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Vortex filament dynamics for  
Gross-Pitaevsky type equations

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# VORTEX FILAMENT DYNAMICS FOR GROSS-PITAEVSKY TYPE EQUATIONS

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ABSTRACT. We study solutions of the Gross-Pitaevsky equation and similar equations in  $m \geq 3$  space dimensions in a certain scaling limit, with initial data  $u_0^\epsilon$  for which the Jacobian  $Ju_0^\epsilon$  concentrates around an (oriented) rectifiable  $m - 2$  dimensional set, say  $\Gamma_0$ , of finite measure. It is widely conjectured that under these conditions, the Jacobian at later times  $t > 0$  continues to concentrate around some codimension 2 submanifold, say  $\Gamma_t$ , and that the family  $\{\Gamma_t\}$  of submanifolds evolves by binormal mean curvature flow. We prove this conjecture when  $\Gamma_0$  is a round  $m - 2$ -dimensional sphere with multiplicity 1. We also prove a number of partial results for more general initial data.

## 1. INTRODUCTION

In this paper we prove some results about the singular limits of solutions  $u^\epsilon : \mathbb{R}^m \times [0, \infty) \rightarrow \mathbb{C} \cong \mathbb{R}^2$ ,  $m \geq 3$  of the Gross-Pitaevsky equation, a nonlinear Schrödinger equation used in the physics literature as a model for the evolution of the wave function associated with a Bose condensate. The equation can be written

$$(1.1) \quad (k_\epsilon)^{-1} i u_t^\epsilon - \Delta u^\epsilon + \frac{1}{\epsilon^2} W'(|u^\epsilon|^2) u^\epsilon = 0, \quad u^\epsilon(\cdot, 0) = u_0^\epsilon$$

where  $k_\epsilon := |\ln \epsilon|^{-1}$  is a scaling factor. The model example for the nonlinearity is  $W(s) = \frac{1}{2}(s - 1)^2$  in 2 or 3 space dimensions. More generally, we consider qualitatively similar nonlinearities satisfying appropriate growth conditions that depend on the dimension  $m$ .

We study solutions with initial data  $u_0^\epsilon$  for which the Jacobian  $Ju_0^\epsilon$  concentrates as  $\epsilon \rightarrow 0$  around an (oriented) rectifiable  $m - 2$  dimensional set, say  $\Gamma_0$ , of finite  $\mathcal{H}^{m-2}$  measure. On the basis of physical arguments and formal asymptotics (see for example [8], [19], [7], [18]) it is conjectured that the Jacobian at later times  $t > 0$  continues to concentrate around some codimension 2 submanifold, say  $\Gamma_t$ , and that the family  $\{\Gamma_t\}$  of submanifolds evolves by binormal mean curvature flow, a geometric evolution problem that we will describe below.

In physical terms, this conjecture states that in  $\mathbb{R}^3$  for example, quantized vortex filaments in a Bose condensate in the incompressible limit evolve by exactly the law of motion that governs vortex filaments in an ideal incompressible fluid in the self-induction approximation.

In this paper we prove this conjecture when the initial vortex filament is a  $m - 2$ -dimensional round sphere with multiplicity 1. We also prove that for quite general initial data there exist *some* limiting measures  $\{\bar{J}_t\}_{t \in \mathbb{R}}$  which are carried by  $m - 2$  dimensional rectifiable sets  $\{\Gamma_t\}$ , and around which the Jacobian concentrates. We show that, for the time scaling chosen in (1.1), these measures evolve continuously in

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certain weak topologies, and that their evolution is nontrivial. Finally, we introduce a notion of a weak solution of the problem of binormal mean curvature flow, and we identify some conditions that would imply that  $\{\bar{J}_t\}_{t \in \mathbb{R}}$  is a weak solution. These conditions hinge on a careful analysis of the relationship between weak limits of the Jacobian  $Ju^\epsilon$  and weak limits of quadratic terms  $k_\epsilon u_{x_i}^\epsilon \cdot u_{x_j}^\epsilon$  for  $i, j = 1, \dots, m$  as  $\epsilon \rightarrow 0$  under appropriate bounds on the total energy.

We briefly sketch the contents of this paper.

Section 2 contains some background material. We show in Section 3 that a family of measure  $\{J_t\}_{t \in \mathbb{R}}$  can be thought of corresponding to a weak binormal mean curvature flow in a very natural sense if

$$(1.2) \quad \frac{d}{dt} \int \phi \cdot J_t(dx) = \int (\phi^{ij} - \phi^{ji})_{x_i x_k} P_{jk}^\perp |J_t|(dx)$$

for all smooth, compactly supported  $\phi = \sum_{i < j} \phi^{ij} e_i \wedge e_j \in C_c^2(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m)$  and a.e.  $t$ . Here  $P^\perp(x)$  is the  $m \times m$  matrix representing projection onto the two-dimensional approximate normal space of  $J_t$ , and  $|J_t|$  is the total variation measure associated with  $J_t$ . We require that each  $J_t$  have a certain nice geometric structure, so that it can be thought of as representing a weak  $m - 2$ -dimensional oriented surface (more precisely an integer multiplicity rectifiable set). This in particular implies that  $P^\perp(x)$  exists almost everywhere.

There is a striking formal similarity between (1.2) and the identity

$$(1.3) \quad \frac{d}{dt} \int \phi \cdot Ju^\epsilon dx = \int (\phi^{ij} - \phi^{ji})_{x_i x_k} k_\epsilon u_{x_j}^\epsilon(t) \cdot u_{x_k}^\epsilon(t) dx$$

which is satisfied by solutions  $u^\epsilon$  of (1.1). The main point of this paper is to identify conditions when one can pass to limits from (1.3) to deduce (1.2).

To do this two things are necessary. The first is to show that one can find a subsequence  $\epsilon_n$  such that  $Ju^{\epsilon_n}(t) \rightarrow \bar{J}_t$  for all  $t$ , where  $\bar{J}_t$  is some measure having the nice geometric structure mentioned above. We carry this out in Section 4. In this we rely heavily on results of the author and H.M. Soner [12], see also [1], which show that if  $\{v^\epsilon\}$  is a family of functions for which an appropriately scaled Ginzburg-Landau energy  $I^\epsilon(v^\epsilon)$  is uniformly bounded, then the Jacobians  $Jv^\epsilon$  converge (after passing to a subsequence) to some measure  $\bar{J}$  having precisely the desired structure. These energy bounds in particular are satisfied by functions  $\{u^\epsilon(t)\}$  obtained by solving (1.1) for appropriate initial data.

Having found  $\bar{J}_t$ , we then prove that

$$(1.4) \quad k_{\epsilon_n} u_{x_j}^{\epsilon_n}(t) \cdot u_{x_k}^{\epsilon_n}(t) dx \rightharpoonup P_{jk}^\perp |\bar{J}_t| \quad \text{weak-}^*, \forall i, j$$

whenever, roughly speaking, energy concentration around  $\bar{J}_t$  is as small as possible. In fact we prove a more precise result that gives quantitative control over the extent to which (1.4) can fail when the small energy concentration condition fails to hold. This is done in Sections 6 (in two space dimensions) and 7 (for dimensions  $m \geq 3$ ).

In Section 5 we apply these estimates to solutions  $u^\epsilon$  of (1.1). The small energy concentration condition is implied by the condition that  $\bar{J}_t$  be as large as possible, given the available energy. The total mass of  $\bar{J}_t$  roughly corresponds to the  $m - 2$ -dimensional Hausdorff measure (counting multiplicity) of the weak surface represented by  $\bar{J}_t$ . This quantity is hard to control directly, but from conservation laws for (1.1) one can quite easily control the  $m - 1$ -dimensional measure of the

area enclosed by a projection of  $\bar{J}_t$  onto any hyperplane. By the isoperimetric inequality this gives lower bounds for the mass of  $\bar{J}_t$ , and we show that these bounds are sharp precisely when the initial singular set is optimal for the isoperimetric inequality, that is, a round sphere of multiplicity one. Thus we can give a complete analysis in this case.

We conclude this introduction by mentioning some related work. The only rigorous work that we know of on this problem is a recent paper by T.C. Lin, [17] that derives the law of motion for vortex filaments in solutions of (1.1) in three space dimensions by linearizing about an approximate solution and using earlier estimates (see for example [16]) on the spectrum of the linearized operator. This result assumes the existence of a smooth solution of the limiting binormal curvature flow, and the analysis does not provide any uniform bounds on the time interval on which it is valid, so that it does not exclude the possibility that its conclusions hold only on a time interval  $[0, t_\epsilon)$  where  $t_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

In two space dimensions, the corresponding problem is to study the dynamics of point vortices in solutions of (1.1) in the singular limit (with a different time scaling). The first rigorous analysis of this problem was given in [5], see also [6]. Some refinements were subsequently established in [15]. These results show that, in a variety of situations, vortex motion in the singular limit is governed by exactly the ODE that describes the motion of classical point vortices in an ideal fluid.

Another related question is the limiting behavior of vortex filaments in solutions of the Ginzburg-Landau heat equation in  $m \geq 3$  space dimensions. F. H. Lin [14] and the author and H.M. Soner [10] independently proved that under appropriate assumptions, the limiting singular set evolves via codimension 2 mean curvature flow, at least as long as the limiting flow remains smooth. More recently Ambrosio and Soner [3] prove that energy measures associated with solutions of the Ginzburg Landau heat equation converge globally in time to a measure that evolves by mean curvature flow in a certain weak sense, if one is allowed to assume that the limiting energy measure satisfies a certain lower density estimate.

Finally, recent work by the author and H.M. Soner [13], [11] investigates the class of functions whose distributional Jacobian exists and is a Radon measure. We show, among other things, that if a function  $u \in W^{1,1}(\mathbb{R}^m; S^1)$  satisfies this condition, then its distributional Jacobian — that is, the collection of all distributional determinants of  $2 \times 2$  submatrices of  $Du$  — is an integer multiplicity measure carried by an oriented rectifiable set of dimension  $m - 2$ . These are related to the compactness results of [12], [1], that characterize limits of Jacobians  $Ju^\epsilon$  for sequences of functions  $u^\epsilon$  that are asymptotically  $S^1$ -valued in a certain precise sense.

## 2. BACKGROUND

**2.1. notation.** We first introduce some notation that we will use throughout this paper.

We use the convention that repeated indices are summed, though we also sometimes explicitly write out summations.

For  $w, v \in \mathbb{C} \cong \mathbb{R}^2$  we write  $v \cdot w$  to denote the *real* inner product:  $v \cdot w = \frac{1}{2}(v\bar{w} + w\bar{v}) = v_i w_i$ . We write  $\det(v, w)$  to denote the determinant of the real  $2 \times 2$  matrix whose columns are  $v$  and  $w$  respectively. Note that  $\det(v, w) = iv \cdot w$ .

We write  $\mathcal{S}^{k \times k}$  to denote the collection of real symmetric  $k \times k$  matrices. If  $M \in \mathcal{S}^{k \times k}$  we write  $M \geq 0$  to mean that  $M$  is nonnegative definite. If  $M = (M_{ij})$

and  $N = (N_{ij})$  are real  $m \times m$  matrices, we write  $M : N$  to denote the inner product  $M : N = M_{ij}N_{ij}$ .

For  $a, b \in \mathbb{R}^k$  we write  $a \otimes b$  to denote the  $k \times k$  matrix whose  $(i, j)$  entry is  $a_i b_j$ . For  $u \in H^1(\mathbb{R}^m; \mathbb{R}^2)$  we write  $Du \otimes Du$  for the  $m \times m$  matrix whose  $(i, j)$  entry is  $u_{x_i} \cdot u_{x_j}$ .

We always work in  $m$  space dimensions,  $m \geq 3$ . When we write  $\mathbb{R}^{m+1}$  we always mean  $\mathbb{R}_x^m \times \mathbb{R}_t$ .

The nonlinearity  $W$  in (1.1) is assumed to be a smooth nonnegative function such that  $W(1) = 0$ ,  $W(s) > 0$  for all  $s \neq 1$ ,  $W''(1) > 0$ , and  $W(s) \leq C(1 + s^\alpha)$  for some  $\alpha < m/m - 2$ . The last condition brings (1.1) within the scope of standard well-posedness theory. The assumptions imply that

$$(2.1) \quad C^{-1}(1-s)^2 \leq W(s) \leq C(1-s)^2 \quad \text{when } 0 \leq s \leq 2.$$

We write  $\mathcal{H}^k$  to denote  $k$ -dimensional Hausdorff measure.

We write  $\{e_i\}_{i=1}^m$  to denote a standard orthonormal basis for the space  $\Lambda_1 \mathbb{R}^m$  of vectors on  $\mathbb{R}^m$ . When considering vectors on  $\mathbb{R}^{m+1}$  we will write the standard basis vector in the  $\mathbb{R}_t$  direction as either  $e_{m+1}$  or  $e_t$ .

Similarly,  $\{e_\alpha\}_{\alpha \in I_{k,m}}$  is an orthonormal basis for  $\Lambda_k \mathbb{R}^m$ , the space of  $k$ -vectors on  $\mathbb{R}^m$ . Here  $I_{k,m}$  is the set of all multiindices of the form  $\alpha = (\alpha_1, \dots, \alpha_k)$  such that  $1 \leq \alpha_1 < \dots < \alpha_k \leq m$ . For such a multiindex,  $e_\alpha := e_{\alpha_1} \wedge \dots \wedge e_{\alpha_k}$ . If  $1 \leq \alpha_1, \dots, \alpha_k \leq m$  are distinct integers not necessarily arranged in increasing order, and  $\pi$  is a permutation on  $k$  elements, then

$$e_{\alpha_1} \wedge \dots \wedge e_{\alpha_k} = \text{sgn}(\pi) e_{\pi(\alpha_1)} \wedge \dots \wedge e_{\pi(\alpha_k)},$$

Here  $\text{sgn}(\pi)$  is the sign of the permutation  $\pi$ .

The space  $\Lambda_k \mathbb{R}^m$  is a real vector space of dimension  $\binom{m}{k}$  and as such is endowed with the Euclidean inner product, which we will write as  $v \cdot w$ . For  $v \in \Lambda_k \mathbb{R}^m$ , we write  $|v|$  to mean the standard Euclidean norm  $(v \cdot v)^{1/2}$ . A *unit* multivector is a multivector with norm  $|v| = 1$ .

We disregard conventions of geometric measure theory and do not distinguish between vectors and covectors; rather we identify  $\Lambda_k \mathbb{R}^m$  with its dual via the inner product. This occasionally leads to unorthodox language but it simplifies our exposition in many ways.

We always write 2-vectorfields in the form  $\phi = \sum_{i < j} \phi^{ij} e_i \wedge e_j$ , and we set  $\phi^{ji} = 0$  whenever  $j \geq i$ .

We say that  $v = \sum_{\alpha \in I_{k,m}} v_\alpha e_\alpha$  is *simple* if there are  $k$  vectors  $v_1, \dots, v_k \in \Lambda_1 \mathbb{R}^m$  such that  $v = v_1 \wedge \dots \wedge v_k$ . If  $P$  is a  $k$ -dimensional subspace of  $\mathbb{R}^m$  spanned by  $\{v_1, \dots, v_k\}$  and  $v \in \Lambda_k \mathbb{R}^m$  is a unit multivector of the form  $v_1 \wedge \dots \wedge v_k$ , then we say that  $v$  *orients*  $P$ .

The Hodge star operator  $\star : \Lambda_k \mathbb{R}^m \rightarrow \Lambda_{m-k} \mathbb{R}^m$  is defined by

$$\star e_\alpha = \text{sgn}(\alpha\beta) e_\beta$$

for the unique  $\beta \in I_{m-k,m}$  such that  $(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{m-k})$  is a permutation of  $(1, \dots, m)$ . Here  $\text{sgn}(\alpha\beta)$  is the sign of the permutation. It is easy to check that

$$e_\alpha \wedge \star e_\alpha = e_1 \wedge \dots \wedge e_m, \quad \star \star e_\alpha = (-1)^{k(m-k)} e_\alpha.$$

Note that if  $M$  is a smooth, oriented codimension  $k$  manifold of  $\mathbb{R}^m$  and  $\nu \in \Lambda_k \mathbb{R}^m$  orients  $(T_x M)^\perp$  at some point  $x \in M$ , then  $\star \nu := \xi$  orients  $T_x M$ .

