j(y)$, $i,j=0,1$, and some of its derivatives are continuous and decaying when $|x-y|\to\infty$. In particular, the initial mean energy density is bounded,

$$E[|\nabla u_0(x)|^2+|v_0(x)|^2]=[\nabla_x\cdot\nabla_y Q_0^{00}(x,y)]|_{y=x}+Q_0^{11}(x,x)\le C<\infty, \quad x\in\mathbb{R}^3. \tag{1.3}$$

Moreover, we assume that the correlation matrix $(Q_0^{ij}(x,y))_{i,j=0,1}$ has the form

$$Q_0^{ij}(x,y)=\left\{\begin{array}{ll} q_-^{ij}(x-y), & x_3,y_3<-a, \\ \\ q_+^{ij}(x-y), & x_3,y_3>a. \end{array}\right. \tag{1.4}$$

Here $q_\pm^{ij}(x-y)$ are the correlation functions of some Borel translation-invariant measures $\mu_\pm$ with zero mean value in $\mathcal{H}$, $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)\in\mathbb{R}^3$, and $a>0$. Therefore, the measure μ_0 is not translation-invariant if $q_-^{ij}\neq q_+^{ij}$. At last, we assume that the initial measure μ_0 fits the mixing condition of Ibragimov-Linnik-Rosenblatt type. Roughly speaking, the random values

$$Y_0(x) \;\; \textit{and} \;\; Y_0(y) \;\; \textit{are asymptotically independent as} \;\; |x-y|\to\infty. \tag{1.5}$$

Our main result is the convergence to the statistic equilibrium, i.e. a weak convergence of the measures μ_t,

$$\mu_t \rightharpoondown \mu_\infty, \quad t\to\infty, \tag{1.6}$$

where μ_∞ is a translation-invariant equilibrium Gaussian measure in the phase space $\mathcal{H}$. A similar convergence hold for negative $t\to-\infty$ since our system is time-reversible. We give some optimal bounds for mixing coefficient of the initial measure μ_0. We construct generic examples of the random initial datum satisfying all assumptions imposed. We get the explicit formulas (2.13)-(2.15) for the correlation functions of the limit measure μ_∞.

We apply our results to the case of the Gibbs measures $\mu_\pm=g_\pm$. Formally

$$g_\pm(du_0,dv_0)=\frac{1}{Z_\pm}e^{-\beta_\pm\int(|\nabla u_0(x)|^2+|v_0(x)|^2)dx}\prod_x du_0(x)dv_0(x), \quad \beta_\pm=T_\pm^{-1}, \tag{1.7}$$

where $T_\pm > 0$ are the corresponding absolute temperatures. We adjust the definition of the Gibbs measures $g_\pm$ in Section 3. The Gibbs measures $g_\pm$ have singular correlation functions and do not satisfy our assumptions (2.10). Respectively, our results can not be applied directly to $g_\pm$. We reduce the problem by a convolution with a smooth function $\theta \in D \equiv C_0^\infty(\mathbb{R}^3)$. We consider Gaussian processes $u_\pm$ corresponding to measures $g_\pm$ and we define the "smoothed" measures $g_\pm^\theta$ as distributions of the convolutions $u_\pm * \theta$. The measures $g_\pm^\theta$ satisfy all our assumptions, and the convergence $g_t^\theta \rightharpoondown g_\infty^\theta$ in $\mathcal{H}^{-\varepsilon}$ follows from our results. This implies a weak convergence of the measures $g_t \rightharpoondown g_\infty$ in some weighted Sobolev space of distributions since θ is arbitrary. We show that the limit energy current for g_∞ is formally

$$\bar{j}_\infty = -\infty \cdot (0, 0, T_+ - T_-),$$

which means the "ultraviolet divergence". This relation is meaningful in the case of smoothed measures g_∞^θ,

$$\bar{j}_\infty^\theta = -C_\theta \cdot (0, 0, T_+ - T_-),$$

if $\theta(x)$ is axially symmetric with respect to Ox_3; $C_\theta > 0$ if $\theta(x) \not\equiv 0$. This corresponds to second law of thermodynamics.

We prove the convergence (1.6) by the strategy of [8, 11, 17, 18] in three steps.
I. The family of measures μ_t, $t \ge 0$, is compact in an appropriate Fréchet space.
II. The correlation functions converge,

$$Q_t^{ij}(x,y) \equiv EY^i(x,t)Y^j(y,t) \to Q_\infty^{ij}(x,y), \quad t \to \infty. \tag{1.8}$$

III. The characteristic functionals converge as $t \to \infty$,

$$\tilde{\mu}_t(\Psi) = \int e^{i<Y,\Psi>}\mu_t(dY) \to e^{-\frac{1}{2}<Q_\infty\Psi,\Psi>}, \quad \forall \Psi \in \mathcal{D}, \tag{1.9}$$

where Q_∞ is the integral operator with the matrix-valued kernel $(Q_\infty^{ij}(x,y))_{i,j=0,1}$.

The compactness **I** follows from the Prokhorov criterion by the method [21]. Namely, we prove a uniform bound for the second moment functions of the measures μ_t, $t \ge 0$. Then the Prokhorov condition follows from the Sobolev embedding theorem by Chebyshev's inequality. We deduce the uniform bound from the explicit expression for the correlation functions $Q_t^{ij}(x,y)$. The expression follows from the Kirchhoff formula for the solutions to (1.1). In particular, for the case $u_0(x) \equiv 0$, we have

$$u(x,t) = \frac{1}{4\pi t} \int\limits_{S_t(x)} v_0(z)dS(z), \tag{1.10}$$

where $dS(z)$ is the Lebesgue measure on the sphere $S_t(x) : |z - x| = t$.

The convergence (1.8) also follows from explicit formulas for $Q_t^{ij}(x,y)$. The formula (1.10) allows to express the correlation functions $Q_t^{ij}(x,y)$ in terms of integrals over spheres of radius t. In the limit, $t \to \infty$, the spheres become the planes. Respectively, $Q_\infty^{ij}(x,y)$ is expressed in terms of integrals of the Radon transform of initial correlation functions $Q_0^{ij}(x,y)$. We reduce the integrals of the Radon transform to some convolutions.

If μ_0 is a Gaussian measure, the convergence (1.9) follows from (1.8). In the case when the measure μ_0 is non-Gaussian, the proof of the convergence (1.9) follows by the method [8]. The method [8] is a development of S.N. Bernstein's "room-corridors" method, and it is suggested by the structure of the Kirchhoff formula (1.10). In [8] the convergence (1.9) is proved for the case of translation invariant measure μ_0. The method uses the convergence of the correlation functions (1.8) and a uniform bound on the moment functions of fourth order. Let us emphasize, however, that the proof of the bound and of (1.9) in [8] do not use explicitly the translation invariance of the measures μ_t.

In conclusion, we extend the main result to the equations with variable coefficients, which are constant outside a finite region. The extension follows immediately from our result for constant coefficients, using method [8]. The method is based on the scattering theory for the solutions of infinite energy, which is constructed in [8].

Remarks i) The dynamics (1.1) is translation invariant, and its Fourier transform has a very simple form. However, we cannot use the Fourier transform for the proof of (1.8) since our main assumption (1.4) is stated in the coordinate space.
ii) Our general proof of the convergence (1.9) in Sections 5 and 6 does not allow a simplification in the case of the Gibbs measure (1.7). This is related to the slow long-range decay of the correlation function $Q_0^{00}(x,y) \sim |x-y|^{-1}$, $|x-y| \to \infty$.
iii) All three steps **I-III** depend drastically on the mixing condition. Simple examples show that all the assertions can fail if the mixing condition breaks down (see [8]).

In Section 2 we state our main result. We apply it to the Gibbs measure in Section 3. We prove the compactness **I** and the convergence **II** in Sections 4 - 6. Section 7 completes the proof of the main result, and Section 8 concerns the variable coefficients. Appendix A concerns the Radon transform and convolution, and Appendix B concerns the Gaussian measures in the weighted Sobolev spaces.

The convergence to statistic equilibrium (2.3) for the wave equation is established for the first time in [8] (see also [11, 12, 17, 18]) in the case of translation-invariant initial measure μ_0 with mixing. This corresponds to our result in the particular case when $T_- = T_+$. Similar result has been proved for the Klein-Gordon equation, [7, 11, 13, 14, 15]. The random process $Y(t)$ is ergodic and mixing (in time) if the initial measure μ_0 coincides with one of the equilibrium limit measures μ_∞, [3, 4, 5, 6].

Let us note that the equation (1.1) describes a continuous n-dimensional family of harmonic oscillators. Therefore, our result is an extension of the results [1, 19] which concern the infinite one-dimensional chains of harmonic oscillators.

2 Main results

Let us describe our results more precisely.

2.1 Notations

We assume that the initial datum Y_0 belongs to the phase space $\mathcal{H}$.

