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Abstract

We study an infinite horizon optimal control problem for a system with two state variables. One of them has the evolution governed by a controlled ordinary differential equation and the other one is related to the latter by a hysteresis relation, represented here by either a play operator or a Prandtl-Ishlinskii operator. By dynamic programming, we derive the corresponding (discontinuous) first order Hamilton-Jacobi equation, which in the first case is of finite dimension and in the second case is of infinite dimension. In both cases we prove that the value function is the only bounded uniformly continuous viscosity solution of the equation.

1 Introduction.

Hysteresis is a common feature of several physical and natural phenomena. It may occur in any input/output relationship between time-dependent quantities. More precisely, we say that an input/output relationship presents hysteresis if: (*memory*) the value of the output at time t does not only depend on the value of the input at the same instant, but also on the whole past history of the input; (*rate-independence)* the value of the output at time t does not depend on the "velocity" of the input but it depends only on the sequence of values reached by the input during its history.

The probably most known example of hysteresis is the one occurring in ferromagnetism: the relationship between magnetic field and magnetization of a ferromagnetic material. Other important examples are in elastoplasticity (stress/strain), filtration through porous media (pressure/saturation), phase transitions (temperature/phase) and also in superconductivity, mechanical damage, shape memory alloys and behavior of thermostats. In biology, hysteresis may occur for instance in the relationship between the concentration of nutrients and the activity of bacteria, in economics between profit and investment.

One way to represent hysteresis effects is the use of the so-called *hysteresis operators*, namely suitable (nonlinear) functionals between spaces of time-dependent functions. We shall restrict ourselves to scalar hysteresis, that is the case where both input and output are scalar time-dependent functions. We give a rather general definition. Let B be a Banach space, $[0, T]$ be a time-interval. A functional

$$\mathcal{F} : C^0([0, T]) \times B \to C^0([0, T]), \quad (u, \xi) \mapsto \mathcal{F}[u, \xi]$$

is said to be a *hysteresis operator* if: *(causality)* for all $\xi \in B$, u, $v \in C^0([0, T])$, $t \in [0, T]$ we have that $u = v$ in $[0, t] \Rightarrow \mathcal{F}[u, \xi](t) = \mathcal{F}[v, \xi](t)$; *(rate-independence)* for all continuous and nondecreasing $\varphi : [0, T] \to [0, T]$, $\xi \in B$, $u \in C^0([0, T])$ we have that $\mathcal{F}[u \circ \varphi, \xi] = \mathcal{F}[u, \xi] \circ \varphi$. The elements of the Banach space B represent the *initial states* of the system. To know the value of the output at certain time, we need to know the previous evolution of the input and the initial values of some (possibly infinite) internal variables. If the internal variables coincide with the value of the output itself then we have $B = I\!\!R$. But there are many cases, where the system evolves other internal variables which are not recognizable by the only observation of the output. Only the knowledge of such internal variables (together with the evolution of the input) gives the knowledge of the evolution of the output.

Now, we describe the control problem. We consider the following controlled dynamical system

$$\begin{cases} y'(t) = f(y(t), w(t), \alpha(t)) \quad w(t) = \mathcal{F}[y, \xi_0](t) \quad t \geq 0, \\ y(0) = y_0 \end{cases} \tag{1.1}$$

where $(y_0, \xi_0) \in \mathcal{O} \subseteq I\!\!R \times B$ is the initial state, $\alpha : [0, +\infty[\to A$ is the measurable control, $y, w \in I\!\!R$, $\mathcal{F}$ is a hysteresis operator, $f : I\!\!R \times I\!\!R \times A \to I\!\!R$ (note that the controlled dynamic "sees" only the "external state" w, but the true state of the system is the couple $(y, \xi) \in I\!\!R \times B$; moreover we can "directly control" only the evolution of the state variable y, the other one depending with hysteresis). We then consider the problem of minimizing a cost functional of Bolza type

$$J(y_0, \xi_0, \alpha) := \int_0^{+\infty} e^{-\lambda t} l(y(t), w(t), \alpha(t)) dt,$$

with $\lambda > 0$ and $l : I\!\!R \times I\!\!R \times A \to I\!\!R$. We define the value function

$$V(y_0, \xi_0) = \inf_{\alpha} J(y_0, \xi_0, \alpha) \quad (y_0, \xi_0) \in \mathcal{O},$$

and want to apply the dynamic programming method in order to derive and to study the (possibly infinite dimensional) corresponding Hamilton-Jacobi equation. We in particular analyze two cases: when the hysteresis operator is a Play operator and when it is a Prandtl-Ishlinskii operator. Both operators are useful for describing several hysteresis relationship, especially in elastoplasticity.

In the first case, the operator is of the so-called *local memory* type, that is without internal variables. Its behavior is described in the Subsection 2.1 (see Fig. 1). The set of initial states $\mathcal{O}$ in (1.1) is the closure of the strip of the (y, w)-plane

$$\Omega_\rho := \Big\{(y,w)\in I\!R^2 | y-\rho < w < y+\rho\Big\},$$

for some $\rho > 0$. We show that the output is solution of a discontinuous ordinary differential equation in $\overline{\Omega}_\rho$, with the discontinuity occurring on the boundary. This leads to the discontinuous Hamilton-Jacobi equation

$$\lambda V + H(y,w,V_y,V_w) = 0 \quad \text{in } \overline{\Omega}_\rho$$

with H given by

$$H(y,w,p_1,p_2) := \sup_{a\in A}\Big\{-p_1 f(y,w,a) \\ -p_2\Big(\chi_{\rho r}(y,w)f(y,w,a)^+ - \chi_{\rho l}(y,w)f(y,w,a)^-\Big) - l(y,w,a)\Big\}.$$

where $\chi_{\rho r}$ (respectively $\chi_{\rho l}$) is the characteristic function of the straight line $w = y-\rho$ (respectively $w = y+\rho$), and $(\cdot)^+$ (respectively $(\cdot)^-$) means the positive (respectively negative) part. By particular properties of the Play operator, we can prove that V is a viscosity subsolution of $\lambda V + H \le 0$ and a supersolution of $\lambda V + H^* \ge 0$, where H^* is the upper semicontinuous envelope of H. This of course is coherent with the definition of viscosity solution for discontinuous Hamiltonians (see Ishii [13]). By the particular form of H, we immediately get that V is a viscosity solution of a boundary value problem of Neumann-type in Ω_ρ. Note that H is continuous in Ω_ρ. For this problem we have of course the uniqueness of the solution. The problem here studied can be contained in the case of optimal control for the so-called Skorokhod problem (or reflecting boundary problem), see Lions-Sznitman [18], Lions [17] and Ishii-Dupuis [10] (see also Ishii [12]). However, here we use the formulation of the dynamic as a discontinuous ordinary differential equation, which is useful for studying the other case when the hysteresis relationship is given by the Prandtl-Ishlinskii operator. Moreover, in the above quoted works on the Skorokhod problem, the authors never mention hysteresis, they are more concerned with the problem of the reflection of a trajectory inside a domain.

In the second case, the operator is of the so-called *nonlocal memory* type. That is, there are internal variables. The operator consists of a superposition of weighted different Plays, each of them labeled by $\rho \in \mathcal{R} \subset]0,+\infty[$. Let μ be a measure on $\mathcal{R}$ and, for every ρ, let $\mathcal{F}_\rho$ be the corresponding Play. The Prandtl-Ishlinskii operator can be defined as

$$\mathcal{F} : C^0([0,T]) \times L^2(\mathcal{R},\mu) \to C^0([0,T]) \quad \mathcal{F}[u,\xi](t) = \int_{\mathcal{R}} \mathcal{F}_\rho[u,\xi(\rho)](t)d\mu.$$

The internal state $\xi \in L^2(\mathcal{R},\mu)$ is the function which maps ρ to the initial state of the corresponding Play. The set of the initial states for the system (1.1) is

$$\mathcal{O} := \Big\{(y,\xi)\in I\!R\times L^2(\mathcal{R},\mu)\Big|(y,\xi(\rho))\in\overline{\Omega}_\rho \ \mu-\text{a.e. } \rho\in\mathcal{R}\Big\}.$$

We use again the representation of the output of the Play as solution of a discontinuous ordinary differential equation. We get a Hamilton-Jacobi equation in $\mathcal{O}$ with Hamiltonian H on $\mathcal{O} \times I\!R \times L^2(\mathcal{R}, \mu)$ given by

$$
\begin{aligned}
&H(y,\xi,p,\psi) := \sup_{a\in A}\Big\{ - f(y,w,a)p \\
&- \int_{\mathcal{R}} \Big[\psi(\rho)\Big(\chi_{\rho r}(y,\xi(\rho))f(y,w,a)^+ - \chi_{\rho l}(y,\xi(\rho))f(y,w,a)^-\Big)\Big]d\mu \\
&\quad -l(y,w,a)\Big\}.
\end{aligned}
$$

where $w = \int_{\mathcal{R}} \xi d\mu$. Of course, the previous Hamiltonian is discontinuous in $\mathcal{O}$, and hence the definition of solution is given by two suitable envelopes of H. Note that $\mathcal{O}$ has no interior in $I\!R \times L^2(\mathcal{R}, \mu)$, however it is an invariant set for the evolution given by (1.1) (more precisely for the evolution of y and of the internal variables). Also in this case we can have a "boundary formulation" of Neumann-type, where the "boundary" is the following subset of $\mathcal{O}$

$$
\mathcal{O}' := \Big\{(y,\xi) \in \mathcal{R}\Big|\mu\left(\{\rho \in \mathcal{R},\ |y - \xi(\rho)| = \rho\}\right) > 0\Big\}.
$$

Note that if $(y,\xi) \notin \mathcal{O}'$, then the Hamiltonian is the following continuous and "finite dimensional" one

$$
H(y,\xi,p,\psi) := \sup_{a\in A}\Big\{ - f(y,w,a)p - l(y,w,a)\Big\}.
$$

Observe that $\mathcal{O}$ has nonempty interior in $I\!R \times L^\infty(\mathcal{R}, \mu)$ only if $\inf \mathcal{R} > 0$, and in that case $\mathcal{O}'$ is strictly contained in the boundary of $\mathcal{O}$ in $I\!R \times L^\infty(\mathcal{R}, \mu)$. Due to the fact that we are working in $I\!R \times L^2(\mathcal{R}, \mu)$ (that is because it is a reflexive space and hence, as usual, suitable for comparison viscosity techniques), the "boundary formulation" is not completely satisfactory since we are not able to force extremal points of suitable functions to be out of $\mathcal{O}'$ or at least to force the continuous equation to hold. Hence, the discontinuous term in the Hamiltonian (i.e. the term containing the integration with respect to μ) may always be different from zero. However, taking a suitably small penalization term (which, as in standard techniques, should be taken for applying a suitable variational principle), we can make the absolute value of the discontinuous term to be arbitrarily small. We can then prove that V is the unique bounded uniformly continuous viscosity solution in $\mathcal{O}$ of the discontinuous equation

$$
\lambda V(y,\xi) + H(y,\xi,V_y(y,\xi),D_\xi V(y,\xi)) = 0,
$$

where V_y is the derivative of V with respect to y and $D_\xi V$ is the Fréchet differential of V with respect to ξ.

