Hyphs and the Ashtekar-Lewandowski measure

by

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Abstract

Properties of the space $\mathcal{A}$ of generalized connections in the Ashtekar framework are investigated.

First a construction method for new connections is given. The new parallel transports differ from the original ones only along paths that pass an initial segment of a fixed path. This is closely related to a new notion of path independence. Although we do not restrict ourselves to the immersive smooth or analytical case, any finite set of paths depends on a finite set of independent paths, a so-called hyph. This generalizes the well-known directedness of the set of smooth webs and that of analytical graphs, respectively.

Due to these propositions, on the one hand, the projections from $\mathcal{A}$ to the lattice gauge theory are surjective and open. On the other hand, an induced Haar measure can be defined for every compact structure group irrespective of the used smoothness category for the paths.
1 Introduction

One of the recent approaches to the quantization of gauge theories, in particular of gravity, is the investigation of generalized connections introduced by Ashtekar et al. in a series of papers, see, e.g., [1, 2, 3]. Mathematically, there are two main ideas: First, every classical (i.e. smooth) connection is uniquely determined by its parallel transports. These are certain elements of the structure group that are in a certain sense smoothly assigned to each path in the (space-time) manifold and that respect the concatenation of paths. Second, quantization here means path integral quantization. Thus, forget – as suggested by the Wiener or Feynman path integral – the smoothness of the connections being the configuration variables. Altogether, a generalized connection is simply defined to be a homomorphism from the groupoid of paths to the structure group.

At first glance this definition seems to be very rigid. But, is there a canonical choice for the groupoid \( P \) of paths? Do we want to restrict ourselves to piecewise analytic or immersed smooth paths? When shall two paths be equivalent? There are lots of ”optimal” choices depending on the concrete problem being under consideration. For instance, for technical reasons piecewise analyticity is beautiful. In this case it is, in particular, impossible that two paths (maps from \([0, 1]\) to the manifold \( M \)) have infinitely many intersection points provided they do not coincide along a whole interval. However, since one of the most important features of gravity is the diffeomorphism invariance, one should admit at least smooth paths. Otherwise, a diffeomorphism will no longer be a map in \( P \). On the other hand, paths that are equal up to the parametrization, i.e. up to a map between their domains \([0, 1]\), should be equivalent. But, which maps from \([0, 1]\) onto itself are reparametrizations? As well, \( \gamma \circ \gamma^{-1} \) are to be equal to the trivial path in the initial point of the path \( \gamma \). This is suggested by the homomorphism property \( h_A(\gamma \circ \gamma^{-1}) = h_A(\gamma)h_A(\gamma)^{-1} = e_G \) of the parallel transports. What are the other purely algebraic relations that \( h_A \) has to fulfill?

As just indicated, two different definitions are on the market for a couple of years. Originally, Ashtekar and Lewandowski had used the piecewise analyticity [2], and later on, Baez and Sawin [5] extended their results to the smooth category. Recently, in a preceding paper [6] we considered a more general case. At the beginning, we only fixed the smoothness category \( C^r, r \in \mathbb{N}^{+} \cup \{\infty\} \cup \{\omega\} \), and decided whether we consider only piecewise immersed paths or not. Furthermore, we proposed two definitions for the equivalence of paths. The first one was – in a certain sense – the minimal one: it identifies \( \gamma \circ \gamma^{-1} \) with the trivial path as well as reparametrized paths. The second one identifies in the immersive case paths that are equal when parametrized w.r.t. the arc length. The main goal of our paper is a preliminary discussion which results are insensitive to the chosen smoothness conditions and which are not.

Foremost, can an induced Haar measure be defined on the space \( A \) of generalized connections in the general case? It is well-known that this is indeed possible in the analytic case using graphs [2] and in the smooth case using webs [3]. What common ideas of these cases can be reused for our problem? Looking at the definition \( A_{(r=\omega)} := \lim \sup \mathcal{A}_r \) and \( \mathcal{A}_{\text{Web}} := \lim \inf \mathcal{A}_w \) we see that the label sets \( \{T\} \) and \( \{w\} \) of the projective limit are in both cases not only projective systems, but also directed systems. This means that, e.g., for every two graphs there is a third graph such that every path in one of the first two graphs is a product of paths (or their inverses) in the third graph. The analogous result holds for the webs. In the analytical case this can be seen very easily [2], for the smooth one we refer to the paper [5] by Baez and
Sawin. In [6] we defined \( \mathcal{A} \) in general by \( \mathcal{A}(r) := \lim_{r \to \infty} \mathcal{A}_r \) whereas, of course, here the graphs are in the smoothness category \( C^r \). This definition has the drawback that the projective label set \( \{\Gamma\} \) is no longer directed. But, nevertheless, note that we have shown [6] in the immersive smooth category that \( \lim_{w} \mathcal{A}_w \) and \( \mathcal{A}_{(\infty)} = \lim_{r} \mathcal{A}_r \) are homeomorphic. Hence that we can hope to find another appropriate label set for the case of arbitrary smoothness that generalizes the notion of webs and that gives a definition of the space of generalized connections which is equivalent to that using graphs.

In the first step we will investigate a condition for the independence of paths. When can one assign parallel transports to paths independently? As we will see, a finite set \( \{\gamma_i\} \) of paths is already independent when every path \( \gamma_i \) contains a point \( v_i \) such that one of the subpaths of \( \gamma_i \) starting in \( v_i \) is non-equivalent to every subpath of the \( \gamma_j \) with \( j < i \). Sets of paths fulfilling this condition will be called hyph. Obviously, the edges of a graph are a hyph as well as the curves of a web. The crucial point is now: For every two hyphs there is a hyph containing them. In other words, the set of hyphs is directed as the set of graphs \( (r = \omega) \) and that of webs \( (r = \infty) \). This ensures the existence of an induced Haar measure in \( \mathcal{A}(r) \) for arbitrary \( r \). Moreover, as a by-product we get an explicit construction for connections that differ from a given one only along paths that are not independent of an arbitrary, but fixed path. This immediately leads to the surjectivity of the projections \( \pi_\Gamma \) from the continuum to the lattice theory as well as that of \( \pi_w \) and \( \pi_v \) projecting to the webs and hyphs, respectively. Furthermore, we prove that \( \pi_\Gamma \) is open. In Section 6 we extend the definition of the Ashtekar-Lewandowski measure to arbitrary smoothness categories. Finally, we discuss in which cases the regular connections form a dense subset in \( \mathcal{A}(r) \).

## 2 Notations

In this section we shall recall the basic definitions and notations introduced in [6]. For further, detailed information we refer the reader to that article.

Let there be given a finite-, but at least two-dimensional manifold \( M \) and a (not necessarily compact) Lie group \( G \). Furthermore we fix an \( r \in \mathbb{N}^+ \cup \{\infty\} \cup \{\omega\} \) and decide whether we work in the category of piecewise immersive maps or not. In the following we will usually say simply \( C^r \) referring to these choices.

A path is a piecewise \( C^r \)-map from \([0, 1]\) to the manifold \( M \). A graph consists of finitely many non-self-intersecting edges whose interiors are disjoint and contain no vertex. Paths in graphs are called simple, and finite products of simple paths are called simple paths. Two finite paths are equivalent if they coincide up to piecewise \( C^r \)-reparametrizations or cancelling or inserting retracings \( \delta \circ \delta^{-1} \). The set of (equivalence classes of) finite paths is denoted by \( \mathcal{P} \).

In what follows, we say simply ”path” instead of ”finite path” and simply ”graph” instead of ”connected graph”.

A generalized connection \( \overline{A} \in \mathcal{A} \) is a homomorphism \( h_\overline{A} : \mathcal{P} \to G \). For every graph with edges \( e_i \in E(\Gamma) \) and vertices \( v_j \in V(\Gamma) \) define the projections

\[
\pi_\Gamma : \mathcal{A} \to \mathcal{A}_\Gamma \equiv G^{#E(\Gamma)}
\]

\[
\overline{A} \mapsto (h_{\overline{A}}(e_1), \ldots, h_{\overline{A}}(e_{#E(\Gamma)}))
\]

to the lattice gauge theory. The topology on \( \mathcal{A} \) is induced using all the \( \pi_\Gamma \) by the topology of each \( G^{#E(\Gamma)} \).
3 A Construction Method for New Connections

Note that in this section we mean by "path" usually not an equivalence class of paths, but a "genuine" path.

The main goal of this section is to provide a method for constructing a connection $\overline{A}$ that only minimally, but significantly differs from a given $\overline{A}$. In detail, we want to define a new connection whose parallel transport along a given path $e$ takes a given group element $g$, but has the same parallel transports as the older one along the other paths. However, this is obviously impossible, because the parallel transports have to obey the homomorphism rule.

How can we find the way out? The idea goes as follows: The only condition a connection has to fulfill as a map from $P$ to $G$ is indeed the homomorphism property. Therefore it should be possible to leave the parallel transports at least along those paths untouched that do not pass any subpath of our given path $e$. Since the generalized connections need not fulfill any continuity condition it does not matter "where" in $e$ the modification should be placed, e.g., whether in the first half or the second or perhaps in the initial point. Since we are looking for minimal variation we try to place the modification into one single point, say, the initial point $e(0)$. This way all paths that do not pass $e(0)$ can keep their parallel transports. This is even true for those paths that though start (or end) in the point $e(0)$, but start (or end) in "another direction" as $e(0)$ does. Hence, we are now left with those paths that pass an initial path of $e$. There we really have to change the parallel transports -- we simply multiply the corresponding factor that changes $\overline{h}(e)$ to $g$ from the left (or its inverse from the right) to the transport of every path that starts (inversely) as $e$. Using a certain decomposition of an arbitrary path we get the desired construction method.