We write  $d$  to denote the exterior derivative in  $\mathbb{R}^m$ . Thus for  $\phi$  of the form  $\phi = \sum_{\alpha \in I_{k,m}} \phi^\alpha e_\alpha$ , we define  $d\phi = \sum_{i,\alpha} \phi_{x_i}^\alpha e_i \wedge e_\alpha$ ; We similarly write  $\mathbf{d}$  to denote the exterior derivative in  $\mathbb{R}^{m+1} \cong \mathbb{R}_x^m \times \mathbb{R}^t$ .

It is convenient to use the duals of Hölder spaces to quantify weak convergence and continuity properties of measures. For this we need to fix some notation. We write

$$\|\phi\|_{\hat{C}^{k,\alpha}} = [D^k \phi]_\alpha \quad \|\phi\|_{C^{k,\alpha}} = \sum_{j=0}^k \|D^j \phi\|_\infty + [D^k \phi]_\alpha.$$

Here  $|D^j \phi|^2 = \sum_{|\gamma|=j} |D^\gamma \phi|^2$  and  $[D^k \phi]_\alpha = \sup_{x \neq y} \frac{|D^k \phi(x) - D^k \phi(y)|}{|x-y|^\alpha}$ . Thus a “hat”, as in  $\hat{C}^{k,\alpha}$ , indicates that we use only the highest-order part of the norm. We use the convention that  $C^0 = \hat{C}^0$ .

We write  $\|\cdot\|_{C^{k,\alpha}*}$  and  $\|\cdot\|_{\hat{C}^{k,\alpha}*}$  to indicate the respective dual norms. So for example, if  $\mu$  is a measure and  $U \subset \mathbb{R}^m$  then

$$\|\mu\|_{\hat{C}_c^{k,\alpha}*}(U) = \sup \left\{ \int \phi \, d\mu : \|\phi\|_{\hat{C}^{k,\alpha}} \leq 1, \phi \text{ has compact support in } U \right\}.$$

We will often write, for example,  $\|J\|_{C_c^{k,\alpha}*}$  rather than  $\|J\|_{C_c^{k,\alpha}*}(\mathbb{R}^m; \Lambda_k \mathbb{R}^m)$  when no confusion can result. Note that  $\|\mu\|_{C^{k,\alpha}*} \leq \|\mu\|_{\hat{C}^{k,\alpha}*}$  for all  $\mu, k, \alpha$ .

Finally, we say that  $\mu_n \rightarrow \mu$  in  $\|\mu\|_{\hat{C}_{loc}^{k,\alpha}*}$  if  $\|\mu_n - \mu\|_{C^{k,\alpha}*}(U) \rightarrow 0$  for every  $U \subset\subset \mathbb{R}^m$ .

**2.2. conserved quantities.** Equation (1.1) has a number of conserved quantities. We define the energy

$$(2.2) \quad E^\epsilon(u^\epsilon) = \frac{1}{2} |Du^\epsilon|^2 + \frac{1}{2\epsilon^2} W(|u^\epsilon|^2)$$

and the linear momentum;

$$(2.3) \quad j(u^\epsilon) = j^k(u^\epsilon) e_k, \quad j^k(u^\epsilon) = iu^\epsilon \cdot u_{x_k}^\epsilon = \det(u^\epsilon, u_{x_k}^\epsilon).$$

We further define the Jacobian

$$(2.4) \quad Ju^\epsilon = \frac{1}{2} d j(u^\epsilon) = \sum_{k < l} J^{kl} u^\epsilon e_k \wedge e_l,$$

where  $J^{kl} u^\epsilon = \frac{1}{2} (j^l(u^\epsilon)_{x_k} - j^k(u^\epsilon)_{x_l}) = \det(u_{x_k}^\epsilon, u_{x_l}^\epsilon)$ .

Note that our Jacobian does not quite agree with the standard Jacobian of geometric measure theory, that is, the factor appearing in the coarea formula. We will refer to the latter as “Federer’s Jacobian”. For functions  $u^\epsilon$  as above, Federer’s Jacobian in our notation is given by  $|Ju^\epsilon|$ , that is, the Euclidean norm of  $Ju^\epsilon$ .

We always assume that the initial data  $u_0^\epsilon$  satisfies  $(u_0^\epsilon - 1) \in H^1(\mathbb{R}^m)$  and has finite energy, ie  $\int_{\mathbb{R}^m} E^\epsilon(u_0^\epsilon) < \infty$ . Under these hypotheses (1.1) is known to have a unique global solution satisfying

$$u^\epsilon(t) - 1 \in H^1(\mathbb{R}^m), \quad \int E^\epsilon(u^\epsilon(t)) \, dx = \int E^\epsilon(u_0^\epsilon) \, dx \quad \text{for all } t.$$

This can be deduced quite easily from standard facts about NLS; for a discussion see Bethuel and Saut [4] Appendix A.

Smooth solutions of (1.1) satisfy

$$(2.5) \quad \frac{d}{dt} j(u^\epsilon) = k_\epsilon \left( 2(u_{x_j}^\epsilon \cdot u_{x_k}^\epsilon)_{x_j} - [2E^\epsilon(u^\epsilon) - iu^\epsilon \cdot u_t]_{x_k} \right) e_k.$$

By taking the exterior derivative of (2.5), we obtain an equation for the evolution of the vorticity.

$$(2.6) \quad \begin{aligned} \frac{d}{dt} J u^\epsilon &= k_\epsilon \sum_{j,k,l} (u_{x_j}^\epsilon \cdot u_{x_l}^\epsilon)_{x_j x_k} e_k \wedge e_l \\ &= k_\epsilon \sum_j \sum_{k < l} \left( (u_{x_j}^\epsilon \cdot u_{x_l}^\epsilon)_{x_j x_k} - (u_{x_j}^\epsilon \cdot u_{x_k}^\epsilon)_{x_j x_l} \right) e_k \wedge e_l. \end{aligned}$$

These identities remain valid in the sense of distributions if the initial data merely satisfies  $u^\epsilon - 1 \in H^1(\mathbb{R}^m)$ . This can be shown by regularizing the initial data to obtain smooth solutions, then passing to limits using standard NLS estimates.

### 2.3. Geometric background.

**2.3.1. rectifiability.** A set  $\Gamma \subset \mathbb{R}^m$  is said to be  $k$ -dimensional rectifiable, for integer  $k < m$ , if  $\Gamma$  can be written in the form  $\Gamma = \cup_{j=0}^\infty \Gamma_j$  where  $\mathcal{H}^k(\Gamma_0) = 0$ , and for each  $j \geq 1$ ,  $\Gamma_j$  is a  $\mathcal{H}^k$ -measurable subset of the image of an injective Lipschitz map  $f_j : U_j \rightarrow \mathbb{R}^m$ , where  $U_j$  is an open subset of  $\mathbb{R}^k$ .

A set  $\Gamma$  is  $k$ -dimensional rectifiable if and only if it has an approximate  $k$ -dimensional tangent space at  $\mathcal{H}^k$  almost every point of its support. For a proof of this fact, as well as the definition of approximate tangent space and related material, consult Simon [21] Section 11 or Giaquinta *et al* [9] Section 2.1.4. We write  $\text{ap}T_x\Gamma$  to denote this approximate tangent space, which is unique.

Whenever a set  $\Gamma \subset \mathbb{R}^m$  is  $k$ -dimensional rectifiable, we can thus define for  $\mathcal{H}^k$  a.e.  $x \in \Gamma$  an  $m \times m$  matrix  $P(x)$  corresponding to projection onto the  $k$ -dimensional approximate tangent space  $\text{ap}T_x\Gamma$ . We will also write  $P^\perp(x)$  to denote projection onto the approximate orthogonal space  $(\text{ap}T_x\Gamma)^\perp$ , so that  $P^\perp(x) = id - P(x)$ .

**2.3.2. first variation and mean curvature.** Suppose that  $\Gamma$  is a  $k$ -dimensional rectifiable subset of  $\mathbb{R}^m$ , and for  $\phi \in C_c^1(\mathbb{R}^m, \mathbb{R}^m)$  define

$$\text{div}_\Gamma \phi(x) = P(x) : D\phi(x)$$

at every  $x \in \Gamma$  where  $\text{ap}T_x\Gamma$  exists. Then  $\text{div}_\Gamma \phi$  is a bounded function which is  $\mathcal{H}^k|_\Gamma$ -measurable, and so  $\int_\Gamma \text{div}_\Gamma \phi(x) \mathcal{H}^k(dx)$  makes sense.

If there exists a  $\mathcal{H}^k$ -measurable function  $\mathbf{H} : \Gamma \rightarrow \mathbb{R}^m$  such that

$$(2.7) \quad \int_\Gamma \text{div}_\Gamma \phi(x) \mathcal{H}^k(dx) = - \int_\Gamma \phi(x) \cdot \mathbf{H}(x) \mathcal{H}^k(dx) \quad \forall \phi \in C_c^1(\mathbb{R}^m; \mathbb{R}^m),$$

then we say that  $\mathbf{H}$  is the mean curvature of  $\Gamma$ . If such a vector field exists, it is uniquely determined up to sets of  $\mathcal{H}^k$ -measure zero, and it coincides with the classical mean curvature (up to null sets) if  $\Gamma$  is smooth. (See Simon [21], Section 16).

The quantities appearing in (2.7) have a natural interpretation. Suppose that, for  $t \in (-\epsilon, \epsilon)$ ,  $t \mapsto \Gamma_t$  is an evolving  $k$ -dimensional rectifiable subset of  $\mathbb{R}^m$ , and



that the velocity of  $\Gamma_t$  at  $t = 0$  is given by the restriction to  $\Gamma_0$  of some smooth vector field  $\phi$ . Then

$$(2.8) \quad \frac{d}{dt} \mathcal{H}^k(\Gamma_t)|_{t=0} = \int_{\Gamma_0} \operatorname{div}_{\Gamma_0} \phi(x) \mathcal{H}^k(dx)$$

See Simon [21] Section 9 for a precise statement and a proof.

**2.3.3. oriented i.m. rectifiable sets.** An oriented integer multiplicity  $k$ -dimensional rectifiable set is a triple  $(\Gamma, \theta, \xi)$ , where  $\Gamma \subset \mathbb{R}^m$  is a  $k$ -dimensional rectifiable set,  $\theta : \Gamma \rightarrow \mathbb{N}$  and  $\xi : \Gamma \rightarrow \Lambda_k \mathbb{R}^m$  are  $\mathcal{H}^k$ -measurable functions, and  $\xi(x)$  orients  $T_x \Gamma$  for  $\mathcal{H}^k$  a.e.  $x \in \Gamma$ . We will write *i.m.* for integer multiplicity, and we will not explicitly mention the dimension  $k$  where there is no possibility of confusion.

A  $k$ -dimensional i.m. rectifiable current on  $\mathbb{R}^m$  is a bounded linear functional on  $C_c^\infty(\mathbb{R}^m; \Lambda_k \mathbb{R}^m)$  that has the form

$$(2.9) \quad T(\phi) = \int_{\Gamma} \phi(x) \cdot \xi(x) \theta(x) \mathcal{H}^k(dx) \quad \forall \phi \in C_c^\infty(\mathbb{R}^m; \Lambda_k \mathbb{R}^m)$$

for some oriented i.m. rectifiable  $(\Gamma, \theta, \xi)$ . For such a current, we define the mass  $\mathbf{M}(T) := \int_{\Gamma} \theta \mathcal{H}^k(dx)$ . When (2.9) holds we will write  $T = \tau(\Gamma, \theta, \xi)$ . We will need one deep fact about integer multiplicity currents, Almgren's optimal isoperimetric inequality, which we will invoke in the proof of Theorem 4.

In our context we will often encounter  $\Lambda_2 \mathbb{R}^m$ -valued measures of the form

$$\int \phi \cdot J = \int_{\Gamma} \phi(x) \cdot \nu(x) \theta(x) \mathcal{H}^k(dx) \quad \forall \phi \in C_c^0(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m).$$

where  $(\Gamma, \theta, \star\nu)$  is an oriented i.m.  $(m-2)$ -dimensional rectifiable set, so that  $\nu(x)$  orients  $(\operatorname{ap}T_x \Gamma)^\perp$  almost everywhere. These arise naturally due to Theorem 1 below. We will write  $J = \pi \theta \nu \mathcal{H}^k|_{\Gamma}$  to describe measures of this form, and when we write expressions like  $\pi \theta \nu \mathcal{H}^k|_{\Gamma}$ , it is always with the understanding that  $(\Gamma, \theta, \star\nu)$  is an oriented i.m. rectifiable set. Given such a measure  $J$ , we write  $|J|$  to indicate the nonnegative scalar measure  $|J| = \pi \theta \mathcal{H}^k|_{\Gamma}$ .

**2.4. Compactness properties.** We define the scaled Ginzburg-Landau functional

$$(2.10) \quad I^\epsilon(u; U) := k_\epsilon \int_U E^\epsilon(u) dx, \quad \text{for } u \in H^1(U; \mathbb{R}^2), \quad U \subset \mathbb{R}^m.$$

We write  $I^\epsilon(u)$  as shorthand for  $I^\epsilon(u; \mathbb{R}^m)$ . As remarked earlier,  $t \mapsto I^\epsilon(u^\epsilon(t))$  is constant for a solution  $u^\epsilon$  of (1.1) with initial data  $u_0^\epsilon$  such that  $u_0^\epsilon - 1 \in H^1(\mathbb{R}^m)$ .

The Jacobian and the Ginzburg-Landau energy are intimately related. In particular, uniform bounds on  $I^\epsilon(u^\epsilon; U)$  for a collection of functions  $\{u^\epsilon\}_{\epsilon>0} \subset H^1(U; \mathbb{R}^2)$ , with  $U \subset \mathbb{R}^m, m \geq 3$  imply that the Jacobians  $\{Ju^\epsilon\}_{\epsilon \in (0,1]}$  are precompact in appropriately weak topologies. The following theorem is established by the author and H.M. Soner in [12], Theorem 5.2.

**Theorem 1.** *Suppose that  $\{u^\epsilon\}_{\epsilon \in (0,1]}$  is a family of functions in  $H^1(\mathbb{R}^m; \mathbb{R}^2)$  such that  $\limsup_{\epsilon \rightarrow 0} I^\epsilon(u^\epsilon) \leq K$ . Then  $\{Ju^\epsilon\}_{\epsilon \in (0,1]}$  is strongly precompact in  $C_{loc}^{0,\gamma^*}$  for all  $\gamma > 0$ . Moreover, if  $\bar{J} = \sum_{i<j} \bar{J}^{ij} e_i \wedge e_j$  is any weak limit of a subsequence  $Ju^{\epsilon_n}$ , then*

(i)  $\bar{J}$  has the form  $\pi \theta \nu \mathcal{H}^{m-2}|_{\Gamma}$  where  $(\Gamma, \theta, \star\nu)$  is some oriented i.m. rectifiable set.

(ii)  $|\bar{J}|(\mathbb{R}^m) \leq \liminf_{n \rightarrow \infty} I^{\epsilon_n}(u^{\epsilon_n})$

(iii)  $d\bar{J} = 0$  in the sense of distributions.

We also show in [12] that, roughly speaking, components of  $\bar{J}$  can naturally be sliced in certain directions, and moreover that slices of  $Ju^\epsilon$  converge to slices of  $\bar{J}$  along appropriate subsequences. We will need this result in Section 7, but the statement is rather technical so we defer it until there.

The compactness assertion stated in [12] is that, if  $U$  is a bounded open subset of  $\mathbb{R}^m$  and  $u^\epsilon \in H^1(U; \mathbb{R}^2)$  satisfy  $I^\epsilon(u^\epsilon; U) \leq C$ , then  $\{Ju^\epsilon\}$  is precompact in  $C_c^{0,\gamma}(U)^*$ . However this immediately implies the result asserted in Theorem 1 above.

The above theorem does not make any assertion about compactness of the functions  $\{u^\epsilon\}$ . These are in fact weakly precompact in  $L^p$  for all  $p < \infty$ , and they may fail to be precompact in any stronger sense.

### 3. WEAK FORMULATION OF BINORMAL MEAN CURVATURE FLOW

In this section we define classical binormal mean curvature flow, and then we give our definition of a weak solution, and we show that any classical solution is a weak solution.

We first introduce some notation.