The results of the paper apply with obvious modification to several other situations: we can replace in both cases the Play with either the Stop or the generalized play or generalized stop (see for instance Visintin [22] for definitions).

An optimal control problem for ordinary differential equation with hysteresis, very similar to the present one, was studied by Brokate in [4]. In that work, the author is concerned in necessary conditions for optimality and he does not apply the dynamical programming technique. Other optimal control problems, with functional dependence in the dynamic, were studied with dynamic programming approach by Soner [20] and Wolenski [23] for the so-called hereditary problem (or delayed problem). However, it seems that this is the first time that the dynamic programming method and the viscosity solution theory are applied to an optimal control problem with hysteresis functional dependence.

We recall that the first studies on mathematical aspects of hysteresis, and in particular on the concept of hysteresis operator, are due to Krasnoselskii and his co-workers (see the monograph Krasnoselskii-Pokrovskii [15]). Other books on the mathematical aspects of hysteresis and applications are Mayergoyz [19], Visintin [22], Brokate-Sprekels [5], Krejci [14].

The theory of viscosity solutions for Hamilton-Jacobi equations goes back to the works of Lions [16] and Crandall-Lions [8] (see also Crandall-Evans-Lions [7]). In particular for the Neumann problem and for the definition of viscosity solutions for discontinuous equations we refer to Lions [17], Ishii [13] and Barles-Lions [3]. For the theory in infinite dimension we refer to the work (first of a series) Crandall-Lions [9]. For a comprehensive account of the theory (in finite dimension) and its application to optimal control problems, we refer to the books Barles [2], Capuzzo Dolcetta-Lions (eds.) [6] and Bardi-Capuzzo Dolcetta [1].

The plan of the paper is as follows. In Section 2 we give the notion of hysteresis operator and introduce the Play and Prandtl-Ishlinskii operators with their main properties. In Section 3 we state the optimal control problem in the general setting of hysteresis operator. In Sections 4 and 5 we respectively study the problem with the Play and the Prandtl-Ishlinskii operators. In Section 6 we give some remarks and extensions to the case of some other hysteresis operators.

2 Hysteresis operators; two examples.

In this section we give the basic notions of hysteresis operators and two examples of such operators with their properties, which we shall use in the next sections. We first argue in a general framework. Let $[0,T]$ be a time interval and B a Banach space.

Definition 2.1. An operator

$$\mathcal{F}:\mathcal{D}\subseteq C^0([0,T])\times B\to C^0([0,T])\quad (u,\xi)\mapsto\mathcal{F}[u,\xi](\cdot)$$

is said to be a *hysteresis operator* if the following two properties are satisfied
a) *(causality)* $\forall(u,\xi),\ (v,\xi)\in\mathcal{D}$, $\forall t\in[0,T]$

$$u_{|[0,t]}=v_{|[0,t]}\Rightarrow\mathcal{F}[u,\xi](t)=\mathcal{F}[v,\xi](t); \tag{2.1}$$

b) *(rate independence)* $\forall(u,\xi)\in\mathcal{D}$, $\forall$ continuous nondecreasing $\varphi:[0,T]\to[0,T]$

$$(u\circ\varphi,\xi)\in\mathcal{D},\quad \mathcal{F}[u\circ\varphi,\xi]=\mathcal{F}[u,\xi]\circ\varphi. \tag{2.2}$$

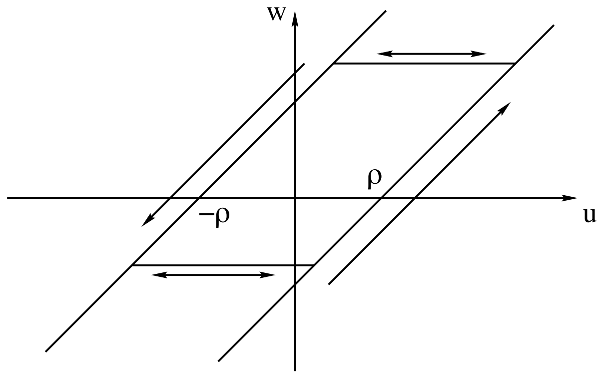

Figure 1: Play operator of hysteresis.

Moreover we suppose that the following further properties are satisfied

c) *(Lipschitz continuity)* there exists $L > 0$ such that $\forall (u, \xi),\ (v, \eta) \in \mathcal{D}$

$$\|\mathcal{F}[u, \xi] - \mathcal{F}[v, \eta]\|_{C^0([0,T])} \leq L \left(\|u - v\|_{C^0([0,T])} + \|\xi - \eta\|_B \right); \tag{2.3}$$

d) *(semigroup property)* there exists another operator $\Phi : \mathcal{D} \to C^0([0, T], B)$ such that $\forall (u, \xi) \in \mathcal{D}$, $\forall t, \tau \in [0, T]$, $t + \tau \leq T$, setting $\xi(t) := \Phi[u, \xi](t)$ we have

$$\mathcal{F}[u, \xi](t + \tau) = \mathcal{F}[u(t + \cdot), \xi(t)](\tau) \tag{2.4}$$

The operator Φ represents the evolution of the *internal variable* $\xi \in B$.

2.1 The Play operator.

For $\rho > 0$ we define the open set

$$\Omega_\rho := \left\{ (u, w) \in I\!\!R^2 \middle| u - \rho < w < u + \rho \right\}.$$

We take $B = I\!\!R$ and $\mathcal{D} = \{(u, w^0) \in C^0([0, T]) \times I\!\!R | (u(0), w^0) \in \overline{\Omega}_\rho\}$. The behavior of the Play operator $w(\cdot) := \mathcal{F}[u, w^0](\cdot)$, with its typical hysteresis loops, can be described by Fig. 1. For instance, supposing that u is piecewise monotone, if $(u(t), w(t)) \in \Omega_\rho$ then w is constant in a neighborhood of t; if $w(t) = u(t) - \rho$ and u is non increasing in $[t, t + \tau]$ (with small τ) then w stays constant in $[t, t + \tau]$; if $w(t) = u(t) - \rho$ and u is nondecreasing in $[t, t + \tau]$ then $w = u - \rho$ in $[t, t + \tau]$; a similar argumentation holds if $w(t) = u(t) + \rho$. Moreover we have $w(0) = w^0$.

Let $I_{[-\rho,\rho]}$ be the indicator function of $[-\rho, \rho]$ (i.e. 0 in $[-\rho, \rho]$ and $+\infty$ otherwise), and $\partial I_{[-\rho,\rho]}$ its subdifferential.

Theorem 2.1 *The output $w = \mathcal{F}[u, w^0]$ of the Play operator $\mathcal{F}$ with input $u \in W^{1,1}(0, T)$ and initial state w^0 is the unique solution $w \in W^{1,1}(0, T)$ of the following differential inclusion*

$$\begin{cases} \dot{w}(t) \in \partial I_{[-\rho,\rho]}\Big(u(t) - w(t)\Big) & a.e.\ t \in (0,T), \\ w(0) = w^0. \end{cases}$$

Moreover, $\mathcal{F}$ can be uniquely extended to a continuous operator from $C^0([0,T]) \times I\!R$ to $C^0([0,T])$ which is a Lipschitz hysteresis operator with Lipschitz constant equal to 1 *and which satisfies the semigroup property (2.4) (with $B = I\!R$ and $\Phi = \mathcal{F}$).*

Theorem 2.2 *For any $p \in [1, +\infty[$, the Play operator is strongly continuous from $W^{1,p}(0,T) \times I\!R$ to $W^{1,p}(0,T)$. Moreover, the following equality holds*

$$(\dot{w}(t))^2 = \dot{u}(t)\dot{w}(t) \quad a.e.\ t \in (0,T). \tag{2.5}$$

For the proofs of the previous two theorems see Visintin [22] and Krejci [14].

Now, we prove a useful property of the output for regular input. Let us call $\gamma_{\rho l}$ and $\gamma_{\rho r}$ respectively the straight lines in the $(u,w)-$plane of equations $w = u + \rho$, $w = u - \rho$ (l stays for "left" and r for "right"). Moreover, for $i \in \{l, r\}$, let us denote by $\chi_{\rho i}(u,w)$ the characteristic function of $\gamma_{\rho i}$ (i.e. 1 on $\gamma_{\rho i}$ and 0 outside). Finally, for $a \in I\!R$ we denote its positive part by $a^+ := \max(a,0)$ and its negative part by $a^- := \max(-a,0)$.