3.1 Hyphs

Before we state and prove the theorem we still need two crucial definitions and a decomposition lemma.

**Definition 3.1** Let $\gamma_1, \gamma_2 \in P$.

We say that $\gamma_1$ and $\gamma_2$ have the same initial segment (shortly: $\gamma_1 \uparrow \gamma_2$) iff there are non-trivial initial paths $\gamma_1'$ and $\gamma_2'$ of $\gamma_1$ and $\gamma_2$, respectively, that coincide up to the parametrization.

We say analogously that the final segment of $\gamma_1$ coincides with the initial segment of $\gamma_2$ (shortly: $\gamma_1 \downarrow \gamma_2$) iff $\gamma_1^{-1} \uparrow \gamma_2$. The definition of $\gamma_1 \uparrow \gamma_2$ and $\gamma_1 \downarrow \gamma_2$ should now be clear.

If the corresponding relations are not fulfilled, we write $\gamma_1 \not\uparrow \gamma_2$ etc.

**Definition 3.2** Let $\gamma$ and $\delta_i, i \in I$, be a paths without self-intersections. $\gamma$ is called independent of $D := \{\delta_i \mid i \in I\}$ if

- there is a $\tau \in [0,1)$ with $\gamma^\tau, + \leftrightarrow_\delta \delta_i(\tau), +$ and $\gamma^\tau, \leftrightarrow_\delta \delta_i(\tau), -$ for all $i \in I$ or
- there is a $\tau \in (0,1]$ with $\gamma^\tau, \leftrightarrow_\delta \delta_i(\tau), +$ and $\gamma^\tau, \leftrightarrow_\delta \delta_i(\tau), -$ for all $i \in I$ holds.\(^1\) The point $\gamma(\tau)$ is then usually called free point of $\gamma$.

\(^1\) $\gamma^\tau, +$ is the subpath of $\gamma$ that corresponds to $\gamma \mid_{[\tau,1]}$; $\gamma^\tau, -$ that for $\gamma \mid_{[0,\tau]}$. Analogously, $\delta^\tau, +$ is the subpath of $\delta$ starting in $x$ supposed $x \in \text{im} \delta$. (See also [6].) If $\gamma(\tau)$ should not be contained in $\text{im} \delta$ then the corresponding relation $\gamma^\tau, \leftrightarrow_\delta \delta_i(\tau), +$ etc. is defined to be fulfilled.
A finite set \( D = \{ \delta_i \} \) of paths without self-intersections is called **lymph** or **moderately independent** iff \( \delta_i \) is independent of \( D_i = \{ \delta_j \mid j < i \} \).

**Lemma 3.1** Let \( \gamma \in \mathcal{P} \) and \( x \in M \). Then \( \gamma^{-1}(\{ x \}) \) is a union of at most finitely many isolated points and finitely many closed intervals in \([0, 1]\).

**Proof** Let \( \gamma \) be (up to the parametrization) equal \( \prod \gamma_i \) with simple \( \gamma'_i \in \mathcal{P} \). Since any \( \gamma'_i \) equals (up to the parametrization) a finite product of edges in graphs and of trivial paths, this is also true for \( \gamma \) itself. Obviously, we can even assume w.l.o.g. that \( \gamma = \prod \gamma_i \) with \( \gamma_i \) being edges in graphs or trivial paths. (Thus, the manner of writing brackets in \( \prod \gamma_i \) does not matter.) The assertion of the lemma is obviously true for any \( \gamma_i \) because an edge in a graph has just been defined as non-self-intersecting and \( \gamma^{-1}_i(\{ x \}) \) is in the case of a trivial path either equal \( \emptyset \) or \([0, 1]\). The case of a general \( \gamma \) is now clear. \( \text{qed} \)

**Corollary 3.2** Let \( x \in M \) be a point. Any \( \gamma \in \mathcal{P} \) can be written (up to parametrization) as a product \( \prod \gamma_i \) with \( \gamma_i \in \mathcal{P} \), such that
- \( \text{int} \ \gamma_i \cap \{ x \} = \emptyset \) or
- \( \text{int} \ \gamma_i = \{ x \} \).

**Proof** Mark on \([0, 1]\) the end points of the closed intervals and the isolated points of \( \gamma^{-1}(\{ x \}) \) outside these intervals. We get finitely many intervals on \([0, 1]\). Each one corresponds to a subpath \( \gamma_i \) of \( \gamma \). Obviously, \( \prod \gamma_i \) is the desired decomposition of \( \gamma \). \( \text{qed} \)

### 3.2 The Construction

How we can state the construction method.

**Construction 3.3** Let \( \overline{A} \in \mathcal{A} \) and \( e \in \mathcal{P} \) be a path without self-intersections. Furthermore, let \( g \in G \).

We now define \( h : \mathcal{P} \rightarrow G \).
- Let \( \gamma \in \mathcal{P} \) be for the moment a path that does not contain the initial point \( e(0) \) of \( e \) as an inner point. Explicitly we have \( \text{int} \ \gamma \cap \{ e(0) \} = \emptyset \).
  - Define
  \[
  h(\gamma) := \begin{cases} 
  g \ h_{\overline{A}}(e)^{-1} \ h_{\overline{A}}(\gamma) \ h_{\overline{A}}(e) \ g^{-1}, & \text{for } \gamma \uparrow \downarrow e \text{ and } \gamma \downarrow \uparrow e \\
  g \ h_{\overline{A}}(e)^{-1} \ h_{\overline{A}}(\gamma), & \text{for } \gamma \uparrow \uparrow e \text{ and } \gamma \downarrow \downarrow e \\
  h_{\overline{A}}(\gamma) \ h_{\overline{A}}(e) \ g^{-1}, & \text{for } \gamma \uparrow \downarrow e \text{ and } \gamma \downarrow \uparrow e \\
  h_{\overline{A}}(\gamma), & \text{else}
  \end{cases}
  \]
- For every trivial path \( \gamma \) set \( h(\gamma) = e_G \).
- Now, let \( \gamma \in \mathcal{P} \) be an arbitrary path. Decompose \( \gamma \) into a finite product \( \prod \gamma_i \) due to Corollary 3.2 such that not any \( \gamma_i \) contains the point \( e(0) \) in the interior supposed \( \gamma_i \) is not trivial. Here, set \( h(\gamma) := \prod h(\gamma_i) \).

**Theorem 3.3** The map \( h : \mathcal{P} \rightarrow G \) from Construction 3.3 is for all \( \overline{A} \), \( e \) and \( g \) a homomorphism, i.e. corresponds to a connection \( \overline{A} \in \mathcal{A} \).
Here, \( \mathcal{P} \) is the set of all equivalence classes of paths.

**Proof**

1. \( h \) is a well-defined mapping from \( \mathcal{P} \) to \( G \).

   - Obviously, \( h(\gamma') = h(\gamma'') \) if \( \gamma' \) and \( \gamma'' \) coincide up to the parametrization. Thus, we can drop the brackets in the following when we construct multiple products of paths.
   - Now, we show \( h(\delta' \circ \delta^*) = h(\delta' \circ \delta \circ \delta^{-1} \circ \delta'') \).
     Decompose \( \delta' \), \( \delta'' \) and \( \delta \) due to Corollary 3.2.
     - \( \delta(0) \neq e(0) \), \( \delta(1) \neq e(0) \) and \( e(0) \in \text{im} \delta \).
     Then the decomposition of \( \delta' \circ \delta'' \) is equal to \( (\prod_{i=1}^{I-1} \delta_i') \gamma_i'' \left( \prod_{i=2}^{I} \delta_i'' \right) \) setting \( \gamma_i'' := \delta_i^I \delta_i^I'' \). The decomposition of \( \delta' \circ \delta \circ \delta^{-1} \circ \delta'' \) is
       \[
       (\prod_{i=1}^{I-1} \delta_i') \gamma_s \left( \prod_{i=2}^{I-I} \delta_i'' \right) \gamma_s \left( \prod_{i=2}^{I-1} \delta_i^{-1} \right) \gamma_s'' \left( \prod_{i=2}^{I} \delta_i'' \right)
       \]
     with \( \gamma_s := \delta_1 \delta_1', \gamma_s := \delta_1 \delta_1', \) and \( \gamma_s'' := \delta_1^{-1} \delta_1'' \). (In the third product the index decreases.)
     A simple calculation shows that the definition above indeed yields the same parallel transport for both paths.
     - The other cases can be proven completely analogously.
   - We have as well \( h(\delta' \circ \delta \circ \delta^{-1}) = h(\delta') = h(\delta \circ \delta^{-1} \circ \delta') \) for all \( \delta' \) and \( \delta \).
   - Since equivalent paths can be transformed into each other by a finite number of just described transformations, we get the well-definedness.

2. \( h \) is a homomorphism, i.e. \( h \) corresponds to a generalized connection.
   Let \( \gamma \) and \( \delta \) be two paths and \( \prod_{i=1}^{I} \gamma_i \) and \( \prod_{j=1}^{J} \delta_j \), respectively, be their decompositions as above. Then the decomposition of \( \gamma \circ \delta \) equals \( \left( \prod_{i=1}^{I-I} \gamma_i \right) \gamma_s \left( \prod_{i=2}^{I} \delta_j \right) \) with \( \gamma_s := \gamma_I \delta_I^{-1} \) supposed.
   - \( \gamma_I(1) \equiv \delta_I(0) \neq e(0) \) or
   - \( \gamma_I(\tau) \) equals \( e(0) \) for all \( \tau \) and so does \( \delta_I(\tau) \).

   Otherwise the decomposition is \( \left( \prod_{i=1}^{I-I} \gamma_i \right) \left( \prod_{j=1}^{J} \delta_j \right) \) and the homomorphism is trivial by the above definition of \( h \) on general paths.