In the following,  $\{\Gamma_t\}_{t \in \mathbb{R}}$  always denotes a family of  $m-2$  dimensional rectifiable subsets of  $\mathbb{R}^m$ ,  $m \geq 3$ . We write  $\Gamma = \cup_t \Gamma_t \times \{t\}$ . We assume that  $\theta : \Gamma \rightarrow \mathbb{N}$  and  $\xi : \Gamma \rightarrow \Lambda_{m-2}\mathbb{R}^m$  are  $\mathcal{H}^{m-1}$ -measurable functions such that  $(\Gamma_t, \theta(\cdot, t), \xi(\cdot, t))$  is an oriented i.m. rectifiable set for every  $t$ . To avoid dealing with boundary conditions, we make the standing assumption that for every  $t$ ,  $(\Gamma_t, \theta, \xi)$  has no boundary in the sense that

$$(3.1) \quad \int_{\Gamma_t} d\phi(x) \cdot \xi(x, t) \theta(x, t) \mathcal{H}^{m-2}(dx) = 0 \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^m; \Lambda_{m-2}\mathbb{R}^m).$$

To define a classical solution, suppose that  $\Gamma_t$  is smooth for every  $t$ , and also that  $\Gamma$  is smooth. Assume also that  $\theta \equiv 1$ . We write  $\boldsymbol{\xi} : \Gamma \rightarrow \Lambda_{m-1}\mathbb{R}^{m+1}$  to denote the unit  $m-1$ -vectorfield that orients  $T_{x,t}\Gamma$ . In (3.5) below we will use Stokes' Theorem, and we fix the relative orientations of  $\xi$  and  $\boldsymbol{\xi}$  such that the signs in (3.5) are correct. Since  $T_x\Gamma_t$  is a subspace of  $T_{x,t}\Gamma$ , we can necessarily write  $\boldsymbol{\xi}$  in the form  $\xi_{m-1} \wedge \xi$ , where  $\xi_{m-1} \in \Lambda_1\mathbb{R}^{m+1}$ ,  $|\xi_{m-1}| = 1$ , and  $\xi_{m-1} \perp T_x\Gamma_t$  in  $\mathbb{R}^{m+1}$ . We assume that  $\xi_{m-1} \cdot e_t$  never vanishes; this amounts to assuming that  $t \mapsto \Gamma_t$  is smooth. We then further assume that  $\xi_{m-1} \cdot e_t$  is always positive; this fixes the orientation of  $\boldsymbol{\xi}$ . So we can write

$$(3.2) \quad \boldsymbol{\xi} = \xi_{m-1} \wedge \xi = \frac{(e_t + V)}{(1 + |V|^2)^{1/2}} \wedge \xi$$

for some  $V : \Gamma \rightarrow \Lambda_1\mathbb{R}^m$  such that  $V(x, t) \in (\text{ap}T_x\Gamma_t)^\perp$  for all  $(x, t) \in \Gamma$ .

Note that  $V(x, t)$  is precisely the nontangential part of the velocity of  $\Gamma_t$  at a point  $x$  in its support.

**Definition 1.** A smooth family of  $m-2$ -dimensional submanifolds  $\{\Gamma_t\}_t$  oriented by multivectors  $\xi \in \Lambda_{m-2}\mathbb{R}^m$  defines a *smooth binormal mean curvature flow* if

$$(3.3) \quad \star V \wedge \xi = \mathbf{H} \quad \text{in } \Gamma,$$

where  $\mathbf{H}$  is the mean curvature vector to  $\Gamma_t$

Note that  $V$  is orthogonal to both  $\mathbf{H}$  and  $T_x\Gamma_t$ , that is, binormal; and also that  $|V| = |\mathbf{H}|$ . Hence the name binormal mean curvature flow.

We next introduce our notion of a weak solution.

**Definition 2.** A family of oriented i.m. rectifiable sets  $\{(\Gamma_t, \theta(\cdot, t), \xi(\cdot, t))\}$  defines a *weak binormal mean curvature flow* if for every  $\phi \in C_c^2(\mathbb{R}^m; \Lambda_{m-2}\mathbb{R}^m)$ , the function  $t \mapsto \int_{\Gamma_t} \phi \cdot \xi \theta \mathcal{H}^{m-2}$  is Lipschitz, and moreover

$$(3.4) \quad \frac{d}{dt} \int_{\Gamma_t} \phi \cdot \xi \theta \mathcal{H}^{m-2} = \int_{\Gamma_t} \operatorname{div}_{\Gamma_t}(\star d\phi) \theta \mathcal{H}^{m-2}$$

for a.e.  $t$ . When this holds we write that  $(\{\Gamma_t\}, \theta, \xi)$  is a weak binormal mean curvature flow.

Note that in order to make sense of the right-hand side of (3.4), all that is needed is that  $\operatorname{div}_{\Gamma_t}$  be well-defined, which as remarked earlier is equivalent to requiring that  $\Gamma_t$  be rectifiable.

In this section we will prove

**Proposition 1.** *Every smooth binormal mean curvature flow is a weak binormal mean curvature flow.*

Here we are considering the smooth oriented manifolds  $\Gamma_t$  as oriented i.m. rectifiable sets with multiplicity  $\theta \equiv 1$ .

This is the very minimum that one can require of a weak solution. One would also like to know, for example, whether a weak solution necessarily coincides with a smooth solution whenever the latter exists.

*Proof.* 1. Recall that we are writing  $d$  to denote the exterior derivative in  $\mathbb{R}^m$ , and  $\mathbf{d}$  the exterior derivative in  $\mathbb{R}^{m+1}$ , and also that the boldface  $\xi$  denotes the space-time tangent multivector to  $\Gamma$ .

Fix  $t \in \mathbb{R}$  and  $\phi \in C_c^\infty(\mathbb{R}^m; \Lambda_{m-2}\mathbb{R}^m)$ , and let  $\tilde{\phi} \in C_c^\infty(\mathbb{R}_x^m \times \mathbb{R}_t; \Lambda_{m-2}\mathbb{R}_x^m)$  be such that  $\tilde{\phi}(\cdot, s) = \phi$  for all  $s$  in an interval containing  $t$ . We write  $\Gamma_{\{s < t\}}$  to denote  $\{(x, s) : x \in \Gamma_s, s < t\}$ . Then by Stokes' Theorem

$$(3.5) \quad \int_{\Gamma_t} \phi \cdot \xi \mathcal{H}^{m-2}(dx) = \int_{\Gamma_{\{s < t\}}} \mathbf{d}\tilde{\phi} \cdot \xi \mathcal{H}^{m-1}(dx ds)$$

For  $(x, t) \in \Gamma$  define  $\pi(x, t) = t$ . Let  $d\pi_{(x,t)}$  denote the induced linear map from  $T_{x,t}\Gamma$  to  $T_t\mathbb{R}$ , where both tangent spaces inherit the ambient Euclidean metrics. By definition Federer's Jacobian  $|J\pi(x, t)|$  is just  $|d\pi_{(x,t)}|$ , and using this and (3.2) one can easily check that  $|J\pi(x, t)| = (1 + |V|^2)^{-1/2}$  for all  $(x, t) \in \Gamma$ . Thus the coarea formula (see Simon [21], Section 10) implies that

$$\int_{\Gamma_{\{s < t\}}} \mathbf{d}\tilde{\phi} \cdot \xi \mathcal{H}^{m-1}(dx ds) = \int_{-\infty}^t \int_{\Gamma_s} \mathbf{d}\tilde{\phi} \cdot \xi (1 + |V|^2)^{1/2} \mathcal{H}^{m-2}(dx) ds.$$

Also, from (3.2) we see that  $\mathbf{d}\tilde{\phi} \cdot \xi (1 + |V|^2)^{1/2} = (d\tilde{\phi} + dt \wedge \tilde{\phi}_t) \cdot ((e_t + V) \wedge \xi) = d\tilde{\phi} \cdot (V \wedge \xi) + \tilde{\phi}_t \cdot \xi$ . So

$$\frac{d}{dt} \int_{\Gamma_t} \phi \cdot \xi \mathcal{H}^{m-2}(dx) = \int_{\Gamma_t} d\phi \cdot (V \wedge \xi) \mathcal{H}^{m-2}(dx).$$

2. Since  $d\phi \cdot (V \wedge \xi) = \star d\phi \cdot \star(V \wedge \xi)$ , the equation (3.3) for a smooth binormal mean curvature flow implies that

$$\frac{d}{dt} \int_{\Gamma_t} \phi \cdot \xi \mathcal{H}^{m-2}(dx) = \int_{\Gamma_t} \mathbf{H} \cdot \star d\phi \mathcal{H}^{m-2}(dx).$$

So the conclusion of the proposition follows from (2.7).  $\square$

It is convenient to reformulate (3.4) as follows.

**Lemma 1.** *Suppose that  $\{(\Gamma_t, \theta(\cdot, t), \xi(\cdot, t))\}_{t \in \mathbb{R}}$  is a family of oriented i.m. codimension two rectifiable sets, and that for each  $t$ ,  $\bar{J}_t$  is the corresponding  $\Lambda_2 \mathbb{R}^m$  valued measure  $J_t = \pi \theta \nu \mathcal{H}^{m-2}|_{\Gamma_t}$ , where  $\star \nu = \xi$ .*

*For every  $t$  and  $\mathcal{H}^{m-2}$  almost every  $x \in \Gamma_t$  let  $P$  denote the matrix corresponding to projection onto  $\text{ap}T_x \Gamma_t$ , and let  $P^\perp = \text{id} - P$ .*

*Then  $\{(\Gamma_t, \theta, \xi)\}$  is a weak binormal mean curvature flow if and only if*

$$(3.6) \quad \frac{d}{dt} \int \phi \cdot J_t(dx) = \int P_{jk}^\perp (\phi^{ij} - \phi^{ji})_{x_i x_k} |J_t|(dx)$$

for all  $\phi \in C_c^2(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m)$  of the form  $\phi = \sum_{i < j} \phi^{ij} e_i \wedge e_j$  and a.e.  $t$ .

*Proof.* If we use the isomorphism  $\star : \Lambda_2 \mathbb{R}^m \cong \Lambda_{m-2} \mathbb{R}^m$  and the definition of  $\text{div}_\Gamma \phi$ , we find that (3.4) is satisfied by  $\{\Gamma_t, \star \nu\}$  if and only if

$$\frac{d}{dt} \int \phi \cdot J_t(dx) = \int P : D(\star d \star \phi) |J_t|(dx)$$

for all  $\phi \in C_c^2(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m)$ . One can check that for  $\phi = \sum_{i < j} \phi^{ij} e_i \wedge e_j$ ,

$$\star d \star \phi = \sum_{i,j=1}^n (\phi^{ji} - \phi^{ij})_{x_i} e_j,$$

where we set  $\phi^{ij} = 0$  if  $i > j$ . As a result,  $P : D(\star d \star \phi) = P_{jk} (\phi^{ji} - \phi^{ij})_{x_i x_k}$ . However, since  $P = \text{id} - P^\perp$  and  $\delta_{jk} (\phi^{ji} - \phi^{ij})_{x_i x_k} \equiv 0$ , we can rewrite

$$\int P_{jk} (\phi^{ji} - \phi^{ij})_{x_i x_k} |J_t|(dx) = \int P_{jk}^\perp (\phi^{ij} - \phi^{ji})_{x_i x_k} |J_t|(dx).$$

□

#### 4. A COMPACTNESS RESULT

In this section we prove a compactness result for  $\{Ju^\epsilon\}_{\epsilon \in (0,1]}$ , where  $u^\epsilon$  is a solution of (1.1) on  $\mathbb{R}^m \times [0, \infty)$ . It follows by combining Theorem 1, which guarantees compactness of  $\{Ju^\epsilon(t)\}$  for every fixed  $t$ , with some simple estimates on the modulus of continuity of  $t \mapsto Ju^\epsilon(t)$  in a weak norm. The latter estimates follow easily from (2.6).

We consider initial data  $u_0^\epsilon$  such that

$$(4.1) \quad u_0^\epsilon - 1 \in H^1(\mathbb{R}^m).$$

We assume in addition that there exists an oriented i.m.  $(m-2)$ -dimensional rectifiable set  $(\Gamma_0, \theta, \xi)$  such that

$$(4.2) \quad Ju_0^\epsilon \rightarrow \bar{J}_0 := \pi \theta \nu \mathcal{H}^{m-2}|_{\Gamma_0} \quad \text{in } C_{\text{loc}}^{0,\gamma^*} \text{ for all } \gamma > 0,$$

where  $\star \nu = \xi$ . We also assume that the energy is asymptotically small in that

$$(4.3) \quad I^\epsilon(u_0^\epsilon) \leq |\bar{J}_0|(\mathbb{R}^m) + o_\epsilon(1).$$

As remarked earlier, the initial value problem is known to be well-posed under these assumptions. Since the energy is conserved, it immediately follows that

$$(4.4) \quad I^\epsilon(u^\epsilon(t)) \leq |\bar{J}_0|(\mathbb{R}^m) + o_\epsilon(1).$$

for all  $t \in \mathbb{R}$ .

*Remark 1.* Alberti, Baldo and Orlandi [1] show that, whenever  $\bar{J}_0$  is a measure of the above form with finite total mass and with vanishing boundary in the sense that  $d\bar{J}_0 = 0$  in the sense of distributions, then there exists a sequence  $\{u^\epsilon\}$  of functions satisfying (4.1), (4.2), and (4.3).

Our main result in this section is

**Theorem 2.** *Suppose that  $u^\epsilon$  is a solution of (1.1) for initial data satisfying (4.1), (4.2), and (4.3). Then given any subsequence  $\epsilon_n$ , there exists a further subsequence (which we still write  $\epsilon_n$ ) and measures  $\{\bar{J}_t\}_{t \in \mathbb{R}}$  of the form  $\bar{J}_t = \pi\theta\nu\mathcal{H}^{m-2}|_{\Gamma_t}$  for some oriented i.m.  $(m-2)$  dimensional rectifiable set  $(\Gamma_t, \theta, \star\nu)$ ; such that  $d\bar{J}_t = 0$  for all  $t$ ;*

$$(4.5) \quad Ju^{\epsilon_n}(t) \rightarrow \bar{J}_t \quad \text{in } C_{loc}^{\alpha*} \text{ for all } \alpha \in (0, 1] \text{ and every } t > 0;$$

$$(4.6) \quad |\bar{J}_t|(\mathbb{R}^m) \leq |\bar{J}_0|(\mathbb{R}^m);$$

and finally,  $t \mapsto \bar{J}_t$  is weak-\* continuous in  $C^{0*}$ , and uniformly Hölder continuous in weaker topologies:

$$(4.7) \quad \|\bar{J}_s - \bar{J}_t\|_{\dot{C}_c^{k, \alpha*}} \leq C(k, \alpha)|t - s|^{(k+\alpha)/2}$$

for  $k = 0, 1$  and  $\alpha \in (0, 1]$ .

*Remark 2.* If we merely assume that  $\limsup_{\epsilon \rightarrow 0} I^\epsilon(u^\epsilon) < \infty$  instead of (4.3), then the theorem remains valid if (4.6) is replaced by  $\bar{J}_t \leq \liminf_{n \rightarrow \infty} I^{\epsilon_n}(u^{\epsilon_n})$ .

*Proof.* 1. We will first use the Arzela-Ascoli Theorem to show that the functions  $\{t \mapsto Ju^\epsilon(t)\}_{\epsilon \in (0, 1]}$  are precompact in  $C(0, T; C_{loc}^{1, 1*})$  for every  $T > 0$ . To do this we need to verify two points: first, that  $\{Ju^\epsilon(t)\}_{\epsilon \in (0, 1]}$  is precompact in  $C_{loc}^{1, 1*}$  for every  $t > 0$ . This is easy, because in view of (4.4),  $\{u^\epsilon(t)\}_{\epsilon > 0}$  satisfies the hypotheses of Theorem 1. Thus for fixed  $t$ ,  $\{Ju^\epsilon(t)\}_{\epsilon \in (0, 1]}$  is precompact in  $C_{loc}^{0, \gamma*}$  for all  $\gamma > 0$ . Precompactness in  $C_{loc}^{1, 1*}$  follows from the obvious fact that the embedding  $C_{loc}^{0, \gamma*} \subset C_{loc}^{1, 1*}$  is continuous.

The second point we need to check is that  $\{t \mapsto Ju^\epsilon(t)\}_{\epsilon \in (0, 1]}$  are equicontinuous as functions into  $C_{loc}^{1, 1*}$ . We show that in fact they are uniformly Lipschitz as functions into  $C_c^{1, 1*}$ , and thus into  $C_{loc}^{1, 1*}$ . To see this, let  $\phi = \sum_{i < j} \phi^{ij} e_i \wedge e_j : \mathbb{R}^m \rightarrow \Lambda_2 \mathbb{R}^m$  be a smooth, compactly supported 2-vectorfield. Then (2.6) and the uniform energy bound (4.4) imply that

$$(4.8) \quad \begin{aligned} \int \phi \cdot (Ju^\epsilon(t_2) - Ju^\epsilon(t_1)) &\leq Ck_\epsilon \int_{t_1}^{t_2} \int |D^2\phi| |Du^\epsilon|^2 dx dt \\ &\leq C|t_1 - t_2| \|\phi\|_{\dot{C}_c^{1, 1}} \end{aligned}$$

In other words,

$$(4.9) \quad \|Ju^\epsilon(t_1) - Ju^\epsilon(t_2)\|_{C_c^{1, 1*}} \leq \|Ju^\epsilon(t_1) - Ju^\epsilon(t_2)\|_{\dot{C}_c^{1, 1*}} \leq C|t_1 - t_2|$$

for all  $t_1, t_2$ .

