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by

Christian Fleischhack

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Christian Fleischhack*

Institut für Theoretische Physik
Universität Leipzig
Augustusplatz 10/11
04109 Leipzig, Germany

Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstraße 22-26
04103 Leipzig, Germany

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Abstract

The bundle structure of the space $\overline{\mathcal{A}}$ of Ashtekar's generalized connections is investigated in the compact case. It is proven that every stratum is a locally trivial fibre bundle. The only stratum being a principal fibre bundle is the generic stratum. Its structure group equals the space $\overline{\mathcal{G}}$ of all generalized gauge transforms modulo the constant center-valued gauge transforms. For abelian gauge theories the generic stratum is globally trivial and equals the total space $\overline{\mathcal{A}}$. However, for a certain class of non-abelian gauge theories – e.g., all $SU(N)$ theories – the generic stratum is nontrivial. This means, there are no global gauge fixings – the so-called Gribov problem. Nevertheless, there is a covering of the generic stratum by trivializations each having total induced Haar measure 1.

*e-mail: Christian.Fleischhack@itp.uni-leipzig.de *or* Christian.Fleischhack@mis.mpg.de

1 Introduction

About 20 years ago, Gribov [9] observed, that for smooth connections in certain gauge theories the Coulomb gauge selects from some gauge orbits more than one single connection. In the language of mathematics this means, the Coulomb gauge is not a global section in the bundle $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$ of all connections over the gauge orbit space. Then, Singer [15] investigated this problem systematically and saw that generally the non-existence of smooth sections in that bundle is a typical feature of gauge theories with non-commutative structure group $\mathbf{G}$. Moreover, he was able to show that even in the subbundle of all irreducible connections[1] there is no such section. Explicitly he proved this for $\mathbf{G} = SU(N)$ and "space-time" manifold $M = S^4$. In the present paper we will investigate, whether there is such a Gribov problem also for generalized connections or not.

What could we expect? First, recall [8] that the gauge orbit of a generalized connection $\overline{A} \in \overline{\mathcal{A}}$ w.r.t. to generalized gauge transforms in $\overline{\mathcal{G}}$ equals $\mathbf{B}(\overline{A}) \big\backslash \overline{\mathcal{G}}$ where $\mathbf{B}(\overline{A})$ is the stabilizer of $\overline{A}$. Now there arise some questions to be answered in this paper.

Question 1 Is $\pi : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\overline{\mathcal{G}}$ a fibre bundle?

In general, the answer is "no"; at least, if we demand, that the fibres $\mathbf{B}(\overline{A}) \big\backslash \overline{\mathcal{G}}$ have to be isomorphic $\overline{\mathcal{G}}$-spaces. This, however, is only true [3] if all stabilizers are $\overline{\mathcal{G}}$-conjugate. But, this implies [8] that all Howe subgroups in $\mathbf{G}$ have to be $\mathbf{G}$-conjugate. This is only possible for abelian $\mathbf{G}$. Indeed, we will prove that in this case π *is* a fibre bundle.

For arbitrary $\mathbf{G}$ we can expect bundle structures only on subsets of $\overline{\mathcal{A}}$ where all connections have conjugate stabilizers. The maximal sets of that kind are exactly the sets of connections having one and the same gauge orbit type t, i.e. the so-called strata $\overline{\mathcal{A}}_{=t}$.

Question 2 Are the strata $\pi : \overline{\mathcal{A}}_{=t} \longrightarrow \overline{\mathcal{A}}_{=t}/\overline{\mathcal{G}}$ fibre bundles?

For actions of compact Lie groups on arbitrary (completely regular) spaces the strata are always fibre bundles. However, for the proof not the Lie property of the acting group is important, but the existence of a slice theorem. In contrast to the former one the latter one is guaranteed also in our case of interest. Hence, we will be able to prove that the strata are indeed fibre bundles. The structure group of the bundle $\overline{\mathcal{A}}_{=t}$ will be – analogously to the case of general transformation groups – equal to $\mathbf{B}(\overline{A}) \big\backslash N(\mathbf{B}(\overline{A}))$, where $\overline{A} \in \overline{\mathcal{A}}_{=t}$ is arbitrary. (Here, $N(\mathbf{B}(\overline{A}))$ denotes the normalizer of $\mathbf{B}(\overline{A})$ w.r.t. $\overline{\mathcal{G}}$.) Moreover, we will see that this group is again (as $\mathbf{B}(\overline{A})$ itself [8]) – at least topologically – uniquely determined by the holonomy group $\mathbf{H}_{\overline{A}}$ of $\overline{A}$.

The next stronger structure after that of a fibre bundle is the structure of a principal fibre bundle.

Question 3 What strata are principal fibre bundles?

For $\overline{\mathcal{A}}_{=t}$ to be a principal fibre bundle, the typical fibre has to be a topological group, i.e., the stabilizer $\mathbf{B}(\overline{A})$ has to be a normal subgroup of $\overline{\mathcal{G}}$. We will show that this is the case in the generic stratum only.

[1] A connection is called irreducible iff its holonomy group equals the total structure group. Obviously, every irreducible connection is generic, but in general not the other way round. See also the discussion at the end of this paper.

Since every stratum is locally trivial as a fibre bundle, the generic stratum is locally trivial, i.e., there are at least local sections in $\overline{\mathcal{A}}_{\rm gen}$.

Question 4 Is $\pi : \overline{\mathcal{A}}_{\rm gen} \longrightarrow \overline{\mathcal{A}}_{\rm gen}/\overline{\mathcal{G}}$ even globally trivial?

For smooth connections – as mentioned above – this is not the case. A necessary condition has been the non-commutativity of the structure group $\mathbf{G}$. We will see that for generalized connections as well, $\pi \mid_{\overline{\mathcal{A}}_{\rm gen}}$ is globally trivial for abelian $\mathbf{G}$, but not globally trivial for a relatively large class of non-abelian $\mathbf{G}$ (among them $\mathbf{G} = SU(N)$).
In the Sobolev case the impact of this Gribov problem to the calculation of functional integrals is enormous. Since for the integration trivial subbundles are of special interest, we have the next natural

Question 5 What "size" can trivial subbundles of $\overline{\mathcal{A}}_{\rm gen}$ have at most?

We will find trivial subbundles that emerge from $\overline{\mathcal{A}}$ by cutting out certain zero subsets. This means, in our case the Gribov problem is completely irrelevant for the calculation of functional integrals (at least if one considers only measures being absolutely continuous w.r.t. the Ashtekar-Lewandowski measure).
Finally, we note that in the following almost all considered bundles are, of course, purely topological bundles and consequently no manifold structure is available.

2 Preliminaries

We recall the basic facts from [8]. For the details we refer the reader to this reference.
Let M be an at least two-dimensional manifold, $m \in M$ some fixed point and $\mathbf{G}$ be a compact Lie group. The set $\overline{\mathcal{A}}$ of generalized connections $\overline{A}$ is defined by $\overline{\mathcal{A}} := \varprojlim_\Gamma \mathbf{G}^{\mathbf{E}(\Gamma)} = \mathrm{Hom}(\mathcal{P}, \mathbf{G})$. Here Γ runs over all finite graphs in M. $\mathbf{E}(\Gamma)$ is the number of edges in Γ, $\mathbf{V}(\Gamma)$ will be that of all vertices. Furthermore, $\mathcal{P}$ denotes the set of all finite paths in M and $\mathcal{HG}$ that of all paths starting and ending in m. The set $\overline{\mathcal{G}}$ of generalized gauge transforms $\overline{g}$ is $\overline{\mathcal{G}} := \varprojlim_\Gamma \mathbf{G}^{\mathbf{V}(\Gamma)} = \mathrm{Maps}(M, \mathbf{G})$ that continuously acts on $\overline{\mathcal{A}}$ via $h_{\overline{A}\circ\overline{g}}(\gamma) = g_{\gamma(0)}^{-1} h_{\overline{A}}(\gamma) g_{\gamma(1)}$ where γ is in $\mathcal{P}$ and $h_{\overline{A}}$ is the homomorphism corresponding to $\overline{A}$. The stabilizer $\mathbf{B}(\overline{A})$ of $\overline{A}$ contains exactly those gauge transforms that fulfill $h_{\overline{A}}(\gamma_x) = g_m^{-1} h_{\overline{A}}(\gamma_x) g_x$ for all $x \in M$ and whose m-component g_m lies in the centralizer $Z(\mathbf{H}_{\overline{A}})$ of the holonomy group of $\overline{A}$, respectively. Here, for all x, γ_x is some fixed path from m to x. We have $\mathbf{B}(\overline{A}) \cong Z(\mathbf{H}_{\overline{A}})$. Now, the orbit type of $\overline{A}$ is defined to be the $\overline{\mathcal{G}}$-conjugacy class of $\mathbf{B}(\overline{A})$, but equivalently it can be defined to be the $\mathbf{G}$-conjugacy class $[Z(\mathbf{H}_{\overline{A}})]$ of the centralizer of the holonomy group of $\overline{A}$. This definition will be used in the following. The types are partially ordered by the natural inclusion-induced ordering of classes of subgroups of $\mathbf{G}$. A stratum $\overline{\mathcal{A}}_{=t}$ is the set of all connections $\overline{A} \in \overline{\mathcal{A}}$ having type t und the generic stratum $\overline{\mathcal{A}}_{\rm gen}$ is the set of all connections having the maximal orbit type $[Z]$ where $Z \equiv Z(\mathbf{G})$ is the center of $\mathbf{G}$. $\overline{\mathcal{A}}_{\rm gen}$ is an open, dense and $\overline{\mathcal{G}}$-invariant subset of $\overline{\mathcal{A}}$ with $\mu_0(\overline{\mathcal{A}}_{\rm gen}) = 1$ for the Ashtekar-Lewandowski measure μ_0. Moreover, there is a slice theorem on $\overline{\mathcal{A}}$. This means, for every $\overline{A} \in \overline{\mathcal{A}}$ there is an so-called slice $\overline{S} \subseteq \overline{\mathcal{A}}$ with $\overline{A} \in \overline{S}$ such that:
- $\overline{S} \circ \overline{\mathcal{G}}$ is an open neighbourhood of $\overline{A} \circ \overline{\mathcal{G}}$ and
- there is an equivariant retraction $F : \overline{S} \circ \overline{\mathcal{G}} \longrightarrow \overline{A} \circ \overline{\mathcal{G}}$ with $F^{-1}(\{\overline{A}\}) = \overline{S}$.

The most important tool for the proof of this theorem has been a so-called reduction mapping $\varphi_{\boldsymbol{\alpha}}$. Note, due to the compactness of $\mathbf{G}$, there is a finite set $\boldsymbol{\alpha}$ of paths starting and ending in m such that $Z(h_{\overline{A}}(\boldsymbol{\alpha})) = Z(\mathbf{H}_{\overline{A}})$. Since $[Z(h_{\overline{A}}(\boldsymbol{\alpha}))]$ is the orbit type of $h_{\overline{A}}(\boldsymbol{\alpha})$ w.r.t. the adjoint action of $\mathbf{G}$ on $\mathbf{G}^{\#\boldsymbol{\alpha}}$, the reduction mapping $\varphi_{\boldsymbol{\alpha}} : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^n$ with $\overline{A}' \longmapsto h_{\overline{A}'}(\boldsymbol{\alpha})$ lifts the slice theorem from $\mathbf{G}^n$ to $\overline{\mathcal{A}}$. The notion of a reduction mapping will be crucial again in the present paper.

3 Bundle Structure of the Strata

In the following, if H is a subgroup of the group G, we denote by $N_G(H)$ the normalizer of H in G. It should be clear in the sequel what bigger group G is used when dealing with H, so that we simply write $N(H)$.

Proposition 3.1 Let $t \in \mathcal{T}$ be a gauge orbit type and $\overline{A} \in \overline{\mathcal{A}}_{=t}$ be some connection. Then the stratum $\overline{\mathcal{A}}_{=t}$ is a fibre bundle with fibre $\mathbf{B}(\overline{A})\backslash \overline{\mathcal{G}}$ and structure group $\mathbf{B}(\overline{A})\backslash N(\mathbf{B}(\overline{A}))$ acting on $\mathbf{B}(\overline{A})\backslash \overline{\mathcal{G}}$ by left translation.

As we mentioned in the introduction the crucial point for the proof is the existence of a slice theorem on $\overline{\mathcal{A}}$, not the Lie property of the acting group. Hence we can reuse the proofs for the fibre bundle property given, e.g., in [3]. And the proposition above is then a consequence of

Proposition 3.2 Let X be a Hausdorff space with a right action of some compact G. Furthermore, assume that a slice theorem holds on X, i.e., for all $x \in X$ there is an $S \subseteq X$ with $x \in S$ such that $S \circ G$ is an open neighbourhood of $x \circ G$ and there is an equivariant retraction $f : S \circ G \longrightarrow x \circ G$ with $f^{-1}(\{x\}) = S$.
Then for all types t holds: The stratum $X_{=t}$ is a fibre bundle with fibre $H\backslash G$ (for any $H \in t$) and structure group $H\backslash N(H)$ acting on $H\backslash G$ by left translation.

For the proof we need the following

Lemma 3.3 Let G be compact, and let H be a closed subgroup. Then the group of all equivariant homeomorphisms from $H\backslash G$ to $H\backslash G$ is isomorphic to $H\backslash N(H)$.

Proof Every such $f : H\backslash G \longrightarrow H\backslash G$ is determined by an $a_f \in G$ via $f(Hg) = Ha_f^{-1}g$ for all $g \in G$. [3] Such an a_f has to fulfill $a_f^{-1}Ha_f \subseteq H$, i.e. lies in $N(H)$. Conversely, every such a determines exactly one f_a of the desired type. Due to $f(H) = Ha_f^{-1}$, the map $f \longmapsto a_f$ is unique up to multiplication of a_f by elements in H. Hence $\psi : \mathrm{Homeo}^G(H\backslash G) \longrightarrow H\backslash N(H)$, $f \longmapsto [a_f^{-1}]_H$, is bijective and by $\psi(f \circ g) = \psi(f)\psi(g)$ even a group isomorphism. **qed**

Proof Proposition 3.2

- Since strata are always invariant w.r.t. the action of G, the existence of a slice theorem on the whole X implies that of slice theorems on all $X_{=t}$.[2] Hence, w.l.o.g.

we assume that X has one single type t, i.e. $X = X_{=t}$. Additionally, we choose some $H \in t$.

- We will construct a bundle chart around every $x \in X$ with $\mathrm{Stab}(x) = H$.[3]
For this, let S be a slice around x with corresponding retraction $f : S \circ G \longrightarrow x \circ G$, and let $s \in S$ and $h \in G_s := \{h \in G \mid s \circ h = s\}$. Then from $x = f(s) = f(s \circ h) = f(s) \circ h = x \circ h$ follows $h \in H$, i.e. $G_s \subseteq H$. Since by assumption all stabilizers on X are conjugate, we get $G_s = H$. Hence, H acts trivially on S. From that we have a chain $S \times {}_H\backslash G \cong S \times_H G \cong S \circ G$ of equivariant homeomorphisms. [3] Furthermore, obviously $S \cong S/H \cong (S \circ G)/G$.[4] Using this identification we get a chart by the following commutative diagram:
$$\chi_S : \quad \begin{tikzcd} S \times {}_H\backslash G \arrow[r, "\cong"] \arrow[dr, "pr_1"'] & S \circ G \arrow[d, "\pi"] \\ & S \end{tikzcd} .$$
- Finally we have to control the transition mappings.
Let χ_S and χ_T be two such chart mappings. We define by
$$\chi_T^{-1}\chi_S : \quad \begin{tikzcd} (S \cap T) \times {}_H\backslash G \arrow[rr, "\cong"] \arrow[dr] & & (S \cap T) \times {}_H\backslash G \arrow[dl] \\ & S \cap T & \end{tikzcd}$$
for all $x \in S \cap T$ an equivariant homeomorphism
$$\vartheta_x : {}_H\backslash G \longrightarrow {}_H\backslash G \text{ by } \chi_T^{-1}\chi_S(x, Hg) = (x, \vartheta_x(Hg)).$$
By the proof of Lemma 3.3, ϑ_x corresponds via $\vartheta_x(H) = Ha_{\vartheta_x}^{-1}$ to a unique $\psi(x) := [a_{\vartheta_x}]^{-1} \in {}_H\backslash N(H)$. Hence, $\chi_S(x, Hg) = \chi_T(x, \psi(x) \circ Hg))$.
It remains only the proof of the continuity of $\psi : S \cap T \longrightarrow {}_H\backslash N(H)$. This, however, is an easy consequence of that of $\chi_T^{-1}\chi_S$ by means of
$$\begin{array}{ccccccc} S \cap T & \xrightarrow{\iota} & (S \cap T) \times {}_H\backslash H & \xrightarrow{\chi_T^{-1}\chi_S} & (S \cap T) \times {}_H\backslash N(H) & \xrightarrow{\mathrm{pr}_2} & {}_H\backslash N(H). \\ x & \longmapsto & (x, H) & \longmapsto & (x, Ha_{\vartheta_x}^{-1}) & \longmapsto & \psi(x) \end{array}$$

qed

[2] If $S \subseteq X$ is a slice around $x \in X_{=t}$, then $S \cap X_{=t}$ is a slice around $x \in X_{=t}$ in $X_{=t}$: $(S \cap X_{=t}) \circ G = (S \circ G) \cap X_{=t}$ is an open neighbourhood von $x \circ G$ and $f\mid_{(S \cap X_{=t}) \circ G}$: $(S \cap X_{=t}) \circ G \longrightarrow x \circ G$ is again an equivariant retraction with $(f\mid_{(S \cap X_{=t}) \circ G})^{-1}(\{x\}) = S \cap X_{=t}$.

[3] Obviously, in every orbit there is at least one such x.

[4] Let $c : S \longrightarrow (S \circ G)/G$ with $c(s) = [s]$.

- c is surjective by construction.
- c is injective, since from $c(s_1) = c(s_2)$ follows first $s_1 \circ g = s_2$ with some $g \in G$, but this (after using f) implies $x \circ g = x$, hence $g \in H = G_s$, thus $s_2 = s_1 \circ g = s_1$.
- c is continuous.
- c is closed, because for closed $U \subseteq S$ (due to the closure of $f^{-1}(\{x\}) = S \subseteq S \circ G$) U is closed in $S \circ G$, hence (due to the compactness of G) $U \circ G$ is closed in $S \circ G$; by the closure of $\pi : S \circ G \longrightarrow (S \circ G)/G$ we get that of c.
- c is open, because it is closed and bijective.

Proof **Proposition 3.1**
Follows immediately from preceding proposition and the existence of a slice theorem on $\overline{\mathcal{A}}$ [8]. **qed**

In the following we will investigate the detailed structure of $\mathbf{B}(\overline{A}) \Big\backslash N(\mathbf{B}(\overline{A}))$, beginning with that of $N(\mathbf{B}(\overline{A}))$ itself.

Proposition 3.4 Let $\overline{A} \in \overline{\mathcal{A}}$ and $\overline{g} \in \overline{\mathcal{G}}$. Furthermore, we again fix for every $x \in M$ a path γ_x from m to x, where γ_m is trivial.
Then we have $\overline{g} \in N(\mathbf{B}(\overline{A}))$ iff
1. $g_m \in N(Z(\mathbf{H}_{\overline{A}}))$ and
2. $g_x \in h_{\overline{A}}(\gamma_x)^{-1}\, Z(Z(\mathbf{H}_{\overline{A}}))\, g_m\, h_{\overline{A}}(\gamma_x)$ for all $x \in M$.

