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**Analytic aspects of the Toda system:
I. A Moser-Trudinger inequality**

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ANALYTIC ASPECTS OF THE TODA SYSTEM: I. A MOSER-TRUDINGER INEQUALITY

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ABSTRACT. In this paper, we analyze solutions of the open Toda system and establish an optimal Moser-Trudinger type inequality for this system. Let Σ be a closed surface with area 1 and $K = (a_{ij})_{N \times N}$ the Cartan matrix for $SU(N+1)$, i.e.,

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

We show that

$$\Phi_M(u) = \frac{1}{2} \sum_{i,j=1}^N \int_{\Sigma} a_{ij} (\nabla u_i \nabla u_j + 2M_i u_j) - \sum_{i=1}^N M_i \log \int_{\Sigma} \exp\left(\sum_{j=1}^N a_{ij} u_j\right)$$

has a lower bound in $(H^1(\Sigma))^N$ if and only if

$$M_j \leq 4\pi, \quad \text{for } j = 1, 2, \dots, N.$$

As a direct consequence, if $M_j < 4\pi$ for $j = 1, 2, \dots, N$, Φ_M has a minimizer u which satisfies

$$-\Delta u_i = M_i \left(\frac{\exp(\sum_{j=1}^N a_{ij} u_j)}{\int_{\Sigma} \exp(\sum_{j=1}^N a_{ij} u_j)} - 1 \right), \quad \text{for } 1 \leq i \leq N.$$

1. INTRODUCTION

Let Σ be a closed surface with area 1. The Moser-Trudinger inequality is

$$\int_{\Sigma} |\nabla u|^2 + 8\pi \int_{\Sigma} u - 8\pi \log \int_{\Sigma} e^u > -C, \text{ for any } u \in H^1(\Sigma), \quad (1.1)$$

for some constant $C > 0$. ((1.1) is a slightly weaker form of the original Moser-Trudinger inequality, see [41, 43, 25, 15, 38].) The inequality (1.1) has been extensively used in many mathematical and physical problems, for instance, in the problem of prescribing Gaussian curvature [7, 8, 11, 9], the mean field equation [5, 6, 31, 18, 42], the model of chemotaxis [46, 28] and the relativistic Abelian Chern-Simons model [27, 29, 4, 44, 16, 17, 19, 38], etc. In all such problems, the corresponding equation is similar to the Liouville equation

$$-\Delta u = M_0 \left(\frac{e^u}{\int_{\Sigma} e^u} - 1 \right), \quad (1.2)$$

for some prescribed constant $M_0 > 0$.

The system-analog of (1.2) is the following system of equations

$$-\Delta u_i = M_i \left(\frac{\exp(\sum_{j=1}^n a_{ij} u_j)}{\int_{\Sigma} \exp(\sum_{j=1}^n a_{ij} u_j)} - 1 \right), \quad 1 \leq i \leq n, \quad (1.3)$$

for a coefficient matrix $A = (a_{ij})_{n \times n}$. Here $M_i > 0$ ($i = 1, 2, \dots, n$) are prescribed constants. When the coefficient matrix admits only nonnegative entries, there is a generalized Moser-Trudinger inequality obtained in [13, 45]

Theorem 1.1. [13, 45] *Let the coefficient matrix A be a positive definite matrix with nonnegative entries. If for any subset $J \subseteq I := \{1, 2, \dots, n\}$*

$$\Lambda_J := 8\pi \sum_{j \in J} M_j - \sum_{i, j \in J} a_{ij} M_i M_j > 0, \quad (1.4)$$

then there is a constant $C > 0$ such that for any $u = (u_1, u_2, \dots, u_n) \in (H^1(\Sigma))^n$

$$\frac{1}{2} \sum_{i=1}^n \int a_{ij} (\nabla u_i \nabla u_j + 2M_i u_j) - \sum_{i=1}^n M_i \log \int_{\Omega} \exp\left(\sum_{j=1}^n a_{ij} u_j\right) \geq -C. \quad (1.5)$$

In fact, the condition that A is positive definite can be removed by using another formulation of the functional J_M , see [13, 45]. Theorem 1.1 is sharp in the sense that if there is a subset $J \subseteq I$ with $\Lambda_J(M) < 0$, then $\inf_{u \in (H^1(\Sigma))^n} J_M(u) = -\infty$. When $n = 1$, the condition (1.4) is equivalent to $a_{11}M_1 < 8\pi$. Therefore, it is natural to conjecture [45] that Theorem 1.1 holds if and only if

$$\Lambda_J(M) \geq 0, \text{ for any } J \subseteq I. \quad (1.6)$$

In [45], a proof of this conjecture was sketched for a special, but interesting case.

Theorem 1.2. *Let A be a symmetric, positive definite row stochastic matrix, i.e.,*

$$a_{ij} \geq 0 \text{ and } \sum_{j=1}^n a_{ij} = 1 \text{ for any } i \in I. \quad (1.7)$$

Then there is a constant $C > 0$ such that

$$J_M(u) \geq -C \quad \text{for any } u \in (H^1(\Sigma))^n$$

if and only if (1.6) holds.

The “only if” part was first proved in [13] in a more general case. It is clear that in general, such an inequality cannot be true if the coefficient matrix A admits some negative entries. However, in many interesting systems arising in Physics and Differential Geometry, there are negative coefficients, for instance, in the Toda system and the relativistic and nonrelativistic non-Abelian Chern-Simons models [32, 20, 24, 26].

In this paper, we want to generalize Theorem 1.2 to the Toda system. Let K denote the Cartan matrix for $SU(N + 1)$, i.e.,

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdot & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

The open $SU(N + 1)$ Toda system (or, 2-dimensional Toda lattice) is

$$-\Delta u_i = \sum_{j=1}^N a_{ij} e^{u_j}, \text{ for } i = 1, 2, \dots, N. \quad (1.8)$$

The popular interpretation of the (one-dimensional) Toda lattice is a Hamiltonian system which describes the motion of N particles moving in a straight line, with “exponential interaction”. The two-dimensional Toda system has a much closer relationship with differential geometry. It can be seen as the Frenet frame of holomorphic curves into $\mathbb{C}\mathbb{P}^N$. For the Toda system and its geometric interpretation see, e.g., [26] and references therein. See also [20].

In this paper, we establish the following Moser-Trudinger type inequality for (1.8).

Theorem 1.3. *Let Σ be a closed surface with area 1 and $A = K$. Define a functional $\Phi^N : (H^1(\Sigma))^N \rightarrow \mathbb{R}$ by*

$$\Phi_M(u) = \frac{1}{2} \sum_{i,j=1}^N \int_{\Sigma} a_{ij} (\nabla u_i \nabla u_j + 2M_i u_j) - \sum_{i=1}^N M_i \log \int_{\Sigma} \exp(\sum_{j=1}^N a_{ij} u_j).$$

Then, the functional has a lower bound if and only if

$$M_j \leq 4\pi, \quad \text{for } j = 1, 2, \dots, N. \quad (1.9)$$

We remark that the condition (1.9) is equivalent to (1.6) in this case. Since the coefficient matrix K admits negative entries, we might encounter the problem that the maximum principle fails. This is the reason why we cannot classify all entire solutions of (1.8) with finite energy yet* (see Sections 2 and 7). Fortunately, when proving Theorem 1.3 we can avoid this problem.

We outline our main idea of the proof of Theorem 1.3 for $N = 2$. We first show that Φ_M has a lower bound if $M_j < 4\pi$ for $j = 1, 2$ (Theorem 4.1). The idea is as follows. Let us define

$$\Lambda = \{M = (M_1, M_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \inf \Phi_M > -\infty\}.$$

From the ordinary Moser-Trudinger inequality (1.1), we know $\Lambda \neq \emptyset$. Theorem 4.1 now is equivalent to $(0, 4\pi) \times (0, 4\pi) \subseteq \Lambda$. If it were false then there exists $M^0 = (M_1^0, M_2^0)$ such that $M^\epsilon = (M_1^0 - \epsilon, M_2^0 - \epsilon) \in \Lambda$,

*We are able to classify all such solutions now.

but $(M_1^0 + \epsilon, M_2^0 + \epsilon) \notin \Lambda$ for any small $\epsilon > 0$. It is easy to show that Φ_{M^ϵ} admits a minimizer $u^\epsilon = (u_1^\epsilon, u_2^\epsilon)$ which satisfies (1.3) for $M^\epsilon = (M_1^0 - \epsilon, M_2^0 - \epsilon)$. If u^ϵ blows up, there are three cases (see Section 4 below). For each case, after rescaling we obtain a “bubble” which is an entire solution of the Liouville equation (3.6) or the Toda system (1.8) with finite energy. A classification result of [12] for the Liouville equation and Corollary 2.6 below imply that in any case one of M_1^0 and M_2^0 is larger than or equal to 4π , a contradiction. However, u^ϵ may not blow up. Ding [14] introduced a trick to deal with such a problem in his study of the ordinary Moser-Trudinger inequality. Following his trick, we perturb the functional Φ_{M^ϵ} a little bit such that the resulting functional F_{M^ϵ} also admits a minimizer \tilde{u}^ϵ which *does* blow up as $\epsilon \rightarrow 0$. \tilde{u}^ϵ satisfies a similar system as (1.3) and has the same “bubble”. Then we are able to use the argument above to get a contradiction.

To prove Theorem 1.3, we consider the sequence of minimizers $u^\epsilon = (u_1^\epsilon, u_2^\epsilon)$ of Φ_{M^ϵ} with $M^\epsilon = M^0 - (\epsilon, \epsilon) = (4\pi - \epsilon, 4\pi - \epsilon)$ for small $\epsilon > 0$. (The existence of the u^ϵ follows from Theorem 4.1.) u^ϵ satisfies a Toda type system. If u^ϵ converges u^0 in $H_2 := H^1 \times H^1$, then it is clear that $\inf_{u \in H_2} \Phi_{M^0}(u) = \Phi_{M^0}(u_0) > -\infty$. Hence, we may assume u^ϵ does not converge in H_2 . Using the analysis developed in Sections 2 and 3, we know there are three possibilities: (a) $|S_1| = 1$ and $S_2 = \emptyset$, (b) $|S_1| = |S_2| = 1$ and $S_1 = S_2$ and (c) $|S_1| = |S_2| = 1$ and $S_1 \cap S_2 = \emptyset$. Here S_j ($j = 1, 2$) is the blow-up set defined in Section 5 below and $|S|$ is the number of points of the finite set S . We use a “local” Pohozaev identity to exclude case (b). This is the crucial point to avoid the aforementioned problem that the maximum principle fails, since the remaining cases are essentially scalar problems. In fact, we can reduce case (a) directly to the corresponding problem of one function—the ordinary Moser-Trudinger inequality. For case (b), we can apply the method developed in [15, 38] to give a lower bound of Φ_{M^0} .

We shall apply Theorem 1.3 in a forthcoming paper [30] to study the relativistic $SU(N + 1)$ non-Abelian Chern-Simons model ([32, 34, 35,

20]), which can be seen as a non-integrable perturbation of the integrable Toda system (1.8). For mathematical aspects of the relativistic non-Abelian Chern-Simons model, see [49, 47][†]. We hope Theorem 1.3 will become a powerful tool for studying problems arising from higher rank models, as the ordinary Moser-Trudinger inequality has become in the Abelian case.