2. Thus given a subsequence  $\{\epsilon_n\}$ , we can pass to a further sequence (still labelled  $\epsilon_n$ ) such that  $Ju^{\epsilon_n}(t)$  converges to some limit  $\bar{J}_t$  in  $C_{loc}^{1, 1*}$ , locally uniformly for  $t > 0$ . Then again appealing to (4.4) and Theorem 1, we deduce that for every  $t$ ,  $\bar{J}_t$  is a measure of the form  $\pi\theta\nu\mathcal{H}^{m-2}|_{\Gamma_t}$ . The same theorem implies that  $d\bar{J}_t = 0$  for all  $t$  and that (4.5) and (4.6) hold.

By passing to limits in (4.9) we find that (4.7) holds for  $k = \alpha = 1$ . The remaining continuity estimates in (4.7) follows by interpolating between the case  $\alpha = k = 1$  and the easy estimate

$$\|\bar{J}_{t_1} - \bar{J}_{t_2}\|_{\hat{C}^{0*}} \leq \|\bar{J}_{t_1}\|_{C_c^{0*}} + \|\bar{J}_{t_2}\|_{C_c^{0*}} \leq 2|\bar{J}_0|(\mathbb{R}^m) \quad \forall t_1, t_2.$$

The relevant interpolation inequality is given in Lemma 2 below.

Finally, the weak-\* continuity of  $t \mapsto \bar{J}_t$  follows directly from the fact that  $\{\bar{J}_t\}_{t \geq 0}$  is uniformly bounded in  $C_c^{0*}$  and hence weak-\* precompact, together with the continuity estimate (4.7) in weaker topologies.  $\square$

**Lemma 2.** *If  $\mu$  is a measure such that  $\|\mu\|_{\hat{C}_c^{1,1*}} < \infty$ , then for  $k = 0, 1$  and  $\alpha \in (0, 1]$*

$$(4.10) \quad \|\mu\|_{\hat{C}_c^{k,\alpha*}} \leq C \|\mu\|_{\hat{C}_c^{1,1*}}^{(k+\alpha)/2} \|\mu\|_{\hat{C}_c^{0*}}^{(2-k-\alpha)/2}.$$

*Proof.* Suppose  $\mu \in \hat{C}_c^{1,1*} \cap C_c^{0*}$ , and fix  $k \in \{0, 1\}$  and  $\alpha \in (0, 1]$ . Fix  $\phi$  such that  $\|\phi\|_{\hat{C}_c^{k,\alpha}} \leq 1$ .

Let  $\eta^\epsilon(x) = \frac{1}{\epsilon^m} \eta(\frac{x}{\epsilon})$ , where  $\eta \in C_c^\infty(\mathbb{R}^n)$  is a nonnegative function such that  $\int \eta dx = 1$ . Define  $\phi^\epsilon = \eta^\epsilon * \phi$ . One easily verifies that

$$\|\phi^\epsilon - \phi\|_{C^0} \leq C\epsilon^{k+\alpha} \|\phi\|_{\hat{C}_c^{k,\alpha}} \leq C\epsilon^{k+\alpha},$$

$$\|\phi\|_{\hat{C}_c^{1,1}} = \|D^2 \phi^\epsilon\|_{C^0} \leq C\epsilon^{-2+k+\alpha} \|\phi\|_{\hat{C}_c^{k,\alpha}} = C\epsilon^{-2+k+\alpha}.$$

Then

$$\begin{aligned} \left| \int \phi d\mu \right| &\leq \left| \int \phi^\epsilon d\mu \right| + \left| \int (\phi - \phi^\epsilon) d\mu \right| \\ &\leq \|\phi^\epsilon\|_{\hat{C}_c^{1,1}} \|\mu\|_{\hat{C}_c^{1,1*}} + \|\phi - \phi^\epsilon\|_{C^0} \|\mu\|_{\hat{C}_c^{0*}} \\ &\leq C\epsilon^{-2+k+\alpha} \|\mu\|_{\hat{C}_c^{1,1*}} + C\epsilon^{k+\alpha} \|\mu\|_{\hat{C}_c^{0*}}. \end{aligned}$$

We now select  $\epsilon = \left( \frac{\|\mu\|_{\hat{C}_c^{1,1*}}}{\|\mu\|_{\hat{C}_c^{0*}}} \right)^{1/2}$  to obtain (4.10).  $\square$

## 5. CONVERGENCE TO BINORMAL MEAN CURVATURE FLOW

In this section we give some conditions which imply that the weak limit of the Jacobians is a weak binormal mean curvature flow. These are based on Theorem 6, which is established in Section 7. As a corollary we show that in general the limiting measures  $\{\bar{J}_t\}$  evolve nontrivially. We also prove that  $\{\bar{J}_t\}$  evolves by the conjectured dynamics in the case where the initial singular submanifold is a multiplicity one sphere.

One criterion for  $\{\bar{J}_t\}$  to evolve by weak binormal mean curvature flow is given in the following theorem, and others are discussed after its proof.

**Theorem 3.** *Suppose that  $u^\epsilon$  is a solution of (1.1) for initial data satisfying (4.1), (4.2), and (4.3). Fix a subsequence  $\epsilon_n$  and measures  $\bar{J}_t$  satisfying the conclusions of Theorem 2.*

*Then  $\{\bar{J}_t\}_{t \in \mathbb{R}}$  is a weak binormal mean curvature flow if  $|\bar{J}_t|(\mathbb{R}^m) = |\bar{J}_0|(\mathbb{R}^m)$  for all  $t$ . In view of (4.6), to prove this it suffices to show that*

$$(5.1) \quad |\bar{J}_t|(\mathbb{R}^m) \geq |\bar{J}_0|(\mathbb{R}^m) \quad \text{for all } t$$

*Remark 3.* The condition that  $t \mapsto |\bar{J}_t|(\mathbb{R}^m)$  be constant is satisfied if  $\{\bar{J}_t\}$  corresponds to a smooth binormal mean curvature flow. This follows from (2.7), (2.8), and the fact that  $V \cdot \mathbf{H} \equiv 0$ . Thus it is not completely unreasonable to imagine that one might be able to verify (5.1) in this weak setting, for  $\bar{J}_t$  constructed as above by passing to limits from solutions of Gross-Pitaevsky equations.

**Corollary 1.** *Assume the hypotheses of Theorem 2, and assume in addition that the distributional mean curvature of  $(\Gamma_0, \theta, \star\nu)$  does not identically vanish, so that*

$$\int (\phi^{ij} - \phi^{ji})_{x_i x_k} P_{jk}^\perp |\bar{J}_0|(dx) \neq 0$$

for some  $\phi \in C_c^2(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m)$ . Then  $t \mapsto \bar{J}_t$  is not constant.

*Proof.* If  $\bar{J}_t \equiv \bar{J}_0$  in any interval containing the origin, clearly (5.1) would be satisfied in that interval, and so  $t \mapsto \bar{J}_t$  would correspond to a weak binormal mean curvature flow in that interval. Then (3.6) implies that

$$0 = \frac{d}{dt} \int \phi \cdot \bar{J}_t \Big|_{t=0} = \int (\phi^{ij} - \phi^{ji})_{x_i x_k} P_{jk}^\perp |\bar{J}_0|(dx)$$

for all  $\phi \in C_c^2(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m)$ , a contradiction.  $\square$

*Proof of Theorem 3.* 1. Fix a sequence  $\epsilon_n$  and measures  $\bar{J}_t$  as in the statement of the theorem. Assume that  $|\bar{J}_t|(\mathbb{R}^m) = |\bar{J}_0|(\mathbb{R}^m)$  for all  $t$ . Fix a test function  $\phi = \sum_{i < j} \phi^{ij} e_i \wedge e_j \in C_c^2(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m)$ . Using (2.6) and reindexing, we find that

$$(5.2) \quad \frac{d}{dt} \int_{\mathbb{R}^m} \phi \cdot J u^{\epsilon_n}(t) dx = k_{\epsilon_n} \int_{\mathbb{R}^m} (\phi^{ij} - \phi^{ji})_{x_k x_i} u_{x_k}^{\epsilon_n} \cdot u_{x_j}^{\epsilon_n}(t) dx.$$

We are using the convention that  $\phi^{ji} = 0$  if  $j \geq i$ .

We know from Theorem 2 that  $t \mapsto \int \phi \cdot \bar{J}_t$  is Lipschitz, and also that

$$\int_{\mathbb{R}^m} \phi \cdot J u^{\epsilon_n}(t) dx \rightarrow \int_{\mathbb{R}^m} \phi \cdot \bar{J}_t(dx)$$

as  $n \rightarrow \infty$ . Thus to pass to limits in (5.2) and deduce that (3.6) holds, it suffices to show that

$$(5.3) \quad \lim_{n \rightarrow \infty} k_{\epsilon_n} \int_{\mathbb{R}^m} (\phi^{ij} - \phi^{ji})_{x_k x_i} u_{x_k}^{\epsilon_n} \cdot u_{x_j}^{\epsilon_n} dx = \int P_{jk}^\perp (\phi^{ij} - \phi^{ji})_{x_k x_i} |\bar{J}_t|(dx)$$

for a.e.  $t$ . Note also that, because  $t \mapsto \int \phi \cdot \bar{J}_t$  is Lipschitz, (5.2) implies that the above limit exists a.e.  $t$ , and we can pass to subsequences freely on the left-hand side of (5.3).

2. Fix some  $t$  and pass to a subsequence (still labelled  $\epsilon_n$ ) for which there exists a matrix-valued measure  $Q_t$  such that  $k_{\epsilon_n} Du^{\epsilon_n}(t) \otimes Du^{\epsilon_n}(t) \rightharpoonup Q_t$  weak-\* in  $C_c^{0*}$ . Then using conservation of energy,

$$\begin{aligned} \text{Tr} Q_t(\mathbb{R}^m) &\leq \limsup_{n \rightarrow \infty} k_{\epsilon_n} \int_{\mathbb{R}^m} |Du^{\epsilon_n}(t)|^2 dx \\ &\leq \limsup_{n \rightarrow \infty} k_{\epsilon_n} \int_{\mathbb{R}^m} 2E^{\epsilon_n}(u^{\epsilon_n}(t)) dx \\ &\leq \limsup_{n \rightarrow \infty} k_{\epsilon_n} \int_{\mathbb{R}^m} 2E^{\epsilon_n}(u^{\epsilon_n}(0)) dx \\ &= 2|\bar{J}_0|(\mathbb{R}^m) = 2|\bar{J}_t|(\mathbb{R}^m) \end{aligned}$$

if (5.1) holds. Then using Theorem 6 we conclude that  $Q_t = P^\perp |\bar{J}_t|$ , which gives (5.3).  $\square$

Theorem 6 gives conditions that ensure that  $Q_t = P^\perp |\bar{J}_t|$ , and thus it can be used to formulate other criteria that guarantee convergence to weak binormal mean curvature flow. For example, assume the hypotheses of Theorem 2, and fix some arbitrary  $t$ . Assume also that after passing to a further subsequence and relabelling as necessary,  $k_\epsilon Du^{\epsilon_n}(t) \otimes Du^{\epsilon_n}(t)$  converges weak-\* to some matrix-valued measure  $Q_t$ . Passing to limits in (5.2), we then can write

$$\begin{aligned} \frac{d}{dt} \int \phi \cdot \bar{J}_t(dx) = & \\ & \int_{\text{supp}|\bar{J}_t|} (\phi^{ij} - \phi^{ji})_{x_i x_k} Q_t^{jk}(dx) + \int_{\mathbb{R}^m \setminus \text{supp}|\bar{J}_t|} (\phi^{ij} - \phi^{ji})_{x_i x_k} Q_t^{jk}(dx). \end{aligned}$$

If  $\eta : \mathbb{R}^m \rightarrow [0, \infty)$  is any smooth function such that  $\eta \equiv 1$  on  $\text{supp}|\bar{J}_t|$ , then one can replace  $\phi$  by  $\eta\phi$  without changing the term of the left-hand side, or the first term on the right. Using this fact one can show that the last term on the right-hand side must vanish. Thus to show that  $\{\bar{J}_t\}$  defines a weak binormal mean curvature flow, we only need to check that  $Q_t = P^\perp |\bar{J}_t|$  on the support of  $|\bar{J}_t|$ . This follows from Theorem 6 if we know that  $\text{Tr} Q_t(\text{supp}|\bar{J}_t|) = 2|\bar{J}_t|(\mathbb{R}^m)$ . One can formulate local conditions, for example in terms of  $\frac{d\text{Tr} Q_t}{d|\bar{J}_t|}$ , that would imply this estimate. This would have to be done rather carefully, because of the possibility that  $\text{supp}|\bar{J}_t|$  is much larger than  $\Gamma_t$ , or that  $Q_t$  can concentrate on smaller-dimensional subsets of  $\Gamma_t$ . We do not do this here, because it is not clear exactly what conditions, if any, one might hope to be able to verify for sequences of solutions  $u^\epsilon$  of the Gross-Pitaevsky equation (1.1).

We now consider initial data such that the initial singular set is a round sphere of multiplicity one. We introduce some notation: For  $x \in \mathbb{R}^m$ ,  $p \in S^{m-1}$  and  $r > 0$  let

$$B_r(x, p) = \{y \in \mathbb{R}^m : |x - y| \leq r, (x - y) \cdot p = 0\}.$$

We equip  $B_r(x, p)$  with an orienting tangent  $m - 1$  vectorfield  $\xi_B \equiv \star p$ . We will also write

$$S_r(x, p) = \{y \in \mathbb{R}^m : |x - y| = r, (x - y) \cdot p = 0\},$$

We endow  $S_r(x, p)$  with the tangent vectorfield  $\xi_S$  that makes it the boundary of  $B_r(x, p)$  in the sense of Stokes' Theorem, and we define a normal 2-vectorfield by requiring that  $\star \nu = \xi_S$ .

**Theorem 4.** *Assume the hypotheses of Theorem 2, and assume moreover that*

$$\bar{J}_0 = \pi \nu \mathcal{H}^{m-2}|_{S_r(x_0, p)}$$

for some  $r > 0$ ,  $x \in \mathbb{R}^m$  and  $p \in S^{m-1}$ . Then

$$Ju^\epsilon(t) \rightarrow \bar{J}_t = \pi \nu \mathcal{H}^{m-2}|_{S_r(x(t), p)}$$

as  $\epsilon \rightarrow 0$  where  $x(t) = x_0 + t \frac{(m-2)p}{r}$ .

*Proof.* 1. By a change of coordinates and a translation we may assume that  $x_0$  is the origin and  $p = e_m$ .

Fix a sequence  $\epsilon_n$  and measures  $\bar{J}_t$  as guaranteed by Theorem 2.



Let  $\phi := x_{m-1}e_{m-1} \wedge e_m$ . Also, let  $\zeta : [0, \infty) \rightarrow [0, 1]$  be a smooth nonincreasing function such that  $\zeta(s) = 1$  if  $s \leq 1$  and  $\zeta(s) = 0$  for  $s \geq 2$ , and define

$$\chi_R(x) := \zeta\left(\frac{|x|}{R}\right).$$

Finally, define  $\phi_R = \chi_R \phi$ . One easily checks that  $\|D^2 \phi_R\|_\infty \leq \frac{C}{R}$ . Thus (4.7) with  $k = \alpha = 1$  implies that

$$(5.4) \quad \left| \int \phi_R \cdot \bar{J}_t - \int \phi_R \cdot \bar{J}_0 \right| \leq C \frac{t}{R}.$$

2. For every  $t$ , let  $T_t$  denote the  $(m-2)$  dimensional integer multiplicity rectifiable current defined by

$$T_t(\star \phi) = \frac{1}{\pi} \int \phi \cdot \bar{J}_t \quad \phi \in C_c^\infty(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m).$$

The fact that  $d\bar{J}_t = 0$  in the sense of distributions implies that  $\partial T_t = 0$ . Thus Almgren's optimal isoperimetric inequality [2] implies that there exists some  $(m-1)$  dimensional i. m. rectifiable current  $Q_t$  such that  $\partial Q_t = T_t$  and

$$(5.5) \quad \begin{aligned} \mathbf{M}(Q_t) &= \min \{ \mathbf{M}(R_t) : R_t \text{ } (m-1) \text{ dim. i.m. rectifiable current, } \partial R_t = T_t \} \\ &\leq ((m-1)\omega_{m-1})^{-1/m-2} \mathbf{M}(T_t)^{m-1/m-2} \end{aligned}$$

where  $\omega_k$  denotes the volume of the unit ball in  $\mathbb{R}^k$ . In [2] it is further shown that the inequality on the right is an equality if and only if  $T_t = \tau(S_r(x, p), 1, \xi_S)$  for some  $r > 0$ ,  $x \in \mathbb{R}^m$ , and  $p \in S^{n-1}$ .