Proof In general, $\overline{g} \in N(\mathbf{B}(\overline{A})) \Longleftrightarrow \overline{g}^{-1}\mathbf{B}(\overline{A})\overline{g} \subseteq \mathbf{B}(\overline{A})$.
So let $\overline{g} \in \overline{\mathcal{G}}$ and $\overline{g}' \in \mathbf{B}(\overline{A})$. Then we have

$$\begin{array}{cl}
 & \overline{g}^{-1}\overline{g}'\overline{g} \in \mathbf{B}(\overline{A}) \\
\Longleftrightarrow & 1.\;\; g_m^{-1}\, g'_m\, g_m \in Z(\mathbf{H}_{\overline{A}}) \\
 & 2.\;\; h_{\overline{A}}(\gamma_x) = (g_m^{-1} g'_m g_m)^{-1}\, h_{\overline{A}}(\gamma_x)\, (g_x^{-1} g'_x g_x) \qquad \forall x \in M \\
\Longleftrightarrow & 1.\;\; g_m^{-1}\, g'_m\, g_m \in Z(\mathbf{H}_{\overline{A}}) \\
 & 2.\;\; g_m h_{\overline{A}}(\gamma_x) g_x^{-1} h_{\overline{A}}(\gamma_x)^{-1} = (g'_m)^{-1}\, g_m h_{\overline{A}}(\gamma_x) g_x^{-1} h_{\overline{A}}(\gamma_x)^{-1}\, g'_m \qquad \forall x \in M. \\
 & \qquad\qquad (\text{since } g'_x = h_{\overline{A}}(\gamma_x)^{-1} g'_m h_{\overline{A}}(\gamma_x))
\end{array}$$

Hence,

$$\begin{array}{cl}
 & \overline{g} \in N(\mathbf{B}(\overline{A})) \\
\Longleftrightarrow & \overline{g}^{-1}\overline{g}'\overline{g} \in \mathbf{B}(\overline{A}) \qquad \forall \overline{g}' \in \mathbf{B}(\overline{A}) \\
\Longleftrightarrow & 1.\;\; g_m \in N(Z(\mathbf{H}_{\overline{A}})) \\
 & 2.\;\; g_m h_{\overline{A}}(\gamma_x) g_x^{-1} h_{\overline{A}}(\gamma_x)^{-1} \in Z(Z(\mathbf{H}_{\overline{A}})) \qquad \forall x \in M \\
\Longleftrightarrow & 1.\;\; g_m \in N(Z(\mathbf{H}_{\overline{A}})) \\
 & 2.\;\; g_x \in h_{\overline{A}}(\gamma_x)^{-1}\, Z(Z(\mathbf{H}_{\overline{A}}))\, g_m h_{\overline{A}}(\gamma_x) \qquad \forall x \in M.
\end{array}$$

qed

Example 1 In the generic stratum we have $Z(\mathbf{H}_{\overline{A}}) = Z(\mathbf{G})$, hence $N(Z(\mathbf{H}_{\overline{A}})) = \mathbf{G}$ and $Z(Z(\mathbf{H}_{\overline{A}})) = \mathbf{G}$. Consequently, $N(\mathbf{B}(\overline{A})) = \overline{\mathcal{G}}$.

Example 2 In the minimal stratum we have $Z(\mathbf{H}_{\overline{A}}) = \mathbf{G}$. Hence $N(Z(\mathbf{H}_{\overline{A}})) = \mathbf{G}$ and $Z(Z(\mathbf{H}_{\overline{A}})) = Z(\mathbf{G})$. By the proposition above we have $\overline{g} \in N(\mathbf{B}(\overline{A}))$ iff first $g_m \in \mathbf{G}$ is arbitrary and second $g_x \in h_{\overline{A}}(\gamma_x)^{-1}\, Z(\mathbf{G})\, g_m h_{\overline{A}}(\gamma_x) = Z(\mathbf{G})\, h_{\overline{A}}(\gamma_x)^{-1}\, g_m h_{\overline{A}}(\gamma_x)$ holds.

The preceding proposition implies

Corollary 3.5 For all $\overline{A} \in \overline{\mathcal{A}}$ we have $N(\mathbf{B}(\overline{A})) \cong N(Z(\mathbf{H}_{\overline{A}})) \times \bigtimes_{x \neq m} Z(Z(\mathbf{H}_{\overline{A}}))$ as an isomorphism between topological spaces.

Proof We see immediately that the map $\Psi_1 : \overline{g} \longmapsto \Big(g_m, (h_{\overline{A}}(\gamma_x) g_x h_{\overline{A}}(\gamma_x)^{-1} g_m^{-1})_{x \neq m}\Big)$ is a desired homeomorphism. **qed**

We emphasize that both subgroups of $\overline{\mathcal{G}}$ are in general *not* isomorphic as topological groups. At least there is no "reasonable" homomorphism. Roughly speaking, the homomorphy property is destroyed by the structure of g_x as a (usually non-commutative) product of g_m with elements in $Z(Z(\mathbf{H}_{\overline{A}}))$.
In order to investigate the structure of ${\mathbf{B}(\overline{A})}\big\backslash N(\mathbf{B}(\overline{A}))$, we recall the form of $\mathbf{B}(\overline{A})$. By [8] $\mathbf{B}(\overline{A})$ and $Z(\mathbf{H}_{\overline{A}}) \times \mathsf{X}_{x\neq m}\{e_\mathbf{G}\}$ are isomorphic topological (even Lie) groups. Heuristically we have a homeomorphism ${\mathbf{B}(\overline{A})}\big\backslash N(\mathbf{B}(\overline{A})) \cong {Z(\mathbf{H}_{\overline{A}})}\big\backslash N(Z(\mathbf{H}_{\overline{A}})) \times \mathsf{X}_{x\neq m} Z(Z(\mathbf{H}_{\overline{A}}))$. The group isomorphy, however, is not to be expected: Since the base centralizer and the holonomy centralizer are isomorphic groups, it is unlikely that there arise isomorphic groups from originally non-isomorphic groups by factorization. Indeed there are examples (generic connections for $\mathbf{G} = SU(2)$) admitting no such "reasonable" group isomorphism. We will discuss this a bit more in detail in Appendix A.

Proposition 3.6 For every $\overline{A} \in \overline{\mathcal{A}}$

$$[\Psi_1] : \begin{array}{rcl} {\mathbf{B}(\overline{A})}\big\backslash N(\mathbf{B}(\overline{A})) & \longrightarrow & {Z(\mathbf{H}_{\overline{A}})}\big\backslash N(Z(\mathbf{H}_{\overline{A}})) \times \mathsf{X}_{x\neq m} Z(Z(\mathbf{H}_{\overline{A}})) \\ [\overline{g}]_{\mathbf{B}(\overline{A})} & \longmapsto & \big([g_m]_{Z(\mathbf{H}_{\overline{A}})}, (h_{\overline{A}}(\gamma_x) g_x h_{\overline{A}}(\gamma_x)^{-1} g_m^{-1})_{x\neq m}\big) \end{array}$$

is a homeomorphism.

Proof
- $[\Psi_1]$ is well-defined.
Let $[\overline{g}_1]_{\mathbf{B}(\overline{A})} = [\overline{g}_2]_{\mathbf{B}(\overline{A})}$, i.e. $\overline{g}_1 = \overline{g}'\overline{g}_2$ with $\overline{g}' \in \mathbf{B}(\overline{A})$. Then $g'_m \in Z(\mathbf{H}_{\overline{A}})$, hence $[g_{1,m}]_{Z(\mathbf{H}_{\overline{A}})} = [g'_m g_{2,m}]_{Z(\mathbf{H}_{\overline{A}})} = [g_{2,m}]_{Z(\mathbf{H}_{\overline{A}})}$. Moreover,
$$\begin{aligned} h_{\overline{A}}(\gamma_x) g_{1,x} h_{\overline{A}}(\gamma_x)^{-1} g_{1,m}^{-1} &= h_{\overline{A}}(\gamma_x) g'_x g_{2,x} h_{\overline{A}}(\gamma_x)^{-1} g_{2,m}^{-1} (g'_m)^{-1} \\ &= h_{\overline{A}}(\gamma_x) g'_x h_{\overline{A}}(\gamma_x)^{-1}\, h_{\overline{A}}(\gamma_x) g_{2,x} h_{\overline{A}}(\gamma_x)^{-1} g_{2,m}^{-1}\, (g'_m)^{-1} \\ &= g'_m\, h_{\overline{A}}(\gamma_x) g_{2,x} h_{\overline{A}}(\gamma_x)^{-1} g_{2,m}^{-1}\, (g'_m)^{-1} \\ &= h_{\overline{A}}(\gamma_x) g_{2,x} h_{\overline{A}}(\gamma_x)^{-1} g_{2,m}^{-1} \end{aligned}$$
due to the properties of $\overline{g}' \in \mathbf{B}(\overline{A})$ [8] and $\overline{g}_i \in N(\mathbf{B}(\overline{A}))$.
- $[\Psi_1]$ is surjective and continuous.
Follows immediately from the commutative diagram
$$\begin{array}{ccc} N(\mathbf{B}(\overline{A})) & \xrightarrow[\cong]{\Psi_1} & N(Z(\mathbf{H}_{\overline{A}})) \times \mathsf{X}_{x\neq m} Z(Z(\mathbf{H}_{\overline{A}})) \\ \downarrow & & \downarrow \\ {\mathbf{B}(\overline{A})}\big\backslash N(\mathbf{B}(\overline{A})) & \xrightarrow{[\Psi_1]} & {Z(\mathbf{H}_{\overline{A}})}\big\backslash N(Z(\mathbf{H}_{\overline{A}})) \times \mathsf{X}_{x\neq m} Z(Z(\mathbf{H}_{\overline{A}})) \end{array}$$
and Corollary 3.5.
- $[\Psi_1]$ is injective.
Let $[\Psi_1]([\overline{g}_1]) = [\Psi_1]([\overline{g}_2])$. Then $g_{1,m} g_{2,m}^{-1} \in Z(\mathbf{H}_{\overline{A}})$ and
$$h_{\overline{A}}(\gamma_x) g_{2,x} h_{\overline{A}}(\gamma_x)^{-1} g_{2,m}^{-1} = h_{\overline{A}}(\gamma_x) g_{1,x} h_{\overline{A}}(\gamma_x)^{-1} g_{1,m}^{-1}.$$
This implies
$$\begin{aligned} g_{2,m} h_{\overline{A}}(\gamma_x) g_{2,x}^{-1} &= g_{1,m} h_{\overline{A}}(\gamma_x) g_{1,x}^{-1} \\ &= g_{1,m} g_{2,m}^{-1}\, g_{2,m} h_{\overline{A}}(\gamma_x) g_{2,x}^{-1} h_{\overline{A}}(\gamma_x)^{-1}\, h_{\overline{A}}(\gamma_x) g_{2,x} g_{1,x}^{-1} \\ &= g_{2,m} h_{\overline{A}}(\gamma_x) g_{2,x}^{-1} h_{\overline{A}}(\gamma_x)^{-1}\, g_{1,m} g_{2,m}^{-1}\, h_{\overline{A}}(\gamma_x) g_{2,x} g_{1,x}^{-1}. \end{aligned}$$
using $\overline{g}_2 \in N(\mathbf{B}(\overline{A}))$ and $g_{1,m} g_{2,m}^{-1} \in Z(\mathbf{H}_{\overline{A}})$. Consequently,
$$h_{\overline{A}}(\gamma_x) = (g_{1,m} g_{2,m}^{-1})^{-1} h_{\overline{A}}(\gamma_x) g_{1,x} g_{2,x}^{-1},$$

thus $\overline{g}_1\overline{g}_2^{-1} \in \mathbf{B}(\overline{A})$.

- $[\Psi_1]^{-1}$ is continuous.
  $N(\mathbf{B}(\overline{A}))$ is a closed subgroup of $\overline{\mathcal{G}}$, hence compact. Thus, $\mathbf{B}(\overline{A})\backslash N(\mathbf{B}(\overline{A}))$ is also compact, hence $[\Psi_1]$ a continuous map of a compact space to a Hausdorff space. This gives the assertion. **qed**

4 Principal Fibre Bundle Structure of the Strata

Now we will find out which strata are even principal fibre bundles. Here we will use a slightly modified definition for principal fibre bundles. Actually, one demands in such a bundle that the structure group acts freely on the fibres, whence all stabilizers are trivial. This will not be the case for generalized connections in general because every holonomy centralizer – being isomorphic to the corresponding stabilizer – contains at least the center of $\mathbf{G}$. Therefore we will factor out exactly these "bugging" parts:

Definition 4.1 A fibre bundle $\pi : X \longrightarrow \pi(X)$ is called **principal fibre bundle** iff all $x \in X$ have the same stabilizer S.
The **structure group** of π is $S\backslash G$.

We note

Proposition 4.1 Let (in the notation above) π be a principal fibre bundle with "typical" stabilizer S. Then S is a normal subgroup in G, i.e., $S\backslash G$ is a topological group.
In a natural manner $S\backslash G$ acts on X. This action is continuous and free.
Moreover, $X/G \cong X/(S\backslash G)$.

This way the definition of a principal fibre bundle above is equivalent to the usual one. Here neither the total nor the base space gets changed. Only the acting group is reduced to its part being essential for the action. By the proposition above we see that the notation "structure *group*" is reasonable. It coincides with the standard definition for fibre bundles.

Proof Let $x \in X$ and $g \in G$. Obviously, the stabilizer of $x \circ g$ equals $G_{x\circ g} = g^{-1}G_x g = g^{-1}Sg$. Since π is a principal fibre bundle, we have $g^{-1}Sg = S$, i.e., S is a normal subgroup.
Obviously, the action $x \circ [g]_S := x \circ g$ is well-defined and continuous. Since from $x \circ [g]_S = x$ follows $x \circ g = x$, hence $g \in S$, the action is free. The homeomorphy of the two quotient spaces is clear as well. **qed**

In order to decide what strata could be principal fibre bundles we have to investigate again the form of the stabilizers, i.e. the base centralizers.

Definition 4.2 The set $\mathbf{B}_Z := \{\overline{g} \in \overline{\mathcal{G}} \mid g_m \in Z(\mathbf{G}) \text{ and } g_x = g_m \quad \forall x \in M\}$ is called **base center**.

Lemma 4.2
- The base center is contained in every base centralizer.
- A base centralizer is a normal subgroup of $\overline{\mathcal{G}}$ iff it equals the base center.

- The base centralizer of a connection equals the base center iff the connection is generic.

Proof
- Let $\overline{A} \in \overline{\mathcal{A}}$ with base centralizer $\mathbf{B}(\overline{A})$. Then the following holds:

$$\overline{g} \in \mathbf{B}(\overline{A}) \iff \begin{array}{ll} 1. & g_m \in Z(\mathbf{H}_{\overline{A}}) \\ 2. & h_{\overline{A}}(\gamma) = g_m^{-1} h_{\overline{A}}(\gamma) g_x \qquad \forall \gamma \in \mathcal{P}_{mx}, x \in M. \end{array}$$

  Since $Z \subseteq Z(U)$ for all $U \subseteq \mathbf{G}$, we have $\mathbf{B}_Z \subseteq \mathbf{B}(\overline{A})$.
- Let $\overline{A} \in \overline{\mathcal{A}}$ with base centralizer $\mathbf{B}(\overline{A})$.

  $\Longrightarrow$ Let $\mathbf{B}(\overline{A})$ be a normal subgroup in $\overline{\mathcal{G}}$.
  Let $\overline{g} \in \mathbf{B}(\overline{A})$ and $g_0 \in \mathbf{G}$. Furthermore, let $x \in M$, $x \neq m$, and $\gamma \in \mathcal{P}_{mx}$ be arbitrary. Additionally choose a $\overline{g}_0 \in \overline{\mathcal{G}}$ with $g_{0,m} = g_0$ and $g_{0,x} = h_{\overline{A}}(\gamma)$. Then by assumption $\overline{g}_0^{-1} \circ \overline{g} \circ \overline{g}_0 \in \mathbf{B}(\overline{A})$, in particular

$$\begin{aligned} h_{\overline{A}}(\gamma) &= g_{0,m}^{-1}\, g_m^{-1}\, g_{0,m}\, h_{\overline{A}}(\gamma)\, g_{0,x}^{-1}\, g_x\, g_{0,x} \\ &= g_0^{-1}\, g_m^{-1}\, g_0\, h_{\overline{A}}(\gamma)\, h_{\overline{A}}(\gamma)^{-1}\, g_x\, h_{\overline{A}}(\gamma) \\ &= g_0^{-1}\, g_m^{-1}\, g_0\, g_x\, h_{\overline{A}}(\gamma), \end{aligned}$$

  hence $g_m\, g_0 = g_0\, g_x$. Since g_0 is arbitrary, we have $g_m = g_x$ for all $x \in M$ and consequently $g_m \in Z$. Thus, $\overline{g} \in \mathbf{B}_Z$.
  We get $\mathbf{B}(\overline{A}) \subseteq \mathbf{B}_Z$, hence by the first part of the present proof the equality.

  $\Longleftarrow$ Let $\mathbf{B}(\overline{A}) = \mathbf{B}_Z$.
  Let now $\overline{g} \in \mathbf{B}_Z$ and $\overline{g}_0 \in \overline{\mathcal{G}}$. Then we have $(\overline{g}_0^{-1} \circ \overline{g} \circ \overline{g}_0)_x = g_{0,x}^{-1}\, g_x\, g_{0,x} = g_{0,x}^{-1}\, g_m\, g_{0,x} = g_{0,x}^{-1}\, g_{0,x}\, g_m = g_m \in Z$, hence, in particular, $(\overline{g}_0^{-1} \circ \overline{g} \circ \overline{g}_0)_x = (\overline{g}_0^{-1} \circ \overline{g} \circ \overline{g}_0)_m$ for all $x \in M$. Thus, $\overline{g}_0^{-1} \circ \overline{g} \circ \overline{g}_0 \in \mathbf{B}_Z$. Hence, $\mathbf{B}(\overline{A}) = \mathbf{B}_Z$ is a normal subgroup.
- $\Longrightarrow$ Let $\overline{A} \notin \overline{\mathcal{A}}_{\mathrm{gen}}$, i.e. $Z(\mathbf{H}_{\overline{A}}) \supset Z$.
  Let $g \in Z(\mathbf{H}_{\overline{A}}) \setminus Z$, and set $\overline{g} := (h_{\overline{A}}(\gamma_x)^{-1}\, g\, h_{\overline{A}}(\gamma_x))_{x \in M}$. Per definitionem we have $\overline{g} \in \mathbf{B}(\overline{A})$, but $\overline{g} \notin \mathbf{B}_Z$.

  $\Longleftarrow$ Let $\overline{A} \in \overline{\mathcal{A}}_{\mathrm{gen}}$, i.e. $Z(\mathbf{H}_{\overline{A}}) = Z$. Now we have

$$\begin{array}{rcll} \overline{g} \in \mathbf{B}(\overline{A}) & \iff & 1. & g_m \in Z(\mathbf{H}_{\overline{A}}) = Z \\ & & 2. & h_{\overline{A}}(\gamma) = g_m^{-1} h_{\overline{A}}(\gamma) g_x \qquad \forall \gamma \in \mathcal{P}_{mx}, x \in M \\ & \iff & 1. & g_m \in Z \\ & & 2. & h_{\overline{A}}(\gamma) = h_{\overline{A}}(\gamma) g_m^{-1} g_x \qquad \forall \gamma \in \mathcal{P}_{mx}, x \in M \\ & \iff & 1. & g_m \in Z \\ & & 2. & g_m = g_x \qquad \forall x \in M \\ & \iff & & \overline{g} \in \mathbf{B}_Z. \end{array}$$

qed

Consequently, only the generic stratum is a principal fibre bundle.