   In the first case we still have to prove \( h(\gamma_I \circ \delta_I) = h(\gamma_I) \circ h(\delta_I) \). But, this can be seen quickly using the homomorphism property of \( h \) and the definition above.

   **qed**

**Remark**

- The theorem just proven is very well suited for the proof of the surjectivity and the openness of \( \pi_T : \mathcal{A} \longrightarrow \mathcal{A}_T \) (see below). In a certain sense it is a generalization of the proposition about the independence of loops in [8, 2].
- This says that (for compact Lie groups with \( \exp(g) = G \)) the holonomies along independent loops are even independent on the level of regular connections.
- For instance, a set of loops is independent if each loop possesses a subpath called free segment that is not passed by any other loop. The independence proposition could be proven modifying suitably a given connection along those free segments, such that the resulting holonomy becomes a certain fixed value.
- In our case we do no longer need the restriction to regular connections. We can instead modify a connection "pointwise", e.g., in the point \( e(0) \) in the construction above.
• In the compact case we will extensively use this theorem in a subsequent paper [7] when we prove a stratification theorem for \( \mathcal{A} \) and \( \mathcal{A}/G \).
• The theorem is valid not only for compact, but also for arbitrary structure groups \( G \).

### 3.3 Consequences

In this subsection we collect some immediate implications given by the construction above. First we consider the case of arbitrarily many paths \( e_i \in E \) that are, first, independent of the corresponding remaining paths in \( E \setminus \{e_i\} \) and, second, whose end points form a finite set containing all the free points. Then the parallel transports can be chosen freely. More precisely, we have

**Proposition 3.4** Let \( \mathcal{A} \subset \mathcal{A} \) and \( I \) be a set. Let \( E := \{e_i \mid i \in I\} \subset \mathcal{P} \) be a set of paths that fulfill the following conditions:

1. \( e_i \) is a path without self-intersections for all \( i \).
2. \( e_i \parallel e_j \) for all \( i \neq j \).
3. \( e_i \parallel e_j \) for all \( i, j \).
4. The set \( V_\gamma := \{e_i(0) \mid i \in I\} \) of all initial points is finite.
5. \( V_\gamma \cap \text{int} e_i = \emptyset \) for all \( i \).

Finally, let there be given a \( g_i \in G \) for all \( i \in I \).

Then, there exists an \( \mathcal{A} \subset \mathcal{A} \) such that

- \( h_{\mathcal{A}}(e_i) = g_i \) for all \( i \in I \) and
- \( h_{\mathcal{A}}(\gamma) = h_{\mathcal{A}}(\gamma) \) for all \( \gamma \) that do not have a subpath \( \gamma' \) that fulfills \( \gamma' \parallel e_i \) or \( \gamma \parallel e_i \) for some \( i \in I \). Especially, this holds for all \( \gamma \) with \( \text{im} \gamma \cap (\bigcup_{i \in I} \text{int} e_i) = \emptyset \).

**Proof** First we observe that it is impossible that \( \gamma \parallel e_i \) and \( \gamma \parallel e_j \) for \( i \neq j \), because this would imply \( e_i \parallel e_j \). Analogously, \( \gamma \not\parallel e_i \) and \( \gamma \not\parallel e_j \) is impossible for \( i \neq j \).

Now we define \( h : \mathcal{P} \to G \) as in Construction 3.3 with some modifications. Let \( \gamma \in \mathcal{P} \). We decompose \( \gamma \) according to the (finite number of) passages of points in \( V_\gamma \). Then we set for every such subpath (again denoted by \( \gamma \))

\[
h(\gamma) := \begin{cases} 
g_i h_{\mathcal{A}}(e_i)^{-1} h_{\mathcal{A}}(\gamma) h_{\mathcal{A}}(e_j) g_j^{-1}, & \text{if } \exists i : \gamma \parallel e_i \text{ and } \exists j : \gamma \parallel e_j \\
g_i h_{\mathcal{A}}(e_i)^{-1} h_{\mathcal{A}}(\gamma), & \text{if } \exists i : \gamma \parallel e_i \text{ and } \forall j : \gamma \not\parallel e_j \\
h_{\mathcal{A}}(\gamma) h_{\mathcal{A}}(e_j) g_j^{-1}, & \text{if } \forall i : \gamma \not\parallel e_i \text{ and } \exists j : \gamma \parallel e_j \\
h_{\mathcal{A}}(\gamma), & \text{else} 
\end{cases}
\]

and extend the definition by homomorphy.

As in Theorem 3.3 one easily proves that \( h \) is a well-defined homomorphism using the observation in the beginning of the present proof. Hence, \( h = h_{\mathcal{A}} \) with some \( \mathcal{A} \subset \mathcal{A} \).

Finally, one sees immediately from the definition of \( h \) that \( h_{\mathcal{A}}(e_i) = g_i \) for all \( i \in I \) and \( h_{\mathcal{A}}(\gamma) = h_{\mathcal{A}}(\gamma) \) for all \( \gamma \) with the properties above. \( \text{qed} \)

The preceding proposition covers both the case of webs and of graphs:

**Corollary 3.5** The assumptions of Proposition 3.4 are fulfilled if \( E \) is the set of all edges of a graph or the set of all curves of a web.
Due to the independence of Corollary 3.8. Proof For finite graphs the proof is trivial. Let therefore be \( E \) the set of all curves of a web. By definition, the conditions 1., 4. and 5. are fulfilled as one easily checks using the definition of a web (cf. [5]).

To prove 2. we assume that \( e_1 \uparrow \uparrow e_2 \) for certain curves \( e_1, e_2 \in E \). Then we know that \( e_1(0) = e_2(0) = p_0 \), i.e., \( e_1 \) and \( e_2 \) belong to one and the same tassel. Suppose now \( \text{im } e_1 \neq \text{im } e_2 \). Then there is w.l.o.g. a \( p \in M \) with \( p \in \text{im } e_1 \setminus \text{im } e_2 \). Then, by the definition of a tassel, in every neighbourhood of \( p_0 \) there is a \( p' \in \text{im } e_1 \setminus \text{im } e_2 \). But this is a contradiction to \( e_1 \uparrow \uparrow e_2 \). Hence, \( \text{im } e_1 = \text{im } e_2 \). Thus, since the \( e_i \) are paths without self-intersections, there is a homeomorphism \( \Pi : [0, 1] \rightarrow [0, 1] \) with \( e_2 = e_1 \circ \Pi \) and \( \Pi(0) = 0 \). Now, due to the consistent parametrization of curves of a tassel we know that there is a positive constant \( k \) with \( \Pi(\tau) = k\tau \) for all \( \tau \in [0, 1] \). Because of \( \Pi(1) = 1 \) we get \( k = 1 \) and \( \Pi = \text{id} \). Thus, \( e_2 = e_1 \).

Finally, condition 3. is fulfilled. In fact, let \( e_i \uparrow \downarrow e_j \). Then we have \( e_i(0) = e_j(1) \). This is obviously impossible by the definition of tassels and webs. \( \text{qed} \)

From the proof we get immediately

Corollary 3.6 The curves of a web form a hyph.

Proof The free point of a curve \( c \) in the web is simply its initial point \( c(0) \). \( \text{qed} \)

Now, we come to the case of arbitrary independent paths leading to the hyphs themselves.

Proposition 3.7 Let \( \overline{A} \in \overline{A} \) and \( C \subseteq \mathcal{P} \) be a set of paths without self-intersections. Now, let \( e \in \mathcal{P} \) be a path without self-intersections and \( g \in \mathcal{G} \) be arbitrary. Furthermore, suppose that \( e \) is independent of \( C \).

Then there is an \( \overline{A} \in \overline{A} \) such that

1. \( h_{\overline{A}}(c) = g \) and
2. \( h_{\overline{A}}(c) = h_{\overline{A}}(e) \) for all \( c \in C \).

Proof Due to the independence of \( e \) w.r.t. \( C \), we have \( e \sim e^\tau- \circ e^\tau+ \) for some \( \tau \in [0, 1] \),\(^2\) such that, w.l.o.g., \( e^+ := e^\tau+ \) is a non-trivial path such that for all subpaths \( c' \) of all the \( c \in C \) we have \( e^+ \uparrow \uparrow c' \) and \( e^+ \downarrow \downarrow c' \). Analogously to Proposition 3.4 there is now an \( \overline{A} \in \overline{A} \) such that with \( e^- := e^\tau- \)

1. \( h_{\overline{A}}(e^+) = (h_{\overline{A}}(e^-))^{-1} g \),
2. \( h_{\overline{A}}(c) = h_{\overline{A}}(e) \) for all \( c \) and
3. \( h_{\overline{A}}(e^-) = h_{\overline{A}}(e^-) \).

The last line follows, because \( e \) is a path without self-intersections, i.e., there cannot exist a subpath \( c' \) of \( e^- \) that is \( \uparrow \uparrow \) or \( \downarrow \downarrow \) to \( e^+ \). Finally, we have \( h_{\overline{A}}(e) = h_{\overline{A}}(e^-) h_{\overline{A}}(e^+) = g \). \( \text{qed} \)

Corollary 3.8 Let \( \overline{A} \in \overline{A} \) be a generalized connection and \( v = \{ e_1, \ldots, e_Y \} \subseteq \mathcal{P} \) be a hyph. Furthermore, let \( g_i \in \mathcal{G}, i = 1, \ldots, Y \), be arbitrary.

Then there is a connection \( \overline{A} \in \overline{A} \) such that \( h_{\overline{A}}(e_i) = g_i \) for all \( i \).