Proposition 2.3 *For every $u \in H^1(0,T)$ and for every admissible initial state $w^0 \in I\!R$ (i.e. $(u,w^0) \in \mathcal{D}$), the output w satisfies the following discontinuous differential equation*

$$\dot{w}(t) = \chi_{\rho r}\Big(u(t), w(t)\Big)(\dot{u}(t))^+ - \chi_{\rho l}\Big(u(t), w(t)\Big)(\dot{u}(t))^- \quad \text{a.e. } t \in (0,T). \tag{2.6}$$

Proof. First of all, let us note that from (2.5), the following easily follows

$$|\dot{w}(t)| \le |\dot{u}(t)| \quad \text{a.e. } t \in (0,T). \tag{2.7}$$

Let $t \in (0,T)$ be such that $\dot{u}(t)$ and $\dot{w}(t)$ exist. Let us first suppose that $|u(t) - w(t)| < \rho$. Then, for $\delta > 0$ sufficiently small we have

$$|h| \le \delta \Rightarrow |u(t+h) - w(t+h)| < \rho \Rightarrow w(t+h) = w(t),$$

and hence $\dot{w}(t) = 0$ which satisfies (2.6). Now, let us suppose that $w(t) = u(t) - \rho$ (the case $w(t) = u(t) + \rho$ can be similarly treated). We claim that

$$\dot{u}(t) \ge 0. \tag{2.8}$$

Indeed, if it is not the case, for some $\delta > 0$ the following holds

$$0 < h \le \delta \Rightarrow u(t-h) > u(t).$$

By the shape of Ω_ρ (see Fig. 1), this implies that $w(t-h) > w(t)$, which is impossible because if δ is small, w cannot decrease in a time interval of length δ in order to reach the value $w(t)$ at time t (w can decrease only if $(u, w) \in \gamma_{\rho l}$). Now, let us note that the following inequality holds

$$\dot{w}(t) = \lim_{h \to 0} \frac{w(t+h) - w(t)}{h} \geq \lim_{h \to 0} \frac{u(t+h) - \rho - u(t) + \rho}{h} = \dot{u}(t). \tag{2.9}$$

By (2.9), using (2.8) and (2.7), we get

$$\dot{w}(t) = \dot{u}(t) = (\dot{u}(t))^+,$$

which is (2.6). □

The Play operator is a so-called *local memory* operator; that is the internal variables coincide with the external one, namely the value of the output.

2.2 The Prandtl-Ishlinskii operator of play-type.

This is an important example of operator with *nonlocal memory*: the value of the output is not completely determined by the initial value of the output and by the evolution of the input. We need the knowledge of some internal variables, namely the elements of a (Banach) space B different from $I\!R$.

The characteristic feature of the Play, when constructed as in the previous section, is the length of the symmetric interval $[-\rho, \rho]$, that is the positive number ρ. Plays corresponding to different ρ give different outputs for the same input. An interesting case for applications is when the output is the "result" of a (possibly infinite) number of weighted Plays working in parallel.

Let $\mathcal{R}$ be a bounded set of $]0, +\infty[$ and μ be a finite Borel measure on $\mathcal{R}$. For every $\rho \in \mathcal{R}$, as in the previous subsection, we consider the set Ω_ρ and the corresponding Play operator, which we call $\mathcal{F}_\rho$. We define the set

$$\mathcal{D} := \left\{ (u, \xi) \in C^0([0,T]) \times L^2(\mathcal{R}, \mu) \middle| (u(0), \xi(\rho)) \in \overline{\Omega}_\rho,\ \mu\text{-a.e. } \rho \in \mathcal{R} \right\},$$

that is we take $B = L^2(\mathcal{R}, \mu)$. We define the Prandtl-Ishlinskii operator $\mathcal{F}$ as

$$\mathcal{F} : \mathcal{D} \subset C^0([0,T]) \times L^2(\mathcal{R}, \mu) \to C^0([0,T]) \quad (u, \xi) \mapsto \int_{\mathcal{R}} \mathcal{F}_\rho[u, \xi(\rho)](\cdot) d\mu.$$

Note that the function ξ gives the initial state of every Play (labeled by ρ). The evolution of the internal variables (namely the output of every Play) is of course given by the evolution of every single Play. That is, at any instant t we have a function

$$\Phi[u, \xi](t) : \mathcal{R} \to I\!R, \quad \rho \mapsto \Phi_\rho[u, \xi](t) := \mathcal{F}_\rho[u, \xi(\rho)](t), \tag{2.10}$$

with of course $\Phi[u, \xi](0) = \xi$.

Theorem 2.4 *The Prandtl-Ishlinskii operator is a Lipschitz continuous hysteresis operator from $C^0([0,T]) \times L^2(\mathcal{R}, \mu)$ to $C^0([0,T])$ and satisfies the semigroup property (2.4) (with Φ defined in (2.10)).*

Proof. The proof is essentially given in Visintin [22]. Here, we only point out that $\Phi[u,\xi](t)$ defined in (2.10) belongs to $L^2(\mathcal{R}, \mu)$ for every t. Indeed, for almost every ρ, the corresponding Play continuously evolves from $\xi(\rho)$ (uniformly with respect to ρ and to $\xi(\rho)$): this guarantees the measurability. Moreover, since $\mathcal{R}$ is bounded, by the definition of Ω_ρ, we have that $\Phi[u,\xi](t)$ is bounded ($u(t) - \rho \leq \Phi_\rho[u,\xi](t) \leq u(t)+\rho$). The Lipschitz continuity follows from the Lipschitz continuity (uniformly with respect to ρ) of every Play (see Theorem 2.1). □

Let us call w the output of $\mathcal{F}$ and w_ρ the output of $\mathcal{F}_\rho$.

Theorem 2.5 *Let $p \in [1, +\infty[$, then $\mathcal{F}$ is strongly continuous from $W^{1,p}(0,T) \times L^2(\mathcal{R}, \mu)$ to $W^{1,p}(0,T)$. Moreover, we have*

$$\dot{w}(t) = \int_{\mathcal{R}} \dot{w}_\rho(t) d\mu \quad a.e.\ t \in (0,T); \tag{2.11}$$

if $u \in W^{1,p}(0,T)$, then the functions $w_{(\cdot)}(\cdot)$ and $\dot{w}_{(\cdot)}(\cdot)$ (which map respectively (t,ρ) to $w_\rho(t) = \mathcal{F}_\rho[u, \xi(\rho)](t)$ and to $\dot{w}_\rho(t) = (d/dt)\left(\mathcal{F}_\rho[u,\xi(\rho)](t)\right)$) satisfy

$$w_{(\cdot)}(\cdot) \in W^{1,p}(0,T; L^2(\mathcal{R}, \mu)), \qquad \frac{d}{dt} w_{(\cdot)}(\cdot) = \dot{w}_{(\cdot)}(\cdot). \tag{2.12}$$

Proof. The proof of the continuity can be found for instance in Brokate-Sprekels [5]. Let us prove (2.12). Indeed $w_{(\cdot)}(\cdot)$ and $\dot{w}_{(\cdot)}(\cdot)$ are measurable and belong to $L^p(0,T; L^2(\mathcal{R}, \mu))$ (since $|w_\rho(t)| \leq |u(t)| + \rho$ and $|\dot{w}_\rho(t)| \leq |\dot{u}(t)|$). For every $\psi \in C_c^1([0,T]; L^2(\mathcal{R}, \mu))$ we have

$$\int_0^T \int_{\mathcal{R}} w_\rho(t) \dot{\psi}(t,\rho) d\mu dt = -\int_0^T \int_{\mathcal{R}} \dot{w}_\rho(t) \psi(t,\rho) d\mu dt.$$

The latter equality, when ψ is independent on ρ, also proves (2.11). □

3 The control problem in a general setting.

In this section, we formulate the optimal control problem and give basic results for the case of a general hysteresis operator, leaving next sections to focus on particular examples of operators and to investigate more deeply the application of dynamic programming.

We consider the following dynamical system

$$\begin{cases} y'(t) = f(y(t), w(t), \alpha(t)) \quad w(t) = \mathcal{F}[y, \xi_0](t) \quad t \geq 0, \\ y(0) = y_0, \end{cases} \tag{3.1}$$

where $(y_0, \xi_0) \in \mathcal{O} \subset I\!R \times B$, with B Banach space; for every $T > 0$, the hysteresis operator (see Definition 2.1)

$$\mathcal{F} : \mathcal{D} \subset C^0([0,T]) \times B \to C^0([0,T])$$

satisfies

$$\begin{array}{l} \text{Lipschitz continuity (2.3); semigroup property (2.4);} \\ \text{strong continuity from } H^1(0,T) \times B \text{ to } H^1(0,T). \end{array} \tag{3.2}$$

The set of initial sates $\mathcal{O}$ can be seen as the subset of $\mathcal{D}$ given by the couples (u, ξ) with u constant function (of course the output of $\mathcal{F}$ for every element of $\mathcal{O}$ is constant); $\alpha : [0, +\infty[\to A$ is a measurable control taking value in the compact set $A \subset I\!R^q$ for some $q \in I\!N$. The function

$$f : I\!R \times I\!R \times A \to I\!R, \quad (y, w, a) \mapsto f(y, w, a),$$

is continuous, bounded and Lipschitz continuous with respect to (y, w) uniformly in a, that is

$$\begin{array}{l} \exists L > 0 \text{ such that } \forall a \in A \ \forall (y_1, w_1), (y_2, w_2) \in I\!R \times I\!R \\ |f(y_1, w_1, a) - f(y_2, w_2, a)| \le L|(y_1, w_1) - (y_2, w_2)| \\ \exists M > 0 \text{ such that } |f(y, w, a)| \le M \ \forall (y, w, a) \in I\!R \times I\!R \times A. \end{array} \tag{3.3}$$

Note that the "external state" of the system (3.1) is (y, w), but the "true" state is the couple (y, ξ) where ξ is the evolution of the internal variables.

Let us define the set of measurable controls

$$\mathcal{A} := \Big\{\alpha : [0, +\infty[\to A \Big| \alpha \text{ is measurable}\Big\},$$

and for every $\alpha \in \mathcal{A}$ let us denote the unique solution of (3.1) (see Proposition 3.1 below) as $(y_{(y_0,\xi_0)}(\cdot; \alpha), w_{(y_0,\xi_0)}(\cdot; \alpha))$.