Proposition 4.3 Let $t \in \mathcal{T}$ be a gauge orbit type. Then we have:
$\pi : \overline{\mathcal{A}}_{=t} \longrightarrow \overline{\mathcal{A}}_{=t}/\overline{\mathcal{G}}$ is a principal fibre bundle iff $t = t_{\max}$, i.e. $\overline{\mathcal{A}}_{=t} = \overline{\mathcal{A}}_{\mathrm{gen}}$.

Proof By the definition of the gauge orbit type, all $Z(\mathbf{H}_{\overline{A}})$ occurring in a fixed stratum $\overline{\mathcal{A}}_{=t}$ are conjugate. As proven in [8] the same is true for the stabilizers $\mathbf{B}(\overline{A})$.

Now π is a principal fibre bundle iff all connections in $\overline{\mathcal{A}}_{=t}$ have the same stabilizer. On the other hand, this is true iff this stabilizer is a normal subgroup in $\overline{\mathcal{G}}$.[5] The lemma above yields the assertion. **qed**

5 Main Theorem on the Structure of the Generic Stratum

In this section we will state the main theorem about the structure of the generic stratum $\overline{\mathcal{A}}_{\text{gen}}$. Here we focus on assertions about trivializations of this principal fibre bundle. Above we have enlarged the usual concept of such bundles a little bit, whence before discussing the main theorem we have to modify slightly the notion of a trivialization as well. Here we apply a definition which is used for general fibre bundles and we only adapt it naturally to the special case of principal fibre bundles.

Definition 5.1 Let X be a Hausdorff space, μ a finite, normalized measure on X. Moreover, let G be a compact topological group acting on X and $\pi : X \longrightarrow X/G$ be a principal fibre bundle.

1. An open, G-invariant set $U \subseteq X$ is called **local trivialization** of the principal fibre bundle π iff there is an equivariant homeomorphism $\chi : U \longrightarrow \pi(U) \times (_S\backslash G)$ with $\pi\mid_U = \mathrm{pr}_1 \circ \chi$. Equivariant means that $(\mathrm{pr}_2 \circ \chi)(x \circ g) = ((\mathrm{pr}_2 \circ \chi)(x)) \cdot [g]_S$ for all $x \in U$ and $g \in G$.[6]
2. A principal fibre bundle is called **locally trivial** iff there is a covering $(U_i)_{i\in I}$ of X by local trivializations U_i.
3. A principal fibre bundle is called **μ-almost globally trivial** iff there is a covering (U_i) as in the previous case where additionally $\mu(U_i) = 1$ for all i.[7]
4. A principal fibre bundle is called **globally trivial** iff X is a local trivialization.

Now we come to the main

Theorem 5.1 We consider the canonical projection $\pi : \overline{\mathcal{A}}_{\text{gen}} \longrightarrow \overline{\mathcal{A}}_{\text{gen}}/\overline{\mathcal{G}}$.

1. π is an μ_0-almost globally trivial principal fibre bundle with structure group $_{\mathbf{B}_Z}\backslash\overline{\mathcal{G}}$.
2. π is globally trivial for abelian $\mathbf{G}$.
3. π is not globally trivial for non-abelian $\mathbf{G}$, supposed $\pi_1^{\text{homotopy}}(\mathbf{G}/Z) \neq 1$ and $\pi_1^{\text{homotopy}}(\mathbf{G}) = 1$.[8]

Here, μ_0 is the Ashtekar-Lewandowski measure [1, 7] and $\mathbf{B}_Z \subseteq \overline{\mathcal{G}}$ is the set of all constant gauge transforms with values in the center Z of $\mathbf{G}$.

[5] Let G be a group acting on X where all stabilizers G_x, $x \in X$, are conjugate. If X is a principal fibre bundle, then $G_x = G_y$ for all $x, y \in X$ and, in particular, $G_x = G_{x\circ g} = g^{-1}G_x g$ for all x. Thus, G_x is a normal subgroup in G. Conversely, let G_x be a normal subgroup. Then, since all stabilizers are conjugate, we have $G_y = g^{-1}G_x g = G_x$ for all $x, y \in X$, i.e., X is a principal fibre bundle.

[6] Often we say χ is a local trivialization as well.

[7] In the following we usually say simply "almost global" instead of "μ-almost global" supposed it is clear which measure μ is meant.

[8] We write π_1^{homotopy} instead of the usual π_1 because we will use the notation π_k for a certain map.

We have already proven that π is a locally trivial principal fibre bundle. Since the proof of the remaining items is quite long und partially rather technical, we sketch it here and divide it afterwards into a sequence of lemmata proven in the next few sections. As for the proof of the stratification of $\overline{\mathcal{A}}$ the so-called reduction mapping [8] will be crucial. By means of that mapping we can lift structures from $\mathbf{G}^k$ to $\overline{\mathcal{A}}$.
How to find appropriate trivializations? As we know [8] there are finitely many $g_i \in \mathbf{G}$ with $Z(\vec{g}) \equiv Z(\{g_1, \ldots, g_k\}) = Z$. Now, choose some graph Γ containing exactly k edges α_i and one single vertex (w.l.o.g. m) and denote the corresponding mapping $\pi_\Gamma : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^k$ by φ. Due to $\alpha_i \in \mathcal{HG}$ the map φ is a reduction mapping and by a corollary in [8] surjective even on $\overline{\mathcal{A}}_{\mathrm{gen}}$. We get the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{A}}_{\mathrm{gen}} & \xrightarrow{\ \ \varphi\ \ } & \mathbf{G}^k \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_k} \\ \overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}} & \xrightarrow{\ [\varphi]\ } & \mathbf{G}^k/\mathrm{Ad} \end{array} .$$

One could now conjecture that the generic stratum should be nontrivial because otherwise one would get a trivialization of $\mathbf{G}^k \longrightarrow \mathbf{G}^k/\mathrm{Ad}$ although in general the latter mapping π_k is not even a bundle mapping. However, this argumentation is incorrect; namely, there is no "gauge invariant" section for φ, i.e. no induced section for $[\varphi]$. But, restricting $\mathbf{G}^k$ in the diagram above to the generic elements (and the three remaining spaces analogously using the given maps) this obstruction disappears. Indeed, we will be able to prove that $\pi : \overline{\mathcal{A}}_{\mathrm{gen}} \longrightarrow \overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}}$ is nontrivial, as far as $\pi_k : (\mathbf{G}^k)_{\mathrm{gen}} \longrightarrow (\mathbf{G}^k)_{\mathrm{gen}}/\mathrm{Ad}$ is nontrivial. This criterion is fulfilled for instance by all $\mathbf{G}$ given in the main theorem.
The bundle $\pi : \overline{\mathcal{A}}_{\mathrm{gen}} \longrightarrow \overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}}$ is in general not globally trivial, but how large are the areas the bundle is trivial over? Using the reduction mapping we will show that the triviality of π_k on $V \subseteq (\mathbf{G}^k)_{\mathrm{gen}}$ implies the triviality of π on $\varphi^{-1}(V) \subseteq \overline{\mathcal{A}}_{\mathrm{gen}}$. As a by-product we get that π is trivial on the whole $\overline{\mathcal{A}}_{\mathrm{gen}} \equiv \overline{\mathcal{A}}$ if $\mathbf{G}$ is abelian. In the general case our task will be the search for domains where π_k is trivial. The idea for that comes from the fact that a smooth principal fibre bundle over a contractible manifold is always trivial. Hence we triangulate $(\mathbf{G}^k)_{\mathrm{gen}}/\mathrm{Ad}$, cut out the lower-dimensional simplices (this is in particular a set of [Lebesgue] measure 0) and get a disjoint union of contractible manifolds. The preimage of that union by $\pi_k \circ \varphi$ yields the desired trivialization $U \subseteq \overline{\mathcal{A}}_{\mathrm{gen}}$ with $\mu_0(U) = 1$.
The only point neglected in the discussion up to now is whether there is such a neighbourhood for all $\overline{A} \in \overline{\mathcal{A}}_{\mathrm{gen}}$. But, this is indeed the case supposed we fix $\boldsymbol{\alpha}$ in such a way that the corresponding reduction mapping $\varphi_{\boldsymbol{\alpha}}$ maps the connection $\overline{A}$ into the generic stratum of $\mathbf{G}^k$ and if we choose such a triangulation of $\varphi(\overline{A})$, where $\varphi(\overline{A})$ is not contained in a lower-dimensional simplex.

6 Almost Global Triviality of $(\mathbf{G}^k)_{\mathrm{gen}}$

In the whole section let $\mathbf{G}$ be a compact Lie group acting on $\mathbf{G}^k$ by conjugation for all $k \in \mathbb{N}_+$. Moreover, let $Z := Z(\mathbf{G})$. Finally, every principal fibre bundle in this section is smooth.

Definition 6.1 Let $k \in \mathbb{N}_+$.
An element $\vec{g} := \{g_1, \ldots, g_k\} \in \mathbf{G}^k$ (and its respective orbit) is called **generic** iff $\mathrm{Typ}(\vec{g}) \equiv [Z(\{g_1, \ldots, g_k\})] = [Z]$.
The set of all generic elements of $\mathbf{G}^k$ is denoted by $(\mathbf{G}^k)_{\mathrm{gen}}$.

Obviously, because of $\mathrm{Typ}(\vec{g}) \leq [Z]$ for all $\vec{g} \in \mathbf{G}^k$ every generic orbit is of maximal type. Furthermore, a Lie group is abelian iff one (and then every) (nontrivial) power consists of generic elements only.[9]

Proposition 6.1 Let $\mathbf{G}$ be a compact Lie group.
Then there is a $k_{\min} \in \mathbb{N}_+$, such that $(\mathbf{G}^k)_{\mathrm{gen}}$ is a non-empty $d\mu_{\mathrm{Haar}}$-almost globally trivial (smooth) principal fibre bundle with structure group $Z \backslash \mathbf{G}$ for all $k \geq k_{\min}$.

Proof
1. Choice of $k_{\min}$
By [8] there is a $k \in \mathbb{N}_+$ such that there is at least one orbit in $\mathbf{G}^k$ with type $[Z]$. Now choose simply $k_{\min}$ to be the minimum of all that k.
In what follows let always $k \geq k_{\min}$.
2. Existence of a generic element in $\mathbf{G}^k$
Let $\vec{g} := \{g_1, \ldots, g_{k_{\min}}\} \in \mathbf{G}^{k_{\min}}$ be a generic element.
Then $\vec{g}_k := \{g_1, \ldots, g_{k_{\min}}, e_{\mathbf{G}}, \ldots, e_{\mathbf{G}}\} \in \mathbf{G}^k$ by
$$\begin{aligned}[Z] \geq \mathrm{Typ}(\vec{g}_k) &= [Z(g_1, \ldots, g_{k_{\min}}, e_{\mathbf{G}}, \ldots, e_{\mathbf{G}})] \\ &= [Z(g_1, \ldots, g_{k_{\min}})] = \mathrm{Typ}(\vec{g}) = [Z]\end{aligned}$$
is generic, too.
3. Bundle structure over $(\mathbf{G}^k)_{\mathrm{gen}}/\mathrm{Ad}$
The adjoint action of $\mathbf{G}$ on $\mathbf{G}^k$ is smooth. Hence, by general arguments [3] the generic stratum $(\mathbf{G}^k)_{\mathrm{gen}}$ is a smooth manifold and $\pi_k : (\mathbf{G}^k)_{\mathrm{gen}} \longrightarrow (\mathbf{G}^k)_{\mathrm{gen}}/\mathrm{Ad}$ is a smooth fibre bundle with fibre and structure group $Z \backslash \mathbf{G}$.[10] Since $Z \backslash \mathbf{G}$ is a Lie group and thus acts naturally on itself by left translation, π_k is even a $Z \backslash \mathbf{G}$-principal fibre bundle on the generic stratum.
4. Choice of a neighbourhood $V_{k,\vec{v}}$ for every fixed $\vec{v} \in (\mathbf{G}^k)_{\mathrm{gen}}$
First we cut out from the n-dimensional smooth manifold $(\mathbf{G}^k)_{\mathrm{gen}}/\mathrm{Ad}$ a neighbourhood B^n of $[\vec{v}]$ diffeomorphic to an n-dimensional ball and get a remaining smooth manifold M. By general arguments [13, 17, 16] there are a simplicial complex K consisting of countably many simplices and a smooth triangulation $f : K \longrightarrow M$. Let K_n denote the set of all n-cells[11] of K. It can be constructed from K by deleting the $(n-1)$-skeleton $K_{\leq n-1}$, i.e. all cells whose dimension is smaller than that of K. Now define $V_{k,\vec{v}} := \pi_k^{-1}(f(K_n) \cup \mathrm{int}\, B^n)$.
5. Properties of $V_{k,\vec{v}}$
   - $V_{k,\vec{v}}$ is Ad-invariant.
   - $\vec{v} \in V_{k,\vec{v}}$.

[9] We have $Z(\vec{e}_{\mathbf{G}}) := Z((e_{\mathbf{G}}, \ldots, e_{\mathbf{G}})) = \mathbf{G}$. If $\mathbf{G}^k$ now consists for $k \in \mathbb{N}_+$ of generic elements only, then $Z = Z(\vec{e}_{\mathbf{G}}) = \mathbf{G}$. Conversely, if $Z = \mathbf{G}$, then $\mathbf{G} \supseteq Z(\vec{g}) \supseteq Z = \mathbf{G}$, hence $Z(\vec{g}) = Z$ for all $\vec{g} \in \mathbf{G}^k$.

[10] Note $N(Z) = \mathbf{G}$.

[11] Note that here a cell $\widehat{\sigma}$ is the *interior* of a simplex σ. Only for dimension 0 the cell shall be a simplex.

- Since f is a smooth triangulation, $f(K_n)$ is a smooth manifold [5] that equals the disjoint union of all $f(\widehat{\sigma})$ where σ is an n-simplex of K. Since f is a homeomorphism and $\widehat{\sigma}$ always contractible, also $f(\widehat{\sigma})$ is always contractible. The contractiblity of $\operatorname{int} B^n$ is trivial. Moreover, $\operatorname{int} B^n$ and $f(K_n)$ are disjoint.
- Hence, as a π_k-preimage $V_{k,\vec{v}}$ is a submanifold of $(\mathbf{G}^k)_{\text{gen}}$, thus of $\mathbf{G}^k$ as well. In particular, $V_{k,\vec{v}}$ is open in $\mathbf{G}^k$ by the continuity of π_k, and $\pi_k(V_{k,\vec{v}})$ is a disjoint union of contractible manifolds.
- We have $V_{k,\vec{v}} = \mathbf{G}^k \setminus \Big(\bigcup_{[H]<[Z]}(\mathbf{G}^k)_{(H)} \cup \bigcup_{\sigma \in K_{\leq n-1}} \pi_k^{-1}(f(\widehat{\sigma})) \cup \pi_k^{-1}(\partial B^n)\Big)$, i.e. $V_{k,\vec{v}}$ emerges from $\mathbf{G}^k$ by deleting the non-generic orbits as well as the π_k-preimages of all f-images of lower-dimensional simplices in $(\mathbf{G}^k_{\text{gen}})/\mathrm{Ad}$ and the boundary of B^n, respectively.[12]
- We have $\mu_{\text{Haar}}(V_{k,\vec{v}}) = 1$.
  It is sufficient to prove that the just eliminated objects have Haar measure 0.
  a) For every closed subgroup H of G with $[H] < [Z]$ the set $(\mathbf{G}^k)_{(H)} \subseteq B \cup E$ is a smooth manifold [3]. Here, B and E are the sets of the singular and the exceptional orbits, respectively, in $\mathbf{G}^k$. We have $\dim(\mathbf{G}^k)_{(H)} < \dim(\mathbf{G}^k)_{(Z)} \equiv \dim(\mathbf{G}^k)_{\text{gen}}$. [3]
  The number of orbit types is finite [14]. Hence, $\mu_{\text{Haar}}(\bigcup_{[H]<[Z]}(\mathbf{G}^k)_{(H)}) \leq \sum_{[H]<[Z]} \mu_{\text{Haar}}((\mathbf{G}^k)_{(H)}) = 0$ because every Haar measure is a Lebesgue measure [6] and the Lebesgue measure of every lower-dimensional manifold is zero [5].
  b) Let $\sigma \subseteq K_{\leq n-1}$. Since f is a homeomorphism, we have $\dim f(\widehat{\sigma}) = \dim \widehat{\sigma} =: m < n$. As above $\pi_k^{-1}(f(\widehat{\sigma}))$ is a submanifold of $(\mathbf{G}^k)_{\text{gen}}$, hence also of $\mathbf{G}^k$ whose codimension is $n - m > 0$. Thus, this set is a zero set, too.
  Together with K, obviously also $K_{\leq n-1}$ a is complex with countably many simplices. Hence,
  $$\mu_{\text{Haar}}\Big(\bigcup_{\sigma \in K_{\leq n-1}} \pi_k^{-1}(f(\widehat{\sigma}))\Big) \leq \sum_{\sigma \in K_{\leq n-1}} \mu_{\text{Haar}}(\pi_k^{-1}(f(\widehat{\sigma}))) = 0.$$
  c) ∂B^n is a smooth submanifold of $(\mathbf{G}^k)_{\text{gen}}/\mathrm{Ad}$ with codimension 1. Thus, $\pi_k^{-1}(\partial B^n)$ is a smooth submanifold of $(\mathbf{G}^k)_{\text{gen}}$ with codimension 1.
  Again, $\mu_{\text{Haar}}(\pi_k^{-1}(\partial B^n)) = 0$.
- $V_{k,\vec{v}}$ is dense in $\mathbf{G}^k$. This follows – because $V_{k,\vec{v}}$ is open with $\mu_{\text{Haar}}(V_{k,\vec{v}}) = 1$ – from the regularity of the Haar measure.

6. Triviality of $\pi_k \mid_{V_{k,\vec{v}}}$
   $\pi_k \mid_{V_{k,\vec{v}}}: V_{k,\vec{v}} \longrightarrow \pi_k(V_{k,\vec{v}}) = f(K_n)$ is a $(Z \backslash \mathbf{G})$-principal fibre bundle over a disjoint union $f(K_n)$ of contractible manifolds. Hence [12], this bundle is trivial, i.e., there is an equivariant homeomorphism $V_{k,\vec{v}} \cong \pi(V_{k,\vec{v}}) \times (Z \backslash \mathbf{G})$.
7. Almost global triviality of $(\mathbf{G}^k)_{\text{gen}}$
   Obviously $\mathcal{V}_k := \{V_{k,\vec{v}} \mid \vec{v} \in (\mathbf{G}^k)_{\text{gen}}\}$ is a non-empty covering of $(\mathbf{G}^k)_{\text{gen}}$ by μ_{Haar}-almost global trivializations. **qed**

[12]Here, $X_{(H)} = \{x \in X \mid \mathrm{Typ}(x) = [H]\}$ if G acts on X and H is a closed subgroup of G.

7 Choice of the Covering of $\overline{\mathcal{A}}_{\mathbf{gen}}$

After we have proven a theorem of the desired type on the level of $\mathbf{G}^k$, we should lift next these assertions using an appropriate mapping to the level of $\overline{\mathcal{A}}$. This again will be the reduction mapping.

