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STABILITY OF QUASICONVEX HULLS AND DEFORMATIONS WITH FINITELY MANY GRADIENTS

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Abstract. We answer a question by Kewei Zhang concerning the existence of sets with stable quasiconvex hulls. As a consequence we confirm a conjecture by John M. Ball about the existence of lipschitz maps using finitely many gradients without any rank-one connection. These functions are obtained using a new argument which unifies the convex integration method and the present Baire category approach to the existence of solutions of partial differential inclusions.

In this short note we address the question of existence of exact and stability of approximate solutions for variational problems of the type

$$\mathcal{F}_K(C) = \inf\left\{\int_\Omega \operatorname{dist}(K, \nabla f(x))\,dx \ ;\ f:\Omega\subset\mathbb{R}^m\to\mathbb{R}^n \text{ lipschitz and } f_{|\partial\Omega}\equiv C\in\mathbb{M}^{n\times m}\right\},$$

where $K\subset\mathbb{M}^{n\times m}$ is a compact set. In [12] the following question, concerning the set $K^{qc} = \{C\ ;\ \mathcal{F}_K(C)=0\}$ of those boundary values which can be realized at arbitrarily small cost, was asked. For which K is K^{qc} stable under small pertubations, or in other words: what are the points of continuity when we consider the map $K\to K^{qc}$ mapping the space of compact sets equipped with the Hausdorff metric into itself? As the map turns quite naturally out to be upper semicontinuous, it is crucial to find sets K for which the so called quasiconvex hull (see [6] for a broader introduction into this subject discussing the related Gradient Young Measures and the underlying duality with quasiconvex functions) K^{qc} does not suddenly decrease under small pertubations of K itself. Here we we give the first general construction of nontrivial examples of this kind.

As finite sets are dense in the Hausdorff metric, we find in this way also generalized "Tartar squares" (see [11] or below for more details stable under small pertubations in the full space dimension. This new stability properties of our configurations allows us a more flexible reiteration of the minimization procedure and in this way we obtain nonaffine lipschitz maps f satisfying $\nabla f(x)\in K$ almost everywhere even for some finite sets K without rank-one connections. This confirms a conjecture by John M.Ball, see [1], about the existence of such maps.

The other contribution of the paper concerns the way such exact solutions are actually constructed. Once our stability result is established, the map can be (and in fact was originally) obtained using the Müller-Šverák result on convex intergration via rank-one convex in-approximations as given e.g. in Theorem 3.2. in [7]. However, we prefer to present also a new, simple, enlightening and in the same time very flexible argument for the existence of solutions to differential inclusions. It essentially shows that for most (in the topological sense) of the lipschitz functions with gradients in a fixed bounded set U the gradient of the functions is stable under small L^∞-pertubations of the function itself. Therefore, the value of the gradient has to be in those parts of U where no rank-one segments pass trough as they would allow to construct and add arbitrarily small "laminations" with nonnegligible gradients. Our approach also shows that there is no basic difference between the Baire category method (see e.g. [4]) and the "convex integration"-approach, e.g. as presented in

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its most flexible way in [8]. In fact, the most striking feature of the "convex integration"-approach is the L^1-convergence of the gradients along minimizing sequences. But from what we just said it is clear that this phenomenon occurs also in the categorial framework - even if this simple fact seemed to be overlooked until now, which unnecessarily reduced the flexibility of the Baire category method.

Our approach to the stability result is in fact a rather geometrical one and uses the notion of rank-one convexity instead of quasiconvexity. Therefore, a few words explaining their relations are in place. It is quite easy to check that if $K = \{A, B\}$ and $\operatorname{rank}(A - B) = 1$ then each $C \in [A, B] = \{\lambda A + (1-\lambda)B \; ; \; \lambda \in [0,1]\}$ satisfies $\mathcal{F}_K(C) = 0$, see e.g. the proof of Proposition 6 for the "lamination construction" of suitable f's. Consequently, the lamination convex hull K^{lc} defined via

$$L_0(K) = K, L_{k+1}(K) = \bigcup\{[A, B] \; ; \; A, B \in L_k(K) \text{ and } \operatorname{rank}(A - B) \leq 1\}, K^{lc} = \bigcup_{k \geq 0} L_k(K)$$