Definition 2.1 *$\mathcal{H} \equiv H^1_{loc}(\mathbb{R}^3) \oplus H^0_{loc}(\mathbb{R}^3)$ is the Fréchet space of $Y \equiv (u(x), v(x))$ with real valued functions $u(x)$, $v(x)$, which is endowed with the local energy seminorms*

$$\|Y\|^2_R = \int\limits_{|x|<R} (|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2)dx < \infty, \quad \forall R > 0. \tag{2.1}$$

The following proposition is well-known [16].

Proposition 2.2 *i) For any $Y_0 \in \mathcal{H}$ there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H})$ to Cauchy problem (1.2).*
ii) The operator $U_t : Y_0 \mapsto Y(t)$ is continuous in $\mathcal{H}$ for any $t \in \mathbb{R}$.

Denote by $H^s_{loc}(\mathbb{R}^3)$, $s \in \mathbb{R}$, the local Sobolev spaces, i.e. the Fréchet spaces of the distributions $u \in D'(\mathbb{R}^3)$ with the finite seminorms

$$\|u\|_{s,R} \equiv \sup_{\|\psi\|_{-s}=1} | < u, \psi > |.$$

Here the $\sup$ is taken over all $\psi \in D$ such that $\psi(x) = 0$ for $|x| > R$, and

$$\|\psi\|^2_{-s} \equiv \int_{\mathbb{R}^3} (1 + |\xi|)^{-2s} |\tilde{\psi}(\xi)|^2 d\xi, \quad \tilde{\psi}(\xi) = F\psi(\xi) = \int e^{i\xi \cdot x}\psi(x)dx.$$

The brackets $< \cdot, \cdot >$ denote the scalar product in $L^2(\mathbb{R}^3)$ or in $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ or their different extensions.

Definition 2.3 *For $\varepsilon > 0$ we denote $\mathcal{H}^{-\varepsilon} \equiv H^{1-\varepsilon}_{loc}(\mathbb{R}^n) \oplus H^{-\varepsilon}_{loc}(\mathbb{R}^n)$.*

Then $\mathcal{H} \subset \mathcal{H}^{-\varepsilon}$ for every $\varepsilon > 0$, and the embedding $\mathcal{H} \subset \mathcal{H}^{-\varepsilon}$ is compact by the Sobolev theorem.

2.2 Random solution

Now we assume that Y_0 in (1.2) is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ where $\mathcal{B}(\mathcal{H})$ is the Borel σ-algebra of subsets in $\mathcal{H}$. We denote by $\mu_0(dY_0)$ a Borel probability measure in $\mathcal{H}$ which is the distribution of the random function Y_0. Then $Y(t) = U_t Y_0$ is a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ due to Proposition 2.2.

Definition 2.4 *μ_t is a Borel probability measure in $\mathcal{H}$ which is the distribution of $Y(t)$:*

$$\mu_t(B) = \mu_0(U_t^{-1}B), \quad \forall B \in \mathcal{B}(\mathcal{H}), \quad t \in \mathbb{R}. \tag{2.2}$$

Our main goal is to derive the convergence of the measures μ_t as $t \to \infty$. We establish the weak convergence of the measures μ_t in the Fréchet spaces $\mathcal{H}^{-\varepsilon}$ with any $\varepsilon > 0$

$$\mu_t \overset{\mathcal{H}^{-\varepsilon}}{\longrightarrow} \mu_\infty, \quad t \to \infty, \tag{2.3}$$

where μ_∞ is a Borel probability measure in the space $\mathcal{H}$. By definition this means the convergence

$$\int f(Y)\mu_t(dY) \to \int f(Y)\mu_\infty(dY), \qquad t \to \infty \tag{2.4}$$

for any bounded continuous functional $f(Y)$ in the space $\mathcal{H}^{-\varepsilon}$.

For the simplicity we assume that $Y_0 = (u_0, v_0)$ is a unit random function in the probability space $(\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu_0)$ and denote by E_0 the corresponding mathematical expectation operator.

Definition 2.5 *The correlation functions of measure μ_t are the distributions*

$$Q_t^{ij}(x,y) \equiv E_0 Y^i(x,t) Y^j(y,t), \quad i,j = 0,1, \tag{2.5}$$

where $Y^i(x,t)$ are the components of $Y(t) = (Y^0(x,t), Y^1(x,t))$.

It means for any $\varphi, \psi \in D$

$$< Q_t^{ij}(x,y), \varphi(x)\psi(y) >= E_0 < Y^i(x,t), \varphi(x) > < Y^j(y,t), \psi(y) > . \tag{2.6}$$

We will denote $\mathcal{D} = D \oplus D$, and

$$< Y, \Psi >=< Y^0, \Psi^0 > + < Y^1, \Psi^1 >$$

for $Y = (Y^0, Y^1) \in \mathcal{H}, \ \Psi = (\Psi^0, \Psi^1) \in \mathcal{D}$. For a Borel probability measure μ in the space $\mathcal{H}$ we denote by $\tilde{\mu}$ the characteristic functional (Fourier transform)

$$\tilde{\mu}(\Psi) \equiv \int \exp(i < Y, \Psi >)\, \mu(dY), \ \ \forall \Psi \in \mathcal{D}.$$

The measure μ is called Gaussian (with zero expectation) if its characteristic functional has the form

$$\tilde{\mu}(\Psi) = e^{-\frac{1}{2}<Q\Psi, \Psi>}, \ \ \Psi \in \mathcal{D},$$

where Q is a linear operator $\mathcal{D} \to \mathcal{D}'$. μ is called translation-invariant if $\forall h \in \mathbb{R}^3$

$$\mu(\hat{h}B) = \mu(B), \quad \forall B \in \mathcal{B}(\mathcal{H}),$$

where $\hat{h}Y(x) = Y(x+h)$.

2.3 Mixing condition

Let $O(r)$ denote the set of all pairs of open subsets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^3$ with $\rho(\mathcal{A}, \mathcal{B}) \geq r$; $\sigma_{i\alpha}(\mathcal{A})$ is the σ-algebra of subsets in $\mathcal{H}$, generated by all linear functionals of the form

$$Y \mapsto < D^\alpha Y^i, \psi >= \int\limits_{\mathbb{R}^3} D^\alpha Y^i(x)\psi(x)\, dx, \ \ |\alpha| \leq 1 - i, \ \ i = 0,1,$$

where $\psi \in D$ with $\operatorname{supp}\psi \subset \mathcal{A}$. We define Ibragimov-Linnik mixing coefficients (see [9]) for $|\alpha| \le 1-i$, $|\beta| \le 1-j$, $i,j = 0,1$, as

$$\phi_{i\alpha,j\beta}(r) \equiv \sup_{(\mathcal{A},\mathcal{B})\in O(r)} \sup_{\substack{A\in\sigma_{i\alpha}(\mathcal{A}),\, B\in\sigma_{j\beta}(\mathcal{B})\\ \mu_0(B)>0}} \frac{|\mu_0(A\cap B)-\mu_0(A)\mu_0(B)|}{\mu_0(B)}.$$

Definition 2.6 *The measure μ_0 fits strong uniform Ibragimov-Linnik mixing condition if*

$$\phi_{i\alpha,j\beta}(r) \to 0, \quad r\to\infty \tag{2.7}$$

for $|\alpha| \le 1-i$, $|\beta| \le 1-j$ and $i,j=0,1$.

We adjust below the rate of this decay.

2.4 Main theorem

Let $\nu_d \in C(0,\infty)$ denote some nonnegative functions ($d=0,1,2$) with the finite integrals,

$$\int_0^\infty (1+r)^{d-1}\nu_d(r)dr = I_d < \infty. \tag{2.8}$$

We also denote $\nu(r) = \nu_2(r)$. We assume the following properties **S0-S5** for the initial measure μ_0.

S0 The measure μ_0 has zero expectation value,

$$E_0 Y_0(x) = 0, \quad x\in \mathbb{R}^3 \tag{2.9}$$

in the sense of distributions, similarly to (2.6).
S1 μ_0 is a Borel measure in the space $\mathcal{H}$ with the correlation functions of the form (1.4).
S2 The following derivatives are continuous and the bounds hold,

$$|D^{\alpha,\beta}_{x,y}Q^{ij}_0(x,y)| \le \left\{\begin{array}{l} C\nu_d(|x-y|) \text{ if } d=0 \text{ or } 1,\\ C\nu_2(|x-y|) \text{ if } 2\le d\le 4,\end{array}\right. \Bigg| \quad d = i+j+|\alpha|+|\beta|. \tag{2.10}$$

S3 The measure μ_0 satisfies the *strong uniform* Ibragimov-Linnik mixing condition, and for $|\alpha| \le 1-i$, $|\beta| \le 1-j$, $i,j=0,1$

$$\phi_{i\alpha,j\beta}(r) \le C\nu_d^2(r), \quad d = i+j+|\alpha|+|\beta|. \tag{2.11}$$

Remark 2.7 i) Condition **S2** implies (1.3).
ii) The mixing condition **S3** is weaker than in [8] where the tranlation-invariant case is considered.
iii) The estimates (2.11) for $d\le 1$ also are weaker than in [8]. On the other hand, the estimates are not required in [8] for $d>2$ and agree with [8] for $d=2$.
iii) The conditions **S2** and **S3** allow various modifications. We choose the variant which allow an application to the case of the Gibbs measures (1.7) (see next section).