Let now $\overline{A} \in \overline{\mathcal{A}}$. As proven in [8] there are finitely many paths α_i in $\mathcal{HG}$, such that $Z(h_{\overline{A}}(\boldsymbol{\alpha})) = Z(\mathbf{H}_{\overline{A}}) = Z$ (w.l.o.g. let $k := \#\boldsymbol{\alpha} \geq k_{\min}$). The corresponding reduction mapping be $\varphi \equiv \varphi_{\boldsymbol{\alpha}} : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^k$, $\varphi(\overline{A}') := h_{\overline{A}'}(\boldsymbol{\alpha})$. The most obvious choice of a neighbourhood of $\overline{A}$ would be $U := \varphi^{-1}(V)$ where V is an element of the covering of $(\mathbf{G}^k)_{\mathrm{gen}}$ above with $\varphi(\overline{A}) \in V$. But, despite of $\mu_{\mathrm{Haar}}(V) = 1$, possibly $\mu_0(U) \neq 1$. This, in particular, holds if φ is not surjective, because, e.g., a path α occurs multiply in $\boldsymbol{\alpha}$ which cannot be excluded a priori. Therefore, for the surjectivity and the measure property we need the proposition below. Before we state this proposition, we recall the notion of a hyph introduced in [7]. First, a path γ without self-intersections is called independent of a (not necessarily finite) set $D := \{\delta_i \mid i \in I\}$ of paths iff there is a point x in the image of γ such that either there is no subpath of the δ_i's starting or ending in x as the subpath of γ starting in x or there is no such subpath that starts or ends as γ's subpath ends. Then x is called free point. Now, the finite (ordered) set $\upsilon = \{c_i\}$ is a hyph iff c_i has no self-intersections and is independent of $\{c_1, \ldots, c_{i-1}\}$ for all i.

Definition 7.1 First we define[13] $\mathbf{V}(\boldsymbol{\delta}) := \{\delta_i(0) \mid i \in I\} \cup \{\delta_i(1) \mid i \in I\}$ to be the set of all initial and terminal points of paths in $\delta = \{\delta_i \mid i \in I\} \subseteq \mathcal{P}$.
A finite subset $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ is called **associated to the hyph** $\upsilon = \{c_i\}$ iff
1. $\#\boldsymbol{\alpha} = \#\upsilon$ and
2. for every $x \in \mathbf{V}(\upsilon) \setminus \{m\}$ there is a path γ_x from m to x such that
   a) $\alpha_i = \gamma_{c_i(0)}\, c_i\, \gamma_{c_i(1)}^{-1}$ for all i (where γ_m is – if occurring – trivial) and
   b) $\boldsymbol{\gamma} \cup \upsilon$ with $\boldsymbol{\gamma} := \{\gamma_x \mid x \in \mathbf{V}(\upsilon) \setminus \{m\}\}$ is a hyph.

More general, $\boldsymbol{\alpha}$ is called associated to a hyph iff there is a hyph υ that $\boldsymbol{\alpha}$ is associated to.

Roughly speaking, such an $\boldsymbol{\alpha}$ is almost a hyph, but $\alpha_i \in \boldsymbol{\alpha}$ can have self-intersections.
The importance of the definition above is clarified by the following two propositions.

Proposition 7.1 Let $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ be associated to a hyph. Then we have:
1. $\varphi_{\boldsymbol{\alpha}} : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^{\#\boldsymbol{\alpha}}$ is surjective.
2. $\mu_0(\varphi_{\boldsymbol{\alpha}}^{-1}(V)) = \mu_{\mathrm{Haar}}(V)$ for all measurable $V \subseteq \mathbf{G}^{\#\boldsymbol{\alpha}}$.

Proof We choose some hyph υ, that $\boldsymbol{\alpha}$ is associated to, and a corresponding $\boldsymbol{\gamma} = \{\gamma_x \mid x \in \mathbf{V}(\upsilon) \setminus \{m\}\}$ as above.
1. Let $(g_1, \ldots, g_{\#\boldsymbol{\alpha}}) \in \mathbf{G}^{\#\boldsymbol{\alpha}}$ be given. Since $\boldsymbol{\gamma} \cup \upsilon$ is a hyph, the corresponding projection $\pi_{\boldsymbol{\gamma}\cup\upsilon} : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^{\#\boldsymbol{\gamma}+\#\upsilon}$ is surjective. [7] In particular, there is an $\overline{A} \in \overline{\mathcal{A}}$ with $h_{\overline{A}}(c_i) = g_i$ for all $i = 1, \ldots, \#\upsilon$ and $h_{\overline{A}}(\gamma_x) = e_{\mathbf{G}}$ for all x. Obviously, $h_{\overline{A}}(\alpha_i) = g_i$ for all i from 1 to $\#\upsilon = \#\boldsymbol{\alpha}$, hence $\varphi_{\boldsymbol{\alpha}}$ is surjective.
2. We have $\varphi_{\boldsymbol{\alpha}} = \pi_{\boldsymbol{\alpha}}^{\upsilon\cup\boldsymbol{\gamma}} \circ \pi_{\upsilon\cup\boldsymbol{\gamma}}$ where $\pi_{\upsilon\cup\boldsymbol{\gamma}}$ is the projection to the parallel transports along the paths in $\upsilon \cup \boldsymbol{\gamma}$ and $\pi_{\boldsymbol{\alpha}}^{\upsilon\cup\boldsymbol{\gamma}} : \mathbf{G}^{\#\upsilon+\#\boldsymbol{\gamma}} \longrightarrow \mathbf{G}^{\#\boldsymbol{\alpha}}$ is the natural projection

[13] If Γ is a graph, $\mathbf{V}(\Gamma)$ is simply the set of all vertices.

induced by the decomposition of α_i into paths in $\upsilon \cup \boldsymbol{\gamma}$. Hence, by the definition of the Ashtekar-Lewandowski measure μ_0 [1, 7] it suffices to show that $\mu_{\text{Haar}}((\pi_{\boldsymbol{\alpha}}^{\upsilon\cup\boldsymbol{\gamma}})^{-1}(V)) = \mu_{\text{Haar}}(V)$ for all measurable $V \subseteq \mathbf{G}^{\#\boldsymbol{\alpha}}$.
This, however, follows analogously to the proof of the well-definedness of μ_0 because every α_i is a product of exactly one c_i and some $\gamma_x \in \boldsymbol{\gamma}$.[14] **qed**

Proposition 7.2 For every $\overline{A} \in \overline{\mathcal{A}}$ there is an $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ such that

1. $\boldsymbol{\alpha}$ is associated to a hyph and
2. $Z(h_{\overline{A}}(\boldsymbol{\alpha})) = Z(\mathbf{H}_{\overline{A}})$.

To avoid cumbersome notation we introduce a partial ordering on the set of all paths.

Definition 7.2 Let $\boldsymbol{\gamma}, \boldsymbol{\delta} \subseteq \mathcal{P}$ be finite sets of paths.
We say $\boldsymbol{\gamma} \geq \boldsymbol{\delta}$ iff every $\delta \in \boldsymbol{\delta}$ can be written as a finite product of paths in $\boldsymbol{\gamma}$ and their inverses. (Here, multiple usage of paths in $\boldsymbol{\gamma}$ is admitted.)

The proposition above follows from the next two lemmata.

Lemma 7.3 Let $\boldsymbol{\beta}$ be a finite subset of $\mathcal{HG}$. Then there is an $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$ such that $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ is associated to a hyph.

Proof
- First we construct as in [7] a hyph $\upsilon = \{c_i\} \subseteq \mathcal{P}$ with $\upsilon \geq \boldsymbol{\beta}$. Let F denote the finite set of all free point of the c_i's.
- Now we show that for every pair of distinct points $x, y \in M$ with $y \notin \text{im}\,(\upsilon)$ there is a path γ without self-intersections connecting x and m, passing y and having the property that $\{\gamma\} \cup \upsilon$ is a hyph again.
For $x, m \notin F$ the statement is trivial[15] because then one chooses an arbitrary path from x to m via y passing no point in F. Let now x or m in F. Then we have to guarantee that γ does not start or end, respectively, as the corresponding subpaths[16] $c_i^{x,\pm}$ and $c_i^{m,\pm}$, respectively, of the c_i having their free point in x and m, respectively. For this, one first chooses a path γ running from x via y to m without passing any point in $F \setminus \{x, m\}$ and modifies afterwards the "ends" of γ such that these finitely many subpaths $c_i^{\cdot,\cdot}$ just do not start or end as the new γ does. Due to the finite number of subpaths being under consideration, this is possible.
Then $\{\gamma\} \cup \upsilon$ is a hyph. Here, the free points of the c_i's remain free per constr. and the free point of γ is, e.g., y.

[14] We have ($k := \#\boldsymbol{\alpha} = \#\upsilon$)

$$\begin{aligned}
\mu_{\text{Haar}}((\pi_{\boldsymbol{\alpha}}^{\upsilon\cup\boldsymbol{\gamma}})^{-1}(V)) &= \int_{\mathbf{G}^{k+\#\boldsymbol{\gamma}}} 1_{(\pi_{\boldsymbol{\alpha}}^{\upsilon\cup\boldsymbol{\gamma}})^{-1}(V)} d\mu_{\text{Haar}}^{k+\#\boldsymbol{\gamma}} \\
&= \int_{\mathbf{G}^{k+\#\boldsymbol{\gamma}}} 1_V \circ \pi_{\boldsymbol{\alpha}}^{\upsilon\cup\boldsymbol{\gamma}}(g_1, \ldots, g_k, g'_{x_1}, \ldots, g'_{x_{\#\boldsymbol{\gamma}}}) d\mu_{\text{Haar}}^{k+\#\boldsymbol{\gamma}} \\
&= \int_{\mathbf{G}^{k+\#\boldsymbol{\gamma}}} 1_V(g'_{c_1(0)} g_1 {g'_{c_1(1)}}^{-1}, \ldots, g'_{c_k(0)} g_k {g'_{c_k(1)}}^{-1}) d\mu_{\text{Haar}}^{k+\#\boldsymbol{\gamma}} \\
&\qquad \text{(corresponding to the decomposition } \alpha_i = \gamma_{c_i(0)} c_i \gamma_{c_i(1)}^{-1}\text{)} \\
&= \int_{\mathbf{G}^k} 1_V(g_1, \ldots, g_k) d\mu_{\text{Haar}}^k \\
&\qquad \text{(translation invariance and normalization of the Haar measure)} \\
&= \mu_{\text{Haar}}(V).
\end{aligned}$$

[15] Note that $\dim M \geq 2$.

[16] $c^{x,\pm}$ is that subpath of c that starts $(+)$ or ends $(-)$ in x.

- In the next step we choose for every $x \in \mathbf{V}(v)$ with $x \neq m$ first a $y_x \notin \mathrm{im}\,(v)$ being distinct from the other $y_{x'}$ and then a path γ_x from x to m passing a $y_x \notin \mathrm{im}\,(v)$ such that $\{\gamma_x\} \cup v$ is a hyph again. Since all y_x are distinct, even $\{\gamma_x \mid x \in \mathbf{V}(v) \setminus \{m\}\} \cup v$ is again a hyph. Furthermore, let γ_m be the trivial path.
- Now let $\alpha_i := \gamma_{c_i(0)}^{-1} c_i \gamma_{c_i(1)}$ for all i. Obviously, $\boldsymbol{\alpha} := \{\alpha_i\}$ is associated to the hyph v.
- Since $v \geq \boldsymbol{\beta}$, every $\beta \in \boldsymbol{\beta}$ can be written as $\prod_j c_{i(j)}^{\epsilon(j)} = \prod_j (\gamma_{c_{i(j)}(0)}^{-1} c_{i(j)} \gamma_{c_{i(j)}(1)})^{\epsilon(j)} = \prod_j \alpha_{i(j)}^{\epsilon(j)}$. The paths $\gamma_{\ldots}$ cancel out each other.
Hence, $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$. **qed**

Lemma 7.4 Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \subseteq \mathcal{HG}$ be finite sets with $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$.
Then for all $\overline{A} \in \overline{\mathcal{A}}$ we have $Z(h_{\overline{A}}(\boldsymbol{\alpha})) \subseteq Z(h_{\overline{A}}(\boldsymbol{\beta}))$.

Proof By assumption every $\beta \in \boldsymbol{\beta}$ is a product of paths in $\boldsymbol{\alpha}$ and their inverses. Then $h_{\overline{A}}(\beta)$ is a product of the $h_{\overline{A}}(\alpha_i)$ and their inverses. Let now $g \in Z(h_{\overline{A}}(\boldsymbol{\alpha}))$. Then g commutes with all finite products of the $h_{\overline{A}}(\alpha_i)$ and their inverses as well, hence, in particular, $g \in Z(h_{\overline{A}}(\beta))$.
The assertion now follows immediately from $Z(h_{\overline{A}}(\boldsymbol{\beta})) = \bigcap_{\beta \in \boldsymbol{\beta}} Z(h_{\overline{A}}(\beta))$. **qed**

Proof Proposition 7.2
First choose a finite $\boldsymbol{\beta}$ in $\mathcal{HG}$ with $Z(h_{\overline{A}}(\boldsymbol{\beta})) = Z(\mathbf{H}_{\overline{A}})$. [8] By Lemma 7.3 there is an $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ associated to a hyph with $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$. Lemma 7.4 yields $Z(\mathbf{H}_{\overline{A}}) \subseteq Z(h_{\overline{A}}(\boldsymbol{\alpha})) \subseteq Z(h_{\overline{A}}(\boldsymbol{\beta})) = Z(\mathbf{H}_{\overline{A}})$, hence the claimed equality. **qed**

Now, we are able to state the desired covering of $\overline{\mathcal{A}}_{\mathrm{gen}}$:

- For all $\overline{A} \in \overline{\mathcal{A}}_{\mathrm{gen}}$ we choose an $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ associated to a hyph with $Z(h_{\overline{A}}(\boldsymbol{\alpha})) = Z(\mathbf{H}_{\overline{A}})$ by Proposition 7.2.
- We denote by $\varphi_{\boldsymbol{\alpha}} : \begin{array}{ccc} \overline{\mathcal{A}} & \longrightarrow & \mathbf{G}^{\#\boldsymbol{\alpha}} \\ \overline{A}' & \longmapsto & h_{\overline{A}'}(\boldsymbol{\alpha}) \end{array}$ the reduction mapping for $\boldsymbol{\alpha}$.
- Choose according to Section 6 an almost global trivialization $V_{\#\boldsymbol{\alpha},\varphi_{\boldsymbol{\alpha}}(\overline{A})}$ of $(\mathbf{G}^{\#\boldsymbol{\alpha}})_{\mathrm{gen}}$ containing $\varphi_{\boldsymbol{\alpha}}(\overline{A})$.
- Let $U_{\overline{A}} := \varphi_{\boldsymbol{\alpha}}^{-1}(V_{\#\boldsymbol{\alpha},\varphi_{\boldsymbol{\alpha}}(\overline{A})}) \subseteq \overline{\mathcal{A}}$ be the preimage of that set.
- Finally let $\mathcal{U} := \{U_{\overline{A}}\}_{\overline{A} \in \overline{\mathcal{A}}_{\mathrm{gen}}}$.

Lemma 7.5 $\mathcal{U}$ is a covering of $\overline{\mathcal{A}}_{\mathrm{gen}}$.

Proof Clear. **qed**

8 Properties of that Covering

In this section we investigate exclusively properties of a fixed chart. For this we fix an arbitrary $\overline{A} \in \overline{\mathcal{A}}$ with corresponding reduction mapping $\varphi := \varphi_{\boldsymbol{\alpha}}$ and set simply $k := \#\boldsymbol{\alpha}$, $V := V_{k,\varphi(\overline{A})} \subseteq (\mathbf{G}^k)_{\mathrm{gen}}$ and $U := U_{\overline{A}} = \varphi^{-1}(V)$.

Lemma 8.1 U is an open, dense, $\overline{\mathcal{G}}$-invariant subset of $\overline{\mathcal{A}}_{\mathrm{gen}}$ with $\mu_0(U) = 1$.

Proof Per constructionem V is an open, dense and Ad-invariant subset of $(\mathbf{G}^k)_{\text{gen}}$ with $\mu_{\text{Haar}}(V) = 1$. Since φ as an equivariant map is minorifying [8], $[Z] = \text{Typ}(\varphi(\overline{A}')) \leq \text{Typ}(\overline{A}') \leq [Z]$, i.e. $\overline{A}' \in \overline{\mathcal{A}}_{\text{gen}}$ for all $\overline{A}' \in U$. Since φ is continuous, $U = \varphi^{-1}(V)$ is open as a subset of $\overline{\mathcal{A}}$, hence due to the openness of $\overline{\mathcal{A}}_{\text{gen}}$ as a subset of $\overline{\mathcal{A}}_{\text{gen}}$ as well. Since $\boldsymbol{\alpha}$ is associated to a hyph, we have $\mu_0(U) = \mu_0(\varphi^{-1}(V)) = \mu_{\text{Haar}}(V) = 1$ by Proposition 7.1. Both statements yield due to the regularity of μ_0 the denseness of U in $\overline{\mathcal{A}}_{\text{gen}}$. Finally, the $\overline{\mathcal{G}}$-invariance of U follows from the Ad-invariance of V. **qed**

Before we show that U is indeed an almost global trivialization of $\overline{\mathcal{A}}_{\text{gen}}$, we still need a construction joining the group structure of $Z\backslash\mathbf{G}$ with that of $\mathbf{B}_Z\backslash\overline{\mathcal{G}}$. This will open the possibility to lift the triviality over V to that over U.

Definition 8.1 Let $* : (Z\backslash\mathbf{G}) \times \overline{\mathcal{G}} \longrightarrow \mathbf{B}_Z\backslash\overline{\mathcal{G}}$ be defined by $[g] * \overline{g} := [(g\, g_x)_{x\in M}]_{\mathbf{B}_Z}$.

Lemma 8.2
- $*$ is well-defined and continuous.
- The restriction of $*$ to $(Z\backslash\mathbf{G}) \times \overline{\mathcal{G}}_0$ is an isomorphism.

We recall $\overline{\mathcal{G}}_0 := \{\overline{g} \in \overline{\mathcal{G}} \mid g_m = e_{\mathbf{G}}\}$.