Proof Use inductively the preceding corollary. Let \( \overline{A}_0 := \overline{A} \). Then for all \( i \) choose an \( \overline{A}_i \) such that \( h_{\overline{A}}(e_i) = g_i \) and \( h_{\overline{A}}(e_j) = h_{\overline{A}_{i-1}}(e_j) \) for all \( j < i \) using the assumed independence of \( e_i \) w.r.t. \( \{ e_j \mid j < i \} \). Finally, set \( \overline{A} := \overline{A}_Y \). \( \overline{A} \) has now the desired property. \( \text{qed} \)

\(^2\)If \( \tau = 0 \) let \( e^\tau- \) be the trivial path and, analogously, \( e^\tau+ \) for \( \tau = 1 \).
3.4 Surjectivity

**Proposition 3.9** \( \pi_T : \mathcal{A} \to \mathcal{A}_T \) is surjective for all graphs \( \Gamma \).
\( \pi_w : \mathcal{A} \to \mathcal{A}_w \) is surjective for all webs \( w \).
\( \pi_v : \mathcal{A} \to \mathcal{A}_v \) is surjective for all hyps \( v \).

For Lie groups with \( \exp(g) = G \) the surjectivity of \( \pi_T \) can also be proven analytically showing that even \( \pi_T : \mathcal{A} \to \mathcal{A}_T \) is surjective. In the case of webs one additionally needs compactness and semi-simplicity of \( G \). But, the proof given here has the advantage that it is completely algebraic and needs no additional assumptions for \( G \). Moreover, it uses the very constructive proposition just proven and is valid also for hyps.

**Proof** Let \( (g_1, \ldots, g_{\#E(\Gamma)}) \in G^{\#E(\Gamma)} \) be given. Now let \( \overline{A} \in \mathcal{A} \) be the trivial connection, i.e. \( h_{\overline{A}}(\gamma) = e_G \) for all \( \gamma \in \mathcal{P} \). By Proposition 3.4 and Corollary 3.5 there is an \( \mathcal{A} \in \mathcal{A} \) with \( h_{\mathcal{A}}(e_i) = g_i \) for all \( i = 1, \ldots, \#E(\Gamma) \).

The proof in the case of webs is completely analogous, the proof for hyps uses Corollary 3.8.

**3.5 Definition of \( \mathcal{A} \) Using Hyps**

In a preceding paper [6] we proved that in the smooth case for a compact and semi-simple structure group \( G \) the spaces \( \mathcal{A}_{(\infty,+)} \) and \( \mathcal{A}_{\text{Web}} \) of generalized connections used here and by Baez and Sawin, respectively, are in fact homeomorphic. Now, we will translate that proof to the case of hyps.

First, we define a partial ordering on the set of hyps: \( \nu_1 \leq \nu_2 \) iff every \( e \in \nu_1 \) equals up to the parametrization a finite product of paths in \( \nu_2 \) and their inverses. Then we can define \( \mathcal{A}_v := \text{Hom}(\mathcal{P}_v, G) \) (\( \mathcal{P}_v \) being the subgroupoid of \( \mathcal{P} \) generated by \( v \)) and

\[
\pi_{v_2}^{v_1} : \mathcal{A}_{v_2} \to \mathcal{A}_{v_1}, \quad h \mapsto h|_{\mathcal{P}_{v_1}}
\]

for \( v_1 \leq v_2 \). We topologize \( \mathcal{A}_v \) identifying it with \( G^{\#v} \). Obviously \( \pi_{v_2}^{v_1} \) is always continuous, surjective and open. So we can define \( \mathcal{A}_{\text{Hyph}} := \lim_{\nu} \mathcal{A}_v \) as the space of generalized connections with the canonical projections

\[
\pi_v : \mathcal{A}_{\text{Hyph}} \to \mathcal{A}_v.
\]

Using the surjectivity of \( \pi_v \), we get

**Proposition 3.10** \( \mathcal{A}_{\text{Hyph}} \) and \( \mathcal{A} \) are homeomorphic in every smoothness category.

The proof is almost literally the same as for \( \mathcal{A}_{\text{Web}} \) and \( \mathcal{A}_{(\infty,+)} \) in [6] and is therefore dropped here.

4 Directedness of the Set of Hyps

In this section we will prove the following.

**Theorem 4.1** The set of all hyps is directed.

\( \pi_v \) is simply the map \( \mathcal{A} \mapsto (h_\mathcal{A}(e_1), \ldots, h_\mathcal{A}(e_Y)) \in G^Y \) where \( e_i \) are the paths in \( v \).
This assertion follows immediately from the more general

**Proposition 4.2** Let $C \subseteq \mathcal{P}$ be a finite set of paths without self-intersections. Then there is a hyp $v$, such that every $c \in C$ equals up to the parametrization a finite product of paths (and their inverses) in $v$.

We will prove this theorem using induction on the number of paths in $C$. If a path $c \in C$ would be independent of the complement $C \setminus \{c\}$, there will be no problems. Therefore, we first consider the other case.

### 4.1 Non-independent Paths

In the following we often decompose paths without self-intersections according to a finite set $P$ of points in the manifold $M$. This means, given some path $e$ we construct non-trivial subpaths $e_i$ such that every $e_i$ starts and ends in $P$ or $e(0)$ or $e(1)$. We obviously need only finitely many $e_i$ and get $e \sim \prod e_i$.

**Lemma 4.3** Let $e$ and $c_j$, $j \in J$, be finitely many paths without self-intersections, such that $e$ is not independent of $C := \{c_j \mid j \in J\}$.

Then there are $\tau_i \in [0, 1]$, $i = 0, \ldots, I$, with $\tau_0 = 0$ and $\tau_I = 1$ such that the following holds: After decomposing every $e$ and $c_j$ into a product of edges $\prod_{i=0}^{I-1} e_i$ and $\prod c_j^I$, respectively, according to the set $\{e(\tau_i)\}$, for every $i = 0, \ldots, I - 1$ one of the following two assertions is true:

1. $e_i \uparrow \downarrow c_k \Rightarrow e_i \sim c_k$
2. $e_i \uparrow \downarrow c_k \Rightarrow (e_i)^{-1} \sim c_k$

Note that here the $\sim$-sign indicates that, e.g. in the first case, $e_i$ and $c_k$ are even equal up to the parametrization.

**Proof** Let $I_{\tau, -j}$, $\tau \in [0, 1]$, contain exactly $\tau$ itself and those $\tau' \in (\tau, 1]$ for that the subpath of $e$ from $\tau$ to $\tau'$ is up to the parametrization equal to some subpath of $c_j$ or $c_j^{-1}$. By assumption for all $\tau \in [0, 1]$ there is a $j$ with $I_{\tau, +j} \neq \{\tau\}$.

Analogously, $I_{\tau, -j}$, $\tau \in [0, 1]$, contains exactly $\tau$ itself and those $\tau' \in [0, \tau)$ for that the subpath of $e$ from $\tau'$ to $\tau$ is up to the parametrization equal to some subpath of $c_j$ or $c_j^{-1}$. Again, by assumption for all $\tau \in (0, 1]$ there is a $j$ with $I_{\tau, -j} \neq \{\tau\}$.

Furthermore, $I_{\tau, \pm j}$ is every time connected.

Now, define $I_{\tau, \pm} := \bigcap_{j \in J, I_{\tau, \pm j} \neq \{\tau\}} I_{\tau, \pm j}$,

as well as $I_{0, -} := \{0\}$ and $I_{1, +} := \{1\}$.

What is the interpretation of such an $I_{\tau, \pm}$? $I_{\tau, +}$, e.g., is that interval in $[0, 1]$ starting in $\tau$ such that every subpath of $c_j$ (or $c_j^{-1}$), that starts in $e(\tau)$ as $e$ does, is even equal (up to the parametrization) to this subpath of $e$ at least from $e(\tau)$ to $e(\tau')$ for every $\tau' \in I_{\tau, \pm}$. However, note, that $I_{\tau, \pm}$ need not be a closed interval.

---

Consequently, for no $c \in C$ there is a path occurring twice in the product for $c$. 

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Observe, that \( I_{	au, \pm} \) is in each case (except for \( I_{0,-} \) and \( I_{1,+} \)) an interval that contains \( \{ \tau \} \) as a proper subset.

2. Now, we construct a sequence \((\tau_i)\) of numbers starting with \( \tau_0 := 0 \) as follows for all \( i \geq 0 \):
   a) \( \tau_{i,+} := \sup I_{\tau_{i,+}} \).
   b) \( \tau_{i+1} := \sup \{ \tau \in [\tau_{i,+}, 1] \mid I_{\tau_{i,+}} \cap I_{\tau,-} \neq \emptyset \} \).
   c) \( \tau_{i+1,-} \) is some number with
      - \( \tau_{i,+} \leq \tau_{i+1,-} \leq \tau_{i+1} \),
      - \( \tau_{i+1,-} \in I_{\tau_{i+1,-}} \) and
      - \( I_{\tau_{i,+}} \cap I_{\tau_{i+1,-}} \neq \emptyset \).
   d) \( \tau_{i+\frac{1}{2}} \) is some number in \( I_{\tau_{i,+}} \cap I_{\tau_{i+1,-}} \).
   e) If \( \tau_{i+1} = 1 \) then stop the procedure.

Observe:
   a) \( \tau_{i,+} > \tau_i \), because \( I_{\tau_{i,+}} \) is a non-trivial interval.
   b) Since \( I_{\tau_{i,+}} \cap I_{\tau_{i+1,-}} \neq \emptyset \) (by definition of \( \tau_{i,+} \)), the set of all numbers \( \tau \) with \( I_{\tau_{i,+}} \cap I_{\tau,-} \neq \emptyset \) and \( \tau \geq \tau_{i,+} \) non-empty. Consequently, it has a supremum \( \tau_{i+1} \geq \tau_{i,+} \).
   c) By choice of \( \tau_{i+1} \) as such a supremum there is a \( \tau' \geq \tau_{i,+} \) with \( \tau' \in I_{\tau_{i+1,-}} \) and \( I_{\tau_{i,+}} \cap I_{\tau',-} \neq \emptyset \). Choose now \( \tau_{i+1,-} := \tau' \).
   d) \( \tau_{i+\frac{1}{2}} \) exists obviously.