For simplicity we only give detailed proofs for the case $N = 2$. The proofs in case $N > 2$ are completely analogous and just require a more complicated notation. In Section 2, we analyze the solutions of (1.8) in \mathbb{R}^2 and obtain a relation between $\int_{\mathbb{R}^2} e^{u_1}$ and $\int_{\mathbb{R}^2} e^{u_2}$ for any entire solutions of (1.8). In Section 3, we analyze the convergence of solutions as in [3]. In Section 4, we first show Φ_M has a lower bound if $M_j < 4\pi$ for all j . Then we show that if Φ_M has a lower bound, $M_j \leq 4\pi$ for all j . We prove Theorem 1.3 in Section 5.

2. ANALYSIS OF THE TODA SYSTEM

In this section, we consider the analysis of solutions of the following system

$$\begin{cases} -\Delta u_1 &= 2e^{u_1} - e^{u_2}, \\ -\Delta u_2 &= -e^{u_1} + 2e^{u_2}, \end{cases} \quad (2.1)$$

which is equivalent to system (1.8) with $N = 2$. Similar results were obtained in [3, 12] for the Liouville equation and in [10, 13] for the Liouville type systems, see also [36].

Lemma 2.1. *Let $u \in C^2(B_3) \cap C^1(\bar{B}_3)$ satisfy*

$$\begin{cases} -\Delta u_1 &= 2e^{u_1} - e^{u_2} && \text{in } B_3, \\ -\Delta u_2 &= -e^{u_1} + 2e^{u_2} && \text{in } B_3, \\ u_j(x_j) &= b_j, && \text{for some } x_j \in B_1, j = 1, 2, \\ u_j &\leq a_0, && \text{in } B_3, \text{ for } j = 1, 2. \end{cases} \quad (2.2)$$

Then there is a constant $C = C(a_0, b_1, b_2)$ such that

$$\min_{x \in B_1} \{u_1, u_2\} \geq C.$$

[†]See also [40].

Proof. Let $w = (w_1, w_2)$ defined by

$$\begin{aligned} w_1(x) &= \frac{1}{2\pi} \int_{B_3} \log|x-y|(2e^{u_1(y)} - e^{u_2(y)})dy \\ w_2(x) &= \frac{1}{2\pi} \int_{B_3} \log|x-y|(2e^{u_2(y)} - e^{u_1(y)})dy \end{aligned}$$

Set $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) = u - w$. Obviously,

$$\Delta \tilde{u}_j = 0 \text{ in } B_2, \text{ for } j = 1, 2 \quad (2.3)$$

and

$$|w_j|(x) \leq c_1(a_0), \text{ for } y \in B_2. \quad (2.4)$$

(2.4) and (2.2) imply that $\tilde{u}_j \leq c_2(a_0)$ and $\tilde{u}_j(x_j) \geq c_3(b_j)$. Now from the Harnack inequality, we have

$$\min_{x \in B_1} \tilde{u}_j(x) \geq -c(a_0, b_1, b_2). \quad (2.5)$$

The Lemma follows from (2.4) and (2.5). \square

Lemma 2.2. *There exists $\gamma_0 > 0$ such that for any $u \in C^2(B_2^+) \cap C^1(\overline{B_2^+})$ satisfying*

$$\begin{cases} -\Delta u_1 = 2e^{u_1} - e^{u_2} & \text{in } B_3, \\ -\Delta u_2 = -e^{u_1} + 2e^{u_2} & \text{in } B_3, \\ \int_{B_2} e^{u_1} < \gamma_0, \\ \int_{B_2} e^{u_2} < \gamma_0, \end{cases} \quad (2.6)$$

we have

$$\max_{\overline{B_{1/4}}} \max\{u_1, u_2\} \leq C_1,$$

for some positive constant C_1 .

Proof. Choosing $\gamma_0 < \frac{4\pi}{3}$, by the Brezis-Merle inequality [3] we have that $\|\Delta u_j\|_{L^p}$ ($j = 1, 2$) is bounded for some $p > 1$. The Lemma follows from the standard elliptic estimates. \square

Note that Lemma 2.2 is true for any $\gamma_0 < 4\pi$, see Lemma 3.2 below.

Proposition 2.3. *Let $u = (u_1, u_2) \in H_{loc}^1(\mathbb{R}^2) \times H_{loc}^1(\mathbb{R}^2)$ be a solution of (2.1) on \mathbb{R}^2 with*

$$\int_{\mathbb{R}^2} e^{u_1} < \infty \text{ and } \int_{\mathbb{R}^2} e^{u_2} < \infty. \quad (2.7)$$

Then u is smooth and satisfies

$$\max_{x \in \mathbb{R}^2} \{u_1(x), u_2(x)\} < \infty.$$

Proof. Let

$$w_j = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\log|x-y| - \log(|y|+1)) e^{u_j(y)} dy$$

and $\alpha_j = \int_{\mathbb{R}^2} e^{u_j}$ for $j = 1, 2$. Set $\tilde{u}_1 = u_1 - 2w_1 + w_2$ and $\tilde{u}_2 = u_2 - 2w_2 + w_1$. Clearly $\Delta \tilde{u}_j = 0$ in \mathbb{R}^2 . By Lemma 2.4 below and Lemma 2.2, \tilde{u}_j ($j = 1, 2$) is bounded from above. Thus, $\tilde{u} \equiv c_j$ for some constants c_1 and c_2 . Now the Lemma follows from Lemma 2.5 and (2.9) below. \square

Using potential analysis as in [12], we have

Lemma 2.4. *For any small $\epsilon > 0$, there is a constant $c_\epsilon > 0$ such that*

$$-(\alpha_j + \epsilon) \log|x| - c_\epsilon \leq w_j(x) \leq -(\alpha_j - \epsilon) \log|x| + c_\epsilon, \text{ for any } x \in \mathbb{R}^2. \quad (2.8)$$

Proof. See [12]. \square

Lemma 2.5. $\beta_1 := 2\alpha_1 - \alpha_2 > 4\pi$ and $\beta_2 := 2\alpha_2 - \alpha_1 > 4\pi$.

Proof.

The previous Lemma implies that

$$-(2\alpha_1 - \alpha_2 + \epsilon) \log|x| - c_\epsilon \leq 2w_1(x) - w_2(x) \leq -(2\alpha_1 - \alpha_2 - \epsilon) \log|x| + c_\epsilon \quad (2.9)$$

Since $u_1 = 2w_1 - w_2 + c_1$, from (2.9) and (2.8), we deduce that $\beta_1 > 4\pi$.

Similarly, we have $\beta_2 > 4\pi$. \square

Corollary 2.6. $\alpha_j > 4\pi$ for $j = 1, 2$.

\square

Lemma 2.7. *Let u be a solution of (2.1) and (2.7). We have,*

$$\begin{aligned} |u_j - \frac{\beta_j}{2\pi} \log(|x| + 1)| &\leq c, \\ \lim_{r \rightarrow \infty} r \frac{\partial u_j}{\partial r} &= -\frac{\beta_j}{2\pi}, \\ \lim_{r \rightarrow \infty} \frac{\partial u_j}{\partial \theta} &= 0, \end{aligned} \quad (2.10)$$

where $x = (r, \theta)$.

Proof. From above, we have

$$\begin{aligned} u_1 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} [\log|x-y| - \log(|y|+1)] (2e^{u_1(y)} - e^{u_2(y)}) dy + c_1, \\ u_2 &= \frac{1}{2\pi} \int_{\mathbb{R}^2} [\log|x-y| - \log(|y|+1)] (2e^{u_2(y)} - e^{u_1(y)}) dy + c_2. \end{aligned}$$

The Lemma follows from potential analysis, see for instance [12]. \square

Now we are in the position to give a relation between $\int e^{u_1}$ and $\int e^{u_2}$.

Proposition 2.8. *Let $\alpha_j = \int_{\mathbb{R}^2} e^{u_j}$ for $j = 1, 2$. We have*

$$\alpha_1^2 + \alpha_2^2 - \alpha_1 \alpha_2 = 4\pi(\alpha_1 + \alpha_2). \quad (2.11)$$

Proof. From equation (2.1), we have the Pohozaev identities as follows:

$$\begin{aligned} -R \int_{\partial B_R} (|\frac{\partial u_1}{\partial r}|^2 - \frac{1}{2} |\nabla u_1|^2) &= 2 \int_{\partial B_R} R e^{u_1} - 4 \int_{B_R} e^{u_1} - \int_{B_R} e^{u_2} x \nabla u_1, \\ -R \int_{\partial B_R} (|\frac{\partial u_2}{\partial r}|^2 - \frac{1}{2} |\nabla u_2|^2) &= 2 \int_{\partial B_R} R e^{u_2} - 4 \int_{B_R} e^{u_2} - \int_{B_R} e^{u_1} x \nabla u_2 \end{aligned}$$

and

$$\begin{aligned} & - \int_{\partial B_R} R \frac{\partial u_1}{\partial n} \frac{\partial u_2}{\partial n} + \int_{B_R} \nabla u_1 \nabla u_2 + \int_{B_R} x \sum_{j=1}^2 \nabla(\nabla_j u_2) \nabla_j u_1 \\ &= - \int_{\partial B_R} R e^{u_2} + 2 \int_{B_R} e^{u_2} + 2 \int_{B_R} e^{u_1} x \nabla u_2, \\ & - \int_{\partial B_R} R \frac{\partial u_1}{\partial n} \frac{\partial u_2}{\partial n} + \int_{B_R} \nabla u_1 \nabla u_2 + \int_{B_R} x \sum_{j=1}^2 \nabla(\nabla_j u_1) \nabla_j u_2 \\ &= - \int_{\partial B_R} R e^{u_1} + 2 \int_{B_R} e^{u_1} + 2 \int_{B_R} e^{u_2} x \nabla u_1. \end{aligned} \quad (2.12)$$

From above, we get the Pohozaev identity for the Toda system (2.1)

$$\begin{aligned} & 3 \int_{\partial B_R} R (e^{u_1} + e^{u_2}) - 6 \int_{B_R} (e^{u_1} + e^{u_2}) = \\ & -2 \sum_{j=1}^2 \int_{\partial B_R} R (|\partial_n u_j|^2 - \frac{1}{2} |\nabla u_j|^2) - 2 \int_{\partial B_R} R (\frac{\partial u_1}{\partial n} \frac{\partial u_2}{\partial n} - \frac{1}{2} \nabla u_1 \nabla u_2) \end{aligned} \quad (2.13)$$

Applying Lemmas 2.5 and 2.7 in (2.13) and letting $R \rightarrow \infty$, we get

$$\beta_1^2 + \beta_2^2 + \beta_1 \beta_2 = 12\pi(\beta_1 + \beta_2),$$

which is equivalent to (2.11). This proves the Lemma. \square

Similar results for systems with non-negative entries were obtained in [10] and [13]. We conjecture that $\alpha_1 = \alpha_2 = 8\pi$. It is also very interesting to classify all solutions of (2.1) with finite energy $\int_{\mathbb{R}^2} e^{u_j} < \infty$ for $j = 1, 2$. When solutions have suitable decay near infinity such that they can be seen as functions on \mathbb{S}^2 , the classification was obtained by differential geometers and physicists, see [23, 20]. In fact, in this case, all solutions can be seen as minimal immersions from \mathbb{S}^2 to \mathbb{CP}^2 which are deformations of the Veronese immersion by the action of the group $PGL(3, \mathbb{C})$, see [23, 2, 48].

3. CONVERGENCE OF SOLUTIONS

In this section, we consider the convergence of solutions of the Toda system. We follow the method developed in [3], but we need to be careful with the use of the maximum principle, since the coefficient matrix has some negative entries.