Proof
- Let $g_1 \sim g_2$, i.e. $g_1 = zg_2$ for a $z \in Z$. Then we have
$$\begin{aligned}
[g_1] * \overline{g} &= [(g_1\, g_x)_{x\in M}]_{\mathbf{B}_Z} \\
&= [(z\, g_2\, g_x)_{x\in M}]_{\mathbf{B}_Z} \\
&= [\overline{z} \circ (g_2\, g_x)_{x\in M}]_{\mathbf{B}_Z} \qquad (\overline{z} \equiv (z)_{x\in M} \in \mathbf{B}_Z) \\
&= [(g_2\, g_x)_{x\in M}]_{\mathbf{B}_Z} \\
&= [g_2] * \overline{g}.
\end{aligned}$$
- The continuity of $*$ follows immediately from the surjectivity and openness of the canonical projection $\mathbf{G} \times \overline{\mathcal{G}} \longrightarrow (Z\backslash\mathbf{G}) \times \overline{\mathcal{G}}$, the continuity of $\mathbf{G} \times \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}$ with $(g, \overline{g}) \longmapsto (gg_x)_{x\in M}$ as well as that of the canonical projection $\overline{\mathcal{G}} \longrightarrow \mathbf{B}_Z\backslash\overline{\mathcal{G}}$ and the commutativity of the corresponding diagram
$$\begin{array}{ccc}
\mathbf{G} \times \overline{\mathcal{G}} & \longrightarrow & \overline{\mathcal{G}} \\
\downarrow & & \downarrow \\
(Z\backslash\mathbf{G}) \times \overline{\mathcal{G}} & \xrightarrow{\;*\;} & \mathbf{B}_Z\backslash\overline{\mathcal{G}}
\end{array}.$$
- $*$ is injective.
Let $[g_1] * \overline{g}_1 = [g_2] * \overline{g}_2$ with $\overline{g}_1, \overline{g}_2 \in \overline{\mathcal{G}}_0$, i.e. $g_{1,m} = g_{2,m} = e_{\mathbf{G}}$. Then per def. $[(g_1 g_{1,x})_{x\in M}]_{\mathbf{B}_Z} = [(g_2 g_{2,x})_{x\in M}]_{\mathbf{B}_Z}$. Thus, there is a $z \in Z$ with $g_1 g_{1,x} = z\, g_2 g_{2,x}$ for all $x \in M$. For $x = m$ we have $g_1 = zg_2$ (and so $[g_1] = [g_2]$) and consequently $\overline{g}_1 = \overline{g}_2$.
- $*$ is surjective.
Let $[\overline{g}]_{\mathbf{B}_Z}$ be given. Choose a representative $\overline{g}' \in [\overline{g}]_{\mathbf{B}_Z}$. Set $g'' := g'_m$ and $g''_x := (g'')^{-1} g'_x$. Then $[g''] * \overline{g}'' = [\overline{g}]_{\mathbf{B}_Z}$. **qed**

Now,

Proposition 8.3 U is a local trivialization of $\overline{\mathcal{A}}_{\text{gen}}$.

Proof We denote the almost global trivialization of $\mathbf{G}^k$ according to Proposition 6.1 by $\psi : V \longrightarrow \pi_k(V) \times ({}_Z\backslash \mathbf{G})$. The projection onto the second component be $\psi_2 : V \longrightarrow {}_Z\backslash \mathbf{G}$. For U, α and φ we use the definitions of the preceding section. Furthermore, let γ_x be for every $x \in M$ some fixed path from m to x. W.l.o.g., γ_m is trivial path.

Now we define the trivialization of $\overline{\mathcal{A}}$:

$$\begin{array}{rrcl}\Psi : & U & \longrightarrow & \pi(U) \times {}_{\mathbf{B}_Z}\backslash \overline{\mathcal{G}}. \\ & h & \longmapsto & ([h], \psi_2(\varphi(h)) * (h(\gamma_x))_{x\in M})\end{array}$$

1. Ψ is well-defined.
   Because of $h \in U$ we have $\varphi(h) \in V$, i.e. $\psi_2(\varphi(h))$ is well-defined.
2. Ψ is surjective.
   Let $([h], [\overline{g}]_{\mathbf{B}_Z}) \in \pi(U) \times {}_{\mathbf{B}_Z}\backslash \overline{\mathcal{G}}$ be given. By Lemma 8.2 there is exactly one $[g'] \in {}_Z\backslash \mathbf{G}$ and some $\overline{g}' \in \overline{\mathcal{G}}_0$ with $[g'] * \overline{g}' = [\overline{g}]_{\mathbf{B}_Z}$. Additionally choose some $h' \in [h]$. Hence, $h' \in U$ and $\varphi(h') \equiv h'(\boldsymbol{\alpha}) \in V$. Let $\vec{g} := \psi^{-1}([h'(\boldsymbol{\alpha})], [g'])$. Since $h'(\boldsymbol{\alpha})$ and $\vec{g}$ are in one and the same orbit w.r.t. Ad, there is a $k \in \mathbf{G}$ with $\vec{g} = k^{-1}h'(\boldsymbol{\alpha})k$. Now, let $\overline{h}(\gamma) := (g'_x)^{-1}\, k^{-1}h'(\gamma_x\gamma\gamma_y^{-1})k\, g'_y$ for all $\gamma \in \mathcal{P}_{xy}$.
   Obviously, $\overline{h}$ is gauge equivalent to h' by means of the gauge transform $(h'(\gamma_x)^{-1}kg'_x)_{x\in M}$. Hence $[\overline{h}] = [h'] = [h]$. Moreover, $\varphi(\overline{h}) \equiv \overline{h}(\boldsymbol{\alpha}) = k^{-1}h'(\boldsymbol{\alpha})k$. Finally,
   $$\begin{aligned}\Psi(\overline{h}) &= ([\overline{h}], \psi_2(\varphi(\overline{h})) * (\overline{h}(\gamma_x))_{x\in M}) \\ &= ([h], \psi_2(k^{-1}h'(\boldsymbol{\alpha})k) * ((g'_m)^{-1}k^{-1}h'(1)kg'_x)_{x\in M}) \\ &= ([h], \psi_2(\vec{g}) * (g'_x)_{x\in M}) \\ &= ([h], [g'] * \overline{g}') \\ &= ([h], [\overline{g}]_{\mathbf{B}_Z}).\end{aligned}$$
3. Ψ is injective.
   Let $\Psi(h_1) = \Psi(h_2)$. Then, in particular, $\psi_2(\varphi(h_1)) * (h_1(\gamma_x))_{x\in M} = \psi_2(\varphi(h_2)) * (h_2(\gamma_x))_{x\in M}$. From the bijectivity of $*$ on $({}_Z\backslash \mathbf{G}) \times \overline{\mathcal{G}}_0$ follows $h_1(\gamma_x) = h_2(\gamma_x)$ for all $x \in M$ and $\psi_2(h_1(\boldsymbol{\alpha})) = \psi_2(h_2(\boldsymbol{\alpha}))$. Since by assumption h_1 and h_2 are gauge equivalent, there is a $\overline{g} \in \overline{\mathcal{G}}$ with $h_1 = h_2 \circ \overline{g}$. In particular, we have $h_1(\boldsymbol{\alpha}) = g_m^{-1}h_2(\boldsymbol{\alpha})g_m$, i.e., $h_1(\boldsymbol{\alpha})$ and $h_2(\boldsymbol{\alpha})$ are contained in the same orbit. Due to $\psi_2(h_1(\boldsymbol{\alpha})) = \psi_2(h_2(\boldsymbol{\alpha}))$ we have $h_2(\boldsymbol{\alpha}) = h_1(\boldsymbol{\alpha}) = g_m^{-1}h_2(\boldsymbol{\alpha})g_m$, i.e. $g_m \in Z(h_2(\boldsymbol{\alpha})) = Z$. Finally we get $g_x = h_2(\gamma_x)^{-1}g_m h_1(\gamma_x) = h_2(\gamma_x)^{-1}h_1(\gamma_x)g_m = g_m$ for all $x \in M$, i.e. $\overline{g} \in \mathbf{B}_Z$. We get $h_1 = h_2 \circ \overline{g} = h_2$.
4. Ψ is continuous.
   It is sufficient to prove that the projections from Ψ onto the two factors are continuous:
   - $\Psi_1 := \mathrm{pr}_1 \circ \Psi$ is equal to $\pi : U \longrightarrow \pi(U)$, hence continuous.
   - $\Psi_2 := \mathrm{pr}_2 \circ \Psi$ is continuous as a concatenation of continuous mappings φ, ψ_2, π_{γ_x} and $*$.[17]
5. Ψ^{-1} is continuous.[18]

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[17] $\pi_{\gamma_x} : \overline{\mathcal{A}} \longrightarrow \mathbf{G}$, $\overline{A} \longmapsto h_{\overline{A}}(\gamma_x)$.

[18] Note that the standard theorem on the continuity of the inverse mapping is not applicable because U is (in general) not a compact set.

Let $\overline{A} \in U$. We show that $\Psi(U')$ is a neighbourhood of $\Psi(\overline{A})$ for every open neighbourhood $U' \subseteq U$ of $\overline{A}$.

- Per constructionem of U we have $\varphi(\overline{A}) \in V$ as well as $[\varphi(\overline{A})] \in V/\mathrm{Ad}$. We know that V/Ad is an open submanifold of $\mathbf{G}^k/\mathrm{Ad}$. Hence[19], there is a compact $\widetilde{W}$ and an open $\widetilde{W}_0$ with $[\varphi(\overline{A})] \in \widetilde{W}_0 \subseteq \widetilde{W} \subseteq V/\mathrm{Ad}$. Since $\mathbf{G}$ is a compact Lie group (whence π_{Ad} is a proper mapping), $W := \pi_{\mathrm{Ad}}^{-1}(\widetilde{W}) \subseteq \mathbf{G}^k$ is compact and $W_0 := \pi_{\mathrm{Ad}}^{-1}(\widetilde{W}_0) \subseteq \mathbf{G}^k$ is open for $\varphi(\overline{A}) \in W_0 \subseteq W \subseteq V$.
- Since $\varphi : \overline{\mathcal{A}} \longrightarrow \mathbf{G}^k$ is continuous, we have
  - $U'_{\overline{A}} := \varphi^{-1}(W) \subseteq \overline{\mathcal{A}}$ is closed, hence compact, and
  - $U'_{0,\overline{A}} := \varphi^{-1}(W_0) \subseteq \overline{\mathcal{A}}$ is open and
  - $\overline{A} \in U'_{0,\overline{A}} \subseteq U'_{\overline{A}} \subseteq U$.
- Per constructionem, $U'_{\overline{A}} \subseteq U$ is $\overline{\mathcal{G}}$-invariant. Thus, $\Psi \mid_{U'_{\overline{A}}}$ is a continuous, bijective mapping. Since $U'_{\overline{A}}$ is compact, $\Psi \mid_{U'_{\overline{A}}}$ becomes a homeomorphism. In particular, $\Psi \mid_{U'_{0,\overline{A}}}: U'_{0,\overline{A}} \longrightarrow \pi(U'_{0,\overline{A}}) \times (\mathbf{B}_Z \backslash \overline{\mathcal{G}})$ is a homeomorphism. Since $\pi : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\overline{\mathcal{G}}$ and $U'_{0,\overline{A}} \subseteq \overline{\mathcal{A}}$ are open, $\Psi(U'_{0,\overline{A}})$ is open (w.r.t. $\Psi(U)$).
- Now, let U' be an arbitrary open neighbourhood of $\overline{A}$ in U. Then $U' \cap U'_{0,\overline{A}}$ is again an open neighbourhood of $\overline{A}$ and, in particular, open itself w.r.t. $U'_{0,\overline{A}}$. Consequently, $\Psi \mid_{U'_{0,\overline{A}}} (U' \cap U'_{0,\overline{A}}) \equiv \Psi(U' \cap U'_{0,\overline{A}})$ is open in $\Psi(U'_{0,\overline{A}})$, hence also in $\Psi(U)$. The bijectivity of Ψ gives us $\Psi(U' \cap U'_{0,\overline{A}}) = \Psi(U') \cap \Psi(U'_{0,\overline{A}})$, i.e., $\Psi(U')$ contains the open set $\Psi(U' \cap U'_{0,\overline{A}})$ and is therefore a neighbourhood of $\Psi(\overline{A})$.

6. Ψ is equivariant.
   We have

$$\begin{aligned}
\Psi(h \circ \overline{g}) &= ([h \circ \overline{g}], \psi_2(\varphi(h \circ \overline{g})) * (((h \circ \overline{g})(\gamma_x))_{x \in M})) \\
&= ([h], \psi_2(\varphi(h) \circ g_m) * ((g_m^{-1} h(\gamma_x) g_x)_{x \in M})) \\
&= ([h], (\psi_2(\varphi(h)) \cdot [g_m]) * ((g_m^{-1} h(\gamma_x) g_x)_{x \in M})) \\
&\qquad (\cdot \text{ denotes the multiplication in } Z \backslash \mathbf{G}.) \\
&= ([h], \psi_2(\varphi(h)) * (h(\gamma_x) g_x)_{x \in M}) \\
&= ([h], (\psi_2(\varphi(h)) * (h(\gamma_x))_{x \in M}) \cdot [\overline{g}]_{\mathbf{B}_Z}) \\
&\qquad (\cdot \text{ now denotes multiplication in } \mathbf{B}_Z \backslash \overline{\mathcal{G}}.) \\
&= \Psi(h) \circ [\overline{g}]_{\mathbf{B}_Z}.
\end{aligned}$$

qed

By Proposition 4.3 and by the Lemmata 7.5 and 8.1 the preceding proposition shows that $\pi : \overline{\mathcal{A}}_{\mathrm{gen}} \longrightarrow \overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}}$ is an almost globally trivial principal fibre bundle with structure group $\mathbf{B}_Z \backslash \overline{\mathcal{G}}$. This way the first item of Theorem 5.1 is proven completely.

9 Triviality of $\overline{\mathcal{A}}$ for Abelian G

For commutative structure groups every connection is generic. Moreover, $\overline{\mathcal{A}}$ is even globally trivial:

[19] Every manifold is regular.

Proposition 9.1 Let $\mathbf{G}$ be commutative compact Lie group.
Then $\overline{\mathcal{A}}$ is a globally trivial principal fibre bundle with structure group $\overline{\mathcal{G}}_{\text{const}}\Big\backslash\overline{\mathcal{G}}$.

Here, $\overline{\mathcal{G}}_{\text{const}}$ denoted the set of all constant gauge transforms. $\overline{\mathcal{G}}_{\text{const}}\Big\backslash\overline{\mathcal{G}}$ is isomorphic zu $\overline{\mathcal{G}}_0$ as a topological group via $[\overline{g}]_{\overline{\mathcal{G}}_{\text{const}}} \longmapsto (g_m^{-1} g_x)_{x\in M}$. Therefore one can (after an appropriate modification of the action) regard $\overline{\mathcal{A}}$ in the abelian case as a $\overline{\mathcal{G}}_0$-principal fibre bundle over $\overline{\mathcal{A}}/\overline{\mathcal{G}}$.

Proof Let

$$\begin{array}{rccl}\Psi : & \overline{\mathcal{A}} & \longrightarrow & \overline{\mathcal{A}}/\overline{\mathcal{G}} \times \overline{\mathcal{G}}_{\text{const}}\Big\backslash\overline{\mathcal{G}}, \\ & h & \longmapsto & ([h],[(h(\gamma_x))_{x\in M}])\end{array}$$

where γ_x is as usual for all $x \in M$ some path from m to x being trivial for $x = m$.
In a commutative group the adjoint action is trivial, hence

$$\overline{\mathcal{A}}/\overline{\mathcal{G}} = \operatorname{Hom}(\mathcal{HG},\mathbf{G})/\operatorname{Ad} = \operatorname{Hom}(\mathcal{HG},\mathbf{G}).$$

1. Ψ is surjective.
   Let $[h] \in \overline{\mathcal{A}}/\overline{\mathcal{G}}$ and $[\overline{g}] \in \overline{\mathcal{G}}_{\text{const}}\Big\backslash\overline{\mathcal{G}}$ be given. As just remarked there is an $h' \in \overline{\mathcal{A}}$ with $h' \mid_{\mathcal{HG}}= [h]$. Now, let $h''(\gamma) := g_x^{-1} h'(\gamma_x\gamma\gamma_y^{-1}) g_y$ for $\gamma \in \mathcal{P}_{xy}$. Obviously, $h'' \in \overline{\mathcal{A}}$ and $\Psi(h'') = ([h],[\overline{g}])$.
2. Ψ is injective.
   Let $h_1, h_2 \in \overline{\mathcal{A}}$ and $\Psi(h_1) = \Psi(h_2)$. Then $[(h_1(\gamma_x))_{x\in M}] = [(h_2(\gamma_x))_{x\in M}]$, hence – γ_m is trivial – also $h_1(\gamma_x) = h_2(\gamma_x)$ for all $x \in M$. The injectivity now follows, because two connections are equal if their holonomies are equal and if their parallel transports coincide for each x along at least one path from m to x.
3. Ψ is obviously continuous.
4. Ψ^{-1} is continuous because $\overline{\mathcal{A}}$ is compact and $\overline{\mathcal{A}}/\overline{\mathcal{G}} \times \overline{\mathcal{G}}_{\text{const}}\Big\backslash\overline{\mathcal{G}}$ is Hausdorff.
5. Ψ ist clearly equivariant. **qed**

10 Criterion for the Non-Triviality of $\overline{\mathcal{A}}_{\text{gen}}$

Now we want to know when the generic stratum in nontrivial. First we state a sufficient condition for the non-triviality of $\overline{\mathcal{A}}_{\text{gen}}$ requiring only a property of $\mathbf{G}$ and find then a class of Lie groups having this property. Finally we discuss some problems arising when we tried to prove that $\overline{\mathcal{A}}_{\text{gen}}$ is nontrivial for *all* non-commutative $\mathbf{G}$.
We start with the sufficient condition for the non-triviality of $\overline{\mathcal{A}}_{\text{gen}}$.

Proposition 10.1 If there is a natural number $k \geq 1$ such that $\pi_k : (\mathbf{G}^k)_{\text{gen}} \longrightarrow (\mathbf{G}^k)_{\text{gen}}/\operatorname{Ad}$ is a nontrivial principal fibre bundle, then $\pi : \overline{\mathcal{A}}_{\text{gen}} \longrightarrow \overline{\mathcal{A}}_{\text{gen}}/\overline{\mathcal{G}}$ is non-trivial.

Proof We show that there is no continuous section $s : \overline{\mathcal{A}}_{\text{gen}} \longrightarrow \overline{\mathcal{A}}_{\text{gen}}/\overline{\mathcal{G}}$.[20]
Suppose, there were such a section. We choose a graph Γ with k edges and exactly one vertex. The set of edges is denoted by $\boldsymbol{\alpha} \subseteq \mathcal{HG}$ and defines the reduction mapping $\varphi := \varphi_{\boldsymbol{\alpha}}$. Our goal is now to construct a section $s_{[\varphi]}$ in the bottom line of the diagram ($U := \varphi^{-1}((\mathbf{G}^k)_{\text{gen}})$)

[20] A section in a bundle is a mapping, whose concetanation with the bundle projection is the identity on the base space.

$$\begin{array}{ccccc}
\overline{\mathcal{A}}_{\text{gen}} & \xleftarrow{\supseteq} & U & \underset{\varphi}{\overset{s_\varphi}{\rightleftarrows}} & (\mathbf{G}^k)_{\text{gen}} \\
s \uparrow \downarrow \pi & & s|_{[U]} \uparrow \downarrow \pi|_U & & \pi_k \downarrow \uparrow s_k \\
\overline{\mathcal{A}}_{\text{gen}}/\overline{\mathcal{G}} & \xleftarrow{\supseteq} & [U] & \underset{s_{[\varphi]}}{\overset{[\varphi]}{\rightleftarrows}} & (\mathbf{G}^k)_{\text{gen}}/\mathrm{Ad}
\end{array}.$$

We will see that then $\varphi \circ s \circ s_{[\varphi]}$ induces a continuous section in $(\mathbf{G}^k)_{\text{gen}}$. But, that does not exist by assumption because a principal fibre bundle is trivial iff it has a global section.

We construct first a section $s_\varphi : (\mathbf{G}^k)_{\text{gen}} \longrightarrow \overline{\mathcal{A}}_{\text{gen}}$. Here we choose for $s_\varphi(\vec{g})$ that connection, that is build by means of the construction method [7] out of the trivial connection if one successively assigns the components of $\vec{g}$ to the k edges in Γ. Clearly, then $\varphi(s_\varphi(\vec{g})) = \vec{g}$ for all $\vec{g} \in \mathbf{G}^k$. It is easy to see[21] from this method, that s_φ is continuous ond obviously maps $(\mathbf{G}^k)_{\text{gen}}$ to $\overline{\mathcal{A}}_{\text{gen}}$.