Thus, the construction above is possible.

Furthermore, we have \( \tau_i \leq \tau_{i+1} \leq \tau_{i+\frac{1}{2}} \leq \tau_{i+1,-} \leq \tau_{i+1} \) and \( \tau_i < \tau_{i+1} \).

3. Now, assume that there is no \( N \in \mathbb{N} \) with \( \tau_N = 1 \). Then \( (\tau_i)_{i \in \mathbb{N}} \) is a strictly increasing sequence with values in \([0, 1]\), i.e. \( \tau_i \to \tau \in (0, 1] \) for \( i \to \infty \), and we have \( \tau_i < \tau \) for all \( i \in \mathbb{N} \).

Let \( \tau' \in I_{\tau,-} \) with \( \tau' < \tau \). Then there is an \( n \in \mathbb{N} \) with \( \tau' \leq \tau_n < \tau \). Now we have \( I_{\tau_{n,+}} \cap I_{\tau,-} \neq \emptyset \), because, e.g., \( \tau_n \) is contained in this set. But, from this we get together the step 2.b) above, that \( \tau \leq \tau_{n+1} \). This is a contradiction to \( \tau > \tau_{n+1} \).

Consequently, there is an \( N \in \mathbb{N} \) with \( \tau_N = 1 \).

4. Now, the desired parameter values are \( \tau_i, \tau_{i+\frac{1}{2}} \) and \( \tau_{i+1,-} \) for \( i = 0, \ldots, N-1 \) as well as \( \tau_N \). Divide the edges \( e \) and \( c_j \) according to the set of all those \( e(\tau_{j,-}) \).

We have (if two subsequent vertices \( e(\tau_{j,-}) \) are equal, we drop the correspondent (trivial) subpaths \( e_- \) and \( c_{-j} \)):
   a) \( e_i \uparrow \uparrow c_k \implies e_i \sim c_k \)
      and \( e_i \uparrow \downarrow c_k \implies e_i \sim (c_k)^{-1} \);
   b) \( e_{i+\frac{1}{2}} \downarrow \uparrow c_k \implies (e_{i+\frac{1}{2}})^{-1} \sim c_k \)
      and \( e_{i+\frac{1}{2}} \downarrow \downarrow c_k \implies (e_{i+\frac{1}{2}})^{-1} \sim (c_k)^{-1} \);
   c) \( e_{i+1,-} \downarrow c_k \implies (e_{i+1,-})^{-1} \sim c_k \)
      and \( e_{i+1,-} \downarrow \downarrow c_k \implies (e_{i+1,-})^{-1} \sim (c_k)^{-1} \).

We only show the first item, the two other ones can be proven analogously.

Let \( e_i \uparrow \uparrow c_k \). Since \( c_k \) is a subpath of a \( c_j \), we have \( I_{\tau_{i+j}} \not\in \{ \tau_i \} \). From \( I_{\tau_{i+j}} \supseteq I_{\tau_{i+1}}\supseteq I_{\tau_{i+1/2}} \) we get now \( e_i \) equals (up to the parametrization) a subpath of \( c_j \) starting in \( e(\tau_i) \). But, since \( c_j \) has no self-intersections and is divided according to \( e(\tau_i) \) and \( e(\tau_{i+\frac{1}{2}}) \) (and other vertices that are not contained in im \( e_i \)), we have \( e_i \) even equals \( c_k \) up to the parametrization.
In the case $e_i \uparrow \downarrow e_k'$ we conclude analogously using $e_i \uparrow (e_k')^{-1}$. qed

4.2 Proof of Proposition 4.2

Proof Proposition 4.2

- First of all we decompose all $c_i$ according to the set $V := \{c_i(0)\}_i \cup \{c_i(1)\}_i$ of all end points. Thus, we get a finite set $C'$ of paths without self-intersections, whereas every $c \in C$ equals up to the parametrization a finite product of paths $c' \in C'$ and their inverses and where no end point of a path $c'$ is contained in the interior of another path in $C'$.

Consequently, we can w.l.o.g. assume that our set $C$ in the proposition is of that type.

- Now, we consider $c_1 \in C$.
  1. In the case that $c_1$ is already independent of $\{c_j \mid j > 1\}$ we need not decompose $c_1$; we simply set $c_{1,1} := c_i$ and $I_i := 1$ for all $i$.
  2. In the other case we use Lemma 4.3 and get certain paths $e_k$ (w.l.o.g. such that $c_1 \sim e_1 \circ \cdots \circ e_{I_1}$) such that every $c_j$ is a product of the $e_k$ (and their inverses) and such that the $e_k$, $k \in [1, I_1]$, are independent of the remaining paths. Now, we set $c_{1,k} := e_k$ for all $k \in [1, I_1]$. Analogously, we define $c_{j,l}$ for $i > 1$ being that $e_k$ that (or whose inverse) is used at the $i$th position in the product for $c_i$, after we cancelled all $e_k$ occurring in $c_1$, and denote the number of factors left by $I_i$.

Per constructionem, $c_{1,l}$ is independent of $\{c_{j,l} \mid i > 1 \lor l \neq l\}$. Note, moreover, that the set of end points of the $c_{j,l}$ is again disjoint to the interiors of these paths. Finally, we set $C_1 := \{c_{j,l} \mid i > 1\}$.

- Now, we decompose the paths $c_{2,l} \in C_1$ (if $I_2 \neq 0$).

We start with $c_{2,1}$. If it is not independent of the $\{c_{j,l} \in C_1 \mid i > 2 \lor l \neq l\}$, then decompose it again by Lemma 4.3 by certain independent paths $e_{k}'$. We get as before $c_{2,1} \sim c_{2,1,1} \circ \cdots \circ c_{2,1,l_1}$ and a certain set $C_{2,1}$ that collects all paths used for the decomposition of $c_{2,1}$ with $i > 2$. But, note that $c_{2,l}$ is not decomposed for $l \neq 1$ by that procedure.

Afterwards, we decompose $c_{2,2}$ (w.r.t. $C_{2,1}$) and so on.

Summa summarum, we get paths $c_{2,l,m}$ with $c_{2,l} \sim \Pi_{m_0} c_{2,l,m}$ and a set $C_2 := C_{2,l_2}$ collecting all the paths that $c_{j,l}$ with $i > 2$ is decomposed into, but that are not used in the decomposition of $c_{2,l}$. By the construction, $c_{2,l,m}$ is independent of $\{c_{2,l,m} \mid l \neq l' \lor m_0 \neq m_0\}' \cup C_2$.

- In the next step, we first collect all paths in $C_2$ that are used for the decomposition of $c_3$. After renumbering these paths by $c_3, \ldots, c_{3,I_3}$ we can again apply the previous step.

- Inductively, we get an ordered set

$$C^* = \{c_{N,1,1}, \ldots, c_{N,I_N,M_N,I_N}; c_{2,1,1,1}, \ldots, c_{2,1,1,2}; c_{1,1}, \ldots, c_{1,1}\}$$

of paths that is by construction moderately independent, consequently a hyphen, and that admits a factorization of every $c_i \in C$ into a product of paths in $C^*$ of the desired type. qed

---

Example: $c_1 = e_1e_2e_3$, $c_2 = e_1^{-1}e_4e_3e_5^{-1}$ and $c_3 = e_2^{-1}$. Then we have $I_1 = 3$, $I_2 = 2$, $I_3 = 0$ and $c_{1,1} = e_1$, $c_{1,2} = e_2$, $c_{1,3} = e_3$, $c_{2,1} = e_4$ and $c_{2,2} = e_5$. 

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4.3 Open Problem

In contrast to the case of graphs or webs we need for the definition of the independence in the case of hyphs an ordering among the paths collected in a hyph. Thus, it would be – at least for technical reasons – desirable to solve the following open problem: Does there exist for every given finite set $C$ of paths a set $E$ of strongly independent paths, such that every path in $C$ is a product of paths in $E$ and their inverses? Strongly independent means here that every path in $C$ is independent of the remaining paths in $C$. We indicate the problems that arised when we tried to prove the following answers:

"Yes": The induction used for the proof of Proposition 4.2 cannot be reused. The problem is the following. Suppose we have decomposed the first path $c_1$ in $C$ w.r.t. to the remaining paths as above. Then we decompose (the subpaths of) the second path $c_2$ in $C$ w.r.t. the others. Now, it is possible that vertices used in this procedure for the division of $c_2$ lie on $c_1$ again. Thus, $c_1$ would now be divided once more – with the effect that sometimes subpaths of $c_1$ are created that do not fulfill the independence condition. (Remember that independence means existence of one point in a path with the independence-of-germs condition above.) Hence, we have to divide the respective path again. But, now we could end up in a never-ending procedure that creates an infinite number of subpaths.

"No": It would be enough to present one counterexample. But, up to now, none of the examples we checked lead to a contradiction.

5 Openness of $\pi_\Gamma$

Proposition 5.1  $\pi_\Gamma : \mathcal{A} \rightarrow \mathcal{A}_\Gamma$ is open for all graphs $\Gamma$.

