Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Rigorous bounds for the Föppl-von Kármán theory of isotropically compressed plates

by

Hafedh Ben Belgacem, Sergio Conti, Antonio DeSimone and Stefan Müller

Preprint no.: 47 2000

Rigorous bounds for the Föppl-von Kármán theory of isotropically compressed plates

Hafedh Ben Belgacem,[1] Sergio Conti,[1]
Antonio DeSimone,[1,2] and Stefan Müller[1]

July 18, 2000

Abstract

We study the Föppl-von Kármán theory for isotropically compressed thin plates in a geometrically linear setting, which is commonly used to model weak buckling of thin films. We consider generic smooth domains with clamped boundary conditions, and obtain rigorous upper and lower bounds on the minimum energy linear in the plate thickness σ. This energy is much lower than previous estimates based on certain dimensional reductions of the problem, which had lead to energies of order $1 + \sigma$ (scalar approximation) or $\sigma^{2/3}$ (two-component approximation).

1 Introduction

The Föppl-von Kármán (FvK) equations of thin-plate elasticity [16, 7, 1, 20] describe stretching and bending of a thin, homogeneous, linearly elastic plate of uniform thickness in terms of a three-component, two-dimensional displacement field. The equations are nonlinear and involve high-order derivatives, hence many simplified forms have been proposed in the literature, which have proved appropriate for the study of different phenomena. These include crumpling of paper, structural failure of steel plates, and telephone-cord delamination in thin films (see e.g. [6, 18, 3], and references therein). In the case of small applied compressive strain, assuming that the plate deviates only slightly from a flat surface, one can expand the strain term to leading order in the in-plane (u_x, u_y) and one out-of-plane (w) components of the displacement field, and replace the bending term with a simple singular perturbation in the out-of-plane component. For isotropic materials and isotropic compression one obtains, after some rescalings,

$$I^{(\sigma)}_{\mathrm{FvK}}[u, w, \Omega] = \int_\Omega \left[\nabla w \otimes \nabla w + \nabla u + (\nabla u)^T - Id\right]^2 + \sigma^2 |\nabla^2 w|^2 \; d^2 r\,, \tag{1.1}$$

[1]Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22-26, D 04103 Leipzig, Germany
[2]Dipartimento di Ingegneria Strutturale, Politecnico di Bari, Italy

where σ is the rescaled plate thickness, and for simplicity the Poisson ratio ν has been set to zero (this entails no loss of generality of the results of this paper, see the Appendix). Clamped boundary conditions result in $u = w = \nabla w = 0$ on $\partial\Omega$. The functional (1.1) is quadratic (and convex) in the two in-plane components u of the displacement field, but the strain energy is not convex in the out-of-plane component w. Gioia and Ortiz [19, 10] observed that good agreement with many experimental observations is obtained by considering