Now we define for $[\vec{g}] \in (\mathbf{G}^k)_{\text{gen}}/\mathrm{Ad}$ a mapping $s_{[\varphi]} : (\mathbf{G}^k)_{\text{gen}}/\mathrm{Ad} \longrightarrow \overline{\mathcal{A}}_{\text{gen}}/\overline{\mathcal{G}}$ by $s_{[\varphi]}([\vec{g}]) := \pi(s_\varphi(\vec{g}))$.

- $s_{[\varphi]}$ is well-defined.
  Let $\vec{g}_2 = \vec{g}_1 \circ g$, $g \in \mathbf{G}$. Then $s_\varphi(\vec{g}_2) = s_\varphi(\vec{g}_1) \circ \overline{g}$, where $\overline{g} \in \overline{\mathcal{G}}$ is that gauge transform having the value g everywhere. Hence $s_{[\varphi]}([\vec{g}_1]) = s_{[\varphi]}([\vec{g}_2])$.
- $s_{[\varphi]}$ is a section.
  As just proven we have $s_{[\varphi]} \circ \pi_k = \pi \circ s_\varphi$, thus
  $$\begin{aligned}
  [\varphi] \circ s_{[\varphi]} &= [\varphi] \circ s_{[\varphi]} \circ \pi_k \circ (\pi_k)^{-1} && \text{(Surjectivity of } \pi_k\text{)} \\
  &= [\varphi] \circ \pi \circ s_\varphi \circ (\pi_k)^{-1} \\
  &= \pi_k \circ \varphi \circ s_\varphi \circ (\pi_k)^{-1} && \text{(Commutativity of projections)} \\
  &= \pi_k \circ (\pi_k)^{-1} && \text{(Section property)} \\
  &= \mathrm{id}_{(\mathbf{G}^k)_{\text{gen}}/\mathrm{Ad}}. && \text{(Surjectivity of } \pi_k\text{)}
  \end{aligned}$$
- $s_{[\varphi]}$ is continuous.
  By the quotient criterion $s_{[\varphi]}$ is continuous iff $s_{[\varphi]} \circ \pi_k$ is continuous. But, the latter one is equal to $\pi \circ s_\varphi$, hence continuous.

Finally we prove that $s_k := \varphi \circ s \circ s_{[\varphi]}$ is a section for π_k: We have $\pi_k \circ s_k = \pi_k \circ \varphi \circ s \circ s_{[\varphi]} = [\varphi] \circ \pi \circ s \circ s_{[\varphi]} = \mathrm{id}_{(\mathbf{G}^k)_{\text{gen}}/\mathrm{Ad}}$, because s and $s_{[\varphi]}$ are sections themselves. The continuity of s_k is clear. Hence, there is a global continuous section in $\pi_k : (\mathbf{G}^k)_{\text{gen}} \longrightarrow (\mathbf{G}^k)_{\text{gen}}/\mathrm{Ad}$. That is a contradiction to the assumption that π_k is a nontrivial bundle.

Therefore, there is no continuous section over the whole $\overline{\mathcal{A}}_{\text{gen}}/\overline{\mathcal{G}}$. **qed**

The crucial question is now what concrete $\mathbf{G}$ and k give nontrivial bundles $\pi_k : (\mathbf{G}^k)_{\text{gen}} \longrightarrow (\mathbf{G}^k)_{\text{gen}}/\mathrm{Ad}$. It is quite easy to see (cf. Appendix C) that in the case of $\mathbf{G} = SU(2)$ the bundle is trivial for $k = 2$ and nontrivial for $k \geq 3$. (For $k = 1$ the generic stratum is empty.) So maybe typically up to some k the bundles are trivial, but nontrivial for bigger k. Is there a k for every non-abelian $\mathbf{G}$ such that π_k is nontrivial?

[21] We have to show that $\pi_{\Gamma'} \circ s_\varphi : \mathbf{G}^k \longrightarrow \mathbf{G}^{\mathbf{E}(\Gamma')}$ is continuous for all graphs Γ'. It is even sufficient to check this for all Γ' being a single edge. Let γ be a (nontrivial) edge. If $m \notin \mathrm{im}\,\gamma$, then $\pi_\gamma \circ s_\varphi(\vec{g}) = e_{\mathbf{G}}$ for all $\vec{g}$. If $m \in \mathrm{im}\,\gamma$, then $\pi_\gamma \circ s_\varphi(\vec{g})$ is some (finite) product of components of $\vec{g}$ and their inverses. In any case $\pi_\gamma \circ s_\varphi$ is continuous, too.

Up to now, we did not find a complete answer. However, the following two propositions give a wide class of groups, for which the bundle is nontrivial starting at some k. In particular, the proposition above is non-empty, i.e., its assumptions can be fulfilled.

Proposition 10.2 Let $\mathbf{G}$ be a non-abelian Lie group with $\pi_1^{\text{homotopy}}(\mathbf{G}/Z) \neq 1$ and $\pi_1^{\text{homotopy}}(\mathbf{G}) = 1$.
Then there is a $k \in \mathbb{N}$ such that $\pi_k : (\mathbf{G}^k)_{\text{gen}} \longrightarrow (\mathbf{G}^k)_{\text{gen}}/\text{Ad}$ is a nontrivial principal fibre bundle.

Proof

- Choose $k' \in \mathbb{N}$ so large that $(\mathbf{G}^{k'})_{\text{gen}}$ is non-empty (cf. Proposition 6.1).
By general arguments one sees [3] that the codimension of all non-generic strata, i.e. all strata whose type is smaller than $[Z]$, is at least 1.
By Corollary B.2 in Appendix B the non-generic strata in $\mathbf{G}^{4k'}$ have at least codimension 4. Let $k := 4k'$.
- Suppose the bundle $\pi_k : (\mathbf{G}^k)_{\text{gen}} \longrightarrow (\mathbf{G}^k)_{\text{gen}}/\text{Ad}$ were trivial.
Then $(\mathbf{G}^k)_{\text{gen}} \cong (\mathbf{G}^k)_{\text{gen}}/\text{Ad} \times {}_Z\backslash \mathbf{G}$, hence in particular
$$\pi_1^{\text{homotopy}}((\mathbf{G}^k)_{\text{gen}}) \cong \pi_1^{\text{homotopy}}((\mathbf{G}^k)_{\text{gen}}/\text{Ad}) \oplus \pi_1^{\text{homotopy}}({}_Z\backslash \mathbf{G}). \qquad (1)$$
Since because of the compactness of $\mathbf{G}$ the number of non-generic strata in $\mathbf{G}^k$ is finite [14] and each one of the strata is a submanifold of $\mathbf{G}^k$ [3] having codimension bigger or equal 3, we have $\pi_1^{\text{homotopy}}((\mathbf{G}^k)_{\text{gen}}) \cong \pi_1^{\text{homotopy}}(\mathbf{G}^k) \cong \pi_1^{\text{homotopy}}(\mathbf{G})^k$.
Consequently, (1) reduces to
$$\pi_1^{\text{homotopy}}(\mathbf{G})^k \cong \pi_1^{\text{homotopy}}((\mathbf{G}^k)_{\text{gen}}/\text{Ad}) \oplus \pi_1^{\text{homotopy}}({}_Z\backslash \mathbf{G}).$$
This, however, is contradiction to the assumptions $\pi_1^{\text{homotopy}}(\mathbf{G})^k = 1$ and $\pi_1^{\text{homotopy}}(\mathbf{G}/Z) \neq 1$.
Hence, π_k is nontrivial. **qed**

Proposition 10.3 The assumptions of the proposition above are fulfilled, in particular, for all semisimple (simply connected) Lie groups whose decomposition into simple Lie groups contains at least one of the factors A_n, B_n, C_n, D_n, E_6 or E_7.

Proof One sees from the following list [10] that just those Lie groups written in the proposition have nontrivial center.

Series	Center Z of the universal covering $\mathbf{G}$
A_n	$\mathbb{Z}_{n+1}$
B_n	$\mathbb{Z}_2$
C_n	$\mathbb{Z}_2$
D_{2n+1}	$\mathbb{Z}_4$
D_{2n}	$\mathbb{Z}_2 + \mathbb{Z}_2$
E_6	$\mathbb{Z}_3$
E_7	$\mathbb{Z}_2$
E_8	1
F_4	1
G_2	1

The assumption now follows from $\pi_1^{\text{homotopy}}({}_Z\backslash \mathbf{G}) = \pi_0^{\text{homotopy}}(Z)$ for $\pi_1^{\text{homotopy}}(\mathbf{G}) = 1$ [11] and the fact that the center of the direct product of groups equals the direct product of the corresponding centers. **qed**

In particular we see that $\overline{\mathcal{A}}$ is nontrivial for all $\mathbf{G} = SU(N)$ $(= A_{N-1},\ N \geq 2)$. However, the corresponding problem, e.g., for $\mathbf{G} = SO(N)$ or the prominent case $\mathbf{G} = E_8 \times E_8$ remains unsolved.
We remark that in general for fixed $\mathbf{G}$ the bundles gets "more nontrivial" when increasing k. Strictly speaking, we have

Proposition 10.4 For every $\mathbf{G}$ the non-triviality of π_k implies that of π_{k+1}.

Proof Let k be chosen such that π_k is nontrivial. Suppose there is a section s_{k+1} for π_{k+1}. We get the following commutative diagram

$$\begin{tikzcd}
(\mathbf{G}^{k+1})_{\mathrm{gen}} \arrow[d, two heads, "\pi_{k+1}"] & U \arrow[l, "\supseteq"'] \arrow[r, two heads, "p_k"] \arrow[d, two heads, "\pi_{k+1}|_U"] & (\mathbf{G}^{k})_{\mathrm{gen}} \arrow[d, two heads, "\pi_k"'] \\
(\mathbf{G}^{k+1})_{\mathrm{gen}}/\mathrm{Ad} & {[U]} \arrow[l, "\supseteq"'] \arrow[r, two heads, "{[p_k]}"] & (\mathbf{G}^{k})_{\mathrm{gen}}/\mathrm{Ad}
\end{tikzcd}.$$

Here $p_k : (\mathbf{G}^{k+1})_{\mathrm{gen}} \longrightarrow (\mathbf{G}^{k})_{\mathrm{gen}}$ is the projection onto the first k coordinates and $[p_k]$ is induced in a natural way. Additionally, we defined $U := p_k^{-1}((\mathbf{G}^{k})_{\mathrm{gen}})$.
Now we reuse the idea for the proof of Proposition 10.1. First we set $i_k : (\mathbf{G}^{k})_{\mathrm{gen}} \longrightarrow U = (\mathbf{G}^{k})_{\mathrm{gen}} \times \mathbf{G}$ with $i_k(\vec{g}) := (\vec{g}, e_{\mathbf{G}})$, where the rhs vector is viewed as an element of $\mathbf{G}^{k+1}$. Obviously, i_k is a continuous section for p_k, that additionally – as can be checked quickly – defines a continuous section for $[p_k]$ via $[i_k]([\vec{g}]) := \pi_{k+1}(i_k(\vec{g}))$. Finally one sees that $s_k := p_k \circ s_{k+1} \mid_{[U]} \circ [i_k]$ is a section for π_k. This, however, is a contradiction to the non-triviality of π_k. **qed**

Let us return again to the proof of Proposition 10.1. There we deduced via φ from the non-triviality of the generic stratum in $\mathbf{G}^k$ that the preimage $\varphi^{-1}((\mathbf{G}^{k})_{\mathrm{gen}})$, hence $\overline{\mathcal{A}}_{\mathrm{gen}}$ as well is a nontrivial bundle. But, besides we know that φ is surjective even as a mapping from $\overline{\mathcal{A}}_{\mathrm{gen}}$ to the whole $\mathbf{G}^k$. [8] It seems to be obvious that one can now deduce from the non-existence of a section in π_k over the whole space $\mathbf{G}^k/\mathrm{Ad}$ (and not only over the generic stratum as above) analogously to the case above the non-existence of a global section in $\overline{\mathcal{A}}_{\mathrm{gen}} \longrightarrow \overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}}$. But, the existence of a section for the whole π_k is rather not to be expected because typically in the case of non-commutative structure groups $\mathbf{G}$ the mapping π_k does not define a fibre bundle. (This can easily be seen because in $\mathbf{G}^k$ there occurs both the orbit [i.e. the fibre] ${}_{Z}\backslash\mathbf{G}$ and the orbit $\mathrm{pt} = {}_{\mathbf{G}}\backslash\mathbf{G}$ being never isomorphic. However, this is not a criterion for the non-existence of a section, but simply just an indication.) Hence, one can guess that $\pi : \overline{\mathcal{A}}_{\mathrm{gen}} \longrightarrow \overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}}$ is surely nontrivial for non-commutative $\mathbf{G}$, at least as far as π_k possesses no section over the whole $(\mathbf{G}^{k})_{\mathrm{gen}}/\mathrm{Ad}$.
Unfortunately, we were not able to prove this up to now. At one point the proof above uses explicitly the fact that only the generic stratum in $\mathbf{G}^k$ is considered – namely, for the definition of $s_{[\varphi]}$. A continuation of that mapping from the generic elements of $(\mathbf{G}^{k})_{\mathrm{gen}}/\mathrm{Ad}$ to the whole $\mathbf{G}^k/\mathrm{Ad}$ is not possible as the next proposition shows:

Proposition 10.5 Let $k \in \mathbb{N}_+$ be some number for that there are both generic and non-generic elements in $\mathbf{G}^k$.
Then there is no continuous mapping $s_\varphi : \mathbf{G}^k \longrightarrow \overline{\mathcal{A}}_{\mathrm{gen}}$, such that $s_{[\varphi]} : \mathbf{G}^k/\mathrm{Ad} \longrightarrow \overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}}$ with $s_{[\varphi]}([\vec{g}]) := \pi(s_\varphi(\vec{g}))$ is well-defined and that $\varphi \circ s_\varphi = \mathrm{id}_{\mathbf{G}^k}$.

We need the following

Lemma 10.6 Let $k, l \in \mathbb{N}_+$ such that there are generic elements in $\mathbf{G}^l$.
Then we have: $\mathbf{G}$ is abelian iff there is a continuous $f : \mathbf{G}^l \longrightarrow \mathbf{G}^k$ with
- $\vec{g}_1 \sim \vec{g}_2 \Longrightarrow (\vec{g}_1, f(\vec{g}_1)) \sim (\vec{g}_2, f(\vec{g}_2))$ and
- $Z(\vec{g}, f(\vec{g})) = Z$ for all $\vec{g} \in \mathbf{G}^l$.

Proof
- Let $\mathbf{G}$ be abelian. Then, e.g., $f(\vec{g}) = e_{\mathbf{G}}$, $\vec{g} \in \mathbf{G}^l$, fulfills the conditions of the lemma.
- Let $\mathbf{G}$ be non-abelian. Suppose there were such an f. Let $\vec{g}_1, \vec{g}_2 \in \mathbf{G}^l$ be equivalent, i.e., let there exist a $g \in \mathbf{G}$ with $\vec{g}_2 = \vec{g}_1 \circ g$. By assumption there is also a $g' \in \mathbf{G}$ with $(\vec{g}_2, f(\vec{g}_2)) = (\vec{g}_1, f(\vec{g}_1)) \circ g' = (\vec{g}_1 \circ g', f(\vec{g}_1) \circ g')$. Hence, $\vec{g}_1 \circ g = \vec{g}_1 \circ g'$, i.e. $g' = g''g$ for some $g'' \in Z(\vec{g}_1)$. Consequently, $f(\vec{g}_1 \circ g) = f(\vec{g}_2) = f(\vec{g}_1) \circ g' = f(\vec{g}_1) \circ g'' \circ g$. In particular, for all generic $\vec{g}_1$ we have $g'' \in Z$, i.e. $f(\vec{g}_1 \circ g) = f(\vec{g}_1) \circ g$. Since the generic elements by assumption[22] form a dense subset in $\mathbf{G}^l$ and f is to be continuous, $f(\vec{g}_1 \circ g) = f(\vec{g}_1) \circ g$ has to hold even for all $\vec{g}_1 \in \mathbf{G}^l$ and all $g \in \mathbf{G}$. Let $\vec{g}$ now be a non-generic element in $\mathbf{G}^l$, i.e., let there exist a $g \in Z(\vec{g}) \setminus Z$. But, now $f(\vec{g}) = f(\vec{g} \circ g) = f(\vec{g}) \circ g$, hence $g \in Z(f(\vec{g})) \cap (Z(\vec{g}) \setminus Z) = (Z(\vec{g}) \cap Z(f(\vec{g}))) \setminus Z = \emptyset$. Therefore all $\vec{g} \in \mathbf{G}^l$ are generic in contradiction to the non-commutativity of $\mathbf{G}$. **qed**

Proof Proposition 10.5
Suppose there exists such a $s_\varphi : \mathbf{G}^k \longrightarrow \overline{\mathcal{A}}_{\text{gen}}$.

1. Let $\vec{g} \in \mathbf{G}^k$ be arbitrary, but fixed. Due to $s_\varphi(\vec{g}) \in \overline{\mathcal{A}}_{\text{gen}}$ there is an $\boldsymbol{\alpha}_{\vec{g}} \subseteq \mathcal{HG}$ with $h_{s_\varphi(\vec{g})}(\boldsymbol{\alpha}_{\vec{g}}) \in (\mathbf{G}^{\#\boldsymbol{\alpha}_{\vec{g}}})_{\text{gen}}$. Since the generic stratum is always open and since together with s_φ and $h_{\boldsymbol{\alpha}_{\vec{g}}}$ also $h_{\boldsymbol{\alpha}_{\vec{g}}} \circ s_\varphi : \mathbf{G}^k \longrightarrow \mathbf{G}^{\#\boldsymbol{\alpha}_{\vec{g}}}$ is continuous, $U_{\vec{g}} := (h_{\boldsymbol{\alpha}_{\vec{g}}} \circ s_\varphi)^{-1}((\mathbf{G}^{\#\boldsymbol{\alpha}_{\vec{g}}})_{\text{gen}})$ defines an open neighbourhood of $\vec{g}$.
2. Varying over all $\vec{g}$ one gets an open covering $\mathcal{U} := \{U_{\vec{g}} \mid \vec{g} \in \mathbf{G}^k\}$ of $\mathbf{G}^k$. Since $\mathbf{G}$ is compact, there are finitely many $\vec{g}_i \in \mathbf{G}^k$ such that $\bigcup_i U_{\vec{g}_i} = \mathbf{G}^k$. Let now $\boldsymbol{\alpha}'$ be the set (the tuple, respectively) of all these $\boldsymbol{\alpha}_{\vec{g}_i}$.
3. We define $f := h_{\boldsymbol{\alpha}'} \circ s_\varphi : \mathbf{G}^k \longrightarrow \mathbf{G}^{\#\boldsymbol{\alpha}'}$ and $f' := (h_{\boldsymbol{\alpha}} \circ s_\varphi, h_{\boldsymbol{\alpha}'} \circ s_\varphi) \equiv (\mathrm{id}_{\mathbf{G}^k}, h_{\boldsymbol{\alpha}'} \circ s_\varphi) : \mathbf{G}^k \longrightarrow \mathbf{G}^{k+\#\boldsymbol{\alpha}'}$. (Recall $\varphi \equiv h_{\boldsymbol{\alpha}}$.)
   We have:
   - f is continuous.
   - Let $\vec{g}', \vec{g}'' \in \mathbf{G}^k$ with $\vec{g}' \sim \vec{g}''$. From that, due to the assumed well-definedness of $s_{[\varphi]}$, $s_\varphi(\vec{g}') \sim s_\varphi(\vec{g}'')$ w.r.t. $\overline{\mathcal{G}}$. Hence, in particular $(\vec{g}', f(\vec{g}')) = f'(\vec{g}'') \sim f'(\vec{g}'') = (\vec{g}'', f(\vec{g}''))$.
   - $Z(f(\vec{g})) = Z$ for all $\vec{g} \in \mathbf{G}^k$.
     Let $\vec{g} \in \mathbf{G}^k$. Then there is an i with $\vec{g} \in U_{\vec{g}_i}$. Thus, we have $h_{\boldsymbol{\alpha}_{\vec{g}_i}}(s_\varphi(\vec{g})) \in (\mathbf{G}^{\#\boldsymbol{\alpha}_{\vec{g}_i}})_{\text{gen}}$, hence $f(\vec{g}) = h_{\boldsymbol{\alpha}'} \circ s_\varphi(\vec{g}) \in (\mathbf{G}^{\#\boldsymbol{\alpha}'})_{\text{gen}}$ due to $\boldsymbol{\alpha}_{\vec{g}_i} \subseteq \boldsymbol{\alpha}'$.