Proof  We have to show: $\pi_\Gamma(V)$ is open for all elements $V$ of a basis of $\mathcal{A}$, i.e., $\pi_\Gamma(\pi_\Gamma^{-1}(W_1) \cap \ldots \cap \pi_\Gamma^{-1}(W_j))$ is open for all graphs $\Gamma_i$ and all elements $W_i$ of a basis of $\mathcal{A}_{\Gamma_i} = G_{\#E(\Gamma_i)}$. But, a basis herof is given by all sets of the type $W_{i,1} \times \cdots \times W_{i,\#E(\Gamma_i)}$ with open $W_{i,j} \subseteq G$. Now we have

$$\pi_\Gamma(\pi_\Gamma^{-1}(W_1) \cap \ldots \cap \pi_\Gamma^{-1}(W_j)) = \pi_\Gamma\left(\bigcap_{i=1}^{j} \bigcap_{j,k=1}^{\#E(\Gamma_i)} \pi_{e_{i,j,k}}^{-1}(W_{i,j,k})\right).$$

(W.l.o.g. we assumed that none of the $\Gamma_i$ consists of a single vertex.)

Let us therefore prove the openness of all sets of the type

$$\pi_\Gamma\left(\bigcap_{j=1}^{f} \pi_{e_{j}}^{-1}(W_j)\right)$$

with edges $e_j$ and open $W_j \subseteq G$. Let us denote the edges of $\Gamma$ by $e_i$ and set $E := \{e_i\}$ and $C := \{e_j\}$.

1. Suppose first that there is an $e \in E$ that is independent of $C$. Then it is obviously independent of $C \cup (E(\Gamma) \setminus \{e\})$. We will show that

$$\pi_\Gamma\left(\bigcap_{j=1}^{f} \pi_{e_{j}}^{-1}(W_j)\right) = \pi_{\Gamma \setminus \{e\}}\left(\bigcap_{j=1}^{f} \pi_{e_{j}}^{-1}(W_j)\right) \times G.$$

"$\subset$" Trivial.

"$\supset$" Let $(\bar{g}, g) \in \pi_{\Gamma \setminus \{e\}}\left(\bigcap_{j=1}^{f} \pi_{e_{j}}^{-1}(W_j)\right) \times G$.

Hence, there is an $\bar{A} \in \bigcap_{j=1}^{f} \pi_{e_{j}}^{-1}(W_j)$ with $\pi_{\Gamma \setminus \{e\}}(\bar{A}) = \bar{g}$. 

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Due to Proposition 3.7 there is an $A \in A$ fulfilling
- $h_{A}(e_{j}) = h_{A}(e_{j})$ for all $e_{j} \neq e$, i.e. $g = \pi_{\Gamma \setminus \{e\}}(A) = \pi_{\Gamma \setminus \{e\}}(A)$,
- $h_{A}(e_{j}) = h_{A}(e_{j})$ for all $j = 1, \ldots, J$, i.e. $A \in \pi_{c_{j}}^{-1}(W_{j})$ for all $j$, and
- $h_{A}(e) = g$.

With this we have $\pi_{T}(A) = (\pi_{\Gamma \setminus \{e\}}(A), \pi_{e}(A)) = (g, g)$, i.e. $$(g, g) \in \pi_{T}(\bigcap_{j=1}^{J} \pi_{c_{j}}^{-1}(W_{j})).$$

2. Successively applying the preceding step we get
$$\pi_{T}(\bigcap_{j=1}^{J} \pi_{c_{j}}^{-1}(W_{j})) = \pi_{T_{0}}(\bigcap_{j=1}^{J} \pi_{c_{j}}^{-1}(W_{j})) \times G^{n}.$$ Here $n$ denotes the number of edges $e$ of $\Gamma$ that are independent of $C$. $T_{0}$ denotes that graph that arises from $\Gamma$ by removing all such edges.

3. Since every edge $e$ in $T_{0}$ is not independent of $C$, we can divide $e_{1}$ and the $c_{j} \in C$ as in Lemma 4.3 and get paths $e_{1,1}, \ldots, e_{1,n_{1}}$ and $c_{j,1}, \ldots, c_{j,m_{j}}$. We collect the $c_{j}$ into $C_{1} \subseteq P$. Since the $e_{j}$ are edges of one and the same graph, $e_{j}$ (for $i > 1$) is still not independent of $C_{1}$. We again use Lemma 4.3, now for decomposing $e_{2}$ and the paths in $C_{1}$. We get paths $e_{2,1}, \ldots, e_{2,m_{2}}$ and a $C_{2} \subseteq P$. Successively, we decompose all $e_{i}$ and $C_{i-1}$ getting $e_{i,k_{i}}$ and $c_{i} \in C' \subseteq P$, such that for every $i$ and $k_{i}$ one of the following two assertions is true:

a) $e_{i,k_{i}} \uparrow \downarrow c_{i} \implies e_{i,k_{i}} \sim c_{i}$ and
   
   $e_{i,k_{i}} \uparrow \downarrow c_{i} \implies e_{i,k_{i}} \sim (c_{i})^{-1}$

b) $e_{i,k_{i}} \uparrow \downarrow c_{i} \implies (e_{i,k_{i}})^{-1} \sim c_{i}$ and
   
   $e_{i,k_{i}} \uparrow \downarrow c_{i} \implies (e_{i,k_{i}})^{-1} \sim (c_{i})^{-1}$.

To reduce the technical efforts we first invert all $e_{i,k_{i}}$ that fulfill the second assertion. Afterwards, we invert $c_{i}$ if it is equivalent to an $(e_{i,k_{i}})^{-1}$. This is possible, because there is at most one such edge $e_{i}$. It is clear, that the $e_{i,k_{i}}$ span a graph $\Gamma' \geq T_{0}$, and we know from the construction that no int $c_{i}$ contains a vertex of $\Gamma'$. Furthermore, every $c_{j}$ is equivalent to a finite product of $c_{j}$ (or its inverse). The factors used for $c_{j}$ (again denoted by $c_{j,t_{j}}$) span a graph $\Gamma_{j}$, as well. Thus, we have $\pi_{T_{0}} = \pi_{\Gamma_{0}} \pi_{\Gamma'}$ and $\pi_{c_{j}} = \pi_{\Gamma_{j}}^{-1}(\pi_{c_{j}}^{-1})^{-1}$.

Finally, $(\pi_{\Gamma_{j}}^{-1})(W_{j})$ is open in $G^{m_{j}}$ by continuity, i.e., a union of sets of the type $W_{j,1} \times \cdots \times W_{j,m_{j}}$. Thus, $\pi_{T_{0}} \left( \bigcap_{j=1}^{J} \pi_{c_{j}}^{-1}(W_{j}) \right)$ is the union of sets of the type $\pi_{\Gamma_{0}} \pi_{\Gamma'} \left( \bigcap_{j=1}^{J} \bigcap_{t_{j}=1}^{m_{j}} \pi_{c_{j,t_{j}}}(W_{j,t_{j}}) \right)$.

4. Due to the openness of $\pi_{\Gamma_{0}}$ (see [6]) it is sufficient to prove the openness of $\pi_{T'} \left( \bigcap_{l=1}^{L} \pi_{e_{l}}^{-1}(W_{l}) \right)$ whenever the following holds:
   a) $\Gamma'$ is a graph and $C' = \{c_{l}\}$ is a finite set of paths without self-intersections, b) $\int_{C' \cap V(\Gamma')} = \emptyset$, c) $(e \uparrow \downarrow c_{l} = e \sim c_{l})$ and $e \uparrow \downarrow c_{l}$ for all $l$ and for every edge $e$ of the graph $\Gamma'$ and d) $W_{l} \subseteq G$ is open for all $l$.

We will prove for non-empty left hand side
$$\pi_{\Gamma'} \left( \bigcap_{l=1}^{L} \pi_{e_{l}}^{-1}(W_{l}) \right) = \bigwedge_{e_{l} \in E(\Gamma')} \left( \bigcap_{c_{l} \in C(e_{l})} W_{l} \right), \quad (1)$$  

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where $C(e_k) \subseteq C'$ contains exactly those $c_l \in C'$ that are (up to the parametrization) equal to $e_k$ or $e_k^{-1}$. Since the right hand side is obviously open, the openness is proven if (1) is.

"$\subseteq$" Let $\bar{g} \in \pi_{\Gamma'}(\prod_{l=1}^{l} \pi_{c_l}^{-1}(W_l))$, i.e., there is an $\bar{A} \in \mathcal{A}$ with $\pi_{c_l}(\bar{A}) = g_k$ for all $k$ and $\pi_{c_l}(\bar{A}) \in W_l$ for all $c_l \in C'$. From this follows $g_k \in W_l$ for all $c_l \in C(e_k)$ and so $\bar{g} \in \mathcal{X}_{c_k \in \pi_{\Gamma'}(\prod_{l=1}^{l} \pi_{c_l}^{-1}(W_l))}$.

"$\supseteq$" Let $\tilde{g} \in \mathcal{X}_{c_k \in \pi_{\Gamma'}(\prod_{l=1}^{l} \pi_{c_l}^{-1}(W_l))}$. Choose an $\bar{A}_0 \in \mathcal{A}$ with $\pi_{c_l}(\bar{A}_0) \in W_l$ for all $c_l$. By assumption every $e_k$ is independent of $C'(\bigcup_k C(e_k))$ and so by Proposition 3.7 there exists an $\bar{A} \in \mathcal{A}$ such that

- $\pi_{c_l}(\bar{A}) = g_k$ for all $k$ and
- $\pi_{c_l}(\bar{A}) = \pi_{c_l}(\bar{A}_0)$ for all $c_l$ that are not equal (up to the parametrization) an $e_k$.