Now let us consider a function

$$l : I\!R \times I\!R \times A \to [0, +\infty[, \quad (y, w, a) \mapsto l(y, w, a),$$

which is continuous, bounded and uniformly continuous with respect to (y, w) uniformly in a, that is

$$\begin{array}{l} \exists \omega : [0, +\infty[\to [0, +\infty[\text{ increasing, continuous, } \omega(0) = 0, \\ \forall a \in A \ \forall (y_1, w_1), (y_2, w_2) \in I\!R \times I\!R \\ |l(y_1, w_1, a) - l(y_2, w_2, a)| \le \omega(|(y_1, w_1) - (y_2, w_2)|) \\ \exists M > 0 \text{ such that } |l(y, w, a)| \le M \ \forall (y, w, a) \in I\!R \times I\!R \times A. \end{array} \tag{3.4}$$

Given $\lambda > 0$, for every initial state $(y_0, \xi_0) \in \mathcal{O}$, we want to minimize the following cost functional over the controls $\alpha \in \mathcal{A}$

$$J(y_0, \xi_0, \alpha) := \int_0^{+\infty} e^{-\lambda t} l\Big(y_{(y_0,\xi_0)}(t; \alpha), w_{(y_0,\xi_0)}(t; \alpha), \alpha(t)\Big) dt,$$

and hence we define the value function

$$V(y_0,\xi_0) := \inf_{\alpha\in\mathcal{A}} J(y_0,\xi_0,\alpha) \tag{3.5}$$

Proposition 3.1 *Let us suppose that (3.2), (3.3) hold. Then, for every initial state $(y_0,\xi_0)\in\mathcal{O}$ and for every control $\alpha\in\mathcal{A}$, there exists a unique solution of the system (3.1) $(y_{(y_0,\xi_0)}(\cdot;\alpha), w_{(y_0,\xi_0)}(\cdot,\alpha))$, such that $y_{(y_0,\xi_0)}(\cdot;\alpha)\in H^1(0,T)$ for every $T>0$. This of course implies that $w_{(y_0,\xi_0)}(\cdot;\alpha)\in H^1(0,T)$ for every $T>0$. Moreover, for every $T>0$, there exists a modulus of continuity ω_T (i.e. an increasing continuous function with $\omega_T(0)=0$) such that for all initial states $(y_1,\xi_1),(y_2,\xi_2)\in\mathcal{O}$ and for every $t\in[0,T]$ the following inequality holds*

$$\begin{aligned}&\left|y_{(y_1,\xi_1)}(t;\alpha)-y_{(y_2,\xi_2)}(t;\alpha)\right|+|w_{(y_1,\xi_1)}(t;\alpha)-w_{(y_2,\xi_2)}(t;\alpha)|\\&\le\omega_T(\max\{|y_1-y_2|,\|\xi_1-\xi_2\|_B\})\end{aligned} \tag{3.6}$$

Proof. Existence is proven by a standard delayed approximation. Indeed, regarding y_0 as a constant function, we define the constant $w_0:=\mathcal{F}[y_0,\xi_0]$. Then, in every subinterval of a partition of $[0,T]$, we solve the problem with a known w equal to the output of $\mathcal{F}$ when the input is the solution y in the previous interval (with suitable initial state). In the first interval we take $w\equiv w_0$. We pass to the limit as the length of the partition goes to zero and we obtain a function y, such that, together with its corresponding output w,

$$y(t)=y_0+\int_0^t f(y(s),w(s),\alpha(s))ds \quad \forall t\in[0,T],$$

which is the integral version of (3.1).

Now we prove (3.6), from which the uniqueness easily follows. In the following we shall write $y_i(\cdot):=y_{(y_i,\xi_i)}(\cdot,\alpha)$, $w_i(\cdot):=w_{(y_i,\xi_i)}(\cdot,\alpha)$, for $i=1,2$, as well as y_i^0, ξ_i^0 for the initial states. We shall use the Lipschitz continuity of f and $\mathcal{F}$. All the involved Lipschitz constants are denoted by L. For $t\in[0,T]$, we have

$$\begin{aligned}&|y_1(t)-y_2(t)|\\&\le|y_1^0-y_2^0|+L(1+L)\int_0^t\|y_1-y_2\|_{C^0([0,\tau])}d\tau+L^2T\|\xi_1^0-\xi_2^0\|_B.\end{aligned} \tag{3.7}$$

The right-hand side of (3.7) is increasing in t. Hence, in the left-hand side, we can replace t by any other $\tau\in[0,t]$. We obtain

$$\begin{aligned}&\|y_1-y_2\|_{C^0([0,t])}\\&\le|y_1^0-y_2^0|+L(1+L)\int_0^t\|y_1-y_2\|_{C^0([0,\tau])}d\tau+L^2T\|\xi_1^0-\xi_2^0\|_B.\end{aligned}$$

Hence, by the Gronwall inequality applied to $t\mapsto\|y_1-y_2\|_{C^0([0,t])}$, we get

$$\|y_1-y_2\|_{C^0([0,t])}\le\Big(|y_1^0-y_2^0|+L^2T\|\xi_1^0-\xi_2^0\|_B\Big)e^{L(1+L)t} \tag{3.8}$$

and, by the Lipschitz continuity of $\mathcal{F}$

$$\|w_1 - w_2\|_{C^0([0,t])} \leq L|y_1^0 - y_2^0|e^{L(1+L)t} + L\Big(1 + L^2 T e^{L(1+L)t}\Big)\|\xi_1^0 - \xi_2^0\|_B. \qquad (3.9)$$

From (3.8)–(3.9), it is easy to find ω_T such that (3.6) holds. $\square$

Theorem 3.2 *Let us suppose that (3.2)–(3.4) hold. Then the value function (3.5) is bounded and uniformly continuous in $\mathcal{O}$.*

Proof. The value function V is bounded by (3.4) and its definition.

The uniform continuity follows from (3.6), by standard techniques (see for instance Bardi-Capuzzo Dolcetta [1]). $\square$

Now, we suppose that the set $\mathcal{O} \subset I\!\!R \times B$ of admissible initial states for (3.1) is an invariant set for the trajectories in $I\!\!R \times B$ of the system. That is, considering the evolution functional Φ for the internal variables, which is defined in the semigroup property (2.4), for every $t \in [0, +\infty[$ and $\alpha \in \mathcal{A}$ the following holds

$$(y, \xi) \in \mathcal{O} \Rightarrow \Big(y_{(y,\xi)}(t, \alpha), \Phi[y_{(y,\xi)}(\cdot, \alpha), \xi](t)\Big) \in \mathcal{O}. \qquad (3.10)$$

The cases that will be studied in the next sections, satisfy such property.

Theorem 3.3 *(DPP: Dynamic programming principle) Let us suppose that (3.10) holds. Then, for every $(y, \xi) \in \mathcal{O}$ and for every $t \in [0, +\infty[$*

$$\begin{aligned} V(y,\xi) = \inf_{\alpha \in \mathcal{A}} \Big\{ & \int_0^t e^{-\lambda s} l(y_{(y,\xi)}(s;\alpha), w_{(y,\xi)}(s;\alpha), \alpha(s))ds \\ & + e^{-\lambda t} V(y_{(y,\xi)}(t;\alpha), \Phi[y_{(y,\xi)}(\cdot;\alpha), \xi](t))\Big\}. \end{aligned}$$

Proof. It follows in a standard way by the semigroup property (2.4) and the uniqueness of the solution of (3.1). $\square$

4 The case of Play.

In this section we suppose that the hysteresis operator $\mathcal{F}$ in the control problem is the Play operator, with $\rho > 0$, described in Section 2. We recall that $B = I\!\!R$ and the internal variables coincide with the external one, namely the output w itself. We have

$$\Omega_\rho := \Big\{(y, w) \in I\!\!R^2 \Big| y - \rho < w < y + \rho\Big\},$$

(see Fig. 1, with u replaced by y). From the definition of the Play, it is easy to see that $\overline{\Omega}_\rho$ is an invariant set for the controlled system. For more clarity, we give the formulation of DPP in this particular case. For any $(y, w) \in \overline{\Omega}_\rho$ and for any $t \geq 0$, we have

$$V(y,w)=\inf_{\alpha\in\mathcal{A}}\Big\{\int_0^t e^{-\lambda s}l(y_{(y,w)}(s;\alpha),w_{(y,w)}(s;\alpha),\alpha(s))ds \\ +e^{-\lambda t}V(y_{(y,w)}(t;\alpha),w_{(y,w)}(t;\alpha))\Big\}. \tag{4.1}$$

For every $(y,w)\in\overline{\Omega}_\rho$, $(p_1,p_2)\in I\!R^2$, we define the following Hamiltonian

$$H(y,w,p_1,p_2):=\sup_{a\in A}\Big\{-p_1f(y,w,a) \\ -p_2\Big(\chi_{\rho r}(y,w)f(y,w,a)^+-\chi_{\rho l}(y,w)f(y,w,a)^-\Big)-l(y,w,a)\Big\}. \tag{4.2}$$

Let us note that H is discontinuous in $\overline{\Omega}_\rho\times I\!R^2$. We consider the following Hamilton-Jacobi equation

$$\lambda u(y,w)+H(y,w,u_y,u_w)=0 \quad \text{in } \overline{\Omega}_\rho, \tag{4.3}$$

where u_y (respectively u_w) is the derivative with respect to y (respectively w).

We denote the lower semicontinuous and upper semicontinuous envelopes of H in $\overline{\Omega}_\rho$ respectively by H_* and H^*.

Theorem 4.1 *Let (3.3), (3.4) hold (note that the Play operator satisfies (3.2)). Then the value function V is a uniformly continuous and bounded viscosity solution in $\overline{\Omega}_\rho$ of the Hamilton-Jacobi equation (4.3). That is, for every $\varphi\in C^1(\overline{\Omega}_\rho)$ such that $(y,w)\in\overline{\Omega}_\rho$ is a local extremal point for $V-\varphi$, the following holds*

$$\textit{(subsolution) if } (y,w) \textit{ is a maximum, then} \\ \lambda u(y,w)+H_*(y,w,\varphi_y(y,w),\varphi_w(y,w))\le 0; \tag{4.4}$$

$$\textit{(supersolution) if } (y,w) \textit{ is a minimum, then} \\ \lambda u(y,w)+H^*(y,w,\varphi_y(y,w),\varphi_w(y,w))\ge 0. \tag{4.5}$$

Proof. The uniformly continuity and the boundedness come from the results of Section 3.