   Due to $Z \subseteq Z(\vec{g}, f(\vec{g})) \subseteq Z(f(\vec{g})) = Z$, f fulfills all assumptions of the preceding lemma, i.e., $\mathbf{G}$ is abelian in contradiction to the supposition. **qed**

Despite these obstacles we close these section with an even stronger

Conjecture 10.7 $\overline{\mathcal{A}}_{\text{gen}}$ is nontrivial for every non-commutative $\mathbf{G}$.

Perhaps, there is even for every non-commutative $\mathbf{G}$ a k such that $(\mathbf{G}^k)_{\text{gen}}$ is nontrivial.

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[22] The generic stratum is always dense or empty. [3]

11 Concluding Remarks

The main result of the present paper is the proof of the existence of a Gribov problem for Ashtekar connections as known for Sobolev connections for two decades. We have shown that for certain structure groups $\mathbf{G}$ the generic stratum is nontrivial. It is remarkable that for the groups given in Proposition 10.3 the generic stratum is nontrivial for *every* base manifold (space-time) M. However, the nontriviality "lives" only on a set of induced Haar measure zero. This yields a rigorous proof for the fact mentioned in [8] that the Faddeev-Popov determinant for the transition from $\overline{\mathcal{A}}$ to $\overline{\mathcal{A}}/\overline{\mathcal{G}}$ is indeed equal to 1 up to a subset of induced Haar measure zero. Moreover, we stated a mostly constructive method for the definition of "large" sections for $\overline{\mathcal{A}}_{\mathrm{gen}} \longrightarrow \overline{\mathcal{A}}_{\mathrm{gen}}/\overline{\mathcal{G}}$ arising from a section of $(\mathbf{G}^k)_{\mathrm{gen}} \longrightarrow (\mathbf{G}^k)_{\mathrm{gen}}/\mathrm{Ad}$. Maybe, e.g. the case $\mathbf{G} = SU(2)$ could be interesting for quantum gravity.
Finally, we say a few words about different notations in the case of classical connections and their relation to the generalized ones. Usually there are three kinds of connections according to their holonomy group $\mathbf{H}_A$:
1. $\mathcal{A}_{\mathrm{irr}} := \{A \in \mathcal{A} \mid \mathbf{H}_A = \mathbf{G}\}$ (often called irreducible),
2. $\mathcal{A}_{\mathrm{gen}} := \{A \in \mathcal{A} \mid Z(\mathbf{H}_A) = Z(\mathbf{G})\}$ (often called generic),
3. $\mathcal{A}_{\mathrm{alm\ gen}} := \{A \in \mathcal{A} \mid Z(\mathbf{H}_A) \text{ is discrete}\}$ (we call it almost generic).

Unfortunately, the notations are sometimes diverging. The corresponding sets fulfill $\mathcal{A}_{\mathrm{irr}} \subseteq \mathcal{A}_{\mathrm{gen}} \subseteq \mathcal{A}_{\mathrm{alm\ gen}} \subseteq \mathcal{A}$ for semi-simple $\mathbf{G}$.[23] As mentioned, e.g., in [2] the inclusions are not always proper. Suppose, e.g., $\mathbf{G} = SU(2)$, then $\mathcal{A}_{\mathrm{irr}} = \mathcal{A}_{\mathrm{alm\ gen}}$ for simply connected base manifolds M and $\mathcal{A}_{\mathrm{alm\ gen}} = \mathcal{A}$ for certain M depending on the topology of the bundle $P = P(M, \mathbf{G})$. Moreover, $\mathcal{A}_{\mathrm{gen}} = \mathcal{A}_{\mathrm{alm\ gen}}$ iff $\mathbf{G}$ is a product of $SU(N_i)$'s only. However, in the case of generalized connections the relation $\overline{\mathcal{A}}_{\mathrm{irr}} \subset \overline{\mathcal{A}}_{\mathrm{gen}}$ is proper for every (at least one-dimensional, but not necessarily semi-simple) Lie group $\mathbf{G}$. This is a simple consequence of the fact that for generalized connections every subgroup H of $\mathbf{G}$ occurs as a holonomy group of some connection. (H need not even be a topological group.) Thus, there are always proper subgroups $H \subset \mathbf{G}$ with $Z(H) = Z(\mathbf{G})$ and $H = \mathbf{H}_{\overline{A}}$ for some $\overline{A} \in \overline{\mathcal{A}}$. Up to now, we do not know whether $\overline{\mathcal{A}}_{\mathrm{irr}}$ is open or closed or whatever in $\overline{\mathcal{A}}_{\mathrm{gen}}$. We also do not know whether $\pi : \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/\overline{\mathcal{G}}$ may be trivial on the irreducible connections. This would be an interesting problem, in particular, because the original paper [15] by Singer on the Gribov ambiguity showed the non-triviality of the bundle of irreducible (Sobolev) connections.

12 Acknowledgements

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[23] If $\mathbf{G}$ is not semi-simple, the center of $\mathbf{G}$ is never discrete [4], hence no centralizer can be discrete. Thus, $\mathcal{A}_{\mathrm{alm\ gen}}$ would be empty. However, the relation between $\mathcal{A}_{\mathrm{irr}}$ and $\mathcal{A}_{\mathrm{gen}}$ survives for arbitrary $\mathbf{G}$.

Appendix

A Group Isomorphy of the Two Normalizers

In this appendix we discuss necessary conditions for the existence of a "reasonable" isomorphism between the two topological groups

$$\mathbf{B}(\overline{A})\Big\backslash N(\mathbf{B}(\overline{A})) \text{ and } Z(\mathbf{H}_{\overline{A}})\Big\backslash N(Z(\mathbf{H}_{\overline{A}})) \times \bigtimes_{x\neq m} Z(Z(\mathbf{H}_{\overline{A}})).$$

In order to reduce the size of the expressions in what follows, we assume w.l.o.g. that the connection $\overline{A} \in \overline{\mathcal{A}}$ being under consideration has the property that $h_{\overline{A}}(\gamma_x) = e_{\mathbf{G}}$ for all $x \in M$ where γ_x is as usual some fixed path from m to x and γ_m is trivial. This, indeed, is no restriction because every $\overline{A}' \in \overline{\mathcal{A}}$ is gauge equivalent to such an $\overline{A}$. For this, one would simply set $\overline{g} := (h_{\overline{A}'}(\gamma_x)^{-1})_{x\in M} \in \overline{\mathcal{G}}$ and $\overline{A} := \overline{A}' \circ \overline{g}$.
In the following we will restrict ourselves to so-called "reasonable" isomorphisms. First we only look for isomorphisms between

$$\mathbf{B}(\overline{A})\Big\backslash N(\mathbf{B}(\overline{A})) \text{ and } Z(\mathbf{H}_{\overline{A}})\Big\backslash N(Z(\mathbf{H}_{\overline{A}})) \times \bigtimes_{x\neq m} Z(Z(\mathbf{H}_{\overline{A}}))$$

that are induced by an isomorphism

$$\psi : N(\mathbf{B}(\overline{A})) \longrightarrow N(Z(\mathbf{H}_{\overline{A}})) \times \bigtimes_{x\neq m} Z(Z(\mathbf{H}_{\overline{A}}))$$

between the two non-factorized spaces. This means, such a factor isomorphism has to be a continuation of the natural isomorphism between the base centralizer and the holonomy centralizer: $\psi(\mathbf{B}(\overline{A})) = Z(\mathbf{H}_{\overline{A}}) \times \bigtimes_{x\neq m}\{e_{\mathbf{G}}\}$. We denote the mapping, that one gets by concatenation of ψ and of the corresponding projection to the x-component of the image space, by ψ_x, $x \in M$. For a "reasonable" isomorphism ψ we demand that first $\psi_m \equiv \pi_m : N(\mathbf{B}(\overline{A})) \longrightarrow N(Z(\mathbf{H}_{\overline{A}}))$ and second $\psi_x(\overline{g})$ depends only on the values of $\overline{g}$ in x and m. We think viewing at Proposition 3.4 these restrictions are natural. We neglect only "wild" isomorphisms, i.e. mappings that mix the points of M. For technical reasons we additionally demand that all ψ_x are smooth as mappings from $\mathbf{G} \times \mathbf{G}$ to $\mathbf{G}$, i.e. are Lie mappings.
Now, let ψ be a "reasonable" isomorphism. We fix a point $x \neq m$ and investigate, how the projection ψ_x of ψ to the point x has to look like.
Since ψ is to be a "reasonable" homomorphism, ψ_x is a map from (g_m, g_x) to $\psi_x(\overline{g}) \equiv \psi_x(g_m, g_x)$. By Proposition 3.4 every $\overline{g} \in N(\mathbf{B}(\overline{A}))$ is just determined by the values of $g_m \in N(Z(\mathbf{H}_{\overline{A}}))$ and of $z_x \in Z(Z(\mathbf{H}_{\overline{A}}))$: we have $g_x = z_x g_m$. Hence, ψ_x is well-defined iff[24]

$$\psi_x(g, zg) \in Z(Z(\mathbf{H}_{\overline{A}})) \text{ for all } g \in N(Z(\mathbf{H}_{\overline{A}})) \text{ and } z \in Z(Z(\mathbf{H}_{\overline{A}})). \tag{2}$$

The homomorphy property implies $\psi(\overline{g}_1)\psi(\overline{g}_2) = \psi(\overline{g}_1\overline{g}_2)$ for all $\overline{g}_i \in N(\mathbf{B}(\overline{A}))$, hence

$$\psi_x(g_1g_2, z_1g_1z_2g_2) = \psi_x(g_1, z_1g_1)\psi_x(g_2, z_2g_2) \text{ for all } g_i \in N(Z(\mathbf{H}_{\overline{A}})) \text{ and } z_i \in Z(Z(\mathbf{H}_{\overline{A}})). \tag{3}$$

Now, we define

$$\chi(z) := \psi_x(1, z) \text{ for } z \in Z(Z(\mathbf{H}_{\overline{A}}))$$

and

$$\varphi(g) := \psi_x(g, g) \text{ for } g \in N(Z(\mathbf{H}_{\overline{A}})).$$

Obviously $\chi : Z(Z(\mathbf{H}_{\overline{A}})) \longrightarrow Z(Z(\mathbf{H}_{\overline{A}}))$ and $\varphi : N(Z(\mathbf{H}_{\overline{A}})) \longrightarrow Z(Z(\mathbf{H}_{\overline{A}}))$ are homomorphisms and we have

$$\psi_x(g, zg) = \chi(z)\varphi(g).$$

[24] In what follows we in general drop the index m in g_m and the index x in z_x.

χ is even an automorphism of $Z(Z(\mathbf{H}_{\overline{A}}))$ because χ is per constr. an injective Lie morphism.[25] The injectivity of χ here is a consequence of our assumption that ψ_m is trivial and $\psi_x(\overline{g})$ does not depend on $g_{x'}$, $x' \neq x, m$.

Now let $g \in N(Z(\mathbf{H}_{\overline{A}}))$ with $\varphi(g) \in Z(Z(Z(\mathbf{H}_{\overline{A}}))) = Z(\mathbf{H}_{\overline{A}})$. Due to the bijectivity of χ we have exactly for those g that $\psi_x(g, zg) = \chi(z)\varphi(g) = \varphi(g)\chi(z) = \psi_x(g, gz)$ for all $z \in Z(Z(\mathbf{H}_{\overline{A}}))$. This implies that $zg = gz$, i.e. $g \in Z(Z(Z(\mathbf{H}_{\overline{A}}))) = Z(\mathbf{H}_{\overline{A}})$. Hence we get

$$\varphi^{-1}(Z(\mathbf{H}_{\overline{A}})) \subseteq Z(\mathbf{H}_{\overline{A}}). \tag{4}$$

Our assumption $\psi(\mathbf{B}(\overline{A})) = Z(\mathbf{H}_{\overline{A}}) \times \bigtimes_{x\neq m}\{e_{\mathbf{G}}\}$ implies now[26] $\varphi(Z(\mathbf{H}_{\overline{A}})) = \{e_{\mathbf{G}}\}$. This again yields $\ker\varphi \subseteq \varphi^{-1}(Z(\mathbf{H}_{\overline{A}})) \subseteq Z(\mathbf{H}_{\overline{A}}) \subseteq \ker\varphi$, hence

$$\ker\varphi = Z(\mathbf{H}_{\overline{A}}). \tag{5}$$

Therefore we have

$$Z(\mathbf{H}_{\overline{A}})\Big\backslash N(Z(\mathbf{H}_{\overline{A}})) \cong \operatorname{im}\varphi \subseteq Z(Z(\mathbf{H}_{\overline{A}})). \tag{6}$$

This, however, cannot always be fulfilled. Let, e.g., be $\mathbf{G} = SU(2)$ and $\overline{A}$ be generic. Then $Z(\mathbf{H}_{\overline{A}}) = Z(SU(2)) = \mathbb{Z}_2$ and $Z(Z(\mathbf{H}_{\overline{A}})) = N(Z(\mathbf{H}_{\overline{A}})) = SU(2)$. We are looking now for a homomorphism $\varphi : SU(2) \longrightarrow SU(2)$ with $\ker\varphi = \mathbb{Z}_2$. By the homomorphism theorem $SO(3) \cong SU(2)/\mathbb{Z}_2 \cong \operatorname{im}\varphi$ is a subgroup of $SU(2)$. This is a contradiction.[27] Hence, in general, there is *no* "reasonable" isomorphism of topological groups between $\mathbf{B}(\overline{A})\Big\backslash N(\mathbf{B}(\overline{A}))$ und $Z(\mathbf{H}_{\overline{A}})\Big\backslash N(Z(\mathbf{H}_{\overline{A}})) \times \bigtimes_{x\neq m} Z(Z(\mathbf{H}_{\overline{A}}))$.

Finally, we discuss two special cases.

- $\chi(z) = z$ and $\varphi(g) = g$.
  The resulting mapping $\psi_x(g, zg) = zg$ just corresponds to the restriction Ψ_0 of the identical map on $\overline{\mathcal{G}}$. This, however, gives a group isomorphism only if φ is indeed a map from $N(Z(\mathbf{H}_{\overline{A}}))$ to $Z(Z(\mathbf{H}_{\overline{A}}))$, i.e., these two spaces are equal.
  This criterion is fulfilled for instance in the generic stratum. And indeed, in this case Ψ_0 is a group isomorphism between $N(\mathbf{B}(\overline{A}))$ and $N(Z(\mathbf{H}_{\overline{A}})) \times \bigtimes_{x\neq m}\{e_{\mathbf{G}}\}$. Nevertheless, Ψ_0 factorizes by condition (5) only then to an isomorphism of the quotient groups if $Z(\mathbf{G}) = Z(\mathbf{H}_{\overline{A}}) = \ker\varphi = \{e_{\mathbf{G}}\}$.
  In the minimal stratum Ψ_0 is in general no longer an isomorphism because at least for non-abelian $\mathbf{G}$ (i.e., if $\overline{\mathcal{A}}_{\text{gen}} \neq \overline{\mathcal{A}}_{=t_{\min}}$) $N(Z(\mathbf{H}_{\overline{A}})) = \mathbf{G}$ is not equal $Z(Z(\mathbf{H}_{\overline{A}})) = Z(\mathbf{G})$.
- $\chi(z) = z$ and $\varphi(g) = e_{\mathbf{G}}$.
  The resulting mapping $\psi_x(g, zg) = z$ corresponds here to the homeomorphism Ψ_1 from Corollary 3.5. In order to turn ψ into a homomorphism, by condition (5) $N(Z(\mathbf{H}_{\overline{A}})) = \ker\varphi = Z(\mathbf{H}_{\overline{A}})$ has to hold.
  This condition is fulfilled in the minimal stratum. (Then, as can be easily checked, Ψ_1 is indeed a group isomorphism.)

[25] In general every injective Lie morphism $f : \mathbf{G} \longrightarrow \mathbf{G}$ is already a homeomorphism, if $\mathbf{G}$ is a compact Lie group. This one sees as follows: im f is compact as a continuous image of a compact set, hence closed. Since the image of a homomorphism in general is a subgroup of the target space, im f is Lie subgroup. By the homomorphism theorem [11] $\operatorname{im} f \cong \mathbf{G}/\ker f \cong \mathbf{G}$. Hence, im f as a subgroup of $\mathbf{G}$ has the same dimension and the same number of connected components as $\mathbf{G}$, and thus is equal to $\mathbf{G}$. Therefore f is continuous and bijective, hence a homeomorphism.

[26] Note that $\mathbf{B}(\overline{A})$ due to the special choice of the $h_{\overline{A}}(\gamma_x)$ consists just of the constant $Z(\mathbf{H}_{\overline{A}})$-valued gauge transforms.

[27] $SO(3)$ as (an isomorphic image of) a subgroup of $SU(2)$ having the identical dimension and the same number of connected components has to be even equal $SU(2)$. But, this is impossible, because $SO(3)$ and $SU(2)$ are non-isomorphic.

On the other hand, Ψ_1 is no isomorphism for generic $\overline{A}$ in the non-abelian case. This is clear, because there we have $N(Z(\mathbf{H}_{\overline{A}})) = \mathbf{G}$, but $Z(\mathbf{H}_{\overline{A}}) = Z(\mathbf{G})$.

We see again that it is in many cases impossible to find a group isomorphism that additionally does not depend explicitly on the respective stratum containing $\overline{A}$.

B Codimension of the Singular Strata

Let G be a compact Lie group acting smoothly on a manifold M and let H be a closed subgroup on G. We denote the stratum corresponding to the type $[H]$ by $M_{(H)} := \{x \in M \mid \mathrm{Typ}(x) = [H]\}$. Every stratum is a smooth submanifold of M. [3]

Proposition B.1 Under the previous assumptions we have

$$\max_{[K]<[H]} \dim(M \times M)_{(K)} \leq 2 \max_{[K]<[H]} \dim M_{(K)}.$$

Here, G acts on $M \times M$ in natural way: $(m_1, m_2) \circ g := (m_1 \circ g, m_2 \circ g)$.