Thus, we have $\pi_{c_l}(\bar{A}) \in W_l$ for all $c_l \in C(e_k)$. Consequently, $\tilde{g} \in \pi_{\Gamma'}(\prod_{l=1}^{l} \pi_{c_l}^{-1}(W_l))$.

$$\text{qed}$$

### 6 Induced Haar Measure

In this section we will show that thanks to the directedness of the set of hyphs an induced Haar measure can be defined for arbitrary smoothness assumption for the paths. Our definition covers that of Ashtekar and Lewandowski for graphs in the analytic category [2] as well as that of Baez and Sawin for webs in the smooth category [5]. Throughout this section, $G$ is a compact Lie group.

#### 6.1 Cylindrical Functions

In this subsection we will investigate the algebra of continuous functions on $\bar{A}$. Particular nice is the dense subalgebra of the so-called cylindrical functions [2, 3]. These are functions depending only on the parallel transports along a finite number of paths.

**Definition 6.1** A function $f \in C(\bar{A})$ is called genuine cylindrical function on $\bar{A}$ iff there is a graph $\Gamma$ and a continuous function $f_\Gamma \in C(\bar{A}_\Gamma)$ with $f = f_\Gamma \circ \pi_{\Gamma'}$. The set of all genuine cylindrical functions is denoted by $\text{Cyl}_0(\bar{A})$.

Obviously, $\text{Cyl}_0(\bar{A})$ is *-invariant. But, since for two finite graphs there need not exist a third one containing both, the sum as well as the product of two cylindrical functions is no longer a cylindrical function in general. Therefore we enlarge the definition above to hyphs.

**Definition 6.2** A function $f \in C(\bar{A})$ is called cylindrical function on $\bar{A}$ iff there is a hyph $v$ and a continuous function $f_v \in C(\bar{A}_v)$ with $f = f_v \circ \pi_v$. The set of all cylindrical functions is denoted by $\text{Cyl}(\bar{A})$.

**Lemma 6.1** $\text{Cyl}(\bar{A})$ is a normed *-algebra containing $\text{Cyl}_0(\bar{A})$.

**Proof** $\text{Cyl}(\bar{A})$ is obviously closed w.r.t. scalar multiplication and involution. It remains to prove that it is closed w.r.t. to addition and multiplication.

Let $f' = f'_v \circ \pi_v$ and $f'' = f''_v \circ \pi_v$. By Theorem 4.1 there is a hyph $v$ with $v \geq v'$, $v''$. Thus we have $f' + f'' = f'_v \circ \pi_v + f''_v \circ \pi_v = (f'_v \circ \pi_v + f''_v \circ \pi_v) \circ \pi_v \in \text{Cyl}(\bar{A})$.

Analogously, $f' \cdot f'' \in \text{Cyl}(\bar{A})$. $\text{qed}$
Proposition 6.2  Cyl(\mathcal{A}) is dense in \( C(\mathcal{A}) \).

Proof  The assertion follows from the Stone-Weierstraß theorem:
- \( 1 \in \text{Cyl}(\mathcal{A}) \), whereas \( 1 : \mathcal{A} \longrightarrow \mathbb{C} \) is the function \( 1(\mathcal{A}) := 1 \).
- \( \text{Cyl}(\mathcal{A}) \) separates the points of \( \mathcal{A} \).

Let \( \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A} \) with \( \mathcal{A}_1 \neq \mathcal{A}_2 \). Thus, there is a graph \( \Gamma \) with \( \pi_\Gamma(\mathcal{A}_1) \neq \pi_\Gamma(\mathcal{A}_2) \).

Since \( \mathcal{A}_\Gamma := G^{\#E(\Gamma)} \) is a manifold, hence completely regular, the continuous functions on \( \mathcal{A}_\Gamma \) separate the points of \( \mathcal{A}_\Gamma \) [9]. This means there is an \( f_\Gamma \in C(\mathcal{A}_\Gamma) \) with \( f_\Gamma(\pi_\Gamma(\mathcal{A}_1)) \neq f_\Gamma(\pi_\Gamma(\mathcal{A}_2)) \).

Due to \( f_\Gamma \circ \pi_\Gamma \in \text{Cyl}(\mathcal{A}) \), \( \text{Cyl}(\mathcal{A}) \) separates the points of \( \mathcal{A} \).  

\( \square \)

6.2 The Induced Haar Measure on \( \mathcal{A} \)

According to the Riesz-Markow theorem measures on a compact Hausdorff space are in one-to-one correspondence to linear, continuous, positive functionals on the function algebra over that space. We get

Proposition 6.3  For every linear, continuous, positive functional \( F \) on \( C(\mathcal{A}) \) there is a unique regular Borel measure \( \mu \) on \( \mathcal{A} \), such that
\[
F : C(\mathcal{A}) \longrightarrow \mathbb{C}, \quad f \mapsto \int f d\mu
\]

Due to the denseness of \( \text{Cyl}(\mathcal{A}) \) in \( C(\mathcal{A}) \) it is sufficient to define an appropriate functional on \( \text{Cyl}(\mathcal{A}) \) and to extend this continuously to a functional on \( C(\mathcal{A}) \). One possibility is to replace the integration of functions \( f_\nu \circ \pi_\nu \) over \( \mathcal{A} \) by the integration of \( f_\nu \) over \( \mathcal{A}_\nu = G^{\#_\nu} \). But, on \( G^{\#_\nu} \) there is a "canonical" measure, the Haar measure. Hence, we define (cf. [2]):

Definition 6.3  Let \( f \in \text{Cyl}(\mathcal{A}) \). Define \( F_0(f) := \int_{\mathcal{A}_\nu} f_\nu d\mu_{\text{Haar}}, \) if \( f_\nu \circ \pi_\nu = f \), and extend \( F_0 \) continuously to a functional \( F \) on \( C(\mathcal{A}) \).

Proposition 6.4  \( F : C(\mathcal{A}) \longrightarrow \mathbb{C} \) is a well-defined, linear, continuous, positive functional on \( C(\mathcal{A}) \).

Furthermore, there is a unique Borel measure \( \mu_0 \) on \( \mathcal{A} \) with \( F(f) = \int f d\mu_0 \) for all \( f \in C(\mathcal{A}) \).

Definition 6.4  The measure \( \mu_0 \) of the preceding proposition is called induced Haar measure or Ashtekar-Lewandowski measure on \( \mathcal{A} \).

Proof  \( F_0 \) is well-defined.

Let \( f \) be cylindrical w.r.t. \( \nu' \) and \( \nu'' \). Then \( f \) is again cylindrical w.r.t. \( \nu \), if \( \nu \) is some hyph containing \( \nu' \) and \( \nu'' \). The existence of such an \( \nu \) is guaranteed by Theorem 4.1. Hence, it is sufficient to prove \( \int_{\mathcal{A}_\nu} f_\nu d\mu_{\text{Haar}} = \int_{\mathcal{A}_{\nu'}} f_{\nu'} d\mu_{\text{Haar}} \) for all \( \nu \geq \nu' \).

Let now \( \nu \geq \nu' \). Then every path \( \epsilon_i' \) of \( \nu' \) can be written as a product \( \prod_k e_j^{\pm 1} \) of paths in \( \nu \) (and their inverses). By the moderate independence of hyphs there is a path \( \epsilon_k(i) \) for every \( i \), such that \( \epsilon_k(i) \) occurs exactly once in the decomposition

\[\text{We prove even } \text{Cyl}(\mathcal{A}) \] separates the points of \( \mathcal{A} \).
of $\epsilon'_i$ and does not occur in that of $\epsilon'_i$ with $i' < i$. Now we have ($n := \#\nu$ and $n' := \#\nu'$)

$$
\int_{\prod_{\nu}} f_{\nu} d\mu_{\text{Haar}} = \int_{G^n} f_{\nu}(g_1, \ldots, g_n) d\mu_{\text{Haar}}
= \int_{G^n} f_{\nu} \left( \prod_{k_{1}} g_{j_{1_{(k_{1})}}}^{+1}, \ldots, \prod_{k_{n'}} g_{j_{1_{(n')}}}^{-1} \right) \prod d\mu_{\text{Haar}}
$$

(f$_{\nu} = f_{\nu'} \circ \pi_{\nu'}$ and decomposition of $\epsilon'_i$)

$$
= \int_{G} \cdots \int_{G} f_{\nu'}(\cdots g_{j_{1_{(1)}}}^{+1}, \ldots, \cdots g_{j_{1_{(n')}}}^{+1}) d\mu_{\text{Haar},1} \cdots d\mu_{\text{Haar},n}
$$

(The dots in $\cdots g_{j_{1_{(1)}}}^{+1}, \ldots, \cdots g_{j_{1_{(n')}}}^{+1}$ denote always a product of $g_{j_{1_{(n')}}}^{+1}$ with

$j \neq K(l')$ for all $l' > l$.)

$$
= \int_{G} \cdots \int_{G} f_{\nu'}(g_1, \ldots, g_{n'}) d\mu_{\text{Haar},1} \cdots d\mu_{\text{Haar},n'}
$$

(Translation and invariance invariance, normalization of the Haar measure)

$$
= \int_{\prod_{\nu'}} f_{\nu'} d\mu_{\text{Haar}}.
$$

- $F_0$ is continuous due to $|F_0(f)| \leq \| f \|$. The last equality follows from the surjectivity of $\pi_\nu$, see Proposition 3.9.
- $F_0$ is obviously linear and positive.
- Hence, $F$ is a well-defined, linear, continuous, positive functional on $C(\mathcal{A})$.
- Due to the Riesz-Markow theorem there is a unique Borel measure $\mu_0$ on $\mathcal{A}$ with $F(f) = \int_{\mathcal{A}} f d\mu_0$.
- $F$ is strictly positive.