For simplicity, we only consider the system on the bounded domain Ω . We have

Theorem 3.1. *Let $u^k = (u_1^k, u_2^k)$ be a sequence of solutions of the following system*

$$\begin{cases} -\Delta u_1^k &= 2e^{u_1^k} - e^{u_2^k} + \psi_1^k, & \text{on } \Omega, \\ -\Delta u_2^k &= 2e^{u_2^k} - e^{u_1^k} + \psi_2^k, & \text{on } \Omega, \end{cases} \quad (3.1)$$

with

$$\int_{\Omega} e^{u_1^k} < C \text{ and } \int_{\Omega} e^{u_2^k} < C \quad (3.2)$$

and

$$\|\psi_1^k\|_{L^p(\Omega)} + \|\psi_2^k\|_{L^p(\Omega)} \leq C,$$

for some constant $C > 0$ and $p > 1$. Let

$$S_j = \{x \in \Sigma \mid \text{there is a sequence } y^\epsilon \rightarrow x \text{ such that } u_j^\epsilon(y^\epsilon) \rightarrow +\infty\}. \quad (3.3)$$

Then, one of the following possibilities happens: (after taking subsequences)

- (1) u^k is bounded in $L_{loc}^\infty(\Omega) \times L_{loc}^\infty(\Omega)$.

- (2) For some $j \in \{1, 2\}$, u_i^k in $L_{loc}^\infty(\Omega)$, but $u_j^k \rightarrow -\infty$ uniformly on any compact subset of Ω for $j \neq i$.
- (3) For some $i \in \{1, 2\}$, $S_i \neq \emptyset$, but $S_j = \emptyset$, for $j \neq i$. In this case, $u_i^k \rightarrow -\infty$ on any compact subset of $\Omega \setminus S_i$, and either, u_j^k is bounded in $L_{loc}^\infty(\Omega)$, or $u_j^k \rightarrow -\infty$ on any compact subset of Ω .
- (4) $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Moreover, u_j^k is either bounded or $\rightarrow -\infty$ on any compact subset of $\Omega \setminus (S_1 \cup S_2)$ for $j = 1, 2$.

Proof. Here, for simplicity we only give a proof of the Theorem when $\psi_1^k = \psi_2^k = 0$.

In view of (3.2), we may assume that there exist two nonnegative bounded measures μ_1 and μ_2 such that

$$e^{u_j^k} \psi \rightarrow \int \psi d\mu_j \text{ as } k \rightarrow \infty,$$

for every smooth function ψ with support in Ω and $j = 1, 2$. A point $x \in \Omega$ is called a γ -regular point with respect to μ_j if there is a function $\psi \in C_c(\Omega)$, $0 \leq \psi \leq 1$, with $\psi = 1$ in a neighborhood of x such that

$$\int_{\Omega} \psi d\mu_j < \gamma.$$

We define

$$\Omega_j(\gamma) = \{x \in \Omega \mid x \text{ is not a } \gamma\text{-regular point with respect to } \mu_j\}.$$

By definition and (3.2), it is clear that $\Omega_1(\gamma)$ and $\Omega_2(\gamma)$ are finite. And $\Omega_j(\gamma)$ is independent of γ for small $\gamma > 0$, see below. We divide the proof into 3 steps.

Step 1. For $j = 1, 2$, $S_j = \Omega_j(\gamma)$ provided $\gamma < \frac{4\pi}{3}$.

First from Lemma 2.2, we know that for any point $x \in \Omega \setminus (\Omega_1(\gamma) \cup \Omega_2(\gamma))$, there is some r_0 such that

$$u_j^{k+} \text{ is bounded in } L^\infty(B_{r_0}(x)) \text{ for } j = 1, 2. \quad (3.4)$$

Here $u^+ = \max\{u, 0\}$. The argument in [3] implies directly that

$$S_1 \cup S_2 = \Omega_1(\gamma) \cup \Omega_2(\gamma).$$

Hence, S_1 and S_2 are both finite. Let $x_0 \in S_1$. Assume by contradiction that $x_0 \notin \Omega_1$. Thus, $\int_{B_\delta(x_0)} e^{u_1^k} \leq \gamma$ for any small constant $\delta > 0$. Note

that $-\Delta u_1^k = 2e^{u_1^k} - e^{u_2^k} \leq 2e^{u_1^k}$. Define $w : B_\delta(x_0) \rightarrow \mathbb{R}$ by

$$\begin{cases} -\Delta w = 2e^{u_1^k}, & \text{in } B_\delta(x_0), \\ w = u_1^k, & \text{on } \partial B_\delta(x_0). \end{cases}$$

The maximum principle implies that $u_1^k \leq w$. Since S_1 is finite, we may assume that u_1^{k+} is uniformly bounded in $L^\infty(\partial B_\delta(x_0))$. In view of $\int_{B_\delta(x_0)} e^{u_1^k} \leq \gamma < \frac{4\pi}{3}$, a result of Brezis and Merle [3] implies that $w^+ \in L^\infty(B_{\frac{\delta}{2}}(x_0))$, which in turn implies that $u_1^{k+} \in L^\infty(B_{\frac{\delta}{2}}(x_0))$, a contradiction. Hence, $S_1 \subset \Omega_1(\gamma)$. $\Omega_1(\gamma) \subset S_1$ follows from the argument in [3]. Similarly, we have $S_2 = \Omega_2(\gamma)$.

Step 2. $S_1 \cup S_2 = \emptyset$ implies (1) and (2). $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$ imply (4).

$S_1 \cup S_2 = \emptyset$ means that $u_1^{\epsilon+}$ and $u_2^{\epsilon+}$ are bounded in $L_{loc}^\infty(\Omega)$. Thus, $e^{u_1^{\epsilon+}}$ and $e^{u_2^{\epsilon+}}$ are bounded in $L_{loc}^p(\Omega)$ for any $p > 1$, which implies that $\mu_1, \mu_2 \in L^1(\Omega) \cap L_{loc}^p(\Omega)$. Applying the Harnack inequality as in [3], we have (1) or (2). The second statement follows similarly.

Step 3. $S_1 \neq \emptyset$ and $S_2 = \emptyset$ imply (3).

We need the following lemma.

Lemma 3.2. *Lemma 2.2 is true for any $\gamma_0 < 4\pi$.*

Proof. We use a blow-up argument to prove this Lemma. Assume by contradiction that Lemma 2.2 were false for some $\gamma_0 < 4\pi$, i.e., there exists a sequence of solutions $u^k = (u_1^k, u_2^k)$ of (2.7) with $\int_\Omega e^{u_j} \leq \gamma_0$ ($j = 1, 2$) for some $\gamma_0 < 4\pi$ such that

$$\max_{\overline{B_{1/4}}} \max\{u_1^k, u_2^k\} \rightarrow \infty, \text{ as } k \rightarrow \infty.$$

Without loss of generality, we may assume that $S_1 = \{x_0\}$ and $S_2 \cap (B_{1/4} \setminus \{x_0\}) = \emptyset$. We may also assume that there exists a sequence of points $\{x_k\} \subset B_{1/4}$ such that

$$u_1^k(x_k) = \max_{\overline{B_{1/4}}} \max\{u_1^k, u_2^k\}.$$

Let $m_k = u_1^k(x_k)$. Define $\tilde{u}_j^k(x) = u_j^k(\lambda_k x + x_k) - m_k$ for $j = 1, 2$ with $\lambda_k = e^{-\frac{1}{2}m_k}$. Clearly, $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$ and $\tilde{u}^k = (\tilde{u}_1^k, \tilde{u}_2^k)$ satisfies

$$\begin{cases} -\Delta \tilde{u}_1^k &= 2e^{\tilde{u}_1^k} - e^{\tilde{u}_2^k} & \text{in } \Omega_k \\ -\Delta \tilde{u}_2^k &= 2e^{\tilde{u}_2^k} - e^{\tilde{u}_1^k} & \text{in } \Omega_k \\ \int_{\Omega_k} e^{\tilde{u}_j^k} &\leq \gamma_0, j = 1, 2, \\ \tilde{u}_j^k(x) &\leq 0 = \tilde{u}_1^k(0), \end{cases} \quad (3.5)$$

where $\Omega_k = \lambda_k^{-1}(B_{1/4} - x_k)$. Applying Lemma 2.1, Lemma 2.2 and (1)-(2) in Theorem 3.1, we have two possibilities:

- (i). \tilde{u}^k converges to $u^0 = (u_1^0, u_2^0)$ in $H_{loc}^1(\mathbb{R}^2) \times H_{loc}^1(\mathbb{R}^2)$ which satisfies the Toda system (2.1) in \mathbb{R}^2 . (In this case \tilde{u}_2^k is bounded in $L_{loc}^\infty(\mathbb{R}^2)$.)
- (ii). \tilde{u}_1^k converges to ξ_0 in $H_{loc}^1(\mathbb{R}^2)$ and \tilde{u}_2^k tends to $-\infty$ uniformly in any compact subset in \mathbb{R}^2 . Moreover, ξ_0 satisfies

$$-\Delta \xi_0 = 2e^{\xi_0} \quad (3.6)$$

with

$$\int_{\mathbb{R}^2} e^{\xi_0} < \infty.$$

In view of Corollary 2.6 and a classification result of (3.6) obtained by Chen-Li [12], in these two cases $\lim_{k \rightarrow \infty} \int_{\Omega} e^{u_1^k} \geq 4\pi$, which is a contradiction. This proves the Lemma. \square

Now we continue to prove Step 3. As in Step 2, we know that either

- (i) u_1^k is bounded on any subset of $\Omega \setminus S_1$, or
- (ii) $u_1^k \rightarrow -\infty$ on any subset of $\Omega \setminus S_1$.

In view of Lemma 3.2, (3) implies that $\int_{B_\delta(p)} e^{u_1^k} \geq 4\pi$ for any $p \in S_1$ and any small $\delta > 0$. Now we can follow the argument in [3] to exclude (i). \square

Remark 3.3. We believe that in case (4) $u_j^k \rightarrow -\infty$ on any compact subset. From the argument of Step 3, one can show this if $S_1 \neq S_2$. In our application (the proof of Theorems 5.1 and 1.3), we can exclude case (4).

Before we start to prove our main results, we remark that

- (1) $\epsilon \rightarrow 0$ always means some sequence $\epsilon_n > 0$ such that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

(2) C denotes a constant independent of ϵ , which may vary from line to line.

(3) Any 2-dimensional surface (Σ, g) is locally conformally flat, i.e., for any $x \in \Sigma$ there is a neighborhood $x \in U \subset \Sigma$ and a conformal factor $\xi : U \rightarrow \mathbb{R}$ such that $g|_U = e^\xi(dx^2 + dy^2)$ in local coordinates. Instead of considering the equation $-\Delta_g = f$ in U , we can consider $-\Delta_0 = e^\xi f$ in a domain of the Euclidean plane. Hence, wlog, we can assume $\xi = 0$, i.e., U is a flat domain. In the following sections, we will assume that near a blow-up point there is a flat neighborhood.

4. A MOSER-TRUDINGER INEQUALITY FOR THE $SU(3)$ TODA SYSTEM

Let K be the Cartan matrix for $SU(3)$, i.e.,

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We have the following Moser-Trudinger inequality.