A simple calculation shows that

$$H^*(y,w,p_1,p_2)=\sup_{a\in A}\Big\{-p_1f(y,w,a) \\ +p_2^-\chi_{\rho r}(y,w)f(y,w,a)^+ + p_2^+\chi_{\rho l}(y,w)f(y,w,a)^- - l(y,w,a)\Big\}, \tag{4.6}$$

To prove that V is subsolution, we prove that (4.4) is satisfied even replacing H_* by H itself (and thus (4.4) a fortiori holds). Take $\varphi\in C^1(\overline{\Omega}_\rho)$ and $(y,w)\in\overline{\Omega}_\rho$ such that $V-\varphi$ has a local maximum in (y,w) with respect to $\overline{\Omega}_\rho$. Let us take $a\in A$ and consider the corresponding trajectory $(y(\cdot;a),w(\cdot;a))$ starting from (y,w). If $f(y,w,a)=0$, then the trajectory is the stationary one and hence it is sufficient to observe that $\lambda V(y,w)\le l(y,w,a)$ by definition of V. On the other hand, if

$f(y,w,a)>0$ (respectively $f(y,w,a)<0$), then for sufficiently small t, y is strictly monotone in $[0,t]$ and hence we can suppose that (2.6) holds everywhere in $[0,t]$. In particular (suppose for instance that $f(y,w,a)>0$) we have

$$\dot{w}(\tau;a)=\chi_{\rho r}\Big(y(\tau;a),w(\tau;a)\Big)f\Big(y(\tau;a),w(\tau;a),a\Big)^{+}\quad\forall\tau\in[0,t],\tag{4.7}$$

where either $\chi_{\rho r}(y(\tau;a),w(\tau;a))=0$ or $\chi_{\rho r}(y(\tau;a),w(\tau;a))=1$ for every $\tau\in[0,t]$ and hence the right-hand side of (4.7) is continuous. Using the continuity of V, DPP (4.1) and the continuity of the Play operator, by standard techniques (see for instance Bardi-Capuzzo Dolcetta [1]) we then obtain

$$\begin{aligned}&\lambda V(y,w)-\varphi_y(y,w)f(y,w,a)\\&-\varphi_w(y,w)\Big(\chi_{\rho r}(y,w)f(y,w,a)^{+}-\chi_{\rho l}(y,w)f(y,w,a)^{-}\Big)-l(y,w,a)\le 0,\end{aligned}$$

and then we conclude.

Let us prove that V is a supersolution of (4.3). Fix $\varphi\in C^1(\overline{\Omega}_\rho)$ and let $(y,w)\in\overline{\Omega}_\rho$ be of minimum for $V-\varphi$. Using again DPP (4.1), for every $\varepsilon>0$ and for every $t>0$ sufficiently small, we find $\alpha\in\mathcal{A}$ (depending on ε and t), such that, denoting the corresponding trajectory by $(y(\cdot),w(\cdot))$,

$$\begin{aligned}&V\Big(y(t),w(t)\Big)-\int_0^t e^{-\lambda s}l\Big(y(s),w(s),\alpha(s)\Big)ds\\&-e^{-\lambda t}V\Big(y(t),w(t)\Big)+\varepsilon t\ge\varphi\Big(y(t),w(t)\Big)-\varphi(y,w).\end{aligned}\tag{4.8}$$

Let us note that

$$\begin{aligned}&\varphi\Big(y(t),w(t)\Big)-\varphi(y,w)=\int_0^t\frac{d}{ds}\varphi\Big(y(s),w(s)\Big)ds\\&=\int_0^t\Big[\varphi_y\Big(y(s),w(s)\Big)f\Big(y(s),w(s),\alpha(s)\Big)\\&+\varphi_w\Big(y(s),w(s)\Big)\Big(\chi_{\rho r}(y(s),w(s))f(y(s),w(s),\alpha(s))^{+}\\&-\chi_{\rho l}(y(s),w(s))f(y(s),w(s),\alpha(s))^{-}\Big)\Big]ds.\end{aligned}\tag{4.9}$$

If $\chi_{\rho r}(y,w)=0$ (respectively $\chi_{\rho l}(y,w)=0$), then for small s, $\chi_{\rho r}(y(s),w(s))=0$ (respectively $\chi_{\rho l}(y(s),w(s))=0$). Then for small s we have the following estimates

$$\begin{aligned}&\varphi_w(y,w)\chi_{\rho r}\Big(y(s),w(s)\Big)f\Big(y,w,\alpha(s)\Big)^{+}\\&\qquad\ge-\varphi_w(y,w)^{-}\chi_{\rho r}(y,w)f\Big(y,w,\alpha(s)\Big)^{+};\\&-\varphi_w(y,w)\chi_{\rho l}\Big(y(s),w(s)\Big)f\Big(y,w,\alpha(s)\Big)^{-}\\&\qquad\ge-\varphi_w(y,w)^{+}\chi_{\rho l}(y,w)f\Big(y,w,\alpha(s)\Big)^{-}.\end{aligned}\tag{4.10}$$

We approximate $f(y(s),w(s),\alpha(s))$, $l(y(s),w(s),\alpha(s))$ and $\nabla\varphi(y(s),w(s))$ respectively by $f(y,w,\alpha(s))$, $l(y,w,\alpha(s))$ and $\nabla\varphi(y,w)$, with an error which is infinitesimal as s goes to zero, independently on α (and of course on ε and t). This can be done by the uniform continuity of f and l and the regularity of φ. Hence, by (4.9)–(4.10), noting that the coefficient of φ_w in the integral in (4.9) is uniformly bounded and adding and subtracting $\pm \int_0^t l(y, w, \alpha(s))ds$ we obtain from (4.8)

$$\begin{aligned}
&\int_0^t \Big[-\varphi_y(y,w) f\big(y,w,\alpha(s)\big) - l\big(y,w,\alpha(s)\big) \\
&+\varphi_w(y,w)^- \chi_{\rho r}(y,w) f\big(y,w,\alpha(s)\big)^+ \\
&+\varphi_w(y,w)^+ \chi_{\rho l}(y,w) f\big(y,w,\alpha(s)\big)^- \Big] ds + V\big(y(s),w(s)\big)(1-e^{-\lambda t}) \\
&\geq -\varepsilon t + o(t),
\end{aligned} \tag{4.11}$$

where $o(t)$ indicates a function $g(t)$ such that $t^{-1}g(t) \to 0$ as $t \to 0$. From (4.11), we obtain the conclusion (see for instance Bardi-Capuzzo Dolcetta [1]). $\square$

It is evident that V is a viscosity solution of the continuous Hamilton-Jacobi equation

$$\lambda V(y,w) + H(y,w,V_y,0) = 0 \quad \text{in } \Omega_\rho. \tag{4.12}$$

On the other hand, on $\partial\Omega_\rho$, if (4.12) is not satisfied in the viscosity sense, then the "discontinuous part" of (4.3) (i.e. the part involving the derivatives with respect to w) has to play a role. This means that the derivative of V with respect to w (i.e. the derivative of the test function), should be not zero and have a suitable sign, in order to obtain the right sign for the Hamiltonian (4.2) or (4.6). Hence, if we define the outward vector to $\partial\Omega_\rho$

$$\zeta_\rho(y,w) = \begin{cases} (0,-1) & \text{if } (y,w) \in \gamma_{\rho r}, \\ (0,1) & \text{if } (y,w) \in \gamma_{\rho l}, \end{cases} \tag{4.13}$$

we have the following formulation for our problem

$$\begin{cases} \lambda V(y,w) + H(y,w,V_y,0) = 0 & \text{in } \Omega_\rho, \\ V_{\zeta_\rho}(y,w) = 0 & \text{on } \partial\Omega_\rho, \end{cases} \tag{4.14}$$

where $V_{\zeta_\rho} := (\partial/\partial\zeta_\rho)V$ and the boundary condition has to be understood in (a strict) viscosity sense

$$\begin{cases} \lambda V + H(y,w,V_y,0) > 0 \Rightarrow V_{\zeta_\rho}(y,w) < 0 & \text{for subsolution,} \\ \lambda V + H(y,w,V_y,0) < 0 \Rightarrow V_{\zeta_\rho}(y,w) > 0 & \text{for supersolution.} \end{cases} \tag{4.15}$$

Theorem 4.2 *The value function V is the only bounded continuous viscosity solution in $\overline{\Omega}_\rho$ of the Hamilton-Jacobi equation (4.3) (or equivalently of the Neumann-type problem (4.14)–(4.15)).*

To prove Theorem 4.2, we prove uniqueness for the reformulated problem as a boundary value problem of Neumann-type.

Proof of the Theorem 4.2. The proof is standard. In our case it is easy because Ω_ρ is a strip and we have a "strict" boundary condition (see (4.15)). We give here a sketch of the proof, which can be useful in the next section. We prove a comparison result: if u is a subsolution and v is a supersolution then $u \leq v$ in $\overline{\Omega}_\rho$.