Define the matrix-valued function

$$Q_\infty(x,y) = \Big(Q^{ij}_\infty(x,y)\Big)_{i,j=0,1} = \Big(q^{ij}_\infty(x-y)\Big)_{i,j=0,1}\ , \tag{2.12}$$

where

$$\begin{aligned}
q^{00}_\infty &= \frac{1}{4}\Big[q^{00}_+ + q^{00}_- - \mathcal{E} * (q^{11}_+ + q^{11}_-) + \mathcal{P} * (q^{01}_+ - q^{01}_- - q^{10}_+ + q^{10}_-)\Big], && (2.13)\\
q^{10}_\infty = -\, q^{01}_\infty &= \frac{1}{4}\Big[q^{10}_+ + q^{10}_- - q^{01}_+ - q^{01}_- \quad + \mathcal{P} * (q^{11}_+ - q^{11}_- - \Delta q^{00}_+ + \Delta q^{00}_-)\Big], && (2.14)\\
q^{11}_\infty = -\Delta q^{00}_\infty &= \frac{1}{4}\Big[q^{11}_+ + q^{11}_- - \Delta(q^{00}_+ + q^{00}_-) + \mathcal{P} * \Delta(q^{10}_+ - q^{10}_- - q^{01}_+ + q^{01}_-)\Big]. && (2.15)
\end{aligned}$$

Here $\mathcal{E}(x) = -\dfrac{1}{4\pi|x|}$ is the fundamental solution of the Laplacian, and $\mathcal{P}(x) = -iF^{-1}\Big[\dfrac{\operatorname{sgn}\xi_3}{|\xi|}\Big]$ where F^{-1} is the inverse Fourier transform. The definition of the convolutions with $\mathcal{P}$ in formulas (2.13)–(2.15) is adjusted in Appendix A (formula (6.14)). Our main result is the following theorem.

Theorem 2.8 *Let* **S0-S3** *hold. Then there exists a Gaussian Borel probability measure* μ_∞ *in* $\mathcal{H}$ *such that*
i) the convergence (2.3) holds for any $\varepsilon > 0$ *.*
ii) The measure μ_∞ *is translation invariant.*
iii) Its characteristic functional has the form

$$\tilde\mu_\infty(\Psi) = e^{-\frac{1}{2}<Q_\infty\Psi,\Psi>}, \qquad \Psi \in \mathcal{D},$$

where Q_∞ *is the operator with the integral kernel* $Q_\infty(x,y)$ *.*

Remark 2.9 Theorem 2.8 holds for the Gaussian initial measures μ_0 without the mixing condition **S3**. This follows from Lemmas 4.1 and 5.1 below.

Theorem 2.8 follows from next Propositions 2.10 and 2.11 by the methods of [11, 17, 21].

Proposition 2.10 *The family of the measures* μ_t *,* $t \ge 0$ *, is compact in the space* $\mathcal{H}^{-\varepsilon}$ *with any* $\varepsilon > 0$ *.*

Proposition 2.11 *The characteristic functionals converge,*

$$\tilde\mu_t(\Psi) \to e^{-\frac{1}{2}<Q_\infty\Psi,\Psi>}, \quad t\to\infty, \ \ \forall\Psi\in\mathcal{D}. \tag{2.16}$$

We prove Proposition 2.10 in Section 4 for a simple particular case, and in Section 6 for general case. We prove Proposition 2.11 in Sections 5, 6 for the Gaussian measures μ_0 , and in Sections 7 for general non-Gaussian μ_0 .

2.5 Examples

2.5.1 Gaussian measures

We construct the Gaussian initial measures μ_0 satisfying **S0–S2**. Let us take the Gaussian measures $\mu_\pm$ in $\mathcal{H}$ with the correlation functions $q_\pm^{ij}(x-y)$ which are zero for $i \neq j$, while for $i = 0, 1$,

$$\left. \begin{array}{rcl} q_\pm^{ii}(z) & = & F^{-1}\tilde{q}_\pm^{ii}(\xi), \\ (1+|\xi|)^s \partial_\xi^\gamma \tilde{q}_\pm^{ii}(\xi) & \in & L^1(\mathbb{R}^3), \quad 0 \le d = 2i + s \le 4, \quad |\gamma| \le 1 + d, \\ \tilde{q}_\pm^{ii}(\xi) & \ge & 0. \end{array} \right| \tag{2.17}$$

Then $\mu_\pm$ satisfy **S0–S2** with the functions $\nu_d(r) = C(1+r)^{-1-d}$ for large enough $C > 0$. Let us take the functions $\zeta_\pm \in C^\infty(\mathbb{R})$ s.t.

$$\zeta_\pm(s) = \begin{cases} 1, & \text{for } \pm s > a, \\ 0, & \text{for } \pm s < -a. \end{cases}$$

Let us introduce (Y_-, Y_+) as a unit random function in probability space $(\mathcal{H}\times\mathcal{H}, \mu_- \times \mu_+)$. Then $Y_\pm$ are the Gaussian independent functions in $\mathcal{H}$. Define μ_0 as a distribution of the random function

$$Y_0(x) = \zeta_-(x_3)Y_-(x) + \zeta_+(x_3)Y_+(x). \tag{2.18}$$

Then correlation functions of the measure μ_0 are

$$Q_0^{ij}(x,y) = q_-^{ij}(x-y)\zeta_-(x_3)\zeta_-(y_3) + q_+^{ij}(x-y)\zeta_+(x_3)\zeta_+(y_3), \quad i,j = 0,1, \tag{2.19}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, $q_\pm^{ij}$ are the correlation functions of the measures $\mu_\pm$. Then **S0** and **S1** hold, and **S2** follows for μ_0 with the same functions $\nu_d(r)$ as for $\mu_\pm$. Let us assume, in addition to (2.17), that

$$q_\pm^{ii}(x) = 0, \quad |x| \ge r_0. \tag{2.20}$$

Then mixing (2.7) holds since $\phi_{i\alpha,j\beta}(r) = 0$, $r \ge r_0$, and **S3** follows. For instance, (2.17) and (2.20) hold if $\tilde{q}_\pm^{ii}(\xi) = f(\xi_1)f(\xi_2)f(\xi_3)$ with

$$f(z) = ((1-\cos(r_0 z))/z^2)^N, \quad z \in \mathbb{R},$$

where $N \ge 0$ is an integer, $2N - s > 1$ ($s = 4 - 2i$).

2.5.2 Non-Gaussian measures

Let us choose some odd nonconstant functions f^0, $f^1 \in C^4(\mathbb{R}^n \times \mathbb{R}^n)$ with bounded derivatives. Let us define $\hat{\mu}_0$ as the distribution of the random function $(f^0(Y(x)), f^1(Y(x)))$, where $Y(x)$ is a random function with the Gaussian distribution μ_0 from previous Example. Then **S0**, **S1** and **S3** hold for $\hat{\mu}_0$ with some appropiate functions ν_d since corresponding mixing coefficients $\hat{\phi}_{i\alpha,j\beta}(r) = 0$ for $r \ge r_0$. Therefore, **S0** implies for corresponding correlation functions $\hat{Q}^{ij}(x,y) = 0$ for $|x-y| \ge r_0$, so **S2** also holds. The measure $\hat{\mu}_0$ is not Gaussian since the functions f^0, f^1 are bounded and nonconstant.

3 Application to Gibbs measures

We apply Theorem 2.8 to the case when $\mu_\pm$ are the Gibbs measures (1.7) corresponding to different positive temperatures $T_- \neq T_+$.

3.1 Gibbs measures

We will define the Gibbs measures $g_\pm$ as the Gaussian measures with the correlation functions (cf. (1.7))

$$q_\pm^{00}(x-y) = -T_\pm \mathcal{E}(x-y), \quad q_\pm^{11}(x-y) = T_\pm \delta(x-y), \quad q_\pm^{01}(x-y) = q_\pm^{10}(x-y) = 0, \quad (3.1)$$

where $x, y \in \mathbb{R}^3$. The correlation functions $q_\pm^{ij}$ do not satisfy condition **S2** because of singularity at $x = y$. The singularity means that the measures $g_\pm$ are not concentrated in the space $\mathcal{H}$. Let us introduce appropriate functional spaces for measures $g_\pm$. First, let us define the weighted Sobolev space with any $s, \alpha \in \mathbb{R}$.

Definition 3.1 *$H_{s,\alpha}(\mathbb{R}^3)$ is the Hilbert space of the distributions $u \in S'(\mathbb{R}^3)$ with the finite norm*

$$\|u\|_{s,\alpha} \equiv \|(1+|x|)^\alpha \Lambda^s u\|_{L_2(\mathbb{R}^3)} < \infty, \quad \Lambda^s u \equiv F^{-1}\Big[(1+|\xi|)^s \tilde{u}(\xi)\Big]. \quad (3.2)$$

Let us fix arbitrary $s, \alpha < -3/2$.