will certainly be contained in M^{qc}. However, as rank-one connections are very fragile under pertubations, we will have to use the more robust "functional" hull defined by duality with rank-one convex functions, i.e. $f : \mathbb{M}^{n\times m} \to \mathbb{R}$ such that $t \in \mathbb{R} \to f(A + t\cdot B)$ is convex whenever $A, B \in \mathbb{M}^{m\times n}$ and $\operatorname{rank}(B) = 1$. Indeed, if we set

$$K^{rc} = \left\{A \in \mathbb{M}^{n\times m} \; ; \; f(A) \leq \sup f(K) \text{ for all rank-one convex } f : \mathbb{M}^{n\times m} \to \mathbb{R}\right\},$$

then obviously $K^{lc} \subset K^{rc}$. But more interesting, we also have $K^{rc} \subset K^{qc}$. Indeed, the already mentioned lamination construction shows that $\mathcal{F}_K$ itself is rank-one convex (as a matter of fact which we will not need here, it is just a multiple of the quasiconvexification of $\operatorname{dist}(K, \cdot)$, see e.g. [3]). The fact that K^{rc} can be nontrivial even if $K = K^{lc}$ was first explicitely noticed in 1983, compare the notes in [10] about the history, considering

$$K = \{\operatorname{diag}(1, -3), \operatorname{diag}(3, 1), \operatorname{diag}(-1, 3), \operatorname{diag}(-3, -1)\} \subset \mathbb{M}^{2\times 2}.$$

The example was independently discovered at several other occassions, references can be found in [6]. However, it was folklore that for this set $\nabla f \in K$ a.e. implies f affine because K^{rc} is too small in $\mathbb{M}^{2\times 2}$ to allow a gradual moving of gradients into this set. (Nevertheless, an elementary calculation using some special symmetries of this particular set K shows that K^{qc} is stable under small pertubations even in the full $\mathbb{M}^{2\times 2}$ dimensions. It seems that the author of [12] overlooked this fact when formulating Remark 3.2 there, but a variant of it was extensively used in [7].) In [2] it will be shown that also in general there is no 4-point configuration K without rank-one connection that allows nonconstant $\nabla f(x) \in K$ a.e. On the contrary, in [5] we will present a special possible 5-point configuration and also some particularly simple (i.e. countably piecewise affine) functions of this kind - however, using more than five gradients.

We set $B(M, r) = \{x \; ; \; \operatorname{dist}(x, M) < r\}$ and $B(x, r) = B(\{x\}, r)$. The oridinary convex hull of K is denoted K^c and $|M|$ is the Lebesgue measure of M.

Our first argument provides some lamination convex hull with nonvoid interior.

Lemma 1. *Let $X \in \mathbb{M}^{n\times m}$ be of rank one. Then for any ε positive there is a set $\mathcal{M}_X \subset B(X, \varepsilon) \cap \{Y \; ; \; \operatorname{rank}(Y) = 1\}$ consisting of at most $4nm$ points such that $X \in \operatorname{int}(\mathcal{M}_X^c)$ and $(\{0\} \cup \mathcal{M}_X)^c = (\{0\} \cup \mathcal{M}_X^c)^c = (\{0\} \cup \mathcal{M}_X^c)^{lc}$. Moreover, for each $f : \mathbb{M}^{n\times m} \to \mathbb{R}$ rank-one convex we have*

$$f(Y) < \max f(\mathcal{M}_X^c) \text{ if } f(0) < f(Y) \text{ and } Y \in (\{0\} \cup \mathcal{M}_X)^c \setminus \mathcal{M}_X^c. \tag{1}$$

Proof. After a suitable transformation using pre- and postmultiplication we can reduce the problem to the case when $X = e_1 \otimes e_1$. Let the $\varepsilon > 0$ be given, we fix a positive $\delta < 1, \varepsilon/4nm$ Our set $\mathcal{M}_X$ will consist of the following points. Gradually, we take all $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$, and in case that $i = 1$ or $j = 1$ we add $M_{i,j}^k = X + (-1)^k \delta^2 e_i \otimes e_j$ for $k = 1, 2$ to our set. Else we put for $k = 1,2$ and $l = 1,2$ $M_{i,j}^{k,l} = X + (-1)^k \delta^2 e_i \otimes e_j + (-1)^l \delta(e_1 \otimes e_j + (-1)^k e_i \otimes e_1)$ into our set and finally we obtain $\mathcal{M}_X$. It is clear that $\mathcal{M}_X \subset B(X,\varepsilon) \cap \{Y \,;\, \mathrm{rank}(Y) = 1\}$ and as for any (i,j) both points $X \pm \delta^2 e_i \otimes e_j$ belong to $\mathcal{M}_X^c$, we also have $X \in \mathrm{int}(\mathcal{M}_X^c) \subset B(X,\varepsilon)$.