Theorem 4.1. *Let Σ be a closed surface with area 1 and $A = K$. For any $M = (M_1, M_2) \in (0, 4\pi) \times (0, 4\pi)$, there is a constant $C > 0$ such that*

$$\begin{aligned} \tilde{\Phi}_M(v) &= \frac{1}{2} \int_\Sigma \{2|\nabla v_1|^2 + 2|\nabla v_2|^2 - 2\nabla v_1 \nabla v_2 + 2(2M_1 - M_2)v_1 + \\ &\quad 2(2M_2 - M_1)v_2\} - M_1 \log \int e^{2v_1 - v_2} - M_2 \log \int e^{2v_2 - v_1} \\ &\geq -C, \end{aligned} \tag{4.1}$$

for any $v \in H_2$, or equivalently,

$$\begin{aligned} \Phi_M(u) &= \frac{1}{3} \int_\Sigma \{|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \nabla u_2 + 3M_1 u_1 + 3M_2 u_2\} \\ &\quad - M_1 \log \int e^{u_1} - M_2 \log \int e^{u_2} \geq -C, \end{aligned} \tag{4.2}$$

for any $u \in H_2$.

The equivalence between $\tilde{\Phi}(v)$ and $\Phi(u)$ can be seen easily from the following equation

$$\begin{aligned} u_1 &= 2v_1 - v_2 \\ u_2 &= 2v_2 - v_1 \end{aligned} \tag{4.3}$$

Here we use the method of Ding [14] to prove Theorem 4.1. This method was introduced in his study of the ordinary Moser-Trudinger

inequality and was applied in [45] to obtain a similar inequality for a system of functions.

Proof. Set

$$\Lambda = \{(M_1, M_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \Phi_M \text{ is bounded from below on } H_2\}.$$

Since A is positive definite, it is easy to see that $\Lambda \neq \emptyset$ from the ordinary Moser-Trudinger inequality [41, 43]. In fact, one can show easily that $(\frac{8\pi}{3}, \frac{8\pi}{3}) \in \Lambda$ and $(\frac{16\pi}{3} + \epsilon, \frac{16\pi}{3} + \epsilon) \notin \Lambda$ ($\epsilon > 0$) from the ordinary Moser-Trudinger inequality and the Hölder inequality. Clearly, Λ preserves a partial order of $\mathbb{R}_+ \times \mathbb{R}_+$, namely, if $(M_1, M_2) \in \Lambda$, then $(M'_1, M'_2) \in \Lambda$ provided that $M'_1 \leq M_1$ and $M'_2 \leq M_2$. The Theorem is equivalent to

$$(0, 4\pi) \times (0, 4\pi) \subset \Lambda. \quad (4.4)$$

Assume by contradiction that (4.4) is false. We may assume that there is a point

$$M^0 = (M_1^0, M_2^0) \in (0, 4\pi) \times (0, 4\pi) \quad (4.5)$$

such that

1. For any $\epsilon > 0$, $M^0 - \epsilon = (M_1^0 - \epsilon, M_2^0 - \epsilon) \in \Lambda$,
2. For any $\epsilon > 0$, $M^0 + \epsilon = (M_1^0 + \epsilon, M_2^0 + \epsilon) \notin \Lambda$.

We first need several lemmas.

Lemma 4.2. *For any M with $M_1 < M_1^0$ and $M_2 < M_2^0$, there exists a constant $c > 0$ such that*

$$\Phi_M(u) \geq c^{-1}(\|\nabla u_1\|^2 + \|\nabla u_2\|^2) - c, \text{ for any } u \in H_2. \quad (4.6)$$

Moreover, Φ_M admits a minimizer $u = (u_1, u_2)$.

Proof. Choose a small number $\delta > 0$ such that $M_1(1 + \delta) < M_1^0$ and $M_2(1 + \delta) < M_2^0$. By the definition of M^0 , we know that $(1 + \delta)M = ((1 + \delta)M_1, (1 + \delta)M_2) \in \Lambda$, i.e., there is a constant $C > 0$ such that

$$\Phi_{(1+\delta)M}(u) \geq -C, \quad \text{for any } u \in H_2.$$

It follows that

$$\begin{aligned} \Phi_M(u) &= \frac{1}{1+\delta} \Phi_{(1+\delta)M} + \frac{\delta}{3(1+\delta)} \int_{\Omega} (|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \nabla u_2) \\ &\geq \frac{\delta}{6(1+\delta)} \int_{\Omega} (|\nabla u_1|^2 + |\nabla u_2|^2) - C. \end{aligned}$$

This inequality means that Φ_M satisfies the coercivity condition. Now it is standard to show that Φ_M admits a minimizer, for Φ_M is weakly lower semi-continuous. \square

Lemma 4.3. *There exists a sequence $u^k \in H_2$ such that $\lim_{k \rightarrow \infty} \|\nabla u^k\|^2 \rightarrow \infty$ and*

$$\lim_{k \rightarrow \infty} \frac{\Phi_{M^0}(u^k)}{\|\nabla u^k\|^2} \leq 0, \quad (4.7)$$

where $\|\nabla u\|^2 = \|\nabla u_1\|^2 + \|\nabla u_2\|^2$.

Proof. Assume by contradiction that for any sequence $u^k \in H_2$ with $\lim_{k \rightarrow \infty} \|\nabla u^k\|^2 \rightarrow \infty$,

$$\lim_{k \rightarrow \infty} \frac{\Phi_{M^0}(u^k)}{\|\nabla u^k\|^2} > \delta > 0.$$

Then we can show that Φ_{M^0} satisfies the coercivity condition,

$$\Phi_{M^0}(u) \geq \frac{\delta}{2} \|\nabla u\|^2 + c, \quad u \in H_2,$$

for some constant $c > 0$, which implies that there is a small $\delta > 0$ such that $M^0 + \delta \in \Lambda$, a contradiction. \square

Lemma 4.4. ([14]) *For any two sequences a_k and b_k satisfying*

$$\lim_{k \rightarrow \infty} a_k \rightarrow +\infty \quad \text{and} \quad d_0 = \lim_{k \rightarrow \infty} \frac{b_k}{a_k} \leq 0, \quad (4.8)$$

there exists a smooth function $F : [1, \infty) \rightarrow [0, \frac{1}{2}]$ satisfying

$$|F'(t)| < 1/2 \quad \text{and} \quad |F'(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (4.9)$$

and

$$F(a_{n_k}) - b_{n_k} \rightarrow +\infty \quad \text{as } k \rightarrow \infty, \quad (4.10)$$

for some subsequence $\{n_k\}$.

Proof. We give the proof for completeness, though it is rather elementary. If there is a subsequence b_{n_k} of b_k with property that $b_{n_k} \leq 0$, we can choose $F(t) = \log t$. So we may assume that $b_k \geq 0$ and $d_0 = 0$. Wlog, we assume more that $\frac{b_k}{a_k}$ is non-increasing. Choose another sequence c_k with

$$\frac{c_k}{a_k} \rightarrow 0 \quad \text{and} \quad \frac{b_k}{c_k} \rightarrow 0.$$

It is easy to find a non-increasing smooth function $F : [0, \infty) \rightarrow \mathbb{R}$ with the property that

$$F(a_k) = c_k \quad \text{and} \quad F'(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Clearly, this function F satisfies all conditions of the Lemma. \square

Applying Lemma 4.4 to sequences

$$a_k = \frac{1}{3} \int_{\Sigma} [|\nabla u_1^k|^2 + |\nabla u_2^k|^2 + \nabla u_1^k \nabla u_2^k] \text{ and } b_k = \Phi_{M^0}(u^k),$$

where u^k is obtained in Lemma 4.3, we can find a function F satisfying (4.9) and (4.10). For any small $\epsilon \geq 0$, define a perturbed functional by

$$I_{\epsilon}(u) = \Phi_{M^0 - \epsilon}(u) - F\left(\frac{1}{3} \int_{\Sigma} [|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \nabla u_2]\right),$$

where $M^0 - \epsilon = (M_1^0 - \epsilon, M_2^0 - \epsilon)$.

Lemma 4.5. *Let $\beta_{\epsilon} = \inf_{u \in H_2} I_{\epsilon}(u)$. We have*

1. *for $\epsilon > 0$, the infimum $\beta_{\epsilon} > -\infty$, and it is achieved by $u^{\epsilon} \in H_2$ satisfying*

$$\begin{cases} (1 - F'(g_{\epsilon}))\Delta u_1^{\epsilon} &= -2(M_1^0 - \epsilon)e^{u_1^{\epsilon}} - 2M_1^0 - \epsilon, \\ &+ (M_2^0 - \epsilon)e^{u_2^{\epsilon}} + M_2^0 - \epsilon \\ (1 - F'(g_{\epsilon}))\Delta u_2^{\epsilon} &= -2(M_2^0 - \epsilon)e^{u_2^{\epsilon}} - 2M_2^0 - \epsilon \\ &+ (M_1^0 - \epsilon)e^{u_1^{\epsilon}} + M_1^0 - \epsilon, \end{cases} \quad (4.11)$$

with

$$\int_{\mathbb{R}^2} e^{u_1^{\epsilon}} = \int_{\mathbb{R}^2} e^{u_2^{\epsilon}} = 1. \quad (4.12)$$

where

$$g_{\epsilon} = \frac{1}{3} \int (|\nabla u_1^{\epsilon}|^2 + |\nabla u_2^{\epsilon}|^2 + \nabla u_1^{\epsilon} \cdot \nabla u_2^{\epsilon}).$$

2. *for $\epsilon = 0$, I_0 has no lower bound, i.e.,*

$$\beta_0 = -\infty. \quad (4.13)$$

Proof. 1. As in Lemma 4.2, we can show that I_{ϵ} ($\epsilon > 0$) satisfies the coercivity condition from the construction of F . It is also easy to check that I_{ϵ} is weakly lower semicontinuous. Therefore, I_{ϵ} has a minimizer which satisfies (4.11).

2. From the construction of F , we have

$$I_0(u^k) = \Phi_{M^0}(u^k) - F(\|\nabla u^k\|^2) = -(F(a^k) - b^k) \rightarrow -\infty,$$

as $k \rightarrow \infty$, where the sequence u^k was obtained in Lemma 4.3. \square

We now continue to prove Theorem 4.1 by considering the sequence u^ϵ obtained in the previous Lemma.

We claim that u^ϵ blows up, i.e.,

$$\max_{x \in \Sigma} \max\{u_1^\epsilon(x), u_2^\epsilon(x)\} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0. \quad (4.14)$$

Otherwise, we can show that u^ϵ converges to some u^0 in H_2 that is a minimizer of I_0 . On the other hand, by Lemma 4.5, I_0 has no minimizer, a contradiction.