Definition 3.2 *$\mathcal{G}_{s,\alpha}$ is the Hilbert space $H_{s+1,\alpha}(\mathbb{R}^3) \oplus H_{s,\alpha}(\mathbb{R}^3)$, with the norm*

$$\|Y\|_{\mathcal{G}} \equiv \|u\|_{s+1,\alpha} + \|v\|_{s,\alpha} < \infty, \quad Y = (u, v).$$

Introduce Gaussian Borel probability measures $g_\pm^0(du)$, $g_\pm^1(dv)$ in spaces $H_{s+1,\alpha}(\mathbb{R}^3)$ and $H_{s,\alpha}(\mathbb{R}^3)$, respectively, with characteristic functionals

$$\left.\begin{aligned} \tilde{g}_\pm^0(\psi) &= \int e^{i<u,\psi>} g_\pm^0(du) = e^{-\frac{<\Delta^{-1}\psi,\psi>}{2\beta_\pm}} \\ \tilde{g}_\pm^1(\psi) &= \int e^{i<v,\psi>} g_\pm^1(dv) = e^{-\frac{<\psi,\psi>}{2\beta_\pm}} \end{aligned}\right| \psi \in D.$$

By the Minlos theorem, [2], the Borel probability measures $g_\pm^0$, $g_\pm^1$ exist in the spaces $H_{s+1,\alpha}(\mathbb{R}^3)$, $H_{s,\alpha}(\mathbb{R}^3)$, respectively, because *formally* (see Appendix B)

$$\int \|u\|^2_{s+1,\alpha} g_\pm^0(du) < \infty, \quad \int \|v\|^2_{s,\alpha} g_\pm^1(dv) < \infty, \quad s, \alpha < -3/2. \quad (3.3)$$

Finally, we define the Gibbs measures $g_\pm(dY)$ as the Borel probability measures $g_\pm^0(du) \times g_\pm^1(dv)$ in $(\mathcal{G}_{s,\alpha}, \mathcal{B}(\mathcal{G}_{s,\alpha}))$. Let $g_0(dY)$ be the Borel probability measure in $(\mathcal{G}_{s,\alpha}, \mathcal{B}(\mathcal{G}_{s,\alpha}))$ which is constructed as in Example of previous section with $\mu_\pm(dY) = g_\pm(dY)$. It satisfies **S0** and **S1** with $q_\pm^{ij}$ from (3.1). However, g_0 does not satisfy **S2**. Therefore, Theorem 2.8 can't be apllied directly to $\mu_0 = g_0$.

Standard methods of pseudodifferential operators imply the following lemma.

Lemma 3.3 *The operators $U_t : Y_0 \mapsto Y(t)$ allow a continuous extention $\mathcal{G}_{s,\alpha} \mapsto \mathcal{G}_{s,\alpha}$.*

3.2 Convergence to equilibrium

Let Y_0 be a random function with the distribution g_0. Denote by g_t the distribution of $U_t Y_0$.

Theorem 3.4 *There exists a Gaussian Borel probability measure g_∞ in $\mathcal{G}_{s,\alpha}$ s.t.*

$$g_t \overset{\mathcal{G}_{s,\alpha}}{\longrightarrow} g_\infty, \quad t \to \infty. \tag{3.4}$$

Proof Denote by g_t^θ the distribution of the random function $(U_t Y_0) * \theta$ with a $\theta \in D$. Obviously, $(U_t Y_0) * \theta = U_t(Y_0 * \theta)$. The distribution g_0^θ of $Y_0 * \theta$ satisfies **S0–S4** with the functions $\nu_d(r) = C(1+r)^{-1-d}$ for large enough $C > 0$. Therefore, Theorem 2.8 implies the convergence (2.3) for $\mu_t = g_t^\theta$ by Remark 2.9:

$$g_t^\theta \overset{\mathcal{H}^{-\varepsilon}}{\longrightarrow} g_\infty^\theta, \ \ t \to \infty, \tag{3.5}$$

where g_∞^θ is a Gaussian measure in $\mathcal{H}$. Next, we prove the following Lemma on the compactness.

Lemma 3.5 *The family of measures g_t, $t \geq 0$, is compact in the Hilbert space $\mathcal{G}_{s,\alpha}$ with any $s, \alpha < -3/2$.*

Proof In the Fourier transform the solution to the problem (1.1) is the function $\tilde{u}(\xi, t) = \tilde{u}_0(\xi) \cos|\xi| t + \tilde{v}_0(\xi) \dfrac{\sin|\xi| t}{|\xi|}$. Then for $Y(t) = (u(\cdot,t), \dot{u}(\cdot,t))$ we get similarly to (10.1)-(10.4)

$$\int \|Y\|^2_{\mathcal{G}_{\overline{s},\overline{\alpha}}} g_t(dY) = \int \|Y(t)\|^2_{\mathcal{G}_{\overline{s},\overline{\alpha}}} g_0(dY_0) \leq C < \infty, \quad t \in \mathbb{R} \tag{3.6}$$

for any $\overline{s}, \overline{\alpha} < -3/2$. Let us choose $\overline{s} > s$ and $\overline{\alpha} > \alpha$. Then the embedding $\mathcal{G}_{\overline{s},\overline{\alpha}} \subset \mathcal{G}_{s,\alpha}$ is compact by the Sobolev Theorem, and Lemma 3.5 follows from the Prokhorov criterion by the method of [21]. □

The convergence (3.4) follows from Lemma 3.5 because the limit measure of any sequence g_{t_n} with $t_n \to \infty$ does not depend on the sequence by (3.5). □

The limit measure g_∞ is Gaussian with the correlation matrix $Q_\infty = (Q_\infty^{ij}(x,y))_{i,j=0,1}$, where

$$Q_\infty^{00}(x,y) \equiv q_\infty^{00}(x-y) = -\frac{1}{2}(T_+ + T_-)\mathcal{E}(x-y), \tag{3.7}$$

$$Q_\infty^{10}(x,y) = -Q_\infty^{01}(x,y) \equiv q_\infty^{10}(x-y) = \frac{1}{2}(T_+ - T_-)\mathcal{P}(x-y), \tag{3.8}$$

$$Q_\infty^{11}(x,y) \equiv q_\infty^{11}(x-y) = \frac{1}{2}(T_+ + T_-)\delta(x-y). \tag{3.9}$$

The identities (3.7)–(3.9) formally follow from (3.1) and (2.13)–(2.15). For the proof we apply (2.13)–(2.15) to the initial measure g_0^θ. □

3.3 Limit energy current density

Let $u(x,t)$ is the random solution to (3.1) with the initial measure μ_0 satisfying **S0–S3**. The mean energy current density is $E_0 j(x,t) = -E_0\dot u(x,t)\nabla u(x,t)$. Therefore, in the limit $t\to\infty$,

$$E_0 j(x,t) \to \overline{j}_\infty = \nabla q^{10}_\infty(0).$$

Respectively, in the case of the "Gibbs" initial measure g_0, the expression (3.8) for the limit correlation function implies *formally* that

$$\overline{j}_\infty = \frac{T_+ - T_-}{2}\nabla\mathcal{P}(0),$$

where $[\nabla\mathcal{P}](z) = -F^{-1}\Big[\frac{\xi\,\mathrm{sgn}\,\xi_3}{|\xi|}\Big](z)$. Hence, formally we have the "ultraviolet diverging" limit mean energy current density,

$$\overline{j}_\infty = -\frac{T_+ - T_-}{2(2\pi)^3}\int\limits_{\mathbb{R}^3}\frac{\xi\,\mathrm{sgn}\,\xi_3}{|\xi|}d\xi = -\infty\cdot(0,0,T_+ - T_-).$$

On the other hand, for the convolution $U_t(Y_0*\theta)$ corresponding limit mean current density is finite,

$$\overline{j}^\theta_\infty = -\frac{T_+ - T_-}{2(2\pi)^3}\int\limits_{\mathbb{R}^3}|\tilde\theta(\xi)|^2\frac{\xi\,\mathrm{sgn}\,\xi_3}{|\xi|}d\xi = -C_\theta\cdot(0,0,T_+ - T_-),$$

if $\theta(x)$ is axially symmetric with respect to Ox_3; $C_\theta>0$ if $\theta(x)\not\equiv 0$.

4 Compactness

Proposition 2.10 follows from the estimate (4.1) below using the Prokhorov criterion [21, Lemma 3.1] by the method of [21].

Lemma 4.1 *Let* **S0**–**S2** *hold. Then for any* $R>1$

$$\sup_{t\ge 0} E_0\|U_tY_0\|^2_R \le C(R) < \infty. \qquad (4.1)$$

Proof $U_tY_0(x) = (u(x,t),\dot u(x,t))$, and by definition (2.1),

$$E_0\|U_tY_0(x)\|^2_R = E_0\int\limits_{|x|<R}|u(x,t)|^2dx + E_0\int\limits_{|x|<R}|\nabla u(x,t)|^2dx + E_0\int\limits_{|x|<R}|\dot u(x,t)|^2dx. \qquad (4.2)$$

We bound for example the first integral in the right hand side of (4.2) in the particular case when $u_0\equiv 0$ almost sure. General case will be considered in Section 6 as well as the bounds for two remaining integrals in (4.2).