Let $f \in C(\mathcal{A})$, $f \neq 0$, and $k := f^* f \in C(\mathcal{A})$. Then $U := k^{-1}(\frac{1}{2} \| k \|, \infty)$ is

open and non-empty. Thus, there is a hyph $\nu$ and an open, non-empty $U_\nu$ with

$\pi_\nu^{-1}(U_\nu) \subseteq U$. Since every open non-empty subset of a compact Lie group has non-vanishing Haar measure, we have

$$
F(f^* f) = \int_{\mathcal{A}} k d\mu_0 \geq \int_{U} \frac{1}{2} \| k \| d\mu_0
\geq \frac{1}{2} \| k \| \int_{\pi_\nu^{-1}(U_\nu)} 1 d\mu_0 = \frac{1}{2} \| k \| \int_{U_\nu} 1 d\mu_{\text{Haar}}
= \frac{1}{2} \| k \| \mu_{\text{Haar}}(U_\nu) > 0.
$$

\text{qed}

Let $U \subseteq G$ be open, non-empty. Then $\{ U g \mid g \in G \}$ is a covering of $G$. Since $G$ is compact, there are

only finitely many $g_i$, such that $\bigcup_{i=1}^{n} U g_i = G$. Due to the translation invariance of the Haar measure we have $\mu(U) = \frac{1}{n} \mu(G) \geq \frac{1}{n} \mu(G) > 0$. 

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7 Discussion

In this paper we investigated for some examples how the theory of generalized connections depends on the chosen smoothness category for the paths used in the construction of $\mathcal{A}$. The most important theorem yields that in every case an induced Haar measure can be defined. But, there are some problems that depend very crucially on the smoothness of the paths. So let us resume the discussion of the beginning of this paper: What could be a good choice of smoothness conditions?

One decisive point is the denseness of the classical (smooth) connections in the space $\mathcal{A}_{(r)}$. In the case of compact structure groups $G$ the denseness has been proven for the immersive smooth $[5, 10]$ and piecewise analytic category $[11]$. However, in the first case $[5]$ the space $\mathcal{A}_{\text{Web}}$ was defined not by $\lim_w \mathcal{A}_w$, but by $\lim_w \mathcal{A}_w$ where $\mathcal{A}_w$ (being a Lie subgroup of $G^w$) denotes the image of the space $\mathcal{A}$ of regular connections under the map $\pi_w \equiv h_{e_1} \times \cdots \times h_{e_w}$. Thus, the denseness follows immediately by the directedness of the set of webs (cf. Appendix B). Supposed, $G$ is in addition semi-simple, Lewandowski and Thiemann $[10]$ proved that $\mathcal{A}_w = \mathcal{A}_w = G^w$ which implies that $\mathcal{A}$ is also dense in our $\mathcal{A}_{(\infty, +)}$. Up to now, we do not know whether this is true for arbitrary Lie groups. However, $\mathcal{A}$ is definitely not dense in the space $\mathcal{A}_{(r)}$ for non-immersed paths. Let, e.g., $\gamma$ be an immersed path without self-intersections and $\gamma' (r) := \gamma(\tau^2)$. Then $\gamma'$ is not equivalent to $\gamma$ (cf. $[6]$) and not an immersion. But, obviously $h_\tau (A) = h_\tau (A')$ for all $A \in \mathcal{A}$. Consider now two elements $g, g' \in G$ and corresponding disjoint open neighbourhoods $U, U' \subseteq G$. We see that $\nu := \{ \gamma, \gamma' \}$ is a hyph and so $\pi_\nu^{-1} (U) \cap \pi_{\gamma'}^{-1} (U') = \pi_\nu^{-1} (U \times U')$ is non-empty and open, but contains no regular $A$. So $\mathcal{A}$ is not dense in $\mathcal{A}_{(r)}$.

Since this is, in fact, very unsatisfactory, we should look for other possibilities for the definition of the set $\mathcal{P}$ for non-immersive paths. The probably easiest way should be to redefine the equivalence relation between paths. Why should non-self-intersecting paths $\gamma$ and $\gamma'$ only be equivalent if they coincide up to a piecewise $C^\infty$-transformation? Perhaps we should use a definition of the following kind: $\gamma \sim \gamma'$ iff $h_A (\gamma) = h_A (\gamma')$ for all $A \in \mathcal{A}$ – maybe at least provided $\text{im} \gamma = \text{im} \gamma'$. This one is quite similar to that used originally in $[1, 2]$. On the one hand, we expect that all the constructions made in this paper and its predecessor $[6]$ will still go through. But, on the other hand, even for that definition we do not see that it saves the desired density property in more cases than described above.

What other questions discussed in the Ashtekar framework could be touched by the choice of $\mathcal{P}$? One area we mentioned above – the diffeomorphism invariance of quantum gravity. Here, obviously, we have to admit at least smooth paths. Another problem is quantum geometry. For instance, the definition of the area operator $[4]$ enforced the usage of at most the analytic category. There one has to calculate sums over intersection points of spin networks with surfaces. But, since there can exist infinitely many such points when working with smooth paths, these sums can be infinite. This problem could be solved if there would exist for every fixed surface $S$ in $M$ a basis of $L_2(\mathcal{A}, \mu_0)$, such that every base element has only finitely many intersection points with $S$. But this seems very unlikely.

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Appendix

A Additional Results for $\overline{A/G}$

In this appendix we give three corollaries about assertions that can be proven not only for $A$, but also for $\overline{A/G}$. For the definition of $\overline{A/G}$ and the used notation we refer to [6].

Corollary A.1 $\pi_{\Gamma} : \overline{A/G} \rightarrow \overline{A/G}_{\Gamma}$ and $\pi_{\Gamma} : \overline{A/G} \rightarrow \overline{A/G}_{\Gamma}$ are surjective for all graphs $\Gamma$.

Proof Let $[h_{\Gamma}] \in \overline{A/G}_{\Gamma} \equiv \overline{A_{\Gamma}/G_{\Gamma}}$. From Proposition 3.9 follows the existence of an $h \in A$ with $\pi_{\Gamma}(h) = h_{\Gamma}$. Then, $([\pi_{\Gamma}(h)])_{\Gamma'} \in \overline{A/G}$ with $\pi_{\Gamma'}(([\pi_{\Gamma}(h)])_{\Gamma'}) = [\pi_{\Gamma}(h)] = [h_{\Gamma}]$. Analogously $\pi_{\Gamma}([h]) = [h_{\Gamma}]$ holds for $[h] := \pi_{\overline{A/G}}(h) \in \overline{A/G}$, whereas $\pi_{\overline{A/G}} : A \rightarrow \overline{A/G}$ is the canonical projection. qed

Corollary A.2 $\pi_{\Gamma} : \overline{A/G} \rightarrow \overline{A_{\Gamma}/G_{\Gamma}} \equiv \overline{A_{\Gamma}/G_{\Gamma}}$ is open for all graphs $\Gamma$.

Proof This assertion comes from the surjectivity and the continuity of $\pi_{\overline{A/G}}$, from the openness of $\pi_{\Gamma} : A \rightarrow A_{\Gamma}$ and $\pi_{\overline{A_{\Gamma}/G_{\Gamma}}}$ as well as from the commutativity of the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_{\overline{A/G}}} & \overline{A/G} \\
\downarrow{\pi_{\Gamma}} & & \downarrow{\pi_{\Gamma}} \\
A_{\Gamma} & \xrightarrow{\pi_{\overline{A_{\Gamma}/G_{\Gamma}}}} & \overline{A_{\Gamma}/G_{\Gamma}}
\end{array}
\]

qed

Every measure on a compact $A$ induces a measure on $\overline{A/G}$ via

Definition A.1 Let $\mu$ be a Borel measure on $A$.

Define $\mu_{\overline{G}}(U) := \mu(\pi_{\overline{A/G}}^{-1}(U))$ for all Borel sets $U$ on $\overline{A/G}$.

Proposition A.3 $\mu_{\overline{G}}$ is a Borel measure on $\overline{A/G}$ for all Borel measures $\mu$ on $A$.

Especially, the induced Haar measure can be transferred from $A$ to $\overline{A/G}$.

B Denseness Lemma for Projective Limits

Lemma B.1 Let $A$ be a set, $X_a$ be a topological space for each $a \in A$ and $\leq$ be a partial ordering on $A$. Let $\pi_{a_2} : X_{a_2} \rightarrow X_{a_1}$ for all $a_1 \leq a_2$ be a continuous and surjective map with $\pi_{a_2} \circ \pi_{a_3} = \pi_{a_1}$ if $a_1 \leq a_2 \leq a_3$. Furthermore, let $\pi_a : \lim_{a' \in A} X_{a'} \rightarrow X_a$ be the usual projection on the $a$-component and $X$ be some subset of $\lim_{a \in A} X_a$. Then $X$ is dense in $\lim_{a \in A} X_a$ if
A is directed, i.e., for any two $a', a'' \in A$ there is an $a \in A$ with $a', a'' \leq a$, and

$\pi_a(X)$ is dense in $X_a$ for all $a \in A$.

**Beweis** Let $U \subseteq \lim_{\to} X_a$ be open and non-empty, i.e., $U \supseteq \bigcap_i \pi_{a_i}^{-1}(V_i) \neq \emptyset$ with open $V_i \subseteq X_{a_i}$ and finitely many $a_i \in A$. Since $A$ is directed, there is an $a \in A$ with $a_i \leq a$ for all $i$ and thus $U \supseteq \pi_a^{-1}\left(\bigcap_i (\pi_{a_i})^{-1}(V_i)\right)$ with non-empty $V := \bigcap_i (\pi_{a_i})^{-1}(V_i) \subseteq X_a$. $V$ is open because $\pi_{a_i}$ is continuous. Since $\pi_a(X)$ is dense in $X_a$ for all $a$, there is an $x \in X$ with $\pi_a(x) \in V$ and so $\pi_{a_i}(x) \in V_i$ for all $i$, hence $x \in U$. \qed

**References**