Let $m_\epsilon = \max_{x \in \Sigma} \max\{u_1^\epsilon(x), u_2^\epsilon(x)\} = \max\{u_1^\epsilon(x^\epsilon), u_2^\epsilon(x^\epsilon)\}$. We have $m_\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$. Define a new sequence v^ϵ by

$$\tilde{u}_1^\epsilon = u_1^\epsilon(e^{-\frac{1}{2}m_\epsilon}x + x^\epsilon) - m_\epsilon \quad \text{and} \quad \tilde{u}_2^\epsilon = u_2^\epsilon(e^{-\frac{1}{2}m_\epsilon}x + x^\epsilon) - m_\epsilon$$

near a small, but fixed neighborhood U of $x_0 \in \Sigma$, where $x_0 = \lim x^\epsilon$. Here, we have abused a little bit the notation. (For simplicity, we consider U as a domain in \mathbb{R}^2 .) Clearly, $\tilde{u}^\epsilon = (\tilde{u}_1^\epsilon, \tilde{u}_2^\epsilon)$ satisfies on $U_\epsilon = m_\epsilon^{\frac{1}{2}}U(\cdot - x^\epsilon)$ that

$$\begin{cases} (1 - F'(\|\nabla u^\epsilon\|^2))\Delta \tilde{u}_1^\epsilon &= -2(M_1^0 - \epsilon)e^{\tilde{u}_1^\epsilon} + 2e^{-\frac{1}{2}m_\epsilon}(M_1^0 - \epsilon) \\ &+ (M_2^0 - \epsilon)e^{\tilde{u}_2^\epsilon} - e^{-\frac{1}{2}m_\epsilon}(M_2^0 - \epsilon), \\ (1 - F'(\|\nabla u^\epsilon\|^2))\Delta \tilde{u}_2^\epsilon &= -2(M_2^0 - \epsilon)e^{\tilde{u}_2^\epsilon} + e^{-\frac{1}{2}m_\epsilon}(M_2^0 - \epsilon) \\ &+ (M_1^0 - \epsilon)e^{\tilde{u}_1^\epsilon} - e^{-\frac{1}{2}m_\epsilon}(M_1^0 - \epsilon), \end{cases} \quad (4.15)$$

Recall that $\|\nabla u^\epsilon\|^2 \rightarrow \infty$ as $\epsilon \rightarrow 0$ and $\lim_{t \rightarrow \infty} F'(t) = 0$. By using the same argument as in the proof of Lemma 3.2, we have that one of M_1 and M_2 is larger than or equal to 4π , which is a contradiction. This completes the proof of Theorem 4.1. \square

Corollary 4.6. *Let Σ be a closed surface with area 1. For any $M = (M_1, M_2)$ with $M_1 < 4\pi$ and $M_2 < 4\pi$, Φ_M admits a minimizer $u = (u_1, u_2)$ which satisfies*

$$\begin{cases} -\Delta u_1 &= 2M_1(e^{u_1} - 1) - M_2(e^{u_2} - 1), \\ -\Delta u_2 &= 2M_2(e^{u_2} - 1) - M_1(e^{u_1} - 1). \end{cases}$$

Proof. Theorem 4.1 implies that Φ_M not only has a lower bound but also satisfies the coercivity condition. Since Φ_M is weakly lower semi-continuous, there is a minimizer that satisfies (4.6) which is the Euler-Lagrange equation of Φ_M . \square

To end this section, we observe that 4π is the best constant.

Proposition 4.7. *If one of M_i is greater than 4π , then Φ_M has no lower bound in H_2 .*

Proof. Wlog, assume that $M_1 > 4\pi$ and Σ contains a flat disk B_{δ_0} for a small constant $\delta_0 > 0$. Let

$$\begin{aligned} u_1^\lambda &= \begin{cases} \log \frac{\lambda^2}{(1+\lambda^2\pi|x|^2)^2} & \text{in } B_{\delta_0} \\ \log \frac{\lambda^2}{(1+\lambda^2\pi|\delta_0|^2)^2} & \text{in } \Sigma \setminus B_{\delta_0} \end{cases} \\ u_2^\lambda &= \begin{cases} -\frac{1}{2} \log \frac{\lambda^2}{(1+\lambda^2\pi|x|^2)^2} & \text{in } B_{\delta_0}, \\ -\frac{1}{2} \log \frac{\lambda^2}{(1+\lambda^2\pi|\delta_0|^2)^2} & \text{in } \Sigma \setminus B_{\delta_0}. \end{cases} \end{aligned}$$

We estimate

$$\begin{aligned} \int |\nabla u_1^\lambda|^2 &= 32\pi \log \lambda + O(1), \\ \int |\nabla u_2^\lambda|^2 &= 8\pi \log \lambda + O(1), \\ \int \nabla u_1^\lambda \cdot \nabla u_2^\lambda &= -16\pi \log \lambda + O(1), \\ \int u_1^\lambda &= -2 \log \lambda + O(1), \\ \int u_2^\lambda &= \log \lambda + O(1), \\ \log \int e^{u_1^\lambda} &= O(1), \\ \log \int e^{u_2^\lambda} &= \log \lambda + O(1), \end{aligned} \tag{4.16}$$

which implies that

$$\Phi_M(u^\lambda) = 2(4\pi - M_1) \log \lambda + O(1).$$

The Proposition follows from the previous formula by letting $\lambda \rightarrow \infty$. \square

5. THE OPTIMAL MOSER-TRUDINGER INEQUALITY

In this section, we prove Theorem 1.3 for $N = 2$. Let $M^0 = (4\pi, 4\pi)$.

Theorem 5.1. *Let Σ be a closed surface with area 1. There is a constant $C > 0$ such that*

$$\begin{aligned}\tilde{\Phi}_{M^0}(v) &= \frac{1}{2} \int_{\Sigma} \{2|\nabla v_1|^2 + 2|\nabla v_2|^2 - 2\nabla v_1 \nabla v_2 + 8\pi v_1 + \\ &\quad 8\pi v_2\} - 4\pi \log \int e^{2v_1 - v_2} - 4\pi \log \int e^{2v_2 - v_1} \\ &\geq -C,\end{aligned}\tag{5.1}$$

for any $v \in H_2$, or equivalently,

$$\begin{aligned}\Phi_{M^0}(u) &= \frac{1}{3} \int_{\Sigma} \{|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \nabla u_2 + 12\pi u_1 + 12\pi u_2\} \\ &\quad - 4\pi \log \int e^{u_1} - 4\pi \log \int e^{u_2} \geq -C,\end{aligned}\tag{5.2}$$

for any $u \in H_2$.

To prove Theorem 5.1, we only have to prove that there is a constant $C > 0$ independent of ϵ such that

$$\Phi_{M^\epsilon}(u) \geq -C, \quad \text{for any } u \in H_2,\tag{5.3}$$

where $M^\epsilon = (4\pi - \epsilon, 4\pi - \epsilon)$ for small constant $\epsilon > 0$.

To show (5.3), first applying Corollary 4.6 we obtain a minimizer $u^\epsilon = (u_1^\epsilon, u_2^\epsilon)$ of Φ_{M^ϵ} with $M^\epsilon = (4\pi - \epsilon, 4\pi - \epsilon)$. Recall that u^ϵ satisfies

$$\begin{cases} -\Delta u_1^\epsilon &= (8\pi - 2\epsilon)e^{u_1^\epsilon} - (4\pi - \epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon) \\ -\Delta u_2^\epsilon &= (8\pi - 2\epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon)e^{u_1^\epsilon} - (4\pi - \epsilon), \end{cases}\tag{5.4}$$

with

$$\int_{\Sigma} e^{u_1^\epsilon} = \int_{\Sigma} e^{u_2^\epsilon} = 1.\tag{5.5}$$

Then, we will show that

$$\Phi_{M^\epsilon}(u^\epsilon) \geq -C,\tag{5.6}$$

for some constant $C > 0$. (5.6) is equivalent to (5.3). If u^ϵ is bounded from above uniformly, by using the analysis developed in Section 3 we can show that u^ϵ converges (by taking subsequences) to u^0 in H_2 strongly. This implies that $\inf_{u \in H_2} \Phi_{M^0}(u) = \Phi_{M^0}(u^0) > -\infty$, and hence (5.6). Hence, we assume that

$$\max\{m_1^\epsilon, m_2^\epsilon\} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.\tag{5.7}$$

where $m_j^\epsilon = \max u_j^\epsilon$. Assume that $m_1^\epsilon \geq m_2^\epsilon$. As in Section 3, we may assume that there exist two nonnegative bounded measures μ_1 and μ_2 such that

$$\int e^{u_j^\epsilon} \psi \rightarrow \int \psi d\mu_j \text{ as } \epsilon \rightarrow 0,$$

for every smooth function $\psi : \Sigma \rightarrow \mathbb{R}$ and $j = 1, 2$. We define, for $j = 1, 2$ and small $\gamma > 0$,

$$S_j = \{x \in \Sigma \mid \text{there is a sequence } y^\epsilon \rightarrow x \text{ such that } u_j^\epsilon(y^\epsilon) \rightarrow +\infty\}$$

and

$$\Omega_j(\gamma) = \{x \in \Omega \mid x \text{ is not a } \gamma\text{-regular point with respect to } \mu_j\}.$$

For the definition of γ -regular point, see Section 3 above. From Section 3, we know that

$$S_j = \Omega_j(\gamma) \quad \text{and} \quad |S_j| < \infty \text{ for } j = 1, 2, \quad (5.8)$$

for any small $\gamma > 0$.

Lemma 5.2. $|S_1| = 1$ and $|S_2| \leq 1$.

Proof. By (5.7), we have $|S_1| \geq 1$. Let $y \in S_1$ and choose $\delta > 0$ so small that $(B_\delta(y) \setminus \{y\}) \cap (S_1 \cup S_2) = \emptyset$. One can use the blow-up argument as in the previous section to show that

$$\lim_{\epsilon \rightarrow 0} \int_{B_\delta(y)} e^{u_1^\epsilon} \geq 1,$$

which implies that $S_1 = \{y\}$ by (5.8), because of (5.4).

Assume by contradiction that $|S_2| \geq 2$. Then there exists $z \in S_2$ but $z \neq y$. Similarly, we can show that there is a small constant $\delta > 0$ satisfying $(B_\delta(z) \setminus \{z\}) \cap (S_1 \cup S_2) = \emptyset$ and

$$\lim_{\epsilon \rightarrow 0} \int_{B_\delta(z)} e^{u_2^\epsilon} \geq 1.$$

It follows, together with (5.5) that

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma \setminus B_\delta(z)} e^{u_2^\epsilon} = 0.$$

By (5.8), we have $S_2 = \{z\}$, a contradiction. \square

Proof of Theorem 5.1. In view of Lemma 5.2, there are three possibilities:

- (a). $|S_2| = 0$.
- (b). $|S_2| = 1$ and $S_1 = S_2$,
- (c). $|S_2| = 1$ and $S_1 \cap S_2 = \emptyset$,

We discuss the sequence u^ϵ case by case. We shall show that case (b) cannot occur, which is crucial to establish our Moser-Trudinger inequality. Case (a) is easy to handle, while case (c) is more delicate.

Case (a). Reduction to the ordinary Moser-Trudinger inequality .

Let $S_1 = \{p\}$. In this case, by Theorem 3.1 and similar arguments given in the proof of Lemma 5.6 in case (c) below, we can show that

- (a1). u_2^ϵ converges to G_2 in $H^{1,q}(\Sigma)$ ($q \in (1, 2)$) and in $C_{loc}^2(\Sigma \setminus \{p\})$, where G_2 satisfies

$$-\Delta G_2 = 8\pi e^{G_2} - 4\pi\delta_p - 4\pi$$

with $\int_\Sigma e^{G_2} = 1$.

- (a2). $u_1^\epsilon - \bar{u}_1^\epsilon$ converges to G_1 in $H^{1,q}(\Sigma)$ ($q \in (1, 2)$) and in $C_{loc}^2(\Sigma \setminus \{p\})$, where G_1 satisfies

$$-\Delta G_1 = 8\pi\delta_p - 4\pi e^{G_1} - 4\pi$$

with $\int_\Sigma G_1 = 0$.

Furthermore, we have the following Lemma.