Finally, we have to verify the remaining inclusion and the estimate for f. For this purpose we proceed by induction with respect to the length of the convex combination involved. It is clear that, for any $Z \in \bigcup_{t\in[0,1]} t\mathcal{M}_X^c = (\{0\} \cup \mathcal{M}_X)^c$ we can choose a shortest representation of the kind

$$Z = \sum_{P\in\mathcal{M}_Z} \lambda_P P \text{ where } \mathcal{M}_Z \subset \mathcal{M}_X, \lambda_P > 0 \text{ and } \sum_{\mathcal{M}_Z} \lambda_P \le 1.$$

Arguing by contradiction, we choose among all $Z \in (\{0\} \cup \mathcal{M}_X)^c$ such that either $Z \notin (\{0\} \cup \mathcal{M}_X^c)^{lc}$ or that (1) fails one matrix, denoted by Z_0, which minimizes the cardinality of $\mathcal{M}_Z$. Obviously, $\mathrm{card}(\mathcal{M}_{Z_0}) > 0$ since $Z_0 \neq 0$. Hence, we can find $P_0 \in \mathcal{M}_{Z_0}$ and set $\mu = \lambda_{P_0}/(1 - \sum_{P\neq P_0} \lambda_P)$. It is easy to check that

$$Z_0 = \mu Z_1 + (1-\mu) Z_2 \text{ with } Z_1 = (1 - \sum_{P\neq P_0} \lambda_P) P_0 + \sum_{P\neq P_0} \lambda_P P \text{ and } Z_2 = \sum_{P\neq P_0} \lambda_P P.$$

Checking the sum of the weights one easily verifies that Z_1 is contained in $\mathcal{M}_X^c$. By minimality of $\mathrm{card}(\mathcal{M}_{Z_0})$ we know $Z_2 \in (\{0\} \cup (\mathcal{M}_X^c))^{lc}$ and $f(Z_2) < \max f(\mathcal{M}_X^c)$ or $f(Z_2) \le f(0)$. Since Z_1, Z_2 differ only by the rank-one matrix $(1 - \sum_{P\neq P_0} \lambda_P) P_0$, we conclude that their convex combination Z_0 belongs to $(\{0\} \cup \mathcal{M}_X^c)^{lc}$ as well. Because $Z_1 - Z_0, Z_0 - Z_2 \neq 0$, rank-one convexity of f implies that $f(Z_0) < \max f(\mathcal{M}_X^c)$ if $f(Z_0) > f(0)$. This contradiction shows that we are done. □

Theorem 2. *Let $U \subset \mathbb{M}^{n\times m}$ be open and bounded. Then for any compact set C in U there is a positive ε such that $C \subset M^{rc}$ whenever the set M fulfills $\partial U \subset B(M,\varepsilon)$.*

Proof. Obviously, the result follows once we know that it is true for C being any closed ball $\bar{B}(X_0,R) \subset U$. For later use in the proof of Proposition 6, we will more specifically show the existence of an $\varepsilon > 0$ such that for each set M satisfying $\partial U \subset B(M,\varepsilon)$ there is $S \subset M$ of the following kind. If $X \in S$, then there exists a set $\mathcal{M}_X$ such that

(i) $\mathcal{M}_X - X \subset \{Y \,;\, \mathrm{rank}(Y) = 1\}$,
(ii) $\mathcal{M}_X$ is a subset of $B(X_0,R)$ of cardinality at most $4nm$
(iii) $(\{X\} \cup \mathcal{M}_X)^c = (\{X\} \cup \mathcal{M}_X^c)^{lc}$, and
(iv) $\bigcup_{X\in S} \mathrm{int}(\{X\} \cup \mathcal{M}_X)^c \supset \partial B(X_0,R)$.