A simple calculation shows that there exists a strictly elliptic constant matrix S such that for every $(y,w) \in \partial\Omega_\rho$ we have

$$\zeta_\rho(y,w) \cdot S\Big((y,w) - (y',w')\Big) \geq 0 \quad \forall (y',w') \in \overline{\Omega}_\rho, \tag{4.16}$$

where $S(y,w)$ denotes the matrix-vector product. Let $g(y)$ be a C^1 positive function on $I\!R$ with bounded derivative and $g(y) \to +\infty$ as $|y| \to +\infty$ (note that $|w| \to +\infty$ whenever $|y| \to +\infty$). For $\delta > 0$ and $\beta > 0$, we define in $\overline{\Omega}_\rho \times \overline{\Omega}_\rho$

$$\begin{aligned}\phi\Big((y_1,w_1),(y_2,w_2)\Big) &:= u(y_1,w_2) - v(y_2,w_2) - \beta(g(y_1)+g(y_2)) \\ &- \frac{S(y_1-y_2,w_1-w_2)\cdot(y_1-y_2,w_1-w_2)}{2\delta},\end{aligned}$$

As usual, let us suppose by contradiction that there exists $(\tilde{y},\tilde{w})$ such that $u(\tilde{y},\tilde{w}) - v(\tilde{y},\tilde{w}) = \eta > 0$. Standard techniques show that, for β sufficiently small, there exists a compact set $K \subset \overline{\Omega}_\rho$ such that, for every $\delta > 0$, $\exists \left((y_1^\delta,w_1^\delta),(y_2^\delta,w_2^\delta)\right) \in K \times K$ point of maximum for ϕ over $\overline{\Omega}_\rho \times \overline{\Omega}_\rho$, and this maximum is strictly positive independently from δ. Using also the strictly ellipticity of S, we have

$$\frac{|(y_1^\delta,w_1^\delta)-(y_2^\delta,w_2^\delta)|^2}{2\delta} \to 0 \quad \text{as } \delta \to 0.$$

By (4.16), if (y_1^δ,w_1^δ) or (y_2^δ,w_2^δ) belong to $\partial\Omega_\rho$, we respectively have

$$\begin{aligned}\zeta_\rho(y_1^\delta,w_1^\delta)\cdot S\left(\frac{(y_1^\delta,w_1^\delta)-(y_2^\delta,w_2^\delta)}{\delta}\right) &\geq 0,\\ \zeta_\rho(y_2^\delta,w_2^\delta)\cdot S\left(\frac{(y_1^\delta,w_1^\delta)-(y_2^\delta,w_2^\delta)}{\delta}\right) &\leq 0,\end{aligned} \tag{4.17}$$

By the definition of ϕ, (y_1^δ,w_1^δ) and (y_2^δ,w_2^δ) are points of maximum and minimum for $u-\varphi_1$ and $v-\varphi_2$ respectively, with φ_i test functions whose derivatives in (y_i^δ,w_i^δ) with respect to ζ_ρ are respectively the left-hand side of the first and second row of (4.17). Hence, we deduce that in the two points the (continuous) Hamilton-Jacobi equation holds, and we conclude in the standard way. $\square$

5 The case of the Prandtl-Ishlinskii model.

In this section we study the control problem when the hysteresis relationship is given by a Prandtl-Ishlinskii model of play-type, as in Section 2. In this case, the "true" state of the system is $(y,\xi) \in I\!R \times L^2(\mathcal{R},\mu)$, where ξ is the infinite dimensional internal variable. We recall that $\mathcal{R}$ is of finite μ-measure and hence $L^\infty(\mathcal{R},\mu) \subset$

$L^p(\mathcal{R},\mu)$, for every $p \in [1,+\infty[$. Moreover $\mathcal{R}$ is supposed to be bounded. Hence the following definition is independent from $p \in [1,+\infty]$: we define the set $\mathcal{O}$ of the initial states for the system (3.1)

$$\mathcal{O} := \left\{(y,\xi) \in I\!R \times L^p(\mathcal{R},\mu) \middle| (y,\xi(\rho)) \in \overline{\Omega}_\rho \ \mu - \text{a.e. } \rho \in \mathcal{R}\right\}.$$

Note that $\mathcal{O}$ is nonempty strongly closed and convex in $I\!R \times L^p(\mathcal{R},\mu)$ for every $p \in [1,+\infty]$ and it has empty interior in $I\!R \times L^p(\mathcal{R},\mu)$, for every $p \in [1,+\infty[$. From the definition of the Prandtl-Ishlinskii model, it easily follows that $\mathcal{O}$ is an invariant set for the system (3.1).

We perform our analysis in the Hilbert space $I\!R \times L^2(\mathcal{R},\mu)$. We use the notation (and the statement) of Theorem 2.5 to indicate by $t \mapsto w_{(\cdot)}(t)$ the evolution of the internal variables. Moreover, for the specific evolution given by the solution of the system (3.1), we use the notation $t \mapsto w_{(\cdot)}(t; y_0, \xi_0, \alpha)$.

Let $a \in A$ be a fixed constant control, then for every $(y_0, \xi_0) \in \mathcal{O}$, we claim that, as function of time with value in $L^2(\mathcal{R},\mu)$,

$$t \mapsto \dot{w}_{(\cdot)}(t; y_0, \xi_0, a) \text{ is continuous in } t = 0. \tag{5.1}$$

To prove the claim, let us define $w_0 = \mathcal{F}[y_0, \xi_0]$ the constant initial output. If $f(y_0, w_0, a) = 0$ then the conclusion is obvious (all is stationary). Let us suppose for instance $f(y_0, w_0, a) > 0$ (the other case being analogous). Hence the solution $y(\cdot)$ is strictly increasing in $[0,t]$ for a small t and thus at $t = 0$ the time derivative of the output of every play exists. By Theorem 2.5 the time derivative of $w_{(\cdot)}(t)$ exists too. For every $\tau \in [0,t]$, we write $w_{(\cdot)}(\tau)$ in place of $w_{(\cdot)}(\tau; y_0, \xi_0, a)$ and define $C_\tau := \{\rho \in \mathcal{R} | y(\tau) - \rho = w_\rho(\tau)\}$ (note that $C_0 = \{\rho \in \mathcal{R} | y_0 - \rho = \xi_0(\rho)\}$). We have

$$\begin{aligned} &\int_{\mathcal{R}} |\dot{w}_{(\cdot)}(\tau) - \dot{w}_{(\cdot)}(0)|^2 d\mu = \int_{C_\tau \setminus C_0} |\dot{y}(\tau)|^2 d\mu \\ &+ \int_{C_0 \setminus C_\tau} |\dot{y}(0)|^2 d\mu + \int_{C_\tau \cap C_0} |\dot{y}(\tau) - \dot{y}(0)|^2 d\mu. \end{aligned} \tag{5.2}$$

The right-hand side of (5.2) tends to zero as $\tau \to 0$. Indeed the first integral is infinitesimal because the integrand function is bounded and the integration set's measure tends to zero by Lemma 5.3 below (since $y(\tau)$ and $w_{(\cdot)}(\tau)$ respectively tend to y_0 and ξ_0 in $I\!R$ and $L^2(\mathcal{R},\mu)$); the second integral is zero because the integration set is empty (for every play corresponding to ρ, if $y_0 - \rho = \xi_0(\rho)$ and y is increasing in $[0,\tau]$, then $y(\tau) - \rho = w_\rho(\tau)$); the third integral tends to zero because the integrand function is infinitesimal.

In the sequel, as usual, we identify $L^2(\mathcal{R},\mu)$ with its dual. We consider the following three Hamiltonians defined in $\mathcal{O} \times I\!R \times L^2(\mathcal{R},\mu)$ (here and in the sequel $w := \int_{\mathcal{R}} \xi d\mu$)

$$\begin{aligned} &H(y,\xi,p,\psi) := \sup_{a \in A} \Big\{ - f(y,w,a)p \\ &- \int_{\mathcal{R}} \Big[\psi(\rho)\Big(\chi_{\rho r}(y,\xi(\rho)) f(y,w,a)^+ - \chi_{\rho l}(y,\xi(\rho)) f(y,w,a)^-\Big)\Big] d\mu \\ &\quad -l(y,w,a)\Big\}, \end{aligned}$$

$$H_-(y,\xi,p,\psi) := \sup_{a\in A}\Big\{ - f(y,w,a)p$$
$$- \int_{\mathcal{R}} \Big[\Big(\psi(\rho)^+\chi_{\rho r}(y,\xi(\rho))f(y,w,a)^+ + \psi(\rho)^-\chi_{\rho l}(y,\xi(\rho))f(y,w,a)^-\Big)\Big]d\mu$$
$$-l(y,w,a)\Big\},$$

and

$$H^+(y,\xi,p,\psi) := \sup_{a\in A}\Big\{ - f(y,w,a)p$$
$$+ \int_{\mathcal{R}} \Big[\Big(\psi(\rho)^-\chi_{\rho r}(y,\xi(\rho))f(y,w,a)^+ + \psi(\rho)^+\chi_{\rho l}(y,\xi(\rho))f(y,w,a)^-\Big)\Big]d\mu$$
$$-l(y,w,a)\Big\}.$$

For every Fréchet differentiable function $\varphi : I\!R \times L^2(\mathcal{R},\mu) \to I\!R$, we denote by φ_y and $D_\xi\varphi$ respectively its derivative with respect to $y \in I\!R$ and its Fréchet differential with respect to $\xi \in L^2(\mathcal{R},\mu)$.

Theorem 5.1 *Let (3.3), (3.4) hold (note that the Prandtl-Ishlinskii operator satisfies (3.2)). Then the value function V is a uniformly continuous and bounded viscosity solution in $\mathcal{O} \subset I\!R \times L^2(\mathcal{R},\mu)$ of*

$$\lambda V(y,\xi) + H(y,\xi,V_y,D_\xi V) = 0 \tag{5.3}$$

That is, for every continuously Fréchet differentiable function $\varphi : I\!R\times L^2(\mathcal{R},\mu) \to I\!R$ such that $V-\varphi$ has in (y,ξ) an extremal point with respect to $\mathcal{O}$, the following holds

$$\begin{cases} \lambda V(y,\xi) + H_-(y,\xi,\varphi_y(y,\xi),D\xi_\varphi(y,\xi)) \le 0 & \textit{if } (y,\xi) \textit{ maximum},\\ \lambda V(y,\xi) + H^+(y,\xi,\varphi_y(y,\xi),D\xi_\varphi(y,\xi)) \ge 0 & \textit{if } (y,\xi) \textit{ minimum}. \end{cases} \tag{5.4}$$

If V satisfies the first (respectively the second) equation of (5.4), then it is said a subsolution (respectively a supersolution).

Proof. The uniform continuity and the boundedness come from the results of Section 3.