Lemma 5.3. *For any small $\epsilon > 0$ there exists a function w_ϵ satisfying*

$$-\Delta w_\epsilon = 2(4\pi - \epsilon)e^{h_\epsilon}e^{w_\epsilon} - 2(4\pi - \epsilon) \text{ in } \Sigma \quad (5.9)$$

such that

- (i). $u_1^\epsilon - w_\epsilon$ and h_ϵ are bounded in $C^1(\Sigma)$,
- (ii). $u_2^\epsilon + \frac{1}{2}w_\epsilon - \frac{1}{2}\bar{w}_\epsilon$ is bounded in $C^1(\Sigma)$.

Proof. For small $\epsilon > 0$, let \tilde{w}_ϵ be a function defined by

$$\begin{cases} -\Delta \tilde{w}_\epsilon &= (4\pi - \epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon) \\ \int_\Sigma \tilde{w}_\epsilon &= 0 \end{cases}$$

Since u_2^ϵ is uniformly bounded from above, \tilde{w}_ϵ is bounded in $C^1(\Sigma)$ by elliptic estimates. Let $w_\epsilon = u_1^\epsilon + \tilde{w}_\epsilon$. Clearly, w_ϵ satisfies (5.9) with $h_\epsilon = -\tilde{w}_\epsilon$, which is bounded in $C^1(\Sigma)$. By definition, $u_1^\epsilon - w_\epsilon = \tilde{w}_\epsilon$.

Thus $u_1^\epsilon - w_\epsilon$ is also bounded in $C^1(\Sigma)$. In view of (5.9), the function $u_2^\epsilon + \frac{1}{2}w_\epsilon - 2\tilde{w}_\epsilon$ is harmonic, thus

$$u_2^\epsilon + \frac{1}{2}w_\epsilon - 2\tilde{w}_\epsilon = c_\epsilon,$$

for some constant c_ϵ . By (a1), \bar{u}_2^ϵ is uniformly bounded. Hence, $c_\epsilon - \frac{1}{2}\bar{w}_\epsilon$ is bounded. This proves the Lemma. \square

We now can reduce our problem to the ordinary Moser-Trudinger inequality. By Lemma 5.3, we have

$$\frac{1}{3} \int_{\Sigma} (|\nabla u_1^\epsilon|^2 + |\nabla u_2^\epsilon|^2 + \nabla u_1^\epsilon \nabla u_2^\epsilon) = \frac{1}{4} \int_{\Sigma} |\nabla w_\epsilon|^2 + O(1)$$

and

$$\int_{\Sigma} (u_1^\epsilon + u_2^\epsilon) = \int_S w_\epsilon + O(1).$$

Thus,

$$\Phi_{M^\epsilon}(u^\epsilon) \geq \frac{1}{4} \int_{\Sigma} (|\nabla w_\epsilon|^2 + 4(4\pi - \epsilon)w_\epsilon) + O(1),$$

which has a lower bound due to the ordinary Moser-Trudinger inequality, see e.g. [25] or [15]. This completes the proof of Theorem 5.1 in case (a). \square

Case (b). We show that case (b) does not happen.

Let $S_1 = S_2 = \{p\}$. Wlog, we assume that $B_\delta(p)$ is a flat disk for small $\delta > 0$. By Lemma 3.2, in this case, we have that $\lim_{\epsilon \rightarrow 0} \int_{B_\delta(p)} e^{u_1^\epsilon} = 1$ for any small $\delta > 0$. By the argument in Step 3 in the proof of Theorem 3.1, we have that $u_1^\epsilon \rightarrow -\infty$ (as $\epsilon \rightarrow 0$) on any compact subset of $\Sigma \setminus \{p\}$. We also have either

- (i) $u_2^\epsilon \rightarrow -\infty$ (as $\epsilon \rightarrow 0$) on any compact subset of $\Sigma \setminus \{p\}$, or
- (ii) u_2^ϵ is uniformly bounded on any compact subset of $\Sigma \setminus \{p\}$.

We first consider case (i). In this case, the same argument given in the proof of Lemma 5.6 implies

Lemma 5.4. *For any $q \in (1, 2)$ and $j = 1, 2$*

$$u_j^\epsilon - \bar{u}_j^\epsilon \text{ converges to } G_p \text{ in } H^{1,q}(\Sigma),$$

where G_p satisfies

$$\begin{aligned} -\Delta G_p &= 4\pi\delta_p - 4\pi, \text{ in } \Sigma, \\ \int_{\Sigma} G_p &= 0 \end{aligned} \quad (5.10)$$

Moreover, $u_j^\epsilon - \bar{u}_j^\epsilon$ converges to G_p in $C_{loc}^2(\Sigma \setminus \{p\})$.

By this Lemma, we know that for any small, but fixed number $r > 0$, we have

$$u_j^\epsilon - \bar{u}_j^\epsilon - G_p \rightarrow 0, \text{ in } C^1(\partial B_r(p)), \quad (5.11)$$

as $\epsilon \rightarrow 0$. For any small $\delta > 0$, there exist $\epsilon_0 > 0$ and $r_0 > 0$ such that for any $\epsilon < \epsilon_0$ and $r < r_0$

$$\int_{B_r(p)} e^{u_j^\epsilon} > 1 - \delta. \quad (5.12)$$

As in Proposition 2.8, we have the following Pohozaev identity for (5.1)

$$\begin{aligned} 6(4\pi - \epsilon) \int_{B_r} (e^{u_1} + e^{u_2}) &= 2 \sum_{j=1}^2 \int_{\partial B_r} r (|\frac{\partial u_j^\epsilon}{\partial n}|^2 - \frac{1}{2} |\nabla u_j^\epsilon|^2) \\ &\quad + 2 \int_{\partial B_r} r (\frac{\partial u_1^\epsilon}{\partial n} \frac{\partial u_2^\epsilon}{\partial n} - \frac{1}{2} \nabla u_1^\epsilon \nabla u_2^\epsilon) \\ &\quad + 3(4\pi - \epsilon) \int_{\partial B_r} r (e^{u_1} + e^{u_2}). \end{aligned} \quad (5.13)$$

It is clear that, letting $\epsilon \rightarrow 0$, the left hand side of (5.13) tends to a number $48\pi(1 - \delta)$, while the right hand side of (5.13) tends to

$$\begin{aligned} 4 \int_{\partial B_r} r (|\frac{\partial G}{\partial n}|^2 - \frac{1}{2} |\nabla G|^2) + 2 \int_{\partial B_r} r (|\frac{\partial G}{\partial n}|^2 - \frac{1}{2} |\nabla G|^2) \\ = 3 \int_{\partial B_r} r |\frac{\partial G}{\partial r}|^2, \end{aligned}$$

which tends to 24π as $r \rightarrow 0$, a contradiction. Hence, case (i) does not happen.

Now we consider case (ii), i.e., u_2^ϵ is bounded on any compact subset of $\Sigma \setminus \{p\}$. Let

$$\sigma_2 = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{B_\delta(p)} e^{u_2^\epsilon}.$$

Since $p \in S_2 = \Omega_2(\delta)$ for small $\delta > 0$, we have that $0 < \sigma_2 < 1$. As in the proof of (i), we first have

Lemma 5.5. *There exists a function $w \in C^2(\Sigma)$ with $\int_{\Sigma} e^w = 1 - \sigma_2$ such that*

1. $u_1^\epsilon - \bar{u}_1^\epsilon$ converges to G_1 in $H^{1,q}(\Sigma) \cap C_{loc}^2(\Sigma \setminus \{p\})$, where G_1 satisfies

$$-\Delta G_1 = 4\pi(2 - \sigma_2)\pi\delta_p - 4\pi e^w - 4\pi \text{ and } \int_{\Sigma} G_1 = 0.$$

2. $u_2^\varepsilon - \bar{u}_2^\varepsilon$ converges to $w + G_2$ in $H^{1,q}(\Sigma) \cap C_{loc}^2(\Sigma \setminus \{p\})$, where G_2 satisfies

$$-\Delta G_2 = 8\pi e^w + 4\pi(\sigma_2 - 1)\delta_p - 4\pi \text{ and } \int_{\Sigma} (G_2 + w) = 0.$$

Then, we apply (5.13) again to get a contradiction. In fact, we can show that in this case its left hand tends to $24\pi(1 + \sigma_2)$ while its right hand tends $24\pi(\sigma_2^2 - \sigma_2 + 1)$, which is impossible if $0 < \sigma_2 < 1$. This implies that case (ii), hence case (b), does not happen either.

Such an argument, using a “local” Pohozaev identity, was used in [46] and [50] for studying the blow up of Liouville type equations.

Case (c). This case is more delicate.

Set $S_1 = \{p_1\}$ and $S_2 = \{p_2\}$. Note that $p_1 \neq p_2$. In view of Theorem 3.1 and the blow-up argument given above, u_j^ε ($j = 1, 2$) tends to $-\infty$ uniformly on any compact subset of $\Sigma \setminus \{p_1, p_2\}$. We first show the following lemma.

Lemma 5.6. *Let \bar{u}_j^ε be the average of u_j^ε ($j = 1, 2$). For any $q \in (1, 2)$, we have*

$$u_j^\varepsilon - \bar{u}_j^\varepsilon \text{ converges to } G_j \text{ in } H^{1,q}(\Sigma),$$

where G_1 and G_2 satisfy

$$\begin{cases} -\Delta G_1 &= 8\pi\delta_{p_1} - 4\pi\delta_{p_2} - 4\pi, \\ -\Delta G_2 &= 8\pi\delta_{p_2} - 4\pi\delta_{p_1} - 4\pi, \\ \int_{\Sigma} G_j &= 0, \text{ for } j = 1, 2 \end{cases} \quad (5.14)$$

where δ_y is the Dirac distribution. Moreover,

$$u_j^\varepsilon - \bar{u}_j^\varepsilon \text{ converges to } G_j \text{ in } C_{loc}^2(\Sigma \setminus \{p_1, p_2\}). \quad (5.15)$$

Proof. First we show that for any $q \in (1, 2)$,

$$\int_{\Sigma} (|\nabla u_1^\varepsilon|^q + |\nabla u_2^\varepsilon|^q) \text{ is bounded.} \quad (5.16)$$

Let $q' > 2$ be determined by $\frac{1}{q'} + \frac{1}{q} = 1$. By definition, we know

$$\|\nabla u_1^\varepsilon\|_{L^q} \leq \sup\left\{ \int_{\Sigma} \nabla u_1^\varepsilon \cdot \nabla \phi \mid \phi \in H^{1,q'}(\Sigma), \int_{\Sigma} \phi = 0 \text{ \& } \|\phi\|_{H^{1,q'}} = 1 \right\}. \quad (5.17)$$

The Sobolev embedding theorem implies that for any ϕ with $\|\phi\|_{L^{q'}} = 1$,

$$\|\phi\|_{L^\infty} < c,$$

for some constant $c > 0$. Hence,

$$\begin{aligned} | \int_\Sigma \nabla u_1^\epsilon \cdot \nabla \phi | &= | \int_\Sigma \phi \Delta u_1^\epsilon | \\ &\leq c \int_\Sigma \{ (8\pi - \epsilon) e^{u_1^\epsilon} + (4\pi - \epsilon) e^{u_2^\epsilon} + (4\pi - \epsilon) \} \\ &\leq c_1. \end{aligned} \tag{5.18}$$

It follows that $\|\nabla u_1^\epsilon\|_{L^q} \leq c_1$. Similarly, we have $\|\nabla u_2^\epsilon\|_{L^q} \leq c_2$. This proves (5.16).