For this purpose, we fix any $Y \in \partial B(X_0,R)$. Since $Y - X_0$ is the sum of rank-one matrices, we find D_Y of rank one such that $\langle Y - X_0, D_Y\rangle > 0$. We define $Y_t = Y + t\cdot D_Y$, then $Y_t \notin \bar{B}(X_0,R)$ whenever t is positive. We choose $t_0 > 0$ such that $X_Y = Y_{t_0} \in \partial U$ and select $P_Y = Y_{t_1} \in B(X_0,R)$ for some t_1 negative but sufficiently close to zero. Finally, we fix $r_Y > 0$ such that $\bar{B}(P_Y, 3r_Y) \subset B(x_0,R)$. Now, Lemma 1 ensures the existence of $\delta_Y \in (0,r_Y)$ and $\mathcal{M}_{X_Y} \subset B(P_Y,r_Y)$ fulfilling the properties (i),(ii), (iii) from above and such that $B(Y,2\delta_Y) \subset (\{X_Y\} \cup \mathcal{M}_{X_Y}^c)^{lc}$. By compactness of $\partial B(X_0,R)$ we find

$$Y_1,\ldots,Y_N \in \partial B(X_0,R) \text{ such that } \bigcup_{i=1}^N B(Y_i,\delta_{Y_i}) \supset \partial B(X_0,R).$$

We also select the desired positive $\varepsilon < \mathrm{dist}(\bar{B}(X_0,R), \mathbb{M}^{n\times m} \setminus U), \min_i \delta_{Y_i}$.

Now we take any set M satisfying $B(M,\varepsilon) \supset \partial U$. Then for each $i \le N$ we find $X_i \in M$ such that $X_{Y_i} \in B(X_i,\varepsilon)$ and set $S = \{X_i \,;\, i \le N\}$. We claim that $\bar{B}(X,R) \subset S^{rc}$ and, moreover, that puting $\mathcal{M}_{X_i} = \mathcal{M}_{X_{Y_i}} + (X_i - X_{Y_i})$ we obtain the sets fulfilling (i), ... ,(iv) from above.

Indeed, (i),(ii), (iii) are clearly satisfied and, as concerns (iv) it is easily verified using the way the Y_i's were choosen and the fact that for each $i \le N$

$$B(Y_i,\delta_{Y_i}) \subset B(Y_i,2\delta_{Y_i}) + (X_i - X_{Y_i}) \subset \mathrm{int}(\{X_{Y_i}\} \cup \mathcal{M}_{X_{Y_i}}^c)^{lc} + (X_i - X_{Y_i}) = \mathrm{int}(\{X_i\} \cup \mathcal{M}_{X_i}^c)^{lc}.$$

Finally, if $\bar{B}(X_0, R)$ would not be contained in S^{rc} then there would be a rank-one convex function $f : \mathbb{M}^{n\times m} \to [0,\infty)$ vanishing on S but attaining value 1 in some $Z_0 \in \bar{B}(X_0, R)$. We can of course even assume $1 = \max(f(\bar{B}(X_0, R)))$ and that $Z_0 \in \partial B(X_0, R)$. Hence due to (iv), we find $i \le N$ such that $Z_0 \in \operatorname{int}(\{X_i\} \cup \mathcal{M}^c_{X_i})^{lc}$. Moreover, because $\mathcal{M}^c_{X_i} \subset B(P_{Y_i}, 2r_{Y_i}) \subset B(X_0, R - r_{Y_i})$, we have $Z_0 \notin \mathcal{M}^c_{X_i}$. Since $f(Z_0) > f(X_i)$ we get from (1) that $f(Z_0) < \max f(\mathcal{M}^c_{X_i}) \le \max(f(\bar{B}(X_0, R)))$. This contradiction finishes our proof. □

Corollary 3. *If $X, Y \in \mathbb{M}^{n\times m}$ and $\operatorname{rank}(X - Y) = 1$ then, for any $\varepsilon > 0$, the set $\bar{B}(\{X, Y\}, \varepsilon)$ is stable, i.e $K \to K^{qc}$ is continuous at $\bar{B}(\{X, Y\}, \varepsilon)$.*

Corollary 4. *For any compact set $C \subset \mathbb{M}^{n\times m}$, $\partial(C^{qc})$ is stable.*

This settles two questions from [12]. Finally, we want to apply the stability results just obtained to construct nonaffine lipschitz mappings using only finitely many gradients without any rank-one connection among them. But first we develop the new existence result, which conversely would allow to prove the originally used in-approximation result in [7].