For clarity we recall here the Dynamic Programming Principle with the notation $w_{(\cdot)}(\cdot;\cdot,\cdot,\cdot)$ for the evolution of the internal variables (see above (5.1)). For every $(y,\xi)\in\mathcal{O}$, for every $t\ge 0$ we have

$$\begin{aligned} V(y,\xi) = \inf_{\alpha\in\mathcal{A}} \Big[\int_0^t e^{-\lambda s} l\Big(y_{(y,\xi)}(s;\alpha), w_{(y,\xi)}(s;\alpha),\alpha(s)\Big)ds + \\ e^{-\lambda t} V\Big(y_{(y,\xi)}(t;\alpha), w_{(\cdot)}(t;y,\xi,\alpha)\Big)\Big]. \end{aligned} \tag{5.5}$$

We prove that V is a subsolution of $\lambda V + H \le 0$, which of course implies $(5.4)_1$. Let $\varphi \in C^1(I\!R\times L^2(\mathcal{R},\mu))$. Using (2.6) and (5.1), for every $(y_0,\xi_0)\in\mathcal{O}$ ($w_0 = \mathcal{F}[y_0,\xi_0]$) and for every $a\in A$ fixed, we have (dropping the notation of the initial state in the trajectories)

$$\lim_{t\to 0}\frac{\varphi\Big(y(t),w_{(\cdot)}(t)\Big)-\varphi(y_0,\xi_0)}{t}=\frac{\partial\varphi}{\partial y}(y_0,\xi_0)f(y_0,w_0,a)+$$
$$\Big\langle D_\xi\varphi(y_0,\xi_0),\dot w_{(\cdot)}(0)\Big\rangle_{L^2(\mathcal{R},\mu),L^2(\mathcal{R},\mu)}=\frac{\partial\varphi}{\partial y}(y_0,\xi_0)f(y_0,w_0,a)+$$
$$\int_{\mathcal{R}}D_\xi\varphi(y_0,\xi_0)\Big(\chi_{\rho r}(y_0,\xi_0(\rho))f(y_0,w_0,a)^+-\chi_{\rho l}(y_0,\xi_0(\rho))f(y_0,w_0,a)^-\Big)d\mu.$$

From this and DPP (5.5) we get $(5.4)_1$ (subsolution) in a standard way.

Now we prove $(5.4)_2$ (supersolution). Let (y_0,ξ_0) be a point of minimum for $V-\varphi$ with respect to $\mathcal{O}$. We follow the proof of the single play case. Take $\varepsilon>0$ and $t>0$ and, via DPP, find a measurable control α such that an inequality as (4.8) holds (with suitable modifications concerning the presence of the internal variables). Using (2.12), we have

$$\begin{aligned}\varphi\Big(y(t),w_{(\cdot)}(t)\Big)-\varphi(y_0,\xi_0)=\int_0^t\Big[\varphi_y\Big(y(s),w_{(\cdot)}(s)\Big)\\ +\int_{\mathcal{R}}D_\xi\varphi\Big(y(s),w_{(\cdot)}(s)\Big)(\rho)\dot w_\rho(s)d\mu\Big]ds,\end{aligned}$$

We approximate $f(y(s),w(s),\alpha(s))$, $l(y(s),w(s),\alpha(s))$, $D_\xi\varphi(y(s),w_{(\cdot)}(s))$ respectively by $f(y_0,w_0,\alpha(s))$, $l(y_0,w_0,\alpha(s))$, $D_\xi\varphi(y_0,\xi_0)$. The committed error is infinitesimal as s goes to zero, independently on α (and of course on ε and t); in particular note that, by the property of Play (see (2.7)), we have $\|w_{(\cdot)}(s)-\xi_0\|_{L^\infty(\mathcal{R},\mu)}\le\int_0^s|\dot y(\tau)|d\tau\le Ms$. For μ-a.e. $\rho\in\mathcal{R}$, there exists $s_\rho>0$ such that for every $0\le s\le s_\rho$, we have (compare with (4.10))

$$\begin{aligned}D_\xi\varphi(y_0,\xi_0)(\rho)\chi_{\rho r}(y(s),w_\rho(s))f(y_0,w_0,\alpha(s))^+\\ \ge -D_\xi\varphi(y_0,\xi_0)^-(\rho)\chi_{\rho r}(y_0,\xi_0(\rho))f(y_0,w_0,\alpha(s))^+,\end{aligned}\tag{5.6}$$

and similarly for the term with $\chi_{\rho l}$. The problem here, is that the amplitude of the s-interval where (5.6) holds depends on ρ. However, we define the subsets C_s, $C_0\subseteq\mathcal{R}$ as for (5.2), and note that if, for some s, $\rho\notin C_s\setminus C_0$, then for that s (5.6) holds. Moreover, as in (5.2), by Lemma 5.3 below, $\mu(C_s\setminus C_0)$ tends to zero as s goes to zero; finally, if $\rho\notin C_0$, then the right-hand side of (5.6) is zero. Hence we have

$$\begin{aligned}&\int_0^t\int_{\mathcal{R}}D_\xi\varphi(y_0,\xi_0)(\rho)\chi_{\rho r}(y(s),w_\rho(s))f(y_0,w_0,\alpha(s))^+d\mu ds\\ &\quad\ge\int_0^t\int_{C_s\setminus C_0}D_\xi\varphi(y_0,\xi_0)(\rho)\chi_{\rho r}(y(s),w_\rho(s))f(y_0,w_0,\alpha(s))^+d\mu ds\\ &\qquad+\int_0^t\int_{\mathcal{R}}-D_\xi\varphi(y_0,\xi_0)^-(\rho)\chi_{\rho r}(y_0,\xi_0(\rho))f(y_0,w_0,\alpha(s))^+d\mu ds,\end{aligned}$$

and the first term of the right-hand side, when divided by t, is still infinitesimal for $t\to 0$. We then conclude, in a standard way. $\square$

Also in this case we can get a "Neumann-type boundary condition". For every $(y,\xi)\in\mathcal{O}$, let us define the sets

$$\mathcal{R}^r_{(y,\xi)} := \left\{\rho\in\mathcal{R}\middle|\xi(\rho)-y=-\rho\right\}, \quad \mathcal{R}^l_{(y,\xi)} := \left\{\rho\in\mathcal{R}\middle|\xi(\rho)-y=\rho\right\}$$

Note that if $\rho\in\partial\mathcal{R}^i_{(y,\xi)}$ (with $i\in\{l,r\}$) ,then $(y,\xi(\rho))\in\partial\Omega_\rho$. Moreover, we define $\mathcal{R}_{(y,\xi)} := \mathcal{R}^r_{(y,\xi)}\cup\mathcal{R}^l_{(y,\xi)}$. Similar argumentations as in the previous section lead to the following "boundary condition in the viscosity sense"

$$\begin{cases} \text{if } \mu\left(\mathcal{R}_{(y,\xi)}\right)=0 \qquad \lambda V(y,\xi)+H(y,\xi,V_y,0)=0; \qquad \text{otherwise:}\\ \mu\left(\left\{\rho\in\mathcal{R}^r_{(y,\xi)}\middle|\text{sign}(\xi(\rho)-y)D_\xi V(y,\xi)(\rho)=0\right\}\right)>0\\ \text{or } \mu\left(\left\{\rho\in\mathcal{R}^l_{(y,\xi)}\middle|\text{sign}(\xi(\rho)-y)D_\xi V(y,\xi)(\rho)=0\right\}\right)>0 \end{cases} \tag{5.7}$$

where $\text{sign}(x)=1$ if $x>0$, $\text{sign}(x)=-1$ if $x<0$. The previous boundary condition is "strict" and has to be understood in the following viscosity sense (in the following the first implication is for subsolution and the second one for supersoltion and both hold if $\mu(\mathcal{R}_{(y,\xi)})>0$)

$$\begin{cases} \lambda V+H(y,\xi,V_y,0)>0\\ \quad\Rightarrow \mu\left(\left\{\rho\in\mathcal{R}^r_{(y,\xi)}\middle|\text{sign}(\xi(\rho)-y)D_\xi V(y,\xi)(\rho)<0\right\}\right)>0\\ \qquad \text{or } \mu\left(\left\{\rho\in\mathcal{R}^l_{(y,\xi)}\middle|\text{sign}(\xi(\rho)-y)D_\xi V(y,\xi)(\rho)<0\right\}\right)>0,\\ \lambda V+H(y,\xi,V_y,0)<0\\ \quad\Rightarrow \mu\left(\left\{\rho\in\mathcal{R}^r_{(y,\xi)}\middle|\text{sign}(\xi(\rho)-y)D_\xi V(y,\xi)(\rho)>0\right\}\right)>0\\ \qquad \text{or } \mu\left(\left\{\rho\in\mathcal{R}^l_{(y,\xi)}\middle|\text{sign}(\xi(\rho)-y)D_\xi V(y,\xi)(\rho)>0\right\}\right)>0. \end{cases} \tag{5.8}$$

However, as we already said in the Introduction, such "boundary formulation" is not enough for usual comparison technique; we still need to use the discontinuous Hamilton-Jacobi equation.

Theorem 5.2 *If $\mathcal{R}\subset]0,+\infty[$ is bounded and has μ-finite measure, then the value function V is the unique bounded and uniformly continuous (with respect to $\mathbb{R}\times L^2(\mathcal{R},\mu)$) viscosity solution of the Hamilton-Jacobi equation (5.3).*

Proof. We are going to prove a comparison result for every bounded and uniformly continuous subsolution u and supersolution v.

As usual, by contradiction, let us suppose that there exists $(\overline{y},\overline{\xi})\in\mathcal{O}$ such that $u(\overline{y},\overline{\xi})-v(\overline{y},\overline{\xi})=\eta>0$. Let $g:\mathbb{R}\to\mathbb{R}$ be a C^1 function such that $g\geq 0$ and $g(y)=|y|$ for $|y|$ sufficiently large (and so g has bounded derivative). Let S be the strictly elliptic symmetric matrix as in (4.16). For every $(y_i,\xi_i)\in\mathcal{O}$, $i=1,2$, we define

$$\|(y_1,\xi_1)-(y_2,\xi_2)\| := \int_{\mathcal{R}} S(y_1-y_2,\xi_1(\rho)-\xi_2(\rho))\cdot(y_1-y_2,\xi_1(\rho)-\xi_2(\rho))d\mu$$

Note that there exist positive constants C_1, C_2 such that

$$C_1(|y_1-y_2|^2+\|\xi_1-\xi_2\|^2_{L^2}) \le \|(y_1,\xi_1)-(y_2\xi_2)\| \le C_2(|y_1-y_2|^2+\|\xi_1-\xi_2\|^2_{L^2}).$$

For (y_2,ξ_2) fixed, the map $(y,\xi)\mapsto \|(y,\xi)-(y_2,\xi_2)\|$ is Fréchet differentiable.