By Theorem 3.1 and Remark 3.3, we can show that

$$e^{u_1^\epsilon} \rightarrow \delta_{p_1} \quad \text{and} \quad e^{u_2^\epsilon} \rightarrow \delta_{p_2} \tag{5.19}$$

in the sense of measures as $\epsilon \rightarrow 0$. Like (5.18), we have

$$\begin{aligned} | \int_\Sigma \nabla (u_1^\epsilon - \bar{u}_1^\epsilon - G_1) \nabla \phi | &\leq c \int_\Sigma | (8\pi - 2\epsilon) e^{u_1^\epsilon} - 8\pi \delta_{p_1} | \\ &\quad + c \int_\Sigma | (4\pi - \epsilon) e^{u_1^\epsilon} - 4\pi \delta_{p_2} | + O(\epsilon) \\ &\leq O(\epsilon). \end{aligned} \tag{5.20}$$

Therefore, we have

$$\|\nabla (u_1^\epsilon - \bar{u}_1^\epsilon - G_1)\|_{L^q} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

It follows that $\|u_1^\epsilon - \bar{u}_1^\epsilon - G_1\|_{L^q} \rightarrow 0$ as $\epsilon \rightarrow 0$. Hence, we have $u_1^\epsilon - \bar{u}_1^\epsilon \rightarrow G_1$ in $H^{1,q}(\Sigma)$. Similarly, $u_2^\epsilon - \bar{u}_2^\epsilon \rightarrow G_2$ in $H^{1,q}(\Sigma)$. Now it is easy to show (5.15). This proves the Lemma. \square

Let γ be a smooth closed curve on Σ with the properties that Σ/γ consists of two disjoint component Σ_1 and Σ_2 and $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$. Now we consider our system in Σ_1 first. As above, we set

$$v_1^\epsilon = \frac{1}{3}(2u_1^\epsilon + u_2^\epsilon) \text{ and } v_2^\epsilon = \frac{1}{3}(2u_2^\epsilon + u_1^\epsilon).$$

Clearly, $(v_1^\epsilon, v_2^\epsilon)$ satisfies

$$\begin{cases} -\Delta v_1^\epsilon &= (4\pi - \epsilon) e^{u_1^\epsilon} - (4\pi - \epsilon), \\ -\Delta v_2^\epsilon &= (4\pi - \epsilon) e^{u_2^\epsilon} - (4\pi - \epsilon). \end{cases} \tag{5.21}$$

Lemma 5.7. $v_2^\epsilon - \frac{1}{2}(2\bar{u}_2^\epsilon + \bar{u}_1^\epsilon)$ is bounded in $C^1(\Sigma_1)$.

Proof. Define a function \tilde{v}_2^ϵ satisfying

$$\begin{cases} \tilde{v}_2^\epsilon &= (4\pi - \epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon), & \text{in } \Sigma_1 \\ \tilde{v}_2^\epsilon &= 0, & \text{on } \gamma. \end{cases}$$

Since u_2^ϵ is bounded from above in Σ_1 , \tilde{v}_2^ϵ is bounded in $C^1(\Sigma_1)$. Now it is easy to see that $v_2^\epsilon - \tilde{v}_2^\epsilon - \frac{1}{3}(2\bar{u}_2^\epsilon + \bar{u}_1^\epsilon)$ is also bounded in $C^1(\Sigma_1)$. Hence the Lemma follows. \square

From Lemma 5.7, we have

$$\begin{aligned} & \frac{1}{3} \int_{\Sigma_1} (|\nabla u_1^\epsilon|^2 + |\nabla u_2^\epsilon|^2 + \nabla u_1^\epsilon \cdot \nabla u_2^\epsilon) + (4\pi - \epsilon) \int_{\Sigma_1} (u_1^\epsilon + u_2^\epsilon) \\ &= \frac{1}{4} \int_{\Sigma_1} |\nabla u_1^\epsilon|^2 + \frac{1}{2}(4\pi - \epsilon) \int_{\Sigma_1} u_1^\epsilon + \frac{1}{2}(4\pi - \epsilon)(2\bar{u}_2^\epsilon + \bar{u}_1^\epsilon)|\Sigma_1| + O(1) \\ &= \frac{1}{4} \int_{\Sigma_1} |\nabla u_1^\epsilon|^2 + (4\pi - \epsilon)(\bar{u}_2^\epsilon + \bar{u}_1^\epsilon)|\Sigma_1| + O(1), \end{aligned} \quad (5.22)$$

where we have used the fact that $\int_{\Sigma_1} u_1^\epsilon - \bar{u}_1^\epsilon$ is bounded, which was implied by Lemma 5.3. Here $|\Sigma_1|$ is the area of Σ_1 . Similarly, we can get

$$\begin{aligned} & \frac{1}{3} \int_{\Sigma_2} (|\nabla u_1^\epsilon|^2 + |\nabla u_2^\epsilon|^2 + \nabla u_2^\epsilon \cdot \nabla u_1^\epsilon) + (4\pi - \epsilon) \int_{\Sigma_2} (u_1^\epsilon + u_2^\epsilon) \\ &= \frac{1}{4} \int_{\Sigma_2} |\nabla u_2^\epsilon|^2 + (4\pi - \epsilon)(\bar{u}_2^\epsilon + \bar{u}_1^\epsilon)|\Sigma_2| + O(1). \end{aligned} \quad (5.23)$$

Hence, to prove Theorem 5.1 in this case, we only need to show the following

Lemma 5.8. *There exists a constant $C > 0$ independent of ϵ such that*

$$\frac{1}{4} \int_{\Sigma_1} |\nabla u_1^\epsilon|^2 + \frac{1}{4} \int_{\Sigma_2} |\nabla u_2^\epsilon|^2 + (4\pi - \epsilon)(\bar{u}_1^\epsilon + \bar{u}_2^\epsilon) > -C.$$

Unlike case (a), we cannot use the ordinary Moser-Trudinger inequality directly. Fortunately, the ideas in the proof of the ordinary Moser-Trudinger inequality in [38, 15] can be applied to show Lemma 5.8. We claim that there is a constant $C > 0$ independent of ϵ such that

$$\frac{1}{4} \int_{\Sigma_1} |\nabla u_1^\epsilon|^2 + (4\pi - \epsilon)\bar{u}_1^\epsilon > -C. \quad (5.24)$$

Let x_j^ϵ be one of the maximum points of u_j^ϵ , i.e., $m_j^\epsilon = u_j^\epsilon(x_j^\epsilon)$ ($j = 1, 2$). Set

$$\tilde{u}_1^\epsilon(x) = u_1^\epsilon(\lambda_1^\epsilon x + x_1^\epsilon) - m_1^\epsilon \quad \text{and} \quad \tilde{u}_2^\epsilon(x) = u_2^\epsilon(\lambda_1^\epsilon x + x_1^\epsilon) - m_1^\epsilon,$$

where $(\lambda_1^\epsilon)^2 = e^{-m_1^\epsilon}$. Since $m_1^\epsilon \rightarrow \infty$, $\lambda_1^\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$. From the discussion above, we know that $\tilde{u}_1^\epsilon(0) = \max \tilde{u}_1^\epsilon = 0$ and $\tilde{u}_2^\epsilon \rightarrow -\infty$ uniformly on any compact domain of Σ_1 .

Lemma 5.9. *We have that \tilde{u}_1^ϵ converges to ϕ_0 in $H_{loc}^1(\mathbb{R}^2)$, where $\phi_0 = -2 \log(1 + \pi|x|^2)$ is a solution of the Liouville equation*

$$-\Delta \phi = 8\pi e^\phi.$$

Proof. Note that $\tilde{u}^\epsilon = (\tilde{u}_1^\epsilon, \tilde{u}_2^\epsilon)$ satisfies

$$\begin{cases} -\Delta \tilde{u}_1^\epsilon &= (8\pi - 2\epsilon)e^{\tilde{u}_1^\epsilon} - (4\pi - \epsilon)e^{\tilde{u}_2^\epsilon} - \lambda_1^\epsilon(4\pi - \epsilon), & \text{in } (\lambda_1^\epsilon)^{-1}B_\delta(x_1^\epsilon) \\ -\Delta \tilde{u}_2^\epsilon &= (8\pi - 2\epsilon)e^{\tilde{u}_2^\epsilon} - (4\pi - \epsilon)e^{\tilde{u}_1^\epsilon} - \lambda_1^\epsilon(4\pi - \epsilon). & \text{in } (\lambda_1^\epsilon)^{-1}B_\delta(x_2^\epsilon) \end{cases} \quad (5.25)$$

Clearly, $(\lambda_1^\epsilon)^{-1}B_\delta(x_1^\epsilon) \rightarrow \mathbb{R}^2$ as $\epsilon \rightarrow 0$. For any large, but fixed constant $R > 0$, we consider \tilde{u}^ϵ on $B_R(0)$. Define ξ_R^ϵ by

$$\begin{cases} -\Delta \xi_R^\epsilon &= -(4\pi - \epsilon)e^{\tilde{u}_2^\epsilon} - (\lambda_1^\epsilon)^2(4\pi - \epsilon) := f_\epsilon & \text{in } B_R(0), \\ \xi_R^\epsilon &= 0, & \text{on } \partial B_R(0). \end{cases}$$

The elliptic estimate implies that $\xi_R^\epsilon \rightarrow 0$ in $L^\infty(B_R(0))$. Set $\tilde{w}_1^\epsilon = \tilde{u}_1^\epsilon - \xi_R^\epsilon$. It is clear that

$$-\Delta \tilde{w}_1^\epsilon = (8\pi - 2\epsilon)e^{\xi_R^\epsilon} e^{\tilde{w}_1^\epsilon}. \quad (5.26)$$

Since $\tilde{u}_1^\epsilon \leq 0$, $\tilde{u}_1^\epsilon(0) = 0$ and ξ_R^ϵ is bounded, \tilde{w}_1^ϵ is bounded from above and $|\tilde{w}_1^\epsilon(0)|$ is bounded. It is easy to show that \tilde{w}_1^ϵ , hence \tilde{u}_1^ϵ , converges in $H^{1,2}(B_R)$. (See, for example, [3].) Now by a diagonal argument, we have that \tilde{u}_1^ϵ converges to ϕ_0 in $H_{loc}^{1,2}(\mathbb{R}^2)$, where ϕ_0 satisfies

$$-\Delta \phi_0 = 8\pi e^{\phi_0}$$

with $\phi_0(0) = 0 = \max \phi_0$. A classification of Chen-Li [12] implies that

$$\phi_0(x) = -2 \log(1 + \pi|x|^2).$$

□

Now we need the following

Lemma 5.10. 1. *For any small $\sigma > 0$, there exist constants $R_\sigma > 0$, $\epsilon_\sigma > 0$ and C_σ such that*

$$u_1^\epsilon(x) \leq -(1 - 3\sigma)m_1^\epsilon - \left(4 - \frac{\epsilon}{\pi}\right)(1 - \sigma) \log|x - x_1^\epsilon| + C_\sigma,$$

for any $x \in B_\delta(x_1^\epsilon) \setminus B_{r_\epsilon}(x_1^\epsilon)$ with $r_\epsilon = (\lambda_1^\epsilon)^{-1} R_\sigma$ and $\epsilon < \epsilon_\sigma$.