Let $S_t(x) = \{z\in\mathbb{R}^3:\ |x-z| = t\}$. Let us assume for a moment that the function $v_0(z)$ is continuous almost sure. Then the Kirchhoff formula (6.1) gives

$$u(x,t) = \frac{1}{4\pi t}\int\limits_{S_t(x)} v_0(z)dS(z), \qquad (4.3)$$

where $dS(z)$ is the Lebesgue measure on the sphere $S_t(x)$. Therefore,

$$E_0|u(x,t)|^2 = \frac{1}{(4\pi t)^2} \int\limits_{S_t(x)\times S_t(x)} Q_0^{11}(z,p)dS(z)dS(p). \tag{4.4}$$

Let us assume for a moment, that

$$Q_0^{ij}(x,y) = 0 \quad \text{for} \quad |x-y| > r_0, \quad i,j = 0,1. \tag{4.5}$$

Then (4.4) implies the uniform bound in $t \in \mathbb{R}$,

$$E_0|u(x,t)|^2 \le \frac{C}{t^2} \int\limits_{\substack{S_t(x)\times S_t(x)\\ |z-p|\le r_0}} dS(z)dS(p) \le I = C_1 r_0^2. \tag{4.6}$$

Hence, the bound for the first integral follows,

$$E_0 \int_{|x|<R} |u(x,t)|^2 dx \le I \int_{|x|<R} dx, \quad t \in \mathbb{R}. \tag{4.7}$$

Next we remove the additional assumption (4.5) by the following known lemma on spherical integral identity, [10].

Lemma 4.2 *For any* $r_0 > 0$ *and* $p \in S_t(x)$ *the identity holds,*

$$\int\limits_{\{z\in S_t(x):\, |z-p|\ge r_0\}} \nu(|z-p|)dS(z) = 2\pi \int_{r_0}^{2t} r\nu(r)dr. \tag{4.8}$$

The lemma with $r_0 = 0$, and **S2** with $d = 2$ imply,

$$E_0|u(x,t)|^2 \le \frac{1}{(4\pi t)^2} \int\limits_{S_t(x)\times S_t(x)} \nu_2(|z-p|)dS(z)dS(p) \le CI_2. \tag{4.9}$$

Then (4.7) follows without assumption (4.5). The assumption on the continuity almost sure of $v_0(x)$ can be removed by a convolution with a function $\theta \in D$. □

5 Convergence of correlation functions

Here we prove the convergence (1.8) of correlation functions $Q_t^{ij}(x,y)$, as $t \to \infty$. This implies the convergence of the characteristic functionals $\tilde{\mu}_t$ in the case of Gaussian measures μ_0, $\mu_\pm$.

Lemma 5.1 *Let* **S0**–**S2** *hold. Then* $\forall i,j = 0,1$
i) For the derivatives in the sense of distributions,

$$\partial_x^\alpha \partial_y^\beta Q_t^{ij}(x,y) \in C(\mathbb{R}^3 \times \mathbb{R}^3), \quad i+j+|\alpha|+|\beta| \le 2, \quad \forall t > 0. \tag{5.1}$$

ii) For $\forall x,y \in \mathbb{R}^3$

$$Q_t^{ij}(x,y) \to Q_\infty^{ij}(x,y), \quad t \to \infty. \tag{5.2}$$

Proof We prove the lemma again for $i = j = 0$ in the particular case when $u_0 \equiv 0$ almost sure. General case is considered in Section 6.

i) Let us assume for a moment that the function $v_0(z)$ is continuous almost sure. Then Kirchhoff formula (4.3) gives

$$Q_t^{00}(x,y) = E_0 u(x,t)u(y,t) = \frac{1}{(4\pi t)^2} \int\limits_{S_t(x)} dS(z) \int\limits_{S_t(y)} Q_0^{11}(z,p)dS(p). \tag{5.3}$$

This integral is a convolution of $Q_0^{11}(x,y)$ in both variables x,y with a distribution of compact support. The convolution of distributions with compact support is commutative. Therefore, the assumption on the continuity almost sure of $v_0(x)$ can be removed by a convolution with a function $\theta \in D$. Further, the convolution commutes with the differentiations. Therefore, (5.1) with $i = j = 0$ follows from (2.10) with $i = j = 1$.

ii) Changing variables $z = x + \omega t$ in the right hand side of (5.3), we get

$$\begin{aligned} &\frac{1}{(4\pi t)^2} \int\limits_{S_t(x)} dS(z) \int\limits_{S_t(y)} Q_0^{11}(z,p)dS(p) \\ = \;& \frac{1}{(4\pi)^2} \int\limits_{|\omega|=1,\omega_3<0} dS(\omega) \int\limits_{S_t(y)} Q_0^{11}(x+\omega t,p)dS(p) + \frac{1}{(4\pi)^2} \int\limits_{|\omega|=1,\omega_3>0} dS(\omega) \int\limits_{S_t(y)} Q_0^{11}(x+\omega t,p)dS(p) \\ = \;& I_-(t,x,y) + I_+(t,x,y). \end{aligned} \tag{5.4}$$

Let us recall that $\nu(r) \equiv \nu_2(r)$.

Definition 5.2 *$C_\nu(\mathbb{R}^3)$ is the space of functions $f(y) \in C(\mathbb{R}^3)$ s.t. $|f(y)| \le C\nu(|y|)$ with a constant $C \in \mathbb{R}$.*

Let us define for $f(y) \in C_\nu(\mathbb{R}^3)$

$$Rf(v) \equiv \frac{1}{(4\pi)^2} \int\limits_{|\omega|=1,\pm\omega_3>0} dS(\omega) \int\limits_{p\cdot\omega=v\cdot\omega} f(p)d^2p, \;\; v \in \mathbb{R}^3. \tag{5.5}$$

Here d^2p is the Lebesgue measure on the plane $p\cdot\omega = v\cdot\omega$. Note, the integrals with $\pm$ are identical and converge due to (2.8). Hence, the operator $R : C_\nu(\mathbb{R}^3) \to C_b(\mathbb{R}^3)$ is continuous. We deduce the convergence (5.2) for $i = j = 0$ from (5.3), (5.4) and next lemmas.

Lemma 5.3 *Let* **S2** *hold. Then for $x, y \in \mathbb{R}^3$,*

$$I_\pm(t,x,y) \to Rq_\pm^{11}(x-y), \quad t \to \infty. \tag{5.6}$$

Lemma 5.4 *Let $f(y) \in C_\nu(\mathbb{R}^3)$. Then*

$$Rf(v) = -\frac{1}{4}(\mathcal{E} * f)(v), \;\; v \in \mathbb{R}^3. \tag{5.7}$$

The proof of Lemma 5.4 see in Appendix A.

Proof of Lemma 5.3. For a moment we assume additionally (4.5). Denote by I_{11} the inner integral entering (5.4):

$$I_{11} \equiv I_{11}(x,y,\omega,t) = \int\limits_{S_t(y)} Q_0^{11}(x+\omega t, p)dS(p).$$

Denote $\overline{p} \equiv y+\omega t \in S_t(y)$. Then (4.5) implies that $Q_0^{11}(x+\omega t, p) = 0$ for $|p-\overline{p}| \geq r_0+2R$, because $|x-y| \leq 2R$. Denote

$$O_{\overline{p}} = \{p \in S_t(y): \ |p-\overline{p}| \leq r_0+2R\}.$$

Denote by $T_{\overline{p}}$ a tangent plane to the sphere $S_t(y)$ at the point $\overline{p}$. Let $B_{\overline{p}}$ denote the orthogonal projection of the domain $O_{\overline{p}}$ onto $T_{\overline{p}}$. For $t > r_0+2R$ the domain $O_{\overline{p}}$ is the image of the map $\mathcal{S}_t : B_{\overline{p}} \to \mathbb{R}^3$ defined by

$$\tau \to \mathcal{S}_t(\tau) = \overline{p} + \tau - s_t(\tau)\omega \equiv y + \omega t + \tau - s_t(\tau)\omega$$

where $s_t(\tau) = t - \sqrt{t^2 - |\tau|^2}$. Changing the variables we get for large t,

$$I_{11} = \int\limits_{B_{\overline{p}}} Q_0^{11}\Big(x+\omega t, y+\omega t+\tau - s_t(\tau)\omega\Big)\sqrt{1+|\nabla s_t(\tau)|^2}d^2\tau. \tag{5.8}$$

At last, we compute the limit of this integral as $t \to \infty$. Uniformly in $\tau \in B_{\overline{p}}$ (and in $|\omega| = 1$), we have

$$\left\{\begin{array}{l} s_t(\tau) \to 0, \\ \\ \sqrt{1+|\nabla s_t(\tau)|^2} = (1-|\tau/t|^2)^{-1/2} \to 1, \end{array}\right. \Bigg| \quad t \to +\infty. \tag{5.9}$$