Theorem 5. *Let U, K be bounded sets in $\mathbb{M}^{n\times m}$, U open and K compact. Assume that for each $\varepsilon > 0$ there is a $\delta = \delta_\varepsilon > 0$ such that for all $A \in U \setminus \bar{B}(K, \varepsilon)$ there exists a lipschitz and countably piecewise affine $\varphi_A : \mathbb{R}^m \to \mathbb{R}^n$ with bounded support which satisfies*

- *$A + \nabla\varphi_A(x) \in U$ for a.e. x*
- *$\int |\nabla\varphi_A(x)| > \delta |\operatorname{spt}(\varphi)|$.*

Given any $A \in U$ and $\Omega \subset \mathbb{R}^m$ bounded open, let $\mathscr{P}$ be the space of all countably piecewise affine lipschitz $f : \Omega \to \mathbb{R}^n$ with $f_{|\partial O} \equiv A$ and $\nabla f(x) \in U$ almost everywhere. Then typical $f \in \overline{\mathscr{P}}^{L_\infty}$, which means all f except those from a set of first Baire category (also called a meager set) in $\overline{\mathscr{P}}^{L_\infty}$, satisfies $\nabla f(x) \in K$.

Proof. As $\overline{\mathscr{P}}^{L_\infty}$ consists of functions with a uniform lipschitz constant, the map $\nabla : f \to \nabla f$ maps the complete metric space $(\overline{\mathscr{P}}^{L_\infty}, \|\cdot\|_\infty)$ into $L^1(\Omega)$. Moreover, considering naturally defined difference quotients it is clear that ∇ is the pointwise limit of continuous functions from $\overline{\mathscr{P}}^{L_\infty}$ into $L^1(\Omega)$. Such a map is called a Baire-one function and it is well known, see e.g. Chapter 7 of [9] for a proof which works in any complete metric space, that a Baire-one function is continuous in residually many points. So it remains to show that $\nabla f(x) \in K$ a.e. if f is a point of continuity of ∇ restricted to $\overline{\mathscr{P}}^{L_\infty}$. But else, we would find a compact set $\tilde{C} \subset \Omega$ and an $\varepsilon > 0$ such that

- $|\tilde{C}| > \varepsilon$, $\nabla f_{|\tilde{C}}$ is continuous, and
- $\operatorname{dist}(\nabla f(x), K) > \varepsilon$ for all $x \in \tilde{C}$.

We pick an $\eta > 0$ such that $\|\nabla f - \nabla g\|_{L^1} < \varepsilon\delta_\varepsilon/4$ whenever $g \in B_\infty(f, \eta) \cap \overline{\mathscr{P}}^{L_\infty}$. By definition of $\overline{\mathscr{P}}^{L_\infty}$ there is a sequence of $f_k \in \mathscr{P}$ with $f_k \rightrightarrows f$ and hence, due to the choice of f, satisfying $\nabla f_k(x) \to \nabla f(x)$ a.e. Consequently, we find $k_0 \in \mathbb{N}$ and another compact $\hat{C} \subset \Omega$ with $|\hat{C}| > \varepsilon$ and $\operatorname{dist}(\nabla f_{k_0}(x), K) > \varepsilon$ for all $x \in \hat{C}$. Of course, we can also assume that $\|f - f_{k_0}\|_\infty < \eta/2$. As f_{k_0} is countably piecewise affine, there are disjoint open subsets $\{G_i\}_1^\infty$ of Ω such that $|\Omega \setminus \bigcup G_i| = 0$ and $f_{k_0|G_i}$ is affine with gradient $A_i \in U$. According to our assumption we pick for each i a function φ_{A_i}. Rescaling this function in both domain and image by the same rate, we do not change the (distribution of the) gradient. Using many such rescaled copies, whose disjoint supports exhaust almost all of G_i, we obtain the following. For each i there is a countably piecewise affine lipschitz map $\varphi_i : G_i \to B(0, \eta/2; \mathbb{R}^n)$ with $\int_{G_i} |\nabla\varphi_i(x)| dx > \delta_\varepsilon |G_i|$, $\varphi_{i|\partial G_i} \equiv 0$ and $A_i + \nabla\varphi_i(x) \in U$ a.e. in G_i. Then it is clear that $g = f_{k_0} + \sum_i \varphi_i$ belongs to $B_\infty(f, \eta) \cap \mathscr{P}$ and the same is true for f_{k_0} itself. However, we have $\|\nabla(f_{k_0} - g)\|_{L^1} \ge \varepsilon\delta_\varepsilon$, and this contradiction to the choice of η shows that $\nabla f(x) \in K$ a.e. indeed. □