Our choice of  $\bar{J}_0$  implies that equality holds in (5.5) when  $t = 0$ , and also that  $Q_0 = \tau(B_r(0, e_m), 1, \star e_m)$ .

The definition of  $T_t$  and the identity  $\partial Q_t = T_t$  imply that

$$(5.6) \quad \frac{1}{\pi} \int \eta \cdot \bar{J}_t = Q_t(d \star \eta) \quad \forall \eta \in C_c^\infty(\mathbb{R}^m; \Lambda_2 \mathbb{R}^m).$$

3. By explicitly differentiating one can verify that  $d \star \phi_R$  converges pointwise and boundedly to  $d \star \phi = e_1 \wedge \dots \wedge e_{m-1} = \star e_m$  as  $R \rightarrow \infty$ . Since  $Q_t$  has finite mass, this implies that

$$(5.7) \quad \mathbf{M}(Q_t) \geq Q_t(d \star \phi) = \lim_{R \rightarrow \infty} Q_t(d \star \phi_R) = \lim_{R \rightarrow \infty} \frac{1}{\pi} \int \phi_R \cdot \bar{J}_t,$$

using (5.6). Now (5.4) implies that

$$\lim_{R \rightarrow \infty} \frac{1}{\pi} \int \phi_R \cdot \bar{J}_t = \lim_{R \rightarrow \infty} \frac{1}{\pi} \int \phi_R \cdot \bar{J}_0.$$

Applying (5.7) at time  $t = 0$  we find that

$$\begin{aligned} \lim_{R \rightarrow \infty} \frac{1}{\pi} \int \phi_R \cdot \bar{J}_0 &= Q_0(d \star \phi) \\ &= \int_{B_r(0, e_m)} d\mathcal{H}^{m-1} \\ &= r^{m-1} \omega_{m-1} \end{aligned}$$

since  $d\star\phi = e_1 \wedge \dots \wedge e_{m-1}$  is just the oriented tangent  $\xi_B$  to  $B_r(0, e_m)$ . Assembling these calculations we find that  $\mathbf{M}(Q_t) \geq r^{m-1}\omega_{m-1}$  for all  $t$ . Thus the isoperimetric inequality (5.5) implies that

$$(5.8) \quad \frac{1}{\pi}|\bar{J}_t|(\mathbb{R}^m) = \mathbf{M}(T_t) \geq (m-1)\omega_{m-1}r^{m-2} = \frac{1}{\pi}|\bar{J}_0|(\mathbb{R}^m)$$

where the last identity follows immediately from our assumption about the explicit form of  $\bar{J}_0$ .

4. The estimate (5.8) implies by Theorem 3 that  $\{\bar{J}_t\}$  is a weak binormal mean curvature flow.

In this case we can easily verify that in fact  $\{\bar{J}_t\}$  is a classical binormal mean curvature flow. First note that (4.6) implies that equality holds in (5.8), and so according to the isoperimetric inequality,  $T_t$  must have the form  $T_t = \tau(S_r(x(t), p(t)), \theta, \xi)$  for some  $x(t), p(t)$ , and moreover  $Q_t = \tau(B_r(x(t), p(t)), \theta, \xi_B)$  where  $\xi_B \equiv \star p(t)$ .

It follows also that equality holds in (5.7), and hence that  $\star d\phi = \star e_m$  identically equals the orienting tangent  $\star p(t)$  to  $Q_t$ , which implies that  $p(t) = e_m$  for all  $t$ .

So we only need to find  $x(t) = (x_1(t), \dots, x_m(t))$ . To do this, for  $l = 1, \dots, m$  let  $\phi_l = (-1)^m x_l x_{m-1} e_{m-1} \wedge e_m$ , and note that

$$\frac{1}{\pi} \int \phi_l \cdot \bar{J}_t = T_t(\star\phi_l) = Q_t(d\star\phi_l).$$

Also, one easily checks that

$$d\star\phi_l = \begin{cases} x_l \star e_m & \text{if } l \neq m-1 \\ 2x_{m-1} \star e_m & \text{if } l = m-1. \end{cases}$$

Thus

$$(5.9) \quad \int \phi_l \cdot \bar{J}_t = K(l)\pi \int_{B_r(x(t), e_m)} x_l \mathcal{H}^{m-1}(dx) = K(l)\pi x_l(t)\omega_{m-1}r^{m-1}$$

where  $K(l) = 2$  if  $l = m-1$  and 1 otherwise. Also, one can verify that for every  $y \in \mathbb{R}^m$ ,

$$(5.10) \quad \pi \int_{S_r(y, e_m)} (\phi_l^{ij} - \phi_l^{ji})_{x_i x_k} P_{jk}^\perp \mathcal{H}^{m-2}(dx) = \begin{cases} 0 & \text{if } l \neq m \\ \pi(m-2)\omega_{m-1}r^{m-2} & \text{if } l = m. \end{cases}$$

This can be done by a straightforward calculation, or else simply by using (5.9) and Proposition 1, which guarantees that every smooth binormal mean curvature flow is a weak binormal mean curvature flow; and the explicit classical solution when the initial surface is a round  $m-2$ -sphere.

Putting (5.9) and (5.10) in the definition of a weak solution yields  $\dot{x}_i \equiv 0$  for all  $i < m$ ,  $\dot{x}_m \equiv (m-2)/r$ , which completes the proof.  $\square$

## 6. LIMITS OF $k_\epsilon Du^\epsilon \otimes Du^\epsilon$

In this section we will prove a result analyzing the relationship between limits of  $Ju^\epsilon$  and limits of  $k_\epsilon Du^\epsilon \otimes Du^\epsilon$  in two space dimensions. The main result of this section is

**Theorem 5.** *Suppose that  $u^\epsilon \in H^1(\Omega; \mathbb{R}^2)$  for  $\epsilon \in (0, 1]$ , where  $\Omega$  is an open subset of  $\mathbb{R}^2$ . Assume that*

$$(6.1) \quad \limsup_{\epsilon \rightarrow 0} k_\epsilon \int_{\Omega} E^\epsilon(u^\epsilon) dx < \infty$$

and let  $\epsilon_n \rightarrow 0$  be a sequence such that

$$(6.2) \quad Ju^{\epsilon_n} \rightarrow \bar{J} = \pi \sum d_i \delta_{a_i} \quad \text{in } C_{loc}^{0, \gamma^*} \forall \gamma > 0; \quad \text{and}$$

$$(6.3) \quad k_{\epsilon_n} Du^{\epsilon_n} \otimes Du^{\epsilon_n} \rightharpoonup Q \in C^{0*}(\Omega; S^{2 \times 2}) \quad \text{weakly in } C^{0*}(\Omega).$$

If we define

$$(6.4) \quad Q_d = Q - \text{id}|\bar{J}|$$

then  $\text{Tr}Q_d \geq 0$ , and there exist constants  $c_1, c_2$  such that

$$(6.5) \quad |Q_d(A)| \leq c_1 (\text{Tr}Q_d(A) |\bar{J}|(A))^{1/2} + c_2 \text{Tr}Q_d(A).$$

for every measurable  $A \subset \Omega$ . In particular, if  $\text{Tr}Q(A) = 2|\bar{J}|(A)$  for some  $A \subset \Omega$ , then in fact  $Q = \text{id}|\bar{J}|$  in  $A$ .

*Remark 4.* Given a sequence of functions satisfying (6.1), then Theorem 3.1 in [12] shows that one can find a subsequence satisfying (6.2). It is clear that one can find a subsequence satisfying (6.3).

*Remark 5.* The estimate (6.5) is sharp in a certain sense, which we illustrate by describing an example. Define

$$u^\epsilon(x) = \begin{cases} e^{i(\theta + \alpha \cos^2(\theta))} & \text{if } r \geq \epsilon \\ \frac{|r|}{\epsilon} e^{i(\theta + \alpha \cos^2(\theta))} & \text{if } r \leq \epsilon \end{cases}$$

where  $(r, \theta)$  are polar coordinates,  $d \in \mathbb{Z}$  and  $\alpha \in \mathbb{R}$ . One can then check by an explicit computation that  $Ju^\epsilon \rightarrow \bar{J} := \pi \delta_0$  and  $k_\epsilon Du^\epsilon \otimes Du^\epsilon \rightarrow Q \delta_0$ , where  $Q = \pi \begin{pmatrix} 1 + \frac{\alpha^2}{2} & -\alpha \\ -\alpha & 1 + \frac{\alpha^2}{2} \end{pmatrix}$ , and so  $Q_d(\{0\}) = \pi \begin{pmatrix} \frac{\alpha^2}{2} & -\alpha \\ -\alpha & \frac{\alpha^2}{2} \end{pmatrix}$ . Further defining

$$v^\epsilon(x) = \prod_{i=1}^d u^\epsilon(x - x_i^\epsilon), \quad x_i^\epsilon := |\ln \epsilon|^{-1} e^{2\pi i/d},$$

one can check that  $Jv^\epsilon \rightarrow \bar{J} := d\pi \delta_0$  and  $k_\epsilon Dv^\epsilon \otimes Dv^\epsilon \rightarrow Q \delta_0$ , with

$$Q_d(\{0\}) = d\pi \begin{pmatrix} \frac{\alpha^2}{2} & -\alpha \\ -\alpha & \frac{\alpha^2}{2} \end{pmatrix}.$$

So  $|Q_d(\{0\})| \sim d\pi(|\alpha| + \alpha^2)$  and  $\text{Tr}Q_d(\{0\}) = d\pi\alpha^2$ . In particular the term  $(\text{Tr}Q_d(\{0\})|\bar{J}|(\{0\}))^{1/2} = d\pi|\alpha|$  is required to bound  $|Q_d(\{0\})|$ .

The following simple lemma helps further explain the content of (6.5), which in a sense asserts that the measure  $Q_d = Q - \text{id}|\bar{J}|$  is not too far from being nonnegative.

**Lemma 3.** *If  $M \in \mathcal{S}^{n \times n}$  is nonnegative definite, then  $\frac{1}{\sqrt{n}} \text{Tr}M \leq |M| \leq \text{Tr}M$ .*

*Proof.* To see this, note that  $|M|^2 = \text{Tr}(M^2)$ , so we need to check that  $\frac{1}{n}(\text{Tr}M)^2 \leq \text{Tr}(M^2) \leq (\text{Tr}M)^2$  when  $M \geq 0$ . Diagonalizing  $M$  and  $M^2$  reduces these to elementary inequalities.  $\square$

We briefly describe our strategy for proving Theorem 5. We define a nonnegative function  $\alpha$  on the collection of nonnegative definite  $2 \times 2$  matrices, and for  $A \subset \Omega$  we define

$$F^\epsilon(A) = \alpha\left(\int_A Du^\epsilon \otimes Du^\epsilon\right).$$

The function  $\alpha$  is defined in such a way that (6.5) becomes equivalent to the statement that

$$F^\epsilon(A) \geq \ln\left(\frac{1}{\epsilon}\right)|\bar{J}|(A) + o(|\ln \epsilon|),$$

so that Theorem 5 reduces to proving a lower bound relating a kind of “energy” to the Jacobian.

We do this by showing, first, that if  $\partial B_r$  is a circle on which  $|u^\epsilon| \sim 1$ , then one can prove a lower bound for  $\alpha(\int_{\partial B_r} Du^\epsilon \otimes Du^\epsilon)$  in terms of  $\deg(u^\epsilon; \partial B_r)$ . This is done in Lemma 5 and really follows from a simple application of the Cauchy-Schwartz inequality. We then show that, if  $x_0$  is a point around which  $Ju^\epsilon$  concentrates as  $\epsilon \rightarrow 0$ , then one can find many circles on which  $|u^\epsilon| \sim 1$  and  $\deg u^\epsilon$  is nonzero. We finally assemble these estimates to show that  $F^\epsilon(U)$  is large for a suitable neighborhood  $U$  of  $x_0$ . In doing this we rely on some properties of the function  $\alpha(\cdot)$ , for example a kind of superadditivity property, see Lemma 4.

We start by defining and investigating  $\alpha(\cdot)$  and a related function. For a nonnegative matrix  $S \in \mathcal{S}^{2 \times 2}$  and for  $\sigma \geq 0$  we define

$$(6.6) \quad g(S, \sigma) := |S - \sigma \text{id}| - c_1 (\sigma \text{Tr}(S - \sigma \text{id}))^{1/2} - c_2 \text{Tr}(S - \sigma \text{id}).$$

We will impose conditions on the constants  $c_i, i = 1, 2$  as we go along. For the moment we only insist that  $c_2 \geq 1$ .

We also define, for nonnegative definite  $S$ ,

$$(6.7) \quad \alpha(S) := \sup\{\tilde{\alpha} \leq \text{Tr} S : g(S, \tilde{\alpha}) \leq 0\}.$$

It is clear that  $\alpha \geq 0$ , and that the supremum in the definition of  $\alpha$  is attained.

It is not hard to check that  $g$  and  $\alpha$  have certain monotonicity properties. Using the fact that  $c_2 \geq 1$  one can easily verify that

$$(6.8) \quad S \mapsto g(S, \sigma) \text{ is decreasing for all } \sigma \geq 0.$$

This implies that

$$(6.9) \quad S \mapsto \alpha(S) \text{ is increasing.}$$

Also, one can check by direct differentiation that  $\sigma \mapsto g(S, \sigma)$  is convex for  $0 \leq \sigma \leq \text{Tr} S$ . If  $g(S, 0) < 0$  (in particular this holds if  $S > 0$ ) this implies that

$$(6.10) \quad g(S, \tilde{\alpha}) < 0 \quad \text{for all } 0 \leq \tilde{\alpha} < \alpha(S).$$

Using (6.8) and (6.10) one deduces that

$$(6.11) \quad \alpha(S) \geq \tilde{\alpha} \text{ iff } \exists M \text{ such that } S \geq \tilde{\alpha} \text{id} + M \text{ and } |M| \leq c_1 (\tilde{\alpha} \text{Tr} M)^{1/2} + c_2 \text{Tr} M.$$

Note also that (6.10) and the (obvious) continuity of  $g$  imply that  $\alpha$  can be defined implicitly by the equation  $g(S, \alpha(S)) = 0$ . The convexity of  $\sigma \mapsto g(S, \sigma)$  and the fact that  $g(S, 0) < 0$  whenever  $S$  is positive definite imply that  $\frac{\partial g}{\partial \sigma}(S, \alpha(S)) > 0$  whenever  $S > 0$ . Thus the implicit function theorem implies that  $\alpha(\cdot)$  is continuous on the cone of nonnegative matrices.

We need one more fact about  $\alpha(\cdot)$ , which we state as

**Lemma 4.** *If  $X$  is a space endowed with a measure  $\mu$ , and  $X \ni x \mapsto S(x) \in \mathcal{S}^{2 \times 2}$  is a  $\mu$ -measurable function such that  $S \geq 0$  a.e., then*

$$(6.12) \quad \alpha \left( \int_X S(x) \mu(dx) \right) \geq \int_X \alpha(S(x)) \mu(dx).$$

In particular,

$$(6.13) \quad \alpha \left( \sum S_i \right) \geq \sum \alpha(S_i).$$

*Proof.* For  $x \in X$  we write  $\alpha(x)$  as shorthand for  $\alpha(S(x))$ , and we define  $M(x) = S(x) - \alpha(x)\text{id}$ . Note that  $|M(x)| = c_1(\alpha(x)\text{Tr}M(x))^{1/2} + c_2\text{Tr}M(x)$  for all  $x$ , by the definition of  $\alpha(\cdot)$ . Writing  $\langle M \rangle$  for  $\int_X M(x) \mu(dx)$ , we thus have

$$\begin{aligned} |\langle M \rangle| &\leq \int_X |M(x)| \mu(dx) \\ &= \int_X c_1(\alpha(x)\text{Tr}M(x))^{1/2} + c_2\text{Tr}M(x) \mu(dx) \\ &\leq c_1 \left( \int_X \alpha(x) \mu(dx) \right)^{1/2} (\text{Tr}\langle M \rangle)^{1/2} + c_2 \text{Tr}\langle M \rangle. \end{aligned}$$

In addition,

$$\int_X S(x) \mu(dx) = \text{id} \int_X \alpha(x) \mu(dx) + \langle M(x) \rangle,$$

so (6.12) follows from (6.11). Finally, (6.13) is an immediate consequence of (6.12).  $\square$

Theorem 5 will follow easily from the following proposition. Because we will encounter many balls in our later arguments, we use the notation  $U$  rather than  $B$  to denote a ball, to avoid overusing the symbol  $B$ .