Consider the case $\omega_3 < 0$ and $\omega_3 > 0$ separately. For $\omega_3 < 0$ and for large enough $t > t(\omega)$,

$$x_3 + \omega_3 t < -a, \quad y_3 + \omega_3 t - s_t(\tau)\omega_3 < -a, \quad \tau \in B_{\overline{p}}.$$

Therefore, (5.8) implies due to **S1** and (5.9),

$$\lim_{t\to\infty} I_{11} = \int\limits_{B_{\overline{p}}} q_-^{11}(x-y-\tau)d^2\tau = \int\limits_{\tau\cdot\omega=0} q_-^{11}(x-y-\tau)d^2\tau.$$

Introducing new variable $p = x-y-\tau$, we obtain the inner integral in the right hand side of (5.5) with $f = q_-^{11}$ and $v = x-y$. Similarly for $\omega_3 > 0$. Lemma 5.3 is proved with additional assumption (4.5). At last, Lemma 4.2 and **S4** give a uniform smallness of integral (5.3) over $|z-p| \geq r_0$ with large r_0. □

6 Correlation functions in general case

We prove Lemmas 4.1 and 5.1 for general case. Let us assume for a moment that $u_0 \in C^1(\mathbb{R}^3)$ and $v_0 \in C(\mathbb{R}^3)$ almost sure. Then we apply the Kirchhoff formula for solution $u(x,t)$ to the Cauchy problem (1.1),

$$u(x,t) = \frac{1}{4\pi t} \int\limits_{S_t(x)} \Big(v_0(z) + \frac{1}{t} u_0(z) + \nabla u_0(z) \frac{z-x}{t} \Big) dS(z). \tag{6.1}$$

It implies similarly to (5.3),

$$Q_t^{00}(x,y) = \frac{1}{(4\pi t)^2} \int\limits_{S_t(x)} dS(z) \int\limits_{S_t(y)} \Big(\Big[Q_0^{11}(z,p) + \nabla_p (\nabla_z Q_0^{00}(z,p) \cdot \frac{z-x}{t}) \cdot \frac{p-y}{t} \Big]$$

$$+ \frac{1}{t} \Big[Q_0^{10}(z,p) + Q_0^{01}(z,p) + \nabla_p Q_0^{00}(z,p) \cdot \frac{p-y}{t} + \nabla_z Q_0^{00}(z,p) \cdot \frac{z-x}{t} + \frac{1}{t} Q_0^{00}(z,p) \Big]$$

$$+ \Big[\nabla_z Q_0^{01}(z,p) \cdot \frac{z-x}{t} + \nabla_p Q_0^{10}(z,p) \cdot \frac{p-y}{t} \Big] \Big) dS(p). \tag{6.2}$$

$$Q_t^{01}(x,y) = \frac{1}{(4\pi t)^2} \int\limits_{S_t(x)} dS(z) \int\limits_{S_t(y)} \Big(\Big[\Delta_p Q_0^{10}(z,p) + \nabla_p (\nabla_z Q_0^{01}(z,p) \cdot \frac{z-x}{t}) \cdot \frac{p-y}{t} \Big]$$

$$+ \frac{1}{t} \Big[\Delta_p Q_0^{00}(z,p) + Q_0^{11}(z,p) + \nabla_p Q_0^{01}(z,p) \cdot \frac{p-y}{t} + \nabla_z Q_0^{01}(z,p) \cdot \frac{z-x}{t} + \frac{1}{t} Q_0^{01}(z,p) \Big]$$

$$+ \Big[\nabla_p Q_0^{11}(z,p) \cdot \frac{p-y}{t} + \nabla_z \Delta_p Q_0^{00}(z,p) \cdot \frac{z-x}{t} \Big] \Big) dS(p). \tag{6.3}$$

$$Q_t^{11}(x,y) = \frac{1}{(4\pi t)^2} \int\limits_{S_t(x)} dS(z) \int\limits_{S_t(y)} \Big(\Big[\Delta_z \Delta_p Q_0^{00}(z,p) + \nabla_p (\nabla_z Q_0^{11}(z,p) \cdot \frac{z-x}{t}) \cdot \frac{p-y}{t} \Big]$$

$$+ \frac{1}{t} \Big[\Delta_z Q_0^{01}(z,p) + \Delta_p Q_0^{10}(z,p) + \nabla_p Q_0^{11}(z,p) \cdot \frac{p-y}{t} + \nabla_z Q_0^{11}(z,p) \cdot \frac{z-x}{t} + \frac{1}{t} Q_0^{11}(z,p) \Big]$$

$$+ \Big[\nabla_z \Delta_p Q_0^{10}(z,p) \cdot \frac{z-x}{t} + \nabla_p \Delta_z Q_0^{01}(z,p) \cdot \frac{p-y}{t} \Big] \Big) dS(p). \tag{6.4}$$

Proof of Lemma 4.1 for general case *Step 1* Any integral entering (6.2)-(6.4) is a convolution of a derivative $D^{\alpha,\beta}_{x,y} Q_0^{kl}(x,y)$ in both variables x, y, with a distribution of compact support, $k + l + |\alpha| + |\beta| \le 4$. Therefore, (5.1) follows from (6.2)-(6.4) and **S2**. The assumption on the C^1-continuity almost sure of $u_0(x)$ and continuity almost sure of $v_0(x)$ can be removed in the same way as in previous section.

Step 2 Now (4.2) implies

$$E_0 \| U_t Y_0(x) \|_R^2 = \int\limits_{|x|<R} Q_t^{00}(x,x) dx + \int\limits_{|x|<R} \nabla_x \cdot \nabla_y Q_t^{00}(x,y)|_{y=x} dx + \int\limits_{|x|<R} Q_t^{11}(x,x) dx. \tag{6.5}$$

(6.2) gives similar expression for $\nabla_x\cdot\nabla_y Q_t^{00}(x,y)$. Then correlation functions $\nabla_x\nabla_y Q_t^{00}(x,y)$ and $Q_t^{11}(x,x)$ can be estimated just by method of the proof of Lemma 4.1 in Section 4, applying **S2** and Lemma 6.1 to all entering integrals. Main observation is that any integral there involves $D^{\alpha,\beta}_{x,y}Q_0^{kl}(x,y)$ only with $k+l+|\alpha|+|\beta|=2,3,4$.

Step 3 $Q_t^{00}(x,y)$ requires a particular attention due to the presence of the terms $D^{\alpha,\beta}_{x,y}Q_0^{kl}(x,y)$ with $k+l+|\alpha|+|\beta|=0,1$. Corresponding contribution is

$$I_t^{00}(x,y)$$

$$=\frac{1}{(4\pi t)^2}\int\limits_{S_t(x)} dS(z)\int\limits_{S_t(y)}\frac{1}{t}\Big[\nabla_p Q_0^{00}(z,p)\cdot\frac{p-y}{t}+\nabla_z Q_0^{00}(z,p)\cdot\frac{z-x}{t}+\frac{1}{t}Q_0^{00}(z,p)\Big]\,dS(p).$$

Lemma 6.1 *The integral* $I_t^{00}(x,y)$ *converges to zero as* $t\to\infty$.

Proof The assumption **S2** implies

$$|I_t^{00}(x,y)|\le\frac{1}{(4\pi t)^2}\int\limits_{S_t(x)} dS(z)\int\limits_{S_t(y)}\frac{1}{t}\Big[2\nu_1(|z-p|)+\frac{1}{t}\nu_0(|z-p|)\Big]\,dS(p). \qquad (6.6)$$

Therefore, Lemma 4.2 implies

$$\begin{aligned}|I_t^{00}(x,y)| &\le \frac{1}{(4\pi t)^2}\int\limits_{S_t(x)} dS(z)\frac{1}{t}\Big[\int_0^{2t}(2r\nu_1(r)+\frac{1}{t}r\nu_0(r))dr\Big]\\ &\le C\int_0^{2t}(\frac{r}{t}\nu_1(r)+\frac{r}{t^2}\nu_0(r))dr. \qquad (6.7)\end{aligned}$$

Now (2.8) implies the convergence to zero by the Lebesgue theorem. □

Lemma 4.1 is proved for general case. □

Proof of Lemma 5.1 for general case *Step 1* In the convolutions of $D^{\alpha,\beta}_{x,y}Q_0^{kl}(x,y)$ with $k+l+|\alpha|+|\beta|=2,3,4$, entering (6.2)-(6.4), the convergence follows just by method of the proof of Lemma 5.1 in Section 5.

Step 2 Convolutions with $k+l+|\alpha|+|\beta|\le 1$ enter only (6.2) and converge to zero by Lemma 6.1.