Proposition 6. *For any $n, m \geq 2$ we can find a finite number of matrices $A_1, \dots, A_N \in \mathbb{M}^{n\times m}$ and an $\varepsilon > 0$ such that the following holds. If $B_i \in B(A_i, \varepsilon)$ for each $i \leq N$ then*

i) $\operatorname{rank}(B_i - B_j) = \min(m, n)$ *for $i \neq j$,*
ii) *there exists a lipschitz map $f : (0,1)^n \to \mathbb{R}^m$ which is nonaffine (but has affine boundary data) and fulfills*

$$\nabla f(x) \in \{B_1, \dots, B_N\} \text{ a.e. in } (0,1)^n.$$

Proof. We consider the open unit ball $B(0,1)$ in $\mathbb{M}^{n\times m}$ and the compact subball $C = \bar{B}(0, 1/2)$. In this situation, we fix an $\varepsilon_1 > 0$ whose existence is stated in the begin of the proof of Theorem 2.

Now we start by choosing any finite set $\mathcal{A}_0 \subset \partial B(0,1)$ with $B(\mathcal{A}_0, \varepsilon_1/2) \supset \partial B(0,1)$. As for any minor M of order $\min(n,m)$ and any $X \in \mathbb{M}^{n\times m}$ the set $\{Y \,;\, M(X-Y) \neq 0\}$ is open and dense in $\mathbb{M}^{n\times m}$, we can certainly find a set $\mathcal{A}_1$ with $\operatorname{card}(\mathcal{A}_1) = \operatorname{card}(\mathcal{A}_0)$, $\mathcal{A}_0 \subset B(\mathcal{A}_1, \varepsilon_1/8)$ and $\operatorname{rank}(A_i - A_j) = \min(m,n)$ if $A_i, A_j \in \mathcal{A}_1$ are different. Since we have to consider only finitely many pairs, $\operatorname{rank}(B_i - B_j) = \min(m,n)$ is preserved provided $\varepsilon_2 > 0$ is small enough and $B_i \in B(A_i, \varepsilon_2)$ for all i. Finally, we claim that $(\mathcal{A}_1, \varepsilon)$, where $\varepsilon = \min(\varepsilon_2, \varepsilon_1/8)$ is the promised pair.

Indeed, if we fix arbitrary $B_i \in B(A_i, \varepsilon)$ then (i) was already verified. Put $M = \{B_i \,;\, i \leq \operatorname{card}(\mathcal{A}_1)\}$, then $\partial B(0,1) \subset B(\mathcal{A}_1, 5\varepsilon_1/8) \subset B(M, \varepsilon_1)$. So, by the choice of ε_1 we find $S \subset M$ which has for $B(X_0, R) = B(0, 1/2)$ all properties (i), ... ,(iv) stated in the proof of Theorem 2. We set $U = B(0,1/2) \cup \bigcup_{X\in S} \operatorname{int}(\mathcal{C}_X)$, where $\mathcal{C}_X = (\{X\} \cup \mathcal{M}_X^c)^{lc} = (\{X\} \cup \mathcal{M}_X)^c$, $K = S$ and finish the proof by showing that the assumptions of Theorem 5 are satisfied. As $0 \in U$ we would then of course obtain f with $\nabla f(x) \in K$ a.e. but with zero boundary data and hence nonaffine.

To verify the assumptions, we first recall the following well-known lamination construction, compare e.g. with the proof of Lemma 4.3 in [6]. Whenever the segment $[A, B] \subset \mathbb{M}^{n\times m}$ has a rank-one direction, then for $C = (A+B)/2$ and any $\eta > 0$ there is a lipschitz piecewise affine $\varphi_C : \mathbb{R}^m \to \mathbb{R}^n$ with bounded support which satisfies

- $C + \nabla\varphi_C(x) \in \{A, B\} \cup B(C, \eta)$ for a.e. x
- $\int |\nabla\varphi_A(x)| > |A - B||\operatorname{spt}(\varphi)|/3$.