**Proposition 2.** *Suppose that  $U \subset \mathbb{R}^2$  is an open ball  $U = B_R(x_0)$  and that  $u^\epsilon \in H^1 \cap C^\infty(U; \mathbb{R}^2)$  for  $\epsilon \in (0, 1]$ . Assume also that*

$$(6.14) \quad \limsup_{\epsilon \rightarrow 0} k_\epsilon \int_U E^\epsilon(u^\epsilon) dx < \infty$$

and let  $\epsilon_n \rightarrow 0$  be a sequence such that

$$(6.15) \quad Ju^{\epsilon_n} \rightarrow \bar{J} = \pi d \delta_{x_0} \quad \text{in } C_c^{0, \gamma^*}(U) \quad \forall \gamma > 0.$$

Let

$$S^n = k_{\epsilon_n} \int_U Du^{\epsilon_n} \otimes Du^{\epsilon_n} dx.$$

Then

$$(6.16) \quad \alpha(S^n) \geq (1 - o(1))\pi d = (1 - o(1))|\bar{J}|(U) \quad \text{as } n \rightarrow \infty.$$

We now assume Proposition 2 and show that it implies Theorem 5 quite easily. we then give the proof of Proposition 2.

*Proof of Theorem 5.* We may assume by an approximation argument that  $u^\epsilon$  is smooth for every  $\epsilon$ .

We need to show that  $g(Q(A), |\bar{J}|(A)) \leq 0$ , for every measurable  $A$ ; this is (6.5). By (6.10), this will follow if we can show that  $\alpha(Q(A)) \geq |\bar{J}|(A)$ . Since  $Q$  and  $\bar{J}$  are Radon measures, it suffices to prove this for all open sets  $V \subset \Omega$ .

Fix any such open set, and let  $\{U_i\}_{i=1}^k$  be a collection of pairwise disjoint open balls contained in  $V$  such that each  $U_i$  is centered at a point of  $\text{supp}|\bar{J}| \cap V$ , and every such point is contained in a ball  $U_i$ . We further assume that  $Q(\partial U_i) = 0$  for every  $i$ ; this is clearly possible, since for any fixed  $a \in \Omega$ ,  $Q(\partial B_s(a))$  is nonzero for at most countably many values of  $s$ . Let  $V_0 = V \setminus (\cup_i U_i)$ , and let  $S_0 = Q(V_0)$ , and  $S_i = Q(U_i)$  for  $i = 1, \dots, k$ . Note that  $Q(V) = \sum_{i=0}^k S_i \geq \sum_{i=1}^k S_i$ .

The weak convergence  $k_\epsilon Du^{\epsilon_n} \otimes Du^{\epsilon_n} \rightharpoonup Q$  and the fact that  $Q(\partial U_i) = 0$  imply that

$$(6.17) \quad \int_{U_i} k_\epsilon Du^{\epsilon_n} \otimes Du^{\epsilon_n} dx = S_i^n \rightarrow S_i = Q(U_i), \quad i = 1, \dots, k.$$

So Lemma 4 implies that

$$\alpha(Q(V)) \geq \alpha\left(\sum_{i=1}^k S_i\right) \geq \sum_{i=1}^k \alpha(S_i).$$

And (6.16), (6.17), and the continuity of  $\alpha(\cdot)$  together yield

$$\sum_{i=1}^k \alpha(S_i) = \sum_{i=1}^k \lim_n \alpha(S_i^n) \geq \sum_{i=1}^k |\bar{J}|(U_i) = |\bar{J}|(V)$$

□

*Remark 6.* If we assume (6.1) and (6.2) but do not assume that  $k_{\epsilon_n} Du^{\epsilon_n} \otimes Du^{\epsilon_n}$  converges to a limit, then Theorem 5 implies that

$$(6.18) \quad \liminf_n \alpha\left(k_{\epsilon_n} \int_V Du^{\epsilon_n} \otimes Du^{\epsilon_n}\right) \geq |\bar{J}|(V)$$

for every open set  $V \subset \Omega$ .

**Lemma 5.** *Assume the hypotheses of Proposition 2. Suppose that  $B \subset U$  is a ball of radius  $r$  such that  $|\deg(u^\epsilon; \partial B)| = d > 0$ . Let  $m = \min_{\partial B} |u^\epsilon|$ . Then*

$$\alpha\left(\int_{\partial B} Du^\epsilon \otimes Du^\epsilon\right) \geq m^2 \pi d^2 / r \geq m^2 \pi d / r.$$

*Proof.* Assume that  $m > 0$  as otherwise there is nothing to prove. We can then write  $u^\epsilon$  locally in a neighborhood of  $\partial B$  in the form  $u^\epsilon = \rho e^{i\phi}$ , for a positive function  $\rho$ . Although  $\phi$  is in general multivalued,  $D\phi$  is a well-defined function near  $\partial B$  taking values in  $\mathbb{R}^2$ , and  $Du^\epsilon = D\rho e^{i\phi} + \rho D\phi e^{i\phi}$ . Then

$$(6.19) \quad Du^\epsilon \otimes Du^\epsilon = D\rho \otimes D\rho + \rho^2 D\phi \otimes D\phi \geq m^2 D\phi \otimes D\phi \quad \text{on } \partial B.$$

Let  $\tau$  be a unit tangent vector field to  $\partial B$  and  $\nu$  a unit normal. Note that

$$(6.20) \quad \int_{\partial B} \tau \otimes \tau = \frac{1}{2} \int_{\partial B} \tau \otimes \tau + \nu \otimes \nu = \frac{1}{2} \int_{\partial B} \text{id} = \pi r \text{id}.$$

Note also that the condition  $|\deg(u^\epsilon, \partial B)| = d$  means precisely that

$$(6.21) \quad \int_{\partial B} D\phi \cdot \tau = 2\pi d,$$

if  $\tau$  is oriented appropriately. Define  $w = D\phi - d\tau/r$ , so that from (6.19), (6.20),

$$(6.22) \quad \int_{\partial B} Du^\epsilon \otimes Du^\epsilon \geq m^2 \int_{\partial B} D\phi \otimes D\phi = m^2 \pi \frac{d^2}{r} \text{id} + M$$

where  $M = m^2 \frac{d}{r} \int_{\partial B} (\tau \otimes w + w \otimes \tau) + m^2 \int_{\partial B} w \otimes w$ . Note that because of (6.21),

$$\text{Tr}M = 2m^2 \frac{d}{r} \int_{\partial B} w \cdot \tau + m^2 \int_{\partial B} |w|^2 = m^2 \int_{\partial B} |w|^2.$$

Since  $|w \otimes \tau + \tau \otimes w| \leq 2|\tau| |w| \leq 2|w|$ , Cauchy-Schwartz thus yields

$$(6.23) \quad \begin{aligned} |M| &\leq \frac{2dm^2}{r} \int_{\partial B} |w| + m^2 \int_{\partial B} |w|^2 \\ &\leq \frac{2dm^2}{r} \left( 2\pi r \int_{\partial B} |w|^2 \right)^{1/2} + m^2 \int_{\partial B} |w|^2 \\ &\leq c_1 \left( \pi m^2 \frac{d^2}{r} \text{Tr}M \right)^{1/2} + c_2 \text{Tr}M, \end{aligned}$$

where for example we can take  $c_1 = 2\sqrt{2}$ ,  $c_2 = 1$ . In view of (6.11), the conclusion of the Lemma follows from (6.22) and (6.23)  $\square$

For  $t \in [0, 1]$  and  $\epsilon \in (0, 1]$  we define

$$U^\epsilon(t) := \{x \in U : |u^\epsilon(x)| \leq t\}.$$

We will write  $\gamma_t^\epsilon := \partial U^\epsilon(t) \cap U$  and for any set  $A \subset \mathbb{R}^2$  we use the notation

$$\mathcal{H}_\infty^1(A) = \inf \{ 2 \sum r_i : A \subset \cup_i B_{r_i}(x_i) \}.$$

It is not hard to check that whenever  $A$  is a subset of  $U$ ,

$$(6.24) \quad \mathcal{H}^1(\partial A \cap U) \geq \mathcal{H}_\infty^1(A).$$

This uses the fact that  $U$  is a ball. We now prove

**Lemma 6.** *Assume the hypotheses of Proposition 2. Then for every  $t \in [0, 1]$ ,*

$$(1-t)^2 \mathcal{H}_\infty^1(U^\epsilon(t)) \leq C\epsilon \int_U E^\epsilon(u^\epsilon) dx.$$

The proof is very similar to one given in Sandier [20].

*Proof.* We write  $|u^\epsilon| = \rho$ , and note that  $|D\rho| \leq |Du^\epsilon|$ . Thus Cauchy's inequality implies that  $E^\epsilon(u^\epsilon) \geq \frac{1}{2}|D\rho|^2 + \frac{1}{2\epsilon^2}W(\rho) \geq \frac{1}{\epsilon}|D\rho|\sqrt{W(\rho)}$ . So by the coarea formula and (2.1),

$$\int_U E^\epsilon(u^\epsilon) dx \geq \frac{1}{C\epsilon} \int_0^\infty \int_{\gamma_t^\epsilon} W(t)^{1/2} d\mathcal{H}^1 dt \geq \frac{1}{C\epsilon} \int_0^1 \mathcal{H}^1(\gamma_t^\epsilon) |1-t| dt.$$

From (6.24) we immediately see that  $\mathcal{H}^1(\gamma_t^\epsilon) \geq \mathcal{H}_\infty^1(U^\epsilon(t))$ , and it is clear that  $t \mapsto \mathcal{H}_\infty^1(U^\epsilon(t))$  is nondecreasing. So for any  $t \in (0, 1)$ ,

$$\int_U E^\epsilon(u^\epsilon) dx \geq \frac{1}{C\epsilon} \mathcal{H}_\infty^1(U^\epsilon(t)) \int_t^1 (1-s) ds,$$

which readily implies the conclusion of the lemma.  $\square$

*Remark 7.* If we define  $\tilde{U}^\epsilon(t) = \{x \in U : |u^\epsilon(x)| \geq t\}$  for  $t > 1$ , then the same argument shows that  $(1-t)^2 \mathcal{H}_\infty^1(\tilde{U}^\epsilon(t)) \leq C\epsilon \int_U E^\epsilon(u^\epsilon) dx$  for all  $t \in (1, 2]$ .

We use the notation

$$F^\epsilon(A) := \alpha \left( \int_A Du^\epsilon \otimes Du^\epsilon \right).$$

The monotonicity of  $\alpha(\cdot)$  implies that

$$(6.25) \quad F^\epsilon(\cup A_i) \geq \sum F^\epsilon(A_i), \quad \text{and } F^\epsilon(A) \leq F^\epsilon(B) \text{ whenever } A \subset B.$$

Recall that we are writing  $k_\epsilon := |\ln \epsilon|^{-1}$ . Let  $B_{\rho_2}(a) \setminus B_{\rho_1}(a)$  be an annulus that is contained in  $U \setminus U^\epsilon(1 - k_\epsilon)$ . Since by definition  $|u^\epsilon| > 1 - k_\epsilon$  in the complement of  $U^\epsilon(1 - k_\epsilon)$ , the degree  $\deg(u^\epsilon; \partial B_\rho(a))$  is well-defined and in fact constant for all  $\rho \in (\rho_1, \rho_2)$ . So if  $|\deg(u^\epsilon; \partial B_{\rho_1}(a))| = d$ , then Lemma 5 and Lemma 4 imply that

$$(6.26) \quad F^\epsilon(B_{\rho_2} \setminus B_{\rho_1}(a)) \geq \int_{\rho_1}^{\rho_2} \alpha \left( \int_{\partial B_\rho} Du^\epsilon \otimes Du^\epsilon \right) d\rho \geq (1 - k_\epsilon)^2 \pi d \ln \frac{\rho_2}{\rho_1}.$$

We will use this fact in the proof of

**Lemma 7.** *Assume the hypotheses of Proposition 2. Then given  $\epsilon \in (0, 1]$ , for every  $\sigma \geq 1$  there exists a collection of pairwise disjoint open balls  $B(\sigma) = \{B_i^\sigma\}$  such that*

$$(6.27) \quad U^\epsilon(1 - k_\epsilon) \subset \cup_i B_i^\sigma;$$

$$(6.28) \quad F^\epsilon(B_i^\sigma) \geq (1 - k_\epsilon)^2 d_i^\sigma \ln \sigma \quad \text{if } B_i^\sigma \subset U;$$

$$(6.29) \quad \sum r_i^\sigma \leq C\sigma\epsilon \left( \ln \frac{1}{\epsilon} + 1 \right)^3.$$

Here  $r_i^\sigma$  denotes the radius of  $B_i^\sigma$  and  $d_i^\sigma := |\deg(u^\epsilon; \partial B_i^\sigma)|$ .

The proof is also very similar to one given in Sandier [20].

We will say the “ $\tau$ -expansion of the ball  $B_\rho(x)$ ” to denote the ball  $B_{\tau\rho}(x)$  ball with the same center and radius expanded by a factor  $\tau$ .

*Proof.* 1. We first consider the case  $\sigma = 1$ . Fix an arbitrary  $\epsilon \in (0, 1]$ . From Lemma 6 and (6.14) we see that  $\mathcal{H}_\infty^1(U^\epsilon(1 - k_\epsilon)) \leq C\epsilon \left( \ln \frac{1}{\epsilon} + 1 \right)^3$ . By the definition of  $\mathcal{H}_\infty^1$ , we can then find a collection of open balls that cover  $U^\epsilon(1 - k_\epsilon)$ , with the sum of their radii bounded by  $C\epsilon \left( \ln \frac{1}{\epsilon} + 1 \right)^3$ . We have assumed that  $u^\epsilon$  is continuous, which implies that  $U^\epsilon(1 - k_\epsilon)$  is compact, and so we can take this collection of balls to be finite. Suppose two balls  $B_i$  and  $B_j$  intersect. We then replace them by a single larger ball  $B' \supset B_i \cap B_j$  whose radius is no greater than the sum of the radii of  $B_i$  and  $B_j$ . This can be repeated until we obtain a collection that is pairwise disjoint, with the same bound on  $\sum r_i$ . This collection has the desired properties for  $\sigma = 1$ .

2. Let  $\Sigma$  denote the set of all numbers  $\sigma \geq 1$  for which the conclusions of the lemma hold. We have shown that  $1 \in \Sigma$ . We now claim that, if  $\sigma_0 \in \Sigma$ , then there exists some  $\delta > 0$  such that  $[\sigma_0, \sigma_0 + \delta) \subset \Sigma$ .

To see this, fix some such  $\sigma_0$ . There exists some  $\delta > 0$  such that the  $\sigma/\sigma_0$  expansions of the balls  $B_i^{\sigma_0}$  are pairwise disjoint for all  $\sigma < \sigma_0 + \delta$ . Taking  $\delta$



smaller if necessary, we can also assume that if  $B_i^{\sigma_0} \subset U$ , then the  $\sigma/\sigma_0$  expansion of  $B_i^{\sigma_0}$  does not intersect  $\partial U$  for all  $\sigma < \sigma_0 + \delta$ .

For all  $\sigma \in (\sigma_0, \sigma_0 + \delta)$  define  $B_i^\sigma$  to be the  $\sigma/\sigma_0$  expansion of  $B_i^{\sigma_0}$  if  $B_i^{\sigma_0} \subset U$ , and if  $B_i^{\sigma_0}$  intersects  $\partial U$  leave  $B_i^{\sigma_0}$  unchanged, that is, define  $B_i^\sigma = B_i^{\sigma_0}$ . It is clear that for every  $\sigma \in (\sigma_0, \sigma_0 + \delta)$  the collection of balls thus obtained satisfies (6.27). To verify that (6.28) holds, fix some  $i$  such that  $B_i^\sigma \subset U$ , and note that the annulus  $B_i^\sigma \setminus B_i^{\sigma_0}$  does not intersect  $U^\epsilon(1 - k_\epsilon) \subset \cup_j B_j^{\sigma_0}$ . So (6.26) implies that

$$F^\epsilon(B_i^\sigma \setminus B_i^{\sigma_0}) \geq d_i^\sigma (1 - k_\epsilon)^2 \ln \frac{\sigma}{\sigma_0}.$$

Since  $B_i^{\sigma_0}$  satisfies (6.28), the above estimate and (6.25) imply that  $B_i^\sigma$  satisfies (6.28). Finally, (6.29) holds because  $r_i^\sigma/\sigma$  is nonincreasing for every  $i$ ; it is either constant or decreasing, depending on whether  $B_i$  is expanded or left unchanged.