Step 3 Let us define for the functions $f\in C^1_\nu=\{f\in L^1_{\rm loc}(\mathbb{R}^3):|\nabla f(z)|\in C_\nu(\mathbb{R}^3)\}$, the operator

$$Pf(v)=\frac{1}{(4\pi)^2}\int\limits_{|\omega|=1,\omega_3>0} dS(\omega)\int\limits_{v\cdot\omega=p\cdot\omega}\nabla f(p)\cdot\omega\, d^2p,\quad v\in\mathbb{R}^3. \qquad (6.8)$$

For instance, the operator P can be applied to $D^\alpha q^{kl}_\pm$ with $k+l+|\alpha|=1,2,3$ since then $q^{kl}_\pm\in C^1_\nu(\mathbb{R}^3)$ by **S2**. Similarly, the operator R can be applied to $D^\alpha q^{kl}_\pm$ with $k+l+|\alpha|=2,3,4$ since then $q^{kl}_\pm\in C_\nu(\mathbb{R}^3)$.

Lemma 6.1 implies that the integrals of the expression $\frac{1}{t}[\dots]$ entering (6.2) vanish in the limit $t\to\infty$. The same holds obviously for all similar expressions entering (6.3)-(6.4). Therefore, the convergence (5.2) follows with limit correlation functions $Q^{ij}_\infty(x,y)=$

$\hat q^{ij}_\infty(x-y)$ where

$$\hat q^{00}_\infty = \quad R\Big[q^{11}_+ + q^{11}_- - \Delta(q^{00}_+ + q^{00}_-)\Big] \quad +P\Big[q^{01}_+ - q^{01}_- - q^{10}_+ + q^{10}_-\Big], \tag{6.9}$$

$$\hat q^{10}_\infty = -\hat q^{01}_\infty = \quad R\Big[\Delta(q^{01}_+ + q^{01}_- - q^{10}_+ - q^{10}_-)\Big] \quad +P\Big[q^{11}_+ - q^{11}_- - \Delta(q^{00}_+ - q^{00}_-)\Big], \tag{6.10}$$

$$\hat q^{11}_\infty = \quad R\Big[\Delta(\Delta(q^{00}_+ + q^{00}_+) - q^{11}_+ - q^{11}_-)\Big] +P\Big[\Delta(q^{10}_+ - q^{10}_- - q^{01}_+ + q^{01}_-)\Big]. \tag{6.11}$$

Step 4 It remains to prove $\hat q^{ij}_\infty = q^{ij}_\infty$. First, let us consider the terms with R entering (6.9)–(6.11). For example, let us prove that $R\Delta q^{00}_+ = q^{00}_+$. Indeed, $\Delta(R\Delta q^{00}_+) = \Delta q^{00}_+$ in the sense of distributions, hence $f(x) \equiv R\Delta q^{00}_+ - q^{00}_+$ is a smooth harmonic function in $\mathbb{R}^3$. On the other hand, $\Delta q^{00}_+ \in C_\nu(\mathbb{R}^3)$ by **S2**. Hence, $g(x) \equiv R\Delta q^{00}_+ \in C_b(\mathbb{R}^3)$, and moreover,

$$g(x) \to 0, \ \ |x| \to \infty. \tag{6.12}$$

Indeed,

$$|\int\limits_{p\cdot\omega = x\cdot\omega} \Delta q^{00}_+(p) d^2p| \le \int\limits_{p\cdot\omega = x\cdot\omega} \nu(|p|) d^2p = 2\pi \int\limits_{x\cdot\omega}^{\infty} r\nu(r)dr \tag{6.13}$$

according to (4.8) with $t=\infty$. This integral is bounded uniformly in $|\omega|=1$ and it converges to zero if $|x|\to\infty$ and $x=|x|\theta$ with $\theta\cdot\omega\ne 0$. Therefore, (6.12) follows from (5.5) by the Lebesgue theorem.

Further, $|f(x)| \le |g(x)| + \nu_0(|x|)$ again by **S2**. $\nu_0(r_n)\to 0$ for some sequence $r_n\to\infty$ due to (2.8). Finally, maximum principle and (6.12) imply for any fixed $x\in\mathbb{R}^3$,

$$|f(x)| \le \max_{|y|=r_n} |g(y)| + \nu_0(r_n) \to 0, \ \ n\to\infty.$$

Therefore, $f(x)\equiv 0$ and $\hat q^{ij}_\infty = q^{ij}_\infty$.

Step 5 Further, let us consider the terms with P entering (6.9)–(6.11). Obviously, Pf is a convolution. We prove the next lemma in Appendix B. Let us recall that $\mathcal{P}(x) = -iF^{-1}\Big[\frac{\operatorname{sgn}\xi_3}{|\xi|}\Big]$.

Lemma 6.2 *For $f\in D$ we have*

$$Pf(z) = \frac{1}{4}\mathcal{P} * f(z), \quad \forall z\in\mathbb{R}^3. \tag{6.14}$$

Let us assume for a moment that all correlation functions $q^{kl}_\pm(z)$ are smooth and decay rapidly as $|z|\to\infty$. Then (6.9)–(6.11) coincide with (2.13)–(2.15) by Lemmas 5.4, 6.2. For general case we assume the formula (6.14) as definition of the convolutions with $\mathcal{P}$, entering (2.13)–(2.15). Lemma 5.1 is proved for general case. □

7 Convergence to equilibrium

Proof of Theorem 2.8 Theorem 2.8 follows from Propositions 2.10 and 2.11. Proposition 2.10 has been proved in Section 4. Proposition 2.11 follows by a development of the "rooms-corridors" method of [8, Section 6 and Appendix A]. The method is based on the Lindeberg Central Limit Theorem and uses as the ingredients the convergence of the correlation functions of the measures μ_t, the mixing condition and the derivation of the Lindeberg condition from the estimates of fourth order moment functions for the measures μ_t. The convergence of the correlation functions is proved in Lemma 5.1. On the other hand, the mixing condition **S3** is weaker than in [8] though **S2** is more strong (see Remark 2.7). Respectively, the method [8] requires a suitable modification for the proof of Proposition 2.11. For instance, the Lemma 6.1 is used.
Remark Let us emphasize that the reasons of Section 6 and Appendix A in [8] do not use explicitly the translation invariance of the measures μ_t.

8 Variable coefficients

We extend all results of previous sections to the case of the wave equations with variable coefficients. We consider the wave equations in $\mathbb{R}^n$ with the initial conditions

$$\left\{ \begin{array}{l} \ddot u(x,\,t)=\sum\limits_{j,k=1}^{n}\partial_j(a_{jk}(x)\partial_k u(x,\,t))-a_0(x)\,u(x,\,t),\quad x\in\mathbb{R}^n,\ t\in\mathbb{R} \\ u|_{t=0}=u_0(x),\quad \dot u|_{t=0}=v_0(x), \end{array}\right. \tag{8.1}$$

where $\partial_j\equiv\dfrac{\partial}{\partial x_j}$. We assume the following properties **E1–E3** of the equation (8.1).

E1 $a_{jk}(x)=\delta_{jk}+\hat a_{jk}(x)$, where $\hat a_{jk}(x)\in D$; also $a_0(x)\in D$.

E2 $a_0(x)\ge 0$, and the hyperbolicity condition holds: $\exists\alpha>0$

$$H(x,\xi)\equiv\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\ge\alpha|\xi|^2,\quad x,\xi\in\mathbb{R}^n. \tag{8.2}$$

E3 Non-trapping condition holds, [20]: for $(x(0),\xi(0))\in\mathbb{R}^n\times\mathbb{R}^n$ with $\xi(0)\neq 0$

$$|x(t)|\to\infty,\quad t\to\infty, \tag{8.3}$$

where $(x(t),\xi(t))$ is a solution to the following Hamiltonian system

$$\dot x(t)=\ H_\xi(x(t),\xi(t)),\quad \dot\xi(t)=-H_x(x(t),\xi(t)).$$

Example. **E1-E3** hold for the case of constant coefficients, $a_{jk}(x)\equiv\delta_{ij}$. For instance, **E3** follows because $\dot\xi(t)\equiv 0\Rightarrow x(t)\equiv\xi(0)t+x(0)$.

We denote as above, $Y(t)\equiv(u(\cdot,t),\dot u(\cdot,t))$, $Y_0\equiv(u_0,v_0)$. Then (8.1) becomes

$$\dot Y(t)=\mathcal{F}_*(Y(t)),\ \ t\in\mathbb{R},\ \ Y(0)=Y_0. \tag{8.4}$$

Proposition 2.2 holds for the solutions to the Cauchy problem (8.4) as well as for (1.2). Let Y_0 in (8.4) be a measurable random function with values in $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$, and μ_0 its distribution, as above. Denote by μ_t the distribution of the solution $Y(t)$ to the problem (8.4). Let us state the extension of main Theorem 2.8. We introduce the appropriate Hilbert spaces of initial datum of the infinite energy. Let δ be an arbitrary positive number.