In fact, it is quite clear that we can without loss of generality assume that $C = 0$, and after a suitable transformation using pre- and postmultiplication that $A = -B = e_1 \otimes e_1$. For fixed $k \geq 2$ we consider $P = [-k, k]^m$ and the auxilary function $h : \mathbb{R} \to \mathbb{R}$ which is 1-periodic and fulfills $h(0) = 0$, $h'(t) = 1$ if $t \in (0, 1/2)$, and $h'(t) = -1$ for $t \in (1/2, 1)$. We set

$$f_k(x) = \min\Big(\min_{i=2,\dots,m}\Big(\sqrt{k} - \frac{|x_i|}{\sqrt{k}}\Big), h(x_1)\Big).$$

As $h \geq 0$ we see that $f_k \geq 0$ on P. We can also define $f_k \equiv 0$ outside P since $f_k \equiv 0$ on ∂P, because in all boundary points either the first minimum vanishes or $|x_1| = k$ which implies $h(x_1) = 0$. As f is affine on finitely many pieces, is also quite easy to see that $\nabla f_k \in \{e_1, -e_1\} \cup \{e_i/\sqrt{k}, -e_i/\sqrt{k} \,;\, i = 2, \dots, m\}$ almost everywhere in P. Moreover, since $\min_{i\geq 2} \sqrt{k} - (|x_i|/\sqrt{k}) \geq 2$ and hence $|\nabla f_k| = 1$ almost everywhere on $(1 - \frac{2}{\sqrt{k}})P$, we see that $\varphi_0(x) = e_1 \cdot f_k(x)$ does the job provided k is sufficiently large.

So, to finish it is enough to check the existence of a $\delta_0 > 0$ such that for each $A \in U \setminus \bar{B}(K, \varepsilon)$ there is a rank-one matrix $D \in \mathbb{M}^{n\times m}$ with $|D| \geq \min(\delta_0, \varepsilon/4nm)$ and $A + tD \in U$ for $|t| \leq 1$.

As $\bar{B}(0, 1/2) \subset \operatorname{int}(U)$, we find δ_0 with $\bar{B}(0, 1/2 + \delta_0) \subset U$ and so we need only to consider $A \in \operatorname{int}(\mathcal{C}_X)$ for some $X \in K$. But then $A - X = \sum_{Y\in\mathcal{M}_X} \lambda_Y (Y - X)$ and as $\operatorname{card}(\mathcal{M}_X) \leq 4nm$ we find $Y_A \in \mathcal{M}_X$ with $\lambda_{Y_A}|Y_A - X| \geq |A - X|/4nm > \varepsilon/4nm$. Obviously, $D = \min(1, \delta_0/\lambda_{Y_A}|Y_A - X|)\lambda_{Y_A}(Y_A - X)$ is a sufficiently large rank-one matrix. Moreover, the following simple geometric observation ensures that $[A - D/2, A + D/2] \subset U$. It says that if $A - X = \sum_{P\in\mathcal{M}_A} \lambda_P (P - X)$ with $\mathcal{M}_A \subset \mathcal{M}_X$, $\sum \lambda_P \leq 1$ and $\lambda_P > 0$ for all $P \in \mathcal{M}_A$ and if $A \in \operatorname{int}(\mathcal{C}_X)$ then $\tilde{B} \in U$ whenever $\tilde{B} = \sum_{P\in\mathcal{M}_A} \tilde{\lambda}_P (P - X)$ with $\sum \tilde{\lambda}_P \leq 1$ and $\tilde{\lambda}_P > 0$ for all P. Indeed, we can of course suppose

$\tilde{B} \notin \mathcal{M}_X^c \subset U$ and hence $\sum \tilde{\lambda}_P < 1$. We fix $\eta > 0$ such that $B(A, \eta) \subset \mathcal{C}_X$ and $\kappa \in (0, 1/2(1 - \sum \tilde{\lambda}_P))$ with $\kappa \max_P \lambda_P < \min_P \tilde{\lambda}_P$. Then we can represent each $C \in B(\tilde{B}, \kappa\eta)$ in the form

$$C - X = \kappa(A + \frac{C - \tilde{B}}{\kappa} - X) + ((\tilde{B} - X) - \kappa(A - X)) \in \kappa(\mathcal{C}_X - X) + ((\tilde{B} - X) - \kappa(A - X)) \subset \mathcal{C}_X - X$$

where the last inclusion is checked easily. Therefore, we are done. □

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