3. Suppose now that  $[\sigma_0, \sigma_1) \subset \Sigma$ . We will show that  $\sigma_1 \in \Sigma$ , thereby completing the proof of the lemma.

To do this, define  $\tilde{B}_i^{\sigma_1}$  as in Step 2, to be the  $\sigma_1/\sigma_0$  expansion of  $B_i^{\sigma_0}$  if  $B_i^{\sigma_0} \subset U$ , and if  $B_i^{\sigma_0}$  intersects  $\partial U$  define  $\tilde{B}_i^{\sigma_1} = B_i^{\sigma_0}$ . These balls have all the required properties, except that in general they need not be pairwise disjoint. So we combine balls to form a new collection that is pairwise disjoint, as in Step 1, without increasing the sum of the radii. Call these balls  $B_i^{\sigma_1}$ . Again (6.27) and (6.29) are easily checked, and (6.28) is a consequence of (6.25) and the fact that  $d_i^{\sigma_1}$  is bounded by the sum of the degrees of the balls from the collection  $\{\tilde{B}_j^{\sigma_1}\}_j$  that were combined to form  $B_i^{\sigma_1}$ .  $\square$

**Lemma 8.** *Assume the hypotheses of Proposition 2. Define*

$$(6.30) \quad \mathcal{G}^{\epsilon_n} := \{s \in (0, R) : \deg(u^{\epsilon_n}; \partial B_s(x_0)) = d\}.$$

Then

$$(6.31) \quad \mathcal{L}^1(\mathcal{G}^{\epsilon_n}) \rightarrow R \quad \text{as } \epsilon_n \rightarrow 0.$$

*Remark 8.* A very similar result appears in [6].

*Proof.* First define

$$\begin{aligned} \mathcal{I}_1^{\epsilon_n} &:= \{s \in (0, R) : 1 - k_{\epsilon_n} < |u^{\epsilon_n}| < 1 + k_{\epsilon_n} \text{ on } \partial B_s(x_0)\} \\ &:= \{s \in (0, R) : \partial B_s(x_0) \cap U^{\epsilon_n}(1 - k_{\epsilon_n}) = \partial B_s(x_0) \cap \tilde{U}^{\epsilon_n}(1 + k_{\epsilon_n}) = \emptyset\}. \end{aligned}$$

Lemma 6 and (6.14) imply that  $\mathcal{H}_\infty^1(U^{\epsilon_n}(1 - k_{\epsilon_n})) \rightarrow 0$  as  $\epsilon_n \rightarrow 0$ . Similarly Remark 7 and (6.14) imply that  $\mathcal{H}_\infty^1(\tilde{U}^{\epsilon_n}(1 + k_{\epsilon_n})) \rightarrow 0$ . It follows that  $\mathcal{L}^1(\mathcal{I}_1^{\epsilon_n}) \rightarrow R$  as  $\epsilon_n \rightarrow 0$ .

If (6.31) is false, we can find a subsequence, still denoted  $\epsilon_n$ , and subsets  $\mathcal{I}_2^{\epsilon_n} \subset \mathcal{I}_1^{\epsilon_n}$  such that  $\mathcal{L}^1(\mathcal{I}_2^{\epsilon_n})$  is bounded away from 0 and either  $\deg(u^{\epsilon_n}; \partial B_s(x_0)) \geq d + 1$  or  $\deg(u^{\epsilon_n}; \partial B_s(x_0)) \leq d - 1$  for all  $s \in \mathcal{I}_2^{\epsilon_n}$ . Assume that the former holds; the other case is similar. We can then define a sequence of test functions of the form  $\eta^n(x) = f^n(|x - x_0|) \in C_c^{0,1}(U)$  such that

$$(f^n)'(s) = \begin{cases} -1 & \text{a.e } s \in \mathcal{I}_2^{\epsilon_n} \\ 0 & \text{a.e } s \notin \mathcal{I}_2^{\epsilon_n} \end{cases}$$

In fact  $f^n(s) := \mathcal{L}^1((s, R) \cap \mathcal{I}_2^{\epsilon_n})$ , and so  $\eta^n(0) = \mathcal{L}^1(\mathcal{I}_2^{\epsilon_n})$ .

Write  $u^{\epsilon_n} = \rho e^{i\phi}$ . Then using the fact that  $Ju^{\epsilon_n} = \frac{1}{2}\nabla \times j(u^{\epsilon_n}) = \frac{1}{2}\nabla \times (\rho^2 D\phi)$  and the coarea formula, one computes that

$$\begin{aligned} \int \eta^n Ju^{\epsilon_n} dx &= \frac{1}{2} \int_{\mathcal{I}_2^{\epsilon_n}} \int_{\partial B_s(x_0)} \rho^2 D\phi \cdot \tau d\mathcal{H}^1 ds \\ &\geq \frac{1}{2} \int_{\mathcal{I}_2^{\epsilon_n}} \left( \pi \deg(u^{\epsilon_n}; \partial B_s(x_0)) - \int_{\partial B_s(x_0)} |(\rho^2 - 1)D\phi| d\mathcal{H}^1 \right) ds. \end{aligned}$$

Since  $\deg(u^{\epsilon_n}; \partial B_s(x_0)) \geq d+1$  for  $s \in \mathcal{I}_2^{\epsilon_n}$ , it is clear that

$$\frac{1}{2} \int_{\mathcal{I}_2^{\epsilon_n}} \pi \deg(u^{\epsilon_n}; \partial B_s(x_0)) ds \geq \pi(d+1) \mathcal{L}^1(\mathcal{I}_2^{\epsilon_n}).$$

Suppose that  $s \in \mathcal{I}_2^{\epsilon_n}$ . Because  $\mathcal{I}_2^{\epsilon_n} \subset \mathcal{I}_1^{\epsilon_n}$ , we have  $1 - k_{\epsilon_n} \leq \rho \leq 1 + k_{\epsilon_n}$  on  $\partial B_s(x_0)$ . Thus (2.1) implies that  $(\rho^2 - 1)^2 \leq CW(\rho^2)$  on  $\partial B_s(x_0)$ , and so

$$\begin{aligned} |(\rho^2 - 1)D\phi| &\leq \frac{\epsilon_n}{2} |D\phi|^2 + \frac{1}{2\epsilon_n} (\rho^2 - 1)^2 \\ &\leq C\epsilon_n \left( \frac{1}{2} \rho^2 |D\phi|^2 + \frac{1}{2\epsilon_n^2} W(\rho^2) \right) \leq C\epsilon_n E^{\epsilon_n}(u^{\epsilon_n}) \end{aligned}$$

on  $\partial B_s(x_0)$ . Hence (using the coarea formula again)

$$\frac{1}{2} \int_{\mathcal{I}_2^{\epsilon_n}} \int_{\partial B_s(x_0)} |(\rho^2 - 1)D\phi| d\mathcal{H}^1 ds \leq C\epsilon_n \int_U E^{\epsilon_n}(u^{\epsilon_n}) \leq K\epsilon_n \left( \ln \frac{1}{\epsilon_n} + 1 \right).$$

So we conclude that  $\int \eta^n Ju^{\epsilon_n} dx \geq \pi(d+1) \mathcal{L}^1(\mathcal{I}_2^{\epsilon_n}) - o(1)$  as  $\epsilon_n \rightarrow 0$ . However, this is impossible, since the weak convergence  $Ju^{\epsilon_n} \rightarrow \bar{J}$  implies that

$$\left| \int \eta^n Ju^{\epsilon_n} dx - \pi d \eta^n(0) \right| = \left| \int \eta^n Ju^{\epsilon_n} dx - \pi d \mathcal{L}^1(\mathcal{I}_2^{\epsilon_n}) \right| \rightarrow 0$$

as  $n \rightarrow \infty$ . □

We now give the

*Proof of Proposition 2.* Recalling the definition of  $F^{\epsilon_n}$ , it will suffice to demonstrate that  $F^{\epsilon_n}(U) \geq d |\ln \epsilon_n| (1 - o(1))$  as  $n \rightarrow \infty$ .

Fix  $n_0$  so large that  $\mathcal{L}^1(\mathcal{G}^{\epsilon_n}) \geq R/2$  for all  $n \geq n_0$ . This is possible by Lemma 8.

Fix some  $n \geq n_0$  and let  $\bar{\sigma} = R/(6C\epsilon_n(|\ln \epsilon_n| + 1)^3)$ , where  $C$  is the constant in (6.29). Consider the collection of balls  $\mathcal{B}(\bar{\sigma})$  given by Lemma 7. The choice of  $\bar{\sigma}$  with (6.29) guarantees that  $\sum r_i^{\bar{\sigma}} \leq R/6$ . It follows that there must be some  $s \in \mathcal{G}^{\epsilon_n}$  such that  $\partial B_s(x_0) \cap B_i^{\bar{\sigma}} = \emptyset$  for all  $i$ . Then the additivity of degree implies that

$$d = \deg(u^{\epsilon_n}; \partial B_s(x_0)) = \sum_{\{i : B_i^{\bar{\sigma}} \subset B_s(x_0)\}} \deg(u^{\epsilon_n}; \partial B_i^{\bar{\sigma}}) \leq \sum d_i^{\bar{\sigma}}.$$

Then (6.25), (6.27), and our choice of  $\bar{\sigma}$  imply that

$$F^{\epsilon_n}(U) \geq \sum F^{\epsilon_n}(B_i^{\bar{\sigma}}) \geq d(1 - k_{\epsilon_n})^2 \ln \bar{\sigma} \geq d |\ln \epsilon_n| (1 - o(1))$$

as  $n \rightarrow \infty$ . □

7. LIMITS OF  $k_\epsilon Du^\epsilon \otimes Du^\epsilon$ , CONTINUED.

In this final section we analyze the relationship between limits of  $Ju^\epsilon$  and  $k_\epsilon Du^\epsilon \otimes Du^\epsilon$  in  $m \geq 3$  space dimensions. The main result of this section is

**Theorem 6.** *Suppose that  $u^\epsilon \in H^1(\Omega; \mathbb{R}^2)$  for  $\epsilon \in (0, 1]$ , where  $\Omega$  is an open subset of  $\mathbb{R}^m$ ,  $m \geq 3$ . Assume that*

$$(7.1) \quad \limsup_{\epsilon \rightarrow 0} k_\epsilon \int_{\Omega} E^\epsilon(u^\epsilon) dx < \infty$$

and let  $\epsilon_n \rightarrow 0$  be a sequence such that

$$(7.2) \quad Ju^{\epsilon_n} \rightarrow \bar{J} \quad \text{in } C_{loc}^{0,\gamma^*}(\Omega) \quad \forall \gamma > 0$$

where  $\bar{J}$  has the form  $\pi\theta\nu\mathcal{H}^{m-2}|_{\Gamma}$  for some oriented i.m. rectifiable  $(\Gamma, \theta, \star\nu)$ ; and

$$(7.3) \quad k_{\epsilon_n} Du^{\epsilon_n} \otimes Du^{\epsilon_n} \rightharpoonup Q \in C^{0*}(\Omega; S^{m \times m}) \quad \text{weakly in } C^{0*}(\Omega).$$

Let  $Q_d := Q - P^\perp|J|$ , where for  $\mathcal{H}^{m-2}$  a.e.  $x \in \Gamma$ ,  $P^\perp(x)$  is the projection onto  $(\text{ap}T_x\Gamma)^\perp$ . Then  $\text{Tr}Q_d \geq 0$ , and there exist constants  $c_1, c_2$  (depending on the dimension  $m$ ) such that

$$(7.4) \quad |Q_d(A)| \leq c_1 (\text{Tr}Q_d(A) |\bar{J}|(A))^{1/2} + c_2 \text{Tr}Q_d(A).$$

for every measurable  $A \subset \Omega$ . In particular, if  $\text{Tr}Q(A) = 2|\bar{J}|(A)$  for some  $A \subset \Omega$ , then in fact  $Q = P^\perp|\bar{J}|$  in  $A$ .

The proof of this result relies on a refinement of Theorem 1 that asserts that, roughly speaking, two-dimensional slices of  $Ju^\epsilon$  converge to two-dimensional slices of  $\bar{J}$ . This will allow us essentially to reduce Theorem 6 to Theorem 5. Before stating this refined compactness theorem we introduce some notation.

We continue to write  $x$  to denote typical points in  $\mathbb{R}^m$ . It will frequently be convenient to decompose  $x \in \mathbb{R}^m$  in the form  $x \cong (y, z) \in \mathbb{R}_y^{m-2} \times \mathbb{R}_z^2$ , where  $y_i = x_i$  for  $i = 1, \dots, m-2$  and  $z_i = x_{m-2+i}$  for  $i = 1, 2$ .

Suppose that  $O \subset U$  is an open subset of the form  $O = O_y \times O_z$ , where  $O_y \subset \mathbb{R}_y^{m-2}$  and  $O_z \subset \mathbb{R}_z^2$ . We say that a measure  $\mu$  on  $O$  is represented by slices  $\mu_y(dz)$  if for Lebesgue almost every  $y \in O_y$  there exists a measure  $\mu_y(d \cdot)$  on  $O_z$  such that  $y \rightarrow \mu_y(dz)$  is weakly measurable and

$$(7.5) \quad \int_O \phi(x) d\mu(x) = \int_{O_y} \int_{O_z} \phi(y, z) \mu_y(dz) dy.$$

We say that  $\mu$  is locally represented by slices  $\mu_y(dz)$  if it is represented by slices on every open set of the above form.

**Theorem 7.** *Suppose that  $\{u^\epsilon\}_{\epsilon \in (0,1]}$  is a family of functions in  $H^1(U; \mathbb{R}^m)$  for  $U \subset \mathbb{R}^2$  such that  $\limsup_{\epsilon \rightarrow 0} I^\epsilon(u^\epsilon) \leq K$ . Then  $\{Ju^\epsilon\}_{\epsilon \in (0,1]}$  is strongly precompact in  $(C^{0,\gamma})^*$  for all  $\gamma > 0$ . Moreover, if  $\bar{J} = \bar{J}^{ij} e_i \wedge e_j$  is any weak limit of a subsequence  $Ju^{\epsilon_n}$  that converges in the above sense, then*

(i)  $\bar{J}$  has the form  $\pi\theta\nu\mathcal{H}^{m-2}|_{\Gamma}$  where  $(\Gamma, \theta, \star\nu)$  is some oriented i.m. rectifiable set.

(ii) For any choice of orthonormal basis for  $\mathbb{R}^m$  (determining a decomposition of  $\mathbb{R}^m$  into  $\mathbb{R}_y^{m-2} \times \mathbb{R}_z^2$ ),  $\bar{J}^z = dz \cdot \bar{J}$  is locally represented by slices  $\bar{J}_y(dz)$ , and these slices have the form  $\bar{J}_y(dz) = \pi \sum_i d_i \delta_{a_i}(dz)$  for integers  $d_i$  and points  $a_i \in \mathbb{R}_z^2$ .

(iii) Suppose that  $O \subset U$  is an open subset of the form  $O = O_y \times O_z$ , where  $O_y \subset \mathbb{R}^{m-2}$  and  $O_z \subset \mathbb{R}^{m-2}$ . Then for a.e.  $y \in O_y$ ,

$$dz \cdot Ju^{\epsilon_{n_k}}(y, \cdot) = \det(u_{z_1}^{\epsilon_{n_k}}, u_{z_2}^{\epsilon_{n_k}})(y, \cdot) \rightarrow \bar{J}_y(d \cdot)$$

in  $C^{0,\gamma}(O_z)^*$  for all  $\gamma > 0$ , whenever  $\epsilon_{n_k}$  is a subsequence such that

$$\limsup k_{\epsilon_{n_k}} \int_{O_z} E^{\epsilon_{n_k}}(u^{\epsilon_{n_k}})(y, z) dz < \infty.$$

$$(iv) |\bar{J}|(\mathbb{R}^m) \leq \liminf_{n \rightarrow \infty} I^{\epsilon_n}(u^{\epsilon_n})$$

This is Theorem 5.2 in [12]. Assertion (iii) is not included in the statement of the theorem in [12] but is established in Steps 3 and 4 of the proof.