Proof Let $(x, y) \in (M \times M)_{(K)}$ for some $[K] < [H]$. Hence, $\mathrm{Stab}(x, y) = \mathrm{Stab}(x) \cap \mathrm{Stab}(y)$ and thus $\mathrm{Stab}(x), \mathrm{Stab}(y) \supseteq \mathrm{Stab}(x, y) = g^{-1}Kg$ for some $g \in G$. Therefore, $[\mathrm{Stab}(x)], [\mathrm{Stab}(y)] \leq [K] < [H]$ and so $x, y \in \bigcup_{[K'] \leq [K]} M_{(K')}$.

Consequently,

$$(M \times M)_{(K)} \subseteq \Bigg(\bigcup_{[K'] \leq [K]} M_{(K')} \Bigg) \times \Bigg(\bigcup_{[K''] \leq [K]} M_{(K'')} \Bigg) = \bigcup_{[K'],[K''] \leq [K]} M_{(K')} \times M_{(K'')}.$$

This implies using the finiteness of the number of orbit types on $M \times M$ [14]

$$\begin{aligned}
\dim(M \times M)_{(K)} &\leq \dim \bigcup_{[K'],[K''] \leq [K]} M_{(K')} \times M_{(K'')} \\
&\leq \max_{[K'],[K''] \leq [K]} \dim M_{(K')} \times M_{(K'')} \\
&= 2 \max_{[K'] \leq [K]} \dim M_{(K')} \\
&\leq 2 \max_{[K'] < [H]} \dim M_{(K')}.
\end{aligned}$$

In particular, we have

$$\max_{[K]<[H]} \dim(M \times M)_{(K)} \leq 2 \max_{[K]<[H]} \dim M_{(K)}.$$

qed

Corollary B.2 We have

$$\max_{[K]<[H]} \mathrm{codim}_{M \times M}(M \times M)_{(K)} \geq 2 \max_{[K]<[H]} \mathrm{codim}_M M_{(K)}.$$

C The Example $\mathbf{G} = SU(2)$

We start with some notation. Every $SU(2)$-matrix A can be written uniquely as $A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ where $a, b \in \mathbb{C}$ are complex numbers fulfilling $|a|^2 + |b|^2 = 1$. Alternatively such a matrix can be seen as a quaternion $A = a + b\mathrm{j} \in \mathbb{H}$. In this case we also describe A by $a_0 + a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k}$ or $a_0 + \vec{a}$ with $\sum a_i^2 = 1$, $a_i \in \mathbb{R}$. We have $SU(2) \cong S^3 \subseteq \mathbb{R}^4$.

C.1 Adjoint Action on $SU(2)$

Let $A, C \in SU(2)$ with $A = a_0 + a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k}$ and $C = c_0 + c_1\mathrm{i} + c_2\mathrm{j} + c_3\mathrm{k}$. It is easy to check that the adjoint action in terms of quaternions is
$C^*AC = a_0 + \vec{a} + 2\Big(\langle\vec{a},\vec{c}\rangle\vec{c} - \langle\vec{c},\vec{c}\rangle\vec{a} + c_0(\vec{a}\times\vec{c})\Big) = a_0 + \vec{a} + 2\Big(c_0(\vec{a}\times\vec{c}) - \vec{c}\times(\vec{a}\times\vec{c})\Big).$
Here, $\langle\cdot,\cdot\rangle$ is the canonical scalar product on $\mathbb{R}^3$.
We determine the stabilizer of A.
We have $C^{-1}AC = C^*AC = A + 2\Big(\langle\vec{a},\vec{c}\rangle\vec{c} - \langle\vec{c},\vec{c}\rangle\vec{a} + c_0(\vec{a}\times\vec{c})\Big)$, hence

$$C \in Z(A) \iff C^{-1}AC = A \iff \langle\vec{a},\vec{c}\rangle\vec{c} - \langle\vec{c},\vec{c}\rangle\vec{a} + c_0(\vec{a}\times\vec{c}) = 0. \tag{7}$$

There are two cases:

1. $\vec{a} = 0$, i.e. $A = \pm\mathbf{1}$.
   Clearly, the rhs of (7) is true for all $C \in SU(2)$, i.e. $Z(A) = SU(2)$.
2. $\vec{a} \neq 0$, i.e. $A \neq \pm\mathbf{1}$.
   Let $C \in Z(A)$. Multiplying the rhs of (7) by $\vec{a}$ we get $\langle\vec{a},\vec{c}\rangle\langle\vec{c},\vec{a}\rangle - \langle\vec{c},\vec{c}\rangle\langle\vec{a},\vec{a}\rangle = 0$. This implies due to $\vec{a} \neq 0$ that $\vec{c} = \mu\vec{a}$ for some $\mu \in \mathbb{R}$. Conversely, every such C is indeed a solution. One easily sees $Z(A) \cong U(1)$.

In the following we interpret a subset X of the three-dimensional ball B^3 also as a subset $X := \{A = a_0 + \vec{a} \in SU(2) \mid \vec{a} \in X\}$ of $SU(2)$.

Lemma C.1 For $A = a_0 + \vec{a} \in SU(2)$ we have $Z(A) = \begin{cases} B^3 & \text{for } a_0^2 = 1 \\ \mathbb{R}\vec{a} \cap B^3 & \text{for } a_0^2 \neq 1 \end{cases}$.

Since every $SU(2)$-matrix can be diagonalized, there is a diagonal matrix in every orbit

Proposition C.2 Every orbit $A \circ SU(2)$ w.r.t. the adjoint action of $SU(2)$ on itself contains a point of the form $a_0 + \sqrt{1 - a_0^2}\,\mathrm{i}$. We have $SU(2)/\mathrm{Ad} \cong [-1, 1]$ by $[A] \longmapsto \frac{1}{2}\mathrm{tr}\ A$.
The orbits are the small spheres with constant real part a_0.

C.2 Adjoint Action on $SU(2)^2$

A crucial point for the investigation of gauge orbit types has been the finiteness lemma for centralizers [8], i.e. every centralizer can be represented as the centralizer of finitely many elements. When we have dealt with the generic stratum we have seen that it is important to generate this way the center of the structure group. How many elements do we need at least for that procedure?

Lemma C.3 We have $Z = \{\pm\mathbf{1}\} \equiv \mathbb{Z}_2$.

Proof $Z = \bigcap_{A\in SU(2)} Z(A) = B^3 \cap \bigcap_{\vec{a}\in B^3} \mathbb{R}\vec{a} = \{\vec{0}\} \equiv \{\pm\mathbf{1}\}$ by Lemma C.1. **qed**

Obviously no single element of $SU(2)$ generates the whole center, but already two elements are sufficient. We only have to guarantee that their centralizers have trivial intersection.

Proposition C.4 There are three orbit types on $SU(2) \times SU(2)$.
Explicitly we have for $A, B \in SU(2)$:

1. Type $SU(2)$
   $Z(A,B) = B^3$ iff $Z(A) = Z(B) = B^3$.
2. Type $U(1)$
   $Z(A,B) = \mathbb{R}\vec{c} \cap B^3$ iff
   a) one centralizer equals B^3 and one equals $\mathbb{R}\vec{c} \cap B^3$ or
   b) two centralizers equal $\mathbb{R}\vec{c} \cap B^3$.
3. Type $\mathbb{Z}_2$ (generic elements)
   $Z(A,B) = \{\vec{0}\}$ iff $Z(A) = \mathbb{R}\vec{a} \cap B^3$ and $Z(B) = \mathbb{R}\vec{b} \cap B^3$ where $\vec{a}$ and $\vec{b}$ are non-collinear.

The dimensions of the respective strata are:

- Type $SU(2)$: 0;
- Type $U(1)$: 3 (case 2.a)) or 4 (case 2.b));
- Type $\mathbb{Z}_2$: 6.

Clearly, $(SU(2)^2)_{\text{gen}}$ is open and dense in $SU(2)^2$. The orbits in the generic stratum are isomorphic to $SU(2)/\mathbb{Z}_2$, i.e. are three-dimensional. Hence the quotient space $(SU(2)^2)_{\text{gen}}/\text{Ad}$ is three-dimensional.
Next, we shall find a continuous section in the generic stratum of $SU(2)^2$. What could be a "natural" element describing an orbit $[(A,B)]$? Let there be given $(A,B) \in (SU(2)^2)_{\text{gen}}$. First we are free to use a matrix C for diagonalizing the first component A. We get (C^*AC, C^*BC). It remains the freedom to act with a second matrix Δ_β, however, keeping C^*AC invariant. Hence, Δ_β has to be a diagonal matrix. On the other hand, Δ_β has to transform the matrix C^*BC. Otherwise, C^*BC would be a diagonal matrix in contradiction to $Z(A) \cap Z(B) = \{\vec{0}\}$. Hence, by an appropriate choice of Δ_β we can make the secondary diagonal of $(C\Delta_\beta)^*B(C\Delta_\beta)$ real. Explicitly we get:

Proposition C.5 In every generic orbit there is a unique element of the form

$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}, \begin{pmatrix} x & \sqrt{1-|x|^2} \\ -\sqrt{1-|x|^2} & x^* \end{pmatrix} \right)$$

where $|\lambda| = 1$, $\text{Im}\,\lambda > 0$ and $|x| < 1$. We call such an element standard form of the orbit (or its elements).
Conversely, every such element defines a unique generic orbit.
Furthermore the mapping $\pi_F : (SU(2)^2)_{\text{gen}} \longrightarrow (SU(2)^2)_{\text{gen}}$ assigning to every element its standard form is continuous.
Explicitly, we have for the standard form of $(A,B) = (a_0 + \vec{a}, b_0 + \vec{b})$:

$$\begin{aligned} \lambda &= a_0 + \parallel \vec{a} \parallel \mathrm{i} \\ x &= b_0 + \frac{\langle \vec{a}, \vec{b} \rangle}{\parallel \vec{a} \parallel}\mathrm{i}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is again the canonical scalar product on $\mathbb{R}^3$.

On the level of quaternions the element above can be written as $(\lambda, x + \sqrt{1-|x|^2}\,\mathrm{j})$.

Proof • Existence
Let $A = a_0 + \vec{a} = a_0 + a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k} \neq \pm\mathbf{1}$. Define[28]

[28] δ can be chosen arbitrarily with norm 1 if $a_2 = a_3 = 0$.

$$\varepsilon := \frac{a_1}{\| \vec{a} \|} \quad \text{and} \quad \delta := \frac{a_2 + a_3\mathrm{i}}{\sqrt{a_2^2 + a_3^2}}$$

as well as

$$C := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i}\sqrt{1+\varepsilon} & \delta\sqrt{1-\varepsilon} \\ -\delta^*\sqrt{1-\varepsilon} & -\mathrm{i}\sqrt{1+\varepsilon} \end{pmatrix}.$$

One easily checks $C^*(A,B)C = \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}, \widetilde{B} \right)$ with an appropriate matrix $\widetilde{B}$, where $|\lambda| = 1$ and $\mathrm{Im}\ \lambda > 0$.[29]

Afterwards we choose a $\beta \in \mathbb{C}$ with

$$\beta^2 = \frac{\widetilde{b}_2 + \widetilde{b}_3\mathrm{i}}{\sqrt{\widetilde{b}_2^2 + \widetilde{b}_3^2}}.$$

Here let $\widetilde{b}_i$ be as usual the quaternionic components of $\widetilde{B}$. Note that the denominator above is always nonzero because $\widetilde{B}$ cannot be a diagonal matrix.

Now, on the one hand, the diagonal matrix $\Delta_\beta := \begin{pmatrix} \beta & 0 \\ 0 & \beta^* \end{pmatrix}$ commutes with C^*AC and, on the other hand, the secondary diagonal in $\Delta_\beta^* \widetilde{B} \Delta_\beta$ is real and positive in the upper right corner, thus, in particular, nonzero. Hence $(C\Delta_\beta)^*(A,B)(C\Delta_\beta)$ is of the desired type.

Furthermore, one checks that λ and x depend indeed in the given manner on A and B.[30]

- Uniqueness[31]

  Suppose, there were two such elements $(A_i, B_i) = (\lambda_i, x_i + \sqrt{1 - |x_i|^2}\mathrm{j})$ fulfilling the conditions above. Then there is a $C \in SU(2)$ with $C^*(A_1,B_1)C = (A_2,B_2)$.

  - $C^*A_1C = A_2$

    Since conjugate elements have always the same real part, we have $\mathrm{Re}\ \lambda_1 = \mathrm{Re}\ \lambda_2$. Hence, $\lambda_1 = \lambda_2$ oder $\lambda_1 = \lambda_2^*$. Since $\mathrm{Im}\ \lambda_i > 0$ by assumption, we get $\lambda_1 = \lambda_2$, thus $A_1 = A_2$.

    Moreover, that is why C is in $Z(A_1)$, hence $C = \mu$, $\mu \in \mathbb{C}$ with $|\mu| = 1$.

  - $C^*B_1C = B_2$

    We have

$$\begin{aligned} x_2 + \sqrt{1 - |x_2|^2}\mathrm{j} &= B_2 \\ &= C^*B_1C \\ &= \mu^*(x_1 + \sqrt{1 - |x_1|^2}\mathrm{j})\mu \\ &= x_1 + \sqrt{1 - |x_1|^2}\mu^*\mu^*\mathrm{j}, \end{aligned}$$

    thus first $x_1 = x_2$. Therefore the expressions containing the roots are equal, and due to $|x_i| < 1$ we have $(\mu^*)^2 = 1$, i.e. $\mu = \pm 1$.

  Thus, $C = \pm\mathbf{1} \in Z$ and hence $(A_1,B_1) = (A_2,B_2)$.

---

[29] Note that $\mathrm{Im}\ \lambda = 0$ is impossible because otherwise $\lambda = \pm 1$, i.e. $A = \pm\mathbf{1}$.

[30] More general, for the action of $C\Delta_\beta$ on a matrix $M = m_0 + \vec{m}$ we have:

$$(C\Delta_\beta)^* M (C\Delta_\beta) = m_0 + \frac{\langle \vec{a}, \vec{m} \rangle}{\| \vec{a} \|}\ \mathrm{i} + \frac{\langle \vec{a} \times \vec{b}, \vec{a} \times \vec{m} \rangle}{\| \vec{a} \times \vec{b} \| \| \vec{a} \|}\ \mathrm{j} + \frac{\langle \vec{a} \times \vec{b}, \vec{m} \rangle}{\| \vec{a} \times \vec{b} \|}\ \mathrm{k}.$$

[31] Here, all expressions are quaternionic.

- Since we assumed $|x| < 1$, every element in the proposition above defines an orbit in the generic stratum.
- The continuity of π_F is clear because of $\parallel \vec{a} \parallel \neq 0$ in the generic stratum.[32]

qed

We denote the space of all standard forms by F. Then F is homeomorphic to the product of the upper open semicircle of $U(1)$ (λ-part) and the upper open hemisphere of S^2 (x-part), hence is homeomorphic to $\mathbb{R}^3$.

Proposition C.6 We have $F \cong (SU(2)^2)_{\text{gen}}/\text{Ad}$.

Proof Let $f : F \longrightarrow (SU(2)^2)_{\text{gen}}/\text{Ad}$ be the concatenation of the embedding ι of F into $(SU(2)^2)_{\text{gen}}$ and the canonical projection π_2:

$$\begin{tikzcd} F \arrow[r, hook, shift left, "\iota"] \arrow[d, "f"'] & (SU(2)^2)_{\text{gen}} \arrow[l, shift left, "\pi_F"] \arrow[dl, "\pi_2"] \\ (SU(2)^2)_{\text{gen}}/\text{Ad} & \end{tikzcd}$$

- f is bijective by Proposition C.5.
- f is continuous as a concatenation of continuous maps.
- f^{-1} is continuous, because at least locally (around every point) there is a continuous section s_2 for π_2 such that locally $f^{-1} = \pi_F \circ s_2$ which implies the local continuity. However, then f^{-1} is globally continuous. **qed**

Consequently, the generic stratum of $SU(2)^2$ is homeomorphic to $\mathbb{R}^3 \times SO(3)$ because $SU(2)/\mathbb{Z}_2 \cong SO(3)$.
We remark that the total orbit space $SU(2)^2/\text{Ad}$ is homeomorphic to the three-ball B^3 where the singular strata are simply its boundary S^2.

C.3 Adjoint Action on $SU(2)^3$

We will show here that the adjoint action on $SU(2)^3$ leads to a nontrivial generic stratum. The argument here is again a pure homotopy argument as in the proof of the more general Proposition 10.2. But, here we will explicitly describe the strata on $SU(2)^3$ and show that the non-generic strata have codimension 4. (By the way, one easily recognizes that the codimension of the non-generic strata on $SU(2)^k$ equals $2(k-1)$.)

Proposition C.7 On $SU(2) \times SU(2) \times SU(2)$ there are three orbit types.
Explicitly we have for $A, B, C \in SU(2)$:

1. Type $SU(2)$
   (Dimension 0) $Z(A,B,C) = B^3$ iff $Z(A) = Z(B) = Z(C) = B^3$;
2. Type $U(1)$
   $Z(A,B,C) = \mathbb{R}\vec{d} \cap B^3$ iff

[32]But, note that the map $A \longrightarrow C$ itself is *not* continuous. This can be seen in the special case that A goes to a diagonal matrix, i.e. for $a_2, a_3 \to 0$ and (here) $a_1 < 0$. Then namely $\varepsilon \to -1$, i.e., C goes as $\frac{a_2 + a_3 \mathrm{i}}{\sqrt{a_2^2 + a_3^2}} \mathrm{j}$, hence is divergent.

a) (Dimension 3) two centralizers equal B^3 and one equals $\mathbb{R}\vec{d} \cap B^3$ or
b) (Dimension 4) one centralizer equals B^3 and two equal $\mathbb{R}\vec{d} \cap B^3$ or
c) (Dimension 5) three centralizers equal $\mathbb{R}\vec{d} \cap B^3$.

3. Type $\mathbb{Z}_2$ (generic elements)
$Z(A, B, C) = \{0\}$ iff
a) (Dimension 6) one centralizer equals B^3 and the remaining two are different and not equal B^3
b) (Dimension 7) two centralizers equal $\mathbb{R}\vec{d} \cap B^3$ and one equals $\mathbb{R}\vec{a} \cap B^3$, but not all are equal, or
c) (Dimension 9) all centralizers are different and not equal B^3.

Now we assume that $(SU(2)^3)_{\text{gen}}$ were trivial. Then
$$(SU(2)^3)_{\text{gen}} \cong (SU(2)^3)_{\text{gen}}/\text{Ad} \times {\mathbb{Z}_2}\diagdown SU(2),$$
hence
$$\pi_1^{\text{homotopy}}((SU(2)^3)_{\text{gen}}) \cong \pi_1^{\text{homotopy}}((SU(2)^3)_{\text{gen}}/\text{Ad}) \times \pi_1^{\text{homotopy}}({\mathbb{Z}_2}\diagdown SU(2)).$$
As we have just seen the codimension of $SU(2)^3 \setminus (SU(2)^3)_{\text{gen}}$ equals 4, i.e., we have $\pi_1^{\text{homotopy}}((SU(2)^3)_{\text{gen}}) = \pi_1^{\text{homotopy}}(SU(2)^3) = \pi_1^{\text{homotopy}}(SU(2))^3 = 1$. On the other hand, ${\mathbb{Z}_2}\diagdown SU(2) = SO(3)$ and $\pi_1^{\text{homotopy}}(SO(3)) = \mathbb{Z}_2$. Hence,
$$1 \cong \pi_1^{\text{homotopy}}((SU(2)^3)_{\text{gen}}/\text{Ad}) \oplus \mathbb{Z}_2,$$
which obviously is a contradiction. Hence, $(SU(2)^3)_{\text{gen}}$ is nontrivial.

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