$$2. m_1^\epsilon + \bar{u}_1^\epsilon \geq O(1).$$

Now from Lemma 5.10, we can finish the proof of our main theorem. From (5.4), we have

$$\begin{aligned} \int_{\Sigma_1} |\nabla u_1^\epsilon|^2 &= \int_{\partial\Sigma_1} u_1^\epsilon \frac{\partial u_1^\epsilon}{\partial n} + 2(4\pi - \epsilon) \int_{\Sigma_1} e^{u_1^\epsilon} u_1^\epsilon \\ &\quad - (4\pi - \epsilon) \int_{\Sigma_1} e^{u_2^\epsilon} u_1^\epsilon - (4\pi - \epsilon) \int_{\Sigma_1} u_1^\epsilon \\ &= \int_{\partial\Sigma_1} u_1^\epsilon \frac{\partial u_1^\epsilon}{\partial n} + 2(4\pi - \epsilon) \int_{\Sigma_1} e^{u_1^\epsilon} m_1^\epsilon - (4\pi - \epsilon) \int_{\Sigma_1} e^{u_2^\epsilon} u_1^\epsilon \\ &\quad + 2(4\pi - \epsilon) \int_{\Sigma_1} e^{u_1^\epsilon} (u_1^\epsilon - m_1^\epsilon) - (4\pi - \epsilon) \int_{\Sigma_1} u_1^\epsilon, \end{aligned}$$

where n is the outer normal of Σ_1 . By Lemma 5.6 and equation (5.4), we have

$$\begin{aligned} \int_{\partial\Sigma_1} u_1^\epsilon \frac{\partial u_1^\epsilon}{\partial n} &= \int_{\partial\Sigma_1} \frac{\partial u_1^\epsilon}{\partial n} \bar{u}_1^\epsilon + \int_{\partial\Sigma_1} \frac{\partial u_1^\epsilon}{\partial n} (u_1^\epsilon - \bar{u}_1^\epsilon) \\ &= \int_{\partial\Sigma_1} \frac{\partial u_1^\epsilon}{\partial n} \bar{u}_1^\epsilon + O(1) \\ &= -\bar{u}_1^\epsilon \{2(4\pi - \epsilon) \int_{\Sigma_1} e^{u_1^\epsilon} - (4\pi - \epsilon) \int_{\Sigma_1} e^{u_2^\epsilon} - (4\pi - \epsilon) |\Sigma_1|\} + O(1). \end{aligned}$$

By Lemmas 5.9 and 5.10, we have

$$\begin{aligned} \int_{\Sigma_1} e^{u_1^\epsilon} (u_1^\epsilon - m_1^\epsilon) &= \int_{\Sigma_1 \setminus B_\delta(x_1^\epsilon)} e^{u_1^\epsilon} (u_1^\epsilon - m_1^\epsilon) + \int_{B_\delta(x_1^\epsilon)} e^{u_1^\epsilon} (u_1^\epsilon - m_1^\epsilon) \\ &= \int_{B_\delta(x_1^\epsilon)} e^{u_1^\epsilon} (u_1^\epsilon - m_1^\epsilon) + O(1) \\ &= \int_{B_{(\lambda_1^\epsilon)^{-1}\delta}(0)} e^{\tilde{u}_1^\epsilon} \tilde{u}_1^\epsilon + O(1) \\ &\leq \int_{B_{(\lambda_1^\epsilon)^{-1}\delta}(0)} C_{\sigma_1} e^{(1-\sigma_1)\tilde{u}_1^\epsilon} + O(1) = O(1), \end{aligned}$$

for a small fixed number $\sigma_1 > 0$. Here, we have used (5.27) below.

Hence, using Lemma 5.10 we get

$$\begin{aligned} \int_{\Sigma_1} |\nabla u_1^\epsilon|^2 &= 2(4\pi - \epsilon)(m_1^\epsilon - \bar{u}_1^\epsilon) \int_{\Sigma_1} e^{u_1^\epsilon} - (4\pi - \epsilon) \left(\int_{\Sigma_1} u_1^\epsilon - |\Sigma| \bar{u}_1^\epsilon \right) + O(1) \\ &\geq 4(4\pi - \epsilon) \bar{u}_1^\epsilon \int_{\Sigma_1} e^{u_1^\epsilon} + O(1) \\ &\geq 4(4\pi - \epsilon) \bar{u}_1^\epsilon + O(1). \end{aligned}$$

This is (5.24). Here we have used the fact that

$$m_1^\epsilon \int_{\Sigma \setminus B_\delta(x_1^\epsilon)} e^{u_1^\epsilon} \rightarrow o(1), \quad (5.27)$$

which can be deduced as follows. From (5.4), we have

$$-\Delta u_1^\epsilon \leq (8\pi - 2\epsilon)e^{u_1^\epsilon} \quad \text{in } \Sigma \setminus B_\delta(x_1^\epsilon).$$

Let h_ϵ satisfy

$$\begin{cases} -\Delta h_\epsilon &= (8\pi - 2\epsilon)e^{u_1^\epsilon}, & \text{in } \Sigma \setminus B_\delta(x_1^\epsilon), \\ h_\epsilon &= 0, & \text{on } \partial B_\delta(x_1^\epsilon). \end{cases}$$

Since u_1^ϵ is uniformly bounded from above in $\Sigma \setminus B_\delta(x_1^\epsilon)$, h_ϵ is uniformly bounded. Now applying the maximum principle to $u_1^\epsilon - h_\epsilon$, together with Lemma 5.10, we can obtain that $u_1^\epsilon(x) \leq -(1 - 3\sigma)m_1^\epsilon + C$ for any $x \notin B_\delta(x_1^\epsilon)$, which implies (5.27).

Similarly, we can show that

$$\frac{1}{4} \int_{\Sigma_2} |\nabla u_2^\epsilon|^2 + 4(\pi - \epsilon)\bar{u}_2^\epsilon > -C,$$

for some constant $C > 0$, which, together with (5.24), proves Lemma 5.8, hence Theorem 5.1.

Now it remains to prove Lemma 5.10.

Proof of Lemma 5.10. Wlog, we assume that $B_{2\delta}(p_1)$ is a flat disk, for some small constant $\delta > 0$, see Remark at the end of Section 3. Consider in $B_{2\delta}(x_1^\epsilon)$

$$-\Delta u_1^\epsilon = 2(4\pi - \epsilon)e^{u_1^\epsilon} - (4\pi - \epsilon)e^{u_2^\epsilon} - (4\pi - \epsilon) =: f_\epsilon. \quad (5.28)$$

Recall that $\tilde{u}_j^\epsilon = u_j^\epsilon(\lambda_1^\epsilon x + x_1^\epsilon) - m_1^\epsilon$ (for $j = 1, 2$). For any small $\sigma > 0$, by Lemma 5.9 we choose $R_\sigma > 0$ and $\epsilon_\sigma > 0$ such that

$$\int_{B_{R_\sigma}(0)} e^{\tilde{u}_1^\epsilon} > 1 - \frac{\sigma}{2} \quad \text{and} \quad \int_{\Sigma_1} e^{u_2^\epsilon} < \frac{\sigma}{2}.$$

Let $\Gamma(x, y) = \frac{1}{2\pi} \log|x - y|$. Using Green's representation, we have

$$\begin{aligned} u_1^\epsilon(y) &= - \int_{B_{2\delta}(x_1^\epsilon)} \Gamma(x - y) f_\epsilon(x) dx \\ &\quad + \int_{\partial B_{2\delta}(x_1^\epsilon)} \left(u_1^\epsilon(x) \frac{\partial \Gamma}{\partial n}(x - y) - \Gamma(x - y) \frac{\partial u_1^\epsilon}{\partial n}(x) \right) ds. \end{aligned} \quad (5.29)$$

Since $\delta > 0$ is fixed, it is easy to check that

$$\int_{\partial B_{2\delta}(x_1^\epsilon)} \Gamma(x-y) \frac{\partial u_1^\epsilon}{\partial n} = O(1)$$

and

$$\int_{\partial B_{2\delta}(x_1^\epsilon)} u_1^\epsilon(x) \frac{\partial \Gamma(x-y)}{\partial n} = \bar{u}_1^\epsilon + O(1).$$

Hence, we have

$$\begin{aligned} u_1^\epsilon(y) - m_1^\epsilon &= u_1^\epsilon(y) - u_1^\epsilon(x_1^\epsilon) \\ &= -\frac{1}{2\pi} \int_{B_{2\delta}(x_1^\epsilon)} \log \frac{|x-y|}{|x-x_1^\epsilon|} f_\epsilon dx + O(1). \end{aligned}$$

Now it is convenient to write the previous equation as follows.

$$\begin{aligned} \tilde{u}_1^\epsilon(y) &= u_1^\epsilon(\lambda_1^\epsilon y + x_1^\epsilon) - m_1^\epsilon \\ &= -\frac{1}{2\pi} \int_{B_{2\delta}(x_1^\epsilon)} \log \frac{|x - (\lambda_1^\epsilon y + x_1^\epsilon)|}{|x - x_1^\epsilon|} f(\lambda_1^\epsilon y + x_1^\epsilon) dx \\ &= -\frac{1}{2\pi} \int_{B_{2\delta(\lambda_1^\epsilon)^{-1}}(0)} \log \frac{|x-y|}{|x|} \tilde{f}_\epsilon(x) dx, \end{aligned}$$

where

$$\tilde{f}_1^\epsilon(x) = 2(4\pi - \epsilon)e^{\tilde{u}_1^\epsilon} - (4\pi - \epsilon)e^{\tilde{u}_1^\epsilon} - (\lambda_1^\epsilon)^2(4\pi - \epsilon) = (\lambda_1^\epsilon)^2 f(\lambda_1^\epsilon x + x_1^\epsilon).$$

Applying the potential analysis, it is easy to show that there is a constant $C_\sigma > 0$ such that

$$\tilde{u}_1^\epsilon(y) \leq -(1-\sigma)\left(4 - \frac{\epsilon}{\pi}\right) \log |y| + C_\sigma,$$

for $|y| \geq R_\sigma$. See, for instance, Lemma 2.4 or [38]. This implies statement 1 of the Lemma.

From (5.29) we have

$$\begin{aligned} m_1^\epsilon = u_1^\epsilon(x_1^\epsilon) &= -\frac{1}{2\pi} \int_{B_{2\delta}(x_1^\epsilon)} \log |x - x_1^\epsilon| f_\lambda(x) dx + \bar{u}_1^\epsilon + O(1) \\ &= -\frac{1}{2\pi} \int_{B_{2\delta(\lambda_1^\epsilon)^{-1}}(0)} \log |\lambda_1^\epsilon x| \tilde{f}_\epsilon(x) dx + \bar{u}_1^\epsilon + O(1) \\ &= -\frac{1}{2\pi} \log \lambda_1^\epsilon \int_{B_{2\delta}(x_1^\epsilon)} f_\epsilon(x) dx + \bar{u}_1^\epsilon + O(1) \\ &= \frac{m_1^\epsilon}{4\pi} \int_{B_{2\delta}(x_1^\epsilon)} f_\epsilon(x) dx + \bar{u}_1^\epsilon + O(1) \\ &\leq 2m_1^\epsilon + \bar{u}_1^\epsilon + O(1). \end{aligned}$$

which implies Statement 2. Here we have used

$$\int_{B_{2\delta}(\lambda_1^{-1}(0))} \log|x|\tilde{f}_\epsilon(x)dx \leq C,$$

for some constant C , which is deduced from Statement 1. Hence, we finish the proof of the Lemma, hence our main theorem. \square

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