We will also need the following easy

**Lemma 9.** *If  $Q$  is a matrix-valued measure and  $\nu$  is a nonnegative measure, then*

$$(7.6) \quad \left| \frac{dQ}{d\nu} \right| = \frac{d|Q|}{d\nu} \quad \nu \text{ almost everywhere.}$$

*Proof.* By examining the definitions one can then check that  $\frac{dQ}{d\nu} = \frac{dQ}{d|Q|} \frac{d|Q|}{d\nu}$ ,  $\nu$  a.e. Since  $|\frac{dQ}{d|Q|}(x)| = 1$  for  $|Q|$  a.e.  $x$ , and thus for  $\nu$  a.e.  $x \in \text{supp} \frac{dQ}{d\nu} \nu = \text{supp} \frac{d|Q|}{d\nu} \nu$ , this implies (7.6).  $\square$

In the remainder of this section we give the

*Proof of Theorem 6.* 1. We first claim that it suffices to show that  $\text{Tr} \frac{dQ_d}{d|\bar{J}|} \geq 0$  and

$$(7.7) \quad \frac{d|Q_d|}{d|\bar{J}|} \leq c_1 \left( \text{Tr} \frac{dQ_d}{d|\bar{J}|} \right)^{1/2} + c_2 \text{Tr} \frac{dQ_d}{d|\bar{J}|}$$

$|\bar{J}|$  almost everywhere.

Indeed, suppose that this estimate holds, and let

$$M := \int_A \frac{dQ_d}{d|\bar{J}|} d|\bar{J}| \leq Q_d(A).$$

Then

$$\begin{aligned} |M| &\leq \int_A \frac{d|Q_d|}{d|\bar{J}|} d|\bar{J}| \\ &\leq \int_A \left[ c_1 \left( \text{Tr} \frac{dQ_d}{d|\bar{J}|} \right)^{1/2} + c_2 \text{Tr} \frac{dQ_d}{d|\bar{J}|} \right] d|\bar{J}| \\ &\leq c_1 \left( \int_A \text{Tr} \frac{dQ_d}{d|\bar{J}|} d|\bar{J}| \right)^{1/2} \left( \int_A d|\bar{J}| \right)^{1/2} + c_2 \int_A \text{Tr} \frac{dQ_d}{d|\bar{J}|} d|\bar{J}| \\ &= c_1 (|\bar{J}|(A) \text{Tr} M)^{1/2} + c_2 \text{Tr} M. \end{aligned}$$

Thus  $g(M, |\bar{J}|(A)) \leq 0$ , in the notation of (6.6). Since  $Q_d(A) \geq M$ , the monotonicity properties of  $g$  imply that  $g(Q_d(A), |\bar{J}|(A)) \leq 0$ , which is (7.4).

2. We write  $\bar{J}$  in the form  $\nu|\bar{J}|$ , where  $|\bar{J}| = \pi\theta\mathcal{H}^{m-2}|_{\Gamma}$ . Recall that  $\nu$  is a  $|\bar{J}|$ -measurable function taking values in  $\Lambda^2\mathbb{R}^m$ , such that  $|\nu(x)| = 1$  for  $|\bar{J}|$ -a.e.  $x \in \mathbb{R}^m$ . In addition,  $\nu$  has the form  $\nu = \nu^1 \wedge \nu^2$  for orthonormal unit vectors

$\nu^i \in \Lambda^1 \mathbb{R}^m$  at  $|\bar{J}|$  a.e.  $x$ . General theorems on differentiation of measures imply that

$$(7.8) \quad \lim_{r \rightarrow 0} \frac{1}{|\bar{J}|(B_r(x))} \int_{B_r(x)} |\nu(x') - \nu(x)| |\bar{J}|(dx') = 0$$

and

$$(7.9) \quad \lim_{r \rightarrow 0} \frac{Q(B_r(x))}{|\bar{J}|(B_r(x))} := \frac{dQ}{d|\bar{J}|}(x) \quad \text{exists}$$

for  $|\bar{J}|$ -a.e.  $x \in \mathbb{R}^m$ . It thus suffices to prove (7.4) at every point  $x$  where (7.8) and (7.9) hold and  $\nu(x) = \nu^1 \wedge \nu^2$  is simple with  $\{\nu^1, \nu^2\}$  spanning  $(\text{ap}T_x \Gamma)^\perp$ .

Fix a point  $x_0$  satisfying these conditions. After a change of basis we can assume that  $\nu(x_0) = \nu^1(x_0) \wedge \nu^2(x_0) = e_{m-1} \wedge e_m$ . We decompose  $\mathbb{R}^m$  as  $\mathbb{R}^{m-2} \times \mathbb{R}^2$ , and we write  $x_0 = (y_0, z_0)$ . We write  $\bar{J} = \sum_{i < j} \bar{J}^{ij} e_i \wedge e_j$  in the new coordinate system. We will focus on the scalar signed measure  $\bar{J}^{m-1, m} = e_{m-1} \wedge e_m \cdot \nu \bar{J}$ , which we will write  $\bar{J}^z$  for short.

If  $r$  is sufficiently small — which we will henceforth assume to be the case — then  $B_r^{m-2}(y_0) \times B_r^2(z_0) \subset \Omega$ , and so according to Theorem 7, for Lebesgue almost every  $y \in B_r^{m-2}(y_0)$  there exists a measure  $\bar{J}_y(dz)$  on  $B_r^2(z_0)$  such that

$$\int_{B_r^m(x_0)} \phi(x) \bar{J}^z(dx) = \int_{B_r^{m-2}(y_0)} \int_{B_r^2(z_0)} \phi(y, z) \bar{J}_y(dz) dy$$

for all  $\phi \in C_c(B_r^m(x_0))$ . Also, Theorem 7 asserts that for a.e.  $y \in B_r^{m-2}$  if  $\epsilon_{n_k}$  is any subsequence such that

$$(7.10) \quad \limsup k_{\epsilon_{n_k}} \int_{B_r^2(z_0)} E^{\epsilon_{n_k}}(u^{\epsilon_{n_k}})(y, z) dz < \infty$$

then

$$(7.11) \quad \det(u_{z_1}^{\epsilon_{n_k}}, u_{z_2}^{\epsilon_{n_k}})(y, z) \rightarrow \bar{J}_y(dz) \text{ in } C^{0, \gamma}(B_r^2(z_0))^* \quad \forall \gamma > 0,$$

Note in addition that by (7.8),

$$(7.12) \quad \bar{J}^z(B_r(x_0)) = \nu(x_0) \cdot \int_{B_r(x_0)} \nu(x) |\bar{J}|(dx) = (1 - o_r(1)) |\bar{J}|(B_r(x_0)).$$

A further consequence of (7.8) is that  $x_0$  is  $\frac{dP^\perp |\bar{J}|}{d|\bar{J}|}$  exists at  $x_0$  and equals  $P_z$ , the  $m \times m$  matrix corresponding to projection onto  $\mathbb{R}_z^2$ .

3. We will write  $\bar{Q}$  as shorthand for  $\frac{dQ}{d|\bar{J}|}(x_0)$ , and similarly  $\bar{Q}_d$  for  $\frac{dQ_d}{d|\bar{J}|}$ . Note that  $\bar{Q}_d = \frac{dQ}{d|\bar{J}|}(x_0) - \frac{dP^\perp |\bar{J}|}{d|\bar{J}|}(x_0) = \bar{Q} - P_z$ , and also that  $\bar{Q}$  is nonnegative definite.

We define submatrices  $R \in \mathcal{S}^{m-2 \times m-2}$  and  $S \in \mathcal{S}^{2 \times 2}$  by

$$(7.13) \quad R_{ij} = \bar{Q}_{ij}, \quad i, j \in 1, \dots, m-2; \quad S_{ij} = \bar{Q}_{(m-2+i)(m-2+j)}, \quad i, j \in 1, 2.$$

Both of these are nonnegative definite. We also define  $R_d$  and  $S_d$  to be the corresponding  $m-2 \times m-2$  and  $2 \times 2$  submatrices of  $\bar{Q}_d$ , so that in fact  $R = R_d$  and  $S_d = S - \text{id}$ .

We now claim that it suffices to prove that

$$(7.14) \quad \alpha(S) \geq 1.$$

Indeed, suppose that this holds. Since  $\bar{Q}$  is nonnegative definite,

$$(7.15) \quad |\bar{Q}_{ij}| \leq (\bar{Q}_{ii} \bar{Q}_{jj})^{1/2}$$

for all  $i, j$ .

If  $\alpha(S) \geq 1$ , then the definition and the monotonicity properties of  $\alpha(\cdot)$  imply that  $|S_d| \leq c_1(\text{Tr}S_d)^{1/2} + c_2\text{Tr}S_d$ . This immediately implies that  $|\bar{Q}_{d,ij}| \leq c_1(\text{Tr}\bar{Q}_d)^{1/2} + c_2\text{Tr}\bar{Q}_d$  if  $i, j \in \{m-1, m\}$ .

If  $i, j \in \{1, \dots, m-2\}$  then (7.15) implies that

$$|\bar{Q}_{d,ij}| = |\bar{Q}_{ij}| \leq \bar{Q}_{ii} + \bar{Q}_{jj} \leq \text{Tr}R \leq \text{Tr}\bar{Q}_d.$$

And if  $i \in \{1, \dots, m-2\}, j \in \{m-1, m\}$  then (7.15) yields

$$\begin{aligned} |\bar{Q}_{d,ij}| &= |\bar{Q}_{ij}| \leq (\text{Tr}R \text{Tr}S_d)^{1/2} \\ &= ((\text{Tr}R) (2 + \text{Tr}S_d))^{1/2} \\ &\leq \sqrt{2}(\text{Tr}R)^{1/2} + (\text{Tr}R \text{Tr}S_d)^{1/2} \\ &\leq \sqrt{2}(\text{Tr}R)^{1/2} + \frac{1}{2}(\text{Tr}R + \text{Tr}S_d) \\ &\leq \sqrt{2}(\text{Tr}\bar{Q}_d)^{1/2} + \frac{1}{2}\text{Tr}\bar{Q}_d. \end{aligned}$$

Thus (7.14) implies (7.7) for certain constants  $c_1, c_2$  that depend on the dimension and in particular are larger than the  $c_i$  appearing in the definition of  $\alpha$ .

4. Let  $D_z u^\epsilon = (u_{z_1}^\epsilon, u_{z_2}^\epsilon)$ , and define

$$S_r^{\epsilon_n} = k_{\epsilon_n} \int_{B_r^m} D_z u^{\epsilon_n} \otimes D_z u^{\epsilon_n} dx.$$

Note that

$$S = \lim_{r \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{|\bar{J}|(B_r)(x_0)} S_r^{\epsilon_n}.$$

This limit exists, since we have chosen  $x_0$  to satisfy (7.9). We rewrite

$$S_r^{\epsilon_n} = \int_{B_r^{m-2}(y_0)} S_r^{\epsilon_n}(y) dy$$

where

$$S_r^{\epsilon_n}(y) = \int_{\{y\} \times B_{r(y)}^2(z_0)} k_{\epsilon_n} D_z u^{\epsilon_n} \otimes D_z u^{\epsilon_n} \mathcal{H}^2(dz) \quad r(y) = (r^2 - |y - y_0|^2)^{1/2}.$$

It is not hard to verify from the definition that that  $\alpha(\lambda M) = \lambda \alpha(M)$  for all  $M \geq 0$  and  $\lambda \geq 0$ , so in view of the continuity of  $\alpha$ , to prove (7.14) we need to show that  $\liminf_{n \rightarrow \infty} \alpha(S_r^{\epsilon_n}) \geq (1 - o_r(1))|\bar{J}|(B_r(x_0))$ . Now Lemma 4 implies that

$$\liminf_{n \rightarrow \infty} \alpha(S_r^{\epsilon_n}) \geq \liminf_{n \rightarrow \infty} \int_{B_r^{m-2}(y_0)} \alpha(S_r^{\epsilon_n}(y)) dy$$

So it will finish the proof when we show that

$$(7.16) \quad \liminf_{n \rightarrow \infty} \int_{B_r^{m-2}(y_0)} \alpha(S_r^{\epsilon_n}(y)) dy \geq (1 - o_r(1))|\bar{J}|(B_r(x_0)).$$

5. For  $y \in B_r^{m-2}(y_0)$  we will write  $j(y) := |\bar{J}_y^z|(B_{r(y)}(z_0))$ . Note that

$$(7.17) \quad \int_{B_r^{m-2}(y_0)} j(y) dy = |\bar{J}^z|(B_r^m(x_0)) = (1 - o_r(1))|\bar{J}|(B_r^m(x_0))$$

using (7.12).

For  $y \in B_r^{m-2}(y_0)$ , we define  $\tilde{\alpha}^{\epsilon_n}(y) := \min\{\alpha(S_r^{\epsilon_n}(y)), j(y)\}$ . We claim that

$$(7.18) \quad \tilde{\alpha}^{\epsilon_n}(\cdot) \text{ converges to } j(\cdot) \text{ in measure on } B_r^{m-2}(y_0) \text{ as } n \rightarrow \infty.$$

To prove this, we show that given any  $\delta > 0$ , we can construct a new sequence  $\{\alpha_\delta^{\epsilon_n}\}_{n \geq 1}$  such that

$$(7.19) \quad \limsup_{n \rightarrow \infty} \mathcal{L}^{m-2}(\{y \in B_r^{m-2}(y_0) : \tilde{\alpha}^{\epsilon_n}(y) \neq \alpha_\delta^{\epsilon_n}(y)\}) \leq \delta$$

and such that  $\alpha_\delta^{\epsilon_n}(y) \rightarrow j(y)$  a.e.  $y$ . Indeed, by Chebychev's inequality and the upper bound on the energies (7.1), given any  $\delta$  we can find some number  $K$  such that  $\limsup_{n \rightarrow \infty} \mathcal{L}^{m-2}(\mathcal{Z}_K^n) \leq \delta$ , for

$$\mathcal{Z}_K^n := \left\{ y \in B_r^{m-2}(y_0) : k_{\epsilon_n} \int_{B_{r(y)}^2(z_0)} E^{\epsilon_n}(u^{\epsilon_n})(y, z) dz \geq K \right\}.$$

We define

$$\tilde{\alpha}_\delta^{\epsilon_n}(y) := \begin{cases} \tilde{\alpha}^{\epsilon_n}(y) & \text{if } y \in B_r^{m-2}(y_0) \setminus \mathcal{Z}_K; \\ j(y) & \text{if } y \in \mathcal{Z}_K; \end{cases}$$

It is clear from the definitions that (7.19) holds.

Fix  $y \in B_r^{m-2}(y_0)$  and consider any subsequence  $n_k$ . Passing to a further subsequence (which we still label  $n_k$ ) we may assume that either  $y \in \mathcal{Z}_K^{n_k}$  or  $y \notin \mathcal{Z}_K^{n_k}$  for all  $k$ . If the former holds then trivially  $\tilde{\alpha}_\delta^{\epsilon_{n_k}}(y) \rightarrow j(y)$ . If  $y \notin \mathcal{Z}_K^{n_k}$  for all  $k$ , then (7.10) is satisfied, and as a result (7.11) holds, unless  $y$  belongs to some exceptional set of measure zero. According to Remark 6, however, (7.10) and (7.11) together imply that

$$\liminf_k \alpha(S_r^{\epsilon_{n_k}}(y)) \geq |\bar{J}_y|(B_{r(y)}(z_0)) = j(y).$$

Since  $\tilde{\alpha}_\delta^{\epsilon_{n_k}}(y) = \tilde{\alpha}^{\epsilon_{n_k}}(y) = \min\{\alpha(S_r^{\epsilon_{n_k}}), j(y)\}$  we conclude that  $\tilde{\alpha}_\delta^{\epsilon_{n_k}}$  converges a.e. to  $j$ , thus establishing (7.18).

6. From (7.18) and Fatou's lemma we deduce that

$$\liminf_n \int_{B_r^{m-2}(y_0)} \alpha(S_r^{\epsilon_n}(y)) dy \geq \liminf_n \int_{B_r^{m-2}(y_0)} \tilde{\alpha}^{\epsilon_n}(y) dy \geq \int_{B_r^{m-2}(y_0)} j(y) dy.$$

Thus (7.16) follows from (7.17), so we have finished the proof.  $\square$

## 8. ACKNOWLEDGMENTS

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