Definition 8.1 *$\mathcal{H}_\delta$ is the Hilbert space of the functions $Y = (u, v) \in \mathcal{H}$ with the finite norm*

$$\|Y\|_\delta^2 = \int e^{-2\delta|x|}(|u(x)|^2 + |\nabla u(x)|^2 + |v(x)|^2)\, dx < \infty.$$

Theorem 8.2 *Let $n \geq 3$ be odd, and* **E1–E3**, **S0–S3** *hold. Then*
i) the convergence (2.3) holds for any $\varepsilon > 0$.
ii) The limit measure μ_∞ is a Gaussian measure on $\mathcal{H}$.
iii) The limit characteristic functional has the form

$$\tilde{\mu}_\infty(\psi) = \exp\left(-\frac{1}{2} < Q_\infty \Omega\Psi,\, \Omega\Psi >\right), \quad \Psi \in \mathcal{D}.$$

Here $\Omega : \mathcal{D} \to \mathcal{H}'_\delta$ is a linear continuous operator for sufficiently small $\delta > 0$, and $Q_\infty : \mathcal{H}'_\delta \to \mathcal{H}_\delta$ is a linear continuous operator with the integral kernel (2.12).

Theorem 8.2 follows immediately from Theorem 2.8, using the method [8]. The method is based on the scattering theory for the solutions of infinite energy, which is constructed in [8].

9 Appendix A. Radon transform

Proof of Lemma 5.4. Since $\int\limits_{p\cdot\omega = z\cdot\omega} f(p)d^2p$ is even function with respect to ω, it suffices to prove next lemma.

Lemma 9.1 *Let (2.8) hold, and $f \in C_\nu(\mathbb{R}^3)$. Then*

$$\frac{1}{(4\pi)^2} \int\limits_{|\omega|=1} dS(\omega) \int\limits_{p\cdot\omega = z\cdot\omega} f(p)d^2p = -\frac{1}{2}\mathcal{E} * f(z), \quad \forall z \in \mathbb{R}^3. \tag{9.1}$$

Proof. Both sides of (9.1) define the continuous operators $C_\nu(\mathbb{R}^3) \mapsto C_b(\mathbb{R}^3)$. Therefore, it suffices to consider $f \in D$. Applying the Fourier transform, we obtain with $\rho = |\xi|$,

$$(\mathcal{E} * f)(z) = \frac{1}{(2\pi)^3} \int \tilde{\mathcal{E}}(\xi)\tilde{f}(\xi)e^{-iz\cdot\xi}d^3\xi = \frac{1}{(2\pi)^3} \int\limits_{|\omega|=1} dS(\omega) \int\limits_0^{+\infty} \rho^2 e^{-i\rho z\cdot\omega}\tilde{\mathcal{E}}(\rho\omega)\tilde{f}(\rho\omega)d\rho. \tag{9.2}$$

We substitute $\tilde{\mathcal{E}}(\rho\omega) = -\dfrac{1}{\rho^2}$ in the right hand side of (9.2) and get

$$(\mathcal{E} * f)(z) = -\frac{1}{(2\pi)^3} \int\limits_{|\omega|=1} dS(\omega) \int\limits_0^{+\infty} e^{-i\rho z\cdot\omega}\tilde{f}(\rho\omega)d\rho. \tag{9.3}$$

Note, that

$$\tilde{f}(\rho\omega) = \int\limits_{-\infty}^{+\infty} e^{i\rho h} f^\sharp(h,\omega)dh, \quad \text{where } f^\sharp(h,\omega) \equiv \int\limits_{y\cdot\omega=h} f(y)d^2y. \tag{9.4}$$

Then from (9.3), (9.4) we have

$$(\mathcal{E} * f)(z) = -\frac{1}{(2\pi)^3} \int\limits_{|\omega|=1} dS(\omega) \frac{1}{2} \int\limits_{-\infty}^{+\infty} e^{-iyz\cdot\omega} dy \int\limits_{-\infty}^{+\infty} e^{i\rho h} f^\sharp(h,\omega)dh$$

$$= -\frac{1}{8\pi^2} \int\limits_{|\omega|=1} dS(\omega) F^{-1}_{y\to(z\cdot\omega)} F_{h\to y} f^\sharp(h,\omega) = -\frac{1}{8\pi^2} \int\limits_{|\omega|=1} f^\sharp(z\cdot\omega,\omega)dS(\omega).$$

Lemma 9.1 is proved. □

Proof of Lemma 6.2. Since $F[\mathcal{P}](\xi) = -\dfrac{i}{|\xi|}\operatorname{sgn}\xi_3$, we have

$$(\mathcal{P} * f)(z) = \frac{1}{(2\pi)^3} \int \tilde{\mathcal{P}}(\xi)\tilde{f}(\xi)e^{-iz\cdot\xi}d^3\xi = -\frac{i}{(2\pi)^3} \int\limits_{|\omega|=1} dS(\omega) \int\limits_{0}^{+\infty} \rho e^{-i\rho z\cdot\omega} \operatorname{sgn}(\omega_3)\tilde{f}(\rho\omega)d\rho$$

$$= -\frac{i}{(2\pi)^3} \int\limits_{|\omega|=1,\omega_3>0} dS(\omega) \int\limits_{0}^{+\infty} \rho e^{-i\rho z\cdot\omega} \tilde{f}(\rho\omega)d\rho + \frac{i}{(2\pi)^3} \int\limits_{|\omega|=1,\omega_3<0} dS(\omega) \int\limits_{0}^{+\infty} \rho e^{-i\rho z\cdot\omega} \tilde{f}(\rho\omega)d\rho. \tag{9.5}$$

In the last integral we change the variables $\omega \to -\omega$, $\rho \to -\rho$, then apply (9.4) and get

$$(\mathcal{P} * f)(z) = -\frac{i}{(2\pi)^3} \int\limits_{|\omega|=1,\omega_3>0} dS(\omega) \int\limits_{-\infty}^{+\infty} \rho e^{-i\rho z\cdot\omega} \tilde{f}(\rho\omega)d\rho$$

$$= -\frac{i}{(2\pi)^3} \int\limits_{|\omega|=1,\omega_3>0} dS(\omega) \int\limits_{-\infty}^{+\infty} e^{-i\rho z\cdot\omega}\ \rho d\rho \int\limits_{-\infty}^{+\infty} e^{i\rho h} f^\sharp(h,\omega)dh. \tag{9.6}$$

Note, that

$$\rho \int\limits_{-\infty}^{+\infty} e^{i\rho h} f^\sharp(h,\omega)dh = i \int\limits_{-\infty}^{+\infty} e^{i\rho h} (\nabla f)^\sharp(h,\omega)\cdot\omega\, dh, \quad \rho \in \mathbb{R}. \tag{9.7}$$

Indeed, applying (9.4) in the both sides of

$$F[\nabla f](\rho\omega)\cdot\omega = -i\rho F[f](\rho\omega),$$

we obtain (9.7). Finally, from (9.7) and (9.6) we get

$$(\mathcal{P} * f)(z) \;=\; \frac{1}{(2\pi)^3} \int\limits_{|\omega|=1,\omega_3>0} dS(\omega) \int\limits_{-\infty}^{+\infty} e^{-i\rho z\cdot\omega} d\rho \int\limits_{-\infty}^{+\infty} e^{i\rho h} (\nabla f)^\sharp(h,\omega)\cdot\omega\, dh$$

$$
\begin{aligned}
&= \frac{1}{4\pi^2} \int\limits_{|\omega|=1, \omega_3>0} F^{-1}_{\rho\to z\cdot\omega} F_{h\to\rho} (\nabla f)^\sharp (h,\omega)\cdot\omega\, dS(\omega) \\
&= \frac{1}{4\pi^2} \int\limits_{|\omega|=1, \omega_3>0} (\nabla f)^\sharp (z\cdot\omega,\omega)\cdot\omega\, dS(\omega) \\
&= 4Pf(z). \qquad (9.8)
\end{aligned}
$$

Lemma 6.2 is proved. □

10 Apendix B. Gaussian measures in Sobolev's spaces

We verify (3.3). Definition (3.2) implies for $u \in H_{s,\alpha}$,

$$
\|u\|^2_{s,\alpha} = \int (1+|x|)^{2\alpha} \Big(\int e^{-ix(\xi-\eta)} (1+|\xi|)^s (1+|\eta|)^s \tilde{u}(\xi)\overline{\tilde{u}}(\eta)\, d\xi\, d\eta \Big) dx. \qquad (10.1)
$$

Let $\mu(du)$ be a Gaussian translation invariant measure in $H_{s,\alpha}$ with a correlation function $Q(x,y) = q(x-y)$. Let us introduce the following correlation function

$$
C(\xi,\eta) \equiv \int \tilde{u}(\xi)\overline{\tilde{u}}(\eta)\mu(du) \qquad (10.2)
$$

in the sense of distributions. Using that $u(x)$ is real, we get

$$
C(\xi,\eta) = F_{x\to\xi} F_{y\to -\eta} Q(x,y) = C_n \delta(\xi-\eta)\tilde{q}(\xi). \qquad (10.3)
$$

Then, integrating (10.1) with respect to the measure $\mu(du)$, we get the expression

$$
\int \|u\|^2_{s,\alpha}\mu(du) = C_n \int (1+|x|)^{2\alpha} dx \int (1+|\xi|)^{2s} \tilde{q}(\xi)\, d\xi. \qquad (10.4)
$$

Applying it to $\tilde{q}(\xi) = 1$ and to $\tilde{q}(\xi) = |\xi|^{-2}$, we get (3.3). □

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