Take $\beta>0$ and $\delta>0$ and any $\psi_i\in L^2(\mathcal{R},\mu)$ and $a_i\in I\!R$, $i=1,2$, with $|a_i|,\|\psi_i\|_{L^2}\le\delta$, and define a function ϕ on $\mathcal{O}\times\mathcal{O}$

$$\begin{aligned}&\phi\Big((y_1,\xi_1),(y_2,\xi_2)\Big) := u(y_1,\xi_1)-v(y_2,\xi_2)-\beta(g(y_1)+g(y_2))\\ &-\frac{\|(y_1,\xi_1)-(y_2,\xi_2)\|}{2\delta}-a_1y_1-a_2y_2-\langle\psi_1,\xi_1\rangle-\langle\psi_2,\xi_2\rangle.\end{aligned}$$

If β and δ are sufficiently small, we have

$$\phi\Big((\overline{y},\overline{\xi}),(\overline{y},\overline{\xi})\Big)\ge\frac{\eta}{2}. \tag{5.9}$$

Note that, if $(y,\xi)\in\mathcal{O}$, then

$$\|\xi\|_{L^2}\le|y|\mu(\mathcal{R})^{\frac12}+\|\mathrm{id}_\rho\|_{L^2},$$

where id_ρ is the identity function on $\mathcal{R}$. Hence, for $|y_i|$ sufficiently large and for $\delta(\mu(\mathcal{R})^{\frac12}+1)<\beta$, denoting by M a bound on u and v, we have

$$\begin{aligned}&\phi\Big((y_1,\xi_1),(y_2,\xi_2)\Big)\le u(y_1,\xi_1)-v(y_2,\xi_2)\\ &-\beta(g(y_1)+g(y_2))-a_1y_1-a_2y_2-\langle\psi_1,\xi_1\rangle-\langle\psi_2,\xi\rangle\le\\ &2M-\beta(|y_1|+|y_2|)+\sigma((\mu(\mathcal{R})^{\frac12}+1)(|y_1|+|y_2|+2\|\mathrm{id}_\rho\|_{L^2})<0\end{aligned} \tag{5.10}$$

Note that (5.9), (5.10) hold independently on small δ (and on $|a_i|$, $\|\psi_i\|_{L^2}\le\delta$). Moreover noting that for $(y,\xi)\in\mathcal{O}$, $|y|$ large implies $\|\xi\|_{L^2}$ large, we can then assume that (5.10) holds out of a convex closed bounded set $\mathcal{O}'\times\mathcal{O}'$ independent on small δ. Hence, see Stegall [21], let us take $|a_i|<\delta$ and $\|\psi_i\|_{L^2}<\delta$ such that ϕ has a maximum in $\mathcal{O}'\times\mathcal{O}'$ and thus on $\mathcal{O}\times\mathcal{O}$. Let $(y_1^\delta,\xi_1^\delta),(y_2^\delta,\xi_2^\delta)$ be such a maximum point. By the uniform continuity of u and v, we have

$$\frac{\|(y_1^\delta,\xi_1^\delta)-(y_2^\delta,\xi_2^\delta)\|}{\delta}\to 0\quad\text{as }\delta\to 0$$

Let us concentrate on the subsolution u. As usual, we have that

$$(y,\xi)\mapsto u(y,\xi)-\beta g(y)-a_1y-\langle\psi_1,\xi\rangle-\frac{\|(y,\xi)-(y_2^\delta,\xi_2^\delta)\|}{2\delta}\;=:(u-\varphi)(y,\xi)$$

has a maximum in $(y_1^\delta,\xi_1^\delta)$ with respect to $\mathcal{O}$ and φ is an admissible test function. Let us take $\varepsilon>0$ and define

$$\mathcal{R}^r_\delta:=\left\{\rho\in\mathcal{R}^r_{(y_1^\delta,\xi_1^\delta)}\middle|\mathrm{sign}(\xi_1^\delta(\rho)-y_1^\delta)D_\xi\varphi(y_1^\delta,\xi_1^\delta)(\rho)\le-\varepsilon\right\}.$$

By the definition of ζ_ρ (4.13) and (4.16), we have $\mathcal{R}^r_\delta \subseteq \left\{\rho \in \mathcal{R}^r_{(y_1^\delta,\xi_1^\delta)} \Big| |\psi_1(\rho)| \geq \varepsilon\right\}$. For the particular choice of ψ_1, we have

$$\mu\left(\left\{\rho \in \mathcal{R}^r_{(y_1^\delta,\xi_1^\delta)} \Big| |\psi_1(\rho)| \geq \varepsilon\right\}\right) \leq C_\varepsilon \delta,$$

with C_ε depending on ε. Hence, for for constants $C_\varepsilon > 0$ and $C > 0$, we have,

$$\begin{aligned}
&\int_{\mathcal{R}^r_{(y_1^\delta,\xi_1^\delta)}} \left(D_\xi \varphi(y_1^\delta,\xi_1^\delta)\right)^+ (\rho) d\mu \leq \varepsilon\mu(\mathcal{R}) + \int_{\mathcal{R}^r_\delta} D_\xi \varphi(y_1^\delta,\xi_1^\delta)(\rho) d\mu \\
&\leq \varepsilon\mu(\mathcal{R}) + \int_{\left\{\rho \in \mathcal{R}^r_{(y_1^\delta,\xi_1^\delta)} \Big| |\psi_1| \geq \varepsilon\right\}} D_\xi \varphi(y_1^\delta,\xi_1^\delta)(\rho) d\mu \leq \mu(\mathcal{R})\varepsilon \\
&\quad + \frac{C_\varepsilon}{\delta}\left(|y_1^\delta - y_2^\delta|\delta + \|\xi_1^\delta - \xi_2^\delta\|_{L^2}\delta^{1/2}\right) + C\delta,
\end{aligned}$$

and the first term of the last row is infinitesimal when $\varepsilon > 0$ is fixed and $\delta \to 0$.

We obtain similar conclusions for the other discontinuous term in H_- and for the similar analysis replacing u by v. Note that the finite dimensional part of H (i.e. the part containing the derivative with respect to y) can be treated in the usual way.

Taking first small ε and small β and finally small δ, using the definition of sub- and supersolution for u and v in $(y_1^\delta,\xi_1^\delta)$ and $(y_2^\delta,\xi_2^\delta)$ respectively, and denoting by $\sigma(\cdot,\cdot)$, $\sigma_\varepsilon(\cdot)$ infinitesimal functions of their argument, we have

$$\begin{aligned}
&\max \phi \leq u(y_1^\delta,\xi_1^\delta) - v(y_2^\delta,\xi_2^\delta) - a_1 y_1^\delta - a_2 y_2^\delta - \langle \psi_1, \xi_1^\delta\rangle - \langle \psi_2, \xi_2^\delta\rangle \\
&\leq \frac{1}{\lambda}(H^+ - H_-) + \sigma(\delta) \leq \sigma(\varepsilon,\beta) + \sigma_\varepsilon(\delta) < \frac{\eta}{2}
\end{aligned}$$

which is a contradiction to (5.9). □

Lemma 5.3 *Let $\Omega \subset I\!R^N$ be of finite μ measure and consider $f_n \to f$ μ-a.e. in Ω and $a_n \to a$ in $I\!R$. Then*

$$\mu\left(\left\{x \in \Omega \Big| f_n(x) = a_n,\ f(x) \neq a\right\}\right) \to 0 \quad \text{as } n \to +\infty.$$

Proof. It easily follows from the finiteness of the measure of Ω and the uppersemicontinuity of μ. □

6 Remarks and extensions.

Remarks 6.1. The result of Section 4 is obviously a special case of the one in Section 5. Indeed, it is enough to take $\mu = \delta_\rho$, the Dirac mass centered in ρ. Moreover, we can also consider the case of a finite sum of Plays which corresponds to μ given by a finite sum of Dirac masses. In this case we get a Neumann-type boundary problem in a finite dimensional set with corners. In particular the set is a sort of indefinite parallelepiped. The Neumann boundary condition has to be read $\min_i V_{w_i} < 0$

for subsolution (respectively $\max_i V_{w_i} > 0$ for supersolution), where the min (respectively the max) is taken over all the derivative of V with respect to the the outward vectors relative to each face occurring as boundary. For such a formulation of Neumann conditions see Dupuis-Ishii [11].

Remarks 6.2. (*Stop*). It is quite obvious that, with suitable easy modifications, the results of Section 4 and 5 also hold in the case when we replace the Play operator with the Stop operator (see for instance Visintin [22] for the definition).

Remarks 6.3. (*Generalized play*). We can also replace the Play with a so-called generalized play (see for instance Visintin [22]). It is constructed as the Play, replacing the two straight lines $w = u + \rho$ and $w = u - \rho$ by two strictly increasing Lipschitz continuous curves $w = \gamma_l(u)$, $w = \gamma_r(u)$, where $\gamma_l(u) > \gamma_r(u)$ for all $u \in I\!R$. The results of Sections 2, 3 and 4 easily apply to this case. In particular an equation as (2.6) holds. We have to be careful to the fact that the matrix S defined in (4.16) is not more constant, but it depends on the point. However, under general regularity assumptions on the two curves, such dependence is Lipschitz and smooth. We can then apply again classical results (see for instance Ishii [12]).

Remarks 6.4. (*Generalized Prandtl-Ishlinskii operator*). This operator is a superposition of a (possibly infinite) number of generalized plays (see for instance Visintin [22]). We are given a set $\mathcal{R}$ of indices (which we call again ρ but they are not the half length of intervals as before) and a measure μ on $\mathcal{R}$. For every ρ we have a couple of curves as in the previous remark, which we call $\gamma_{\rho l}$ and $\gamma_{\rho r}$. The output is given by the integral with respect to μ of the outputs of the generalized plays labeled by ρ. Under rather general hypotheses on the family of couple of curves (such as for instance “uniform regularity”of the curves with respect the indices) we can perform an analysis as in Section 5 and obtain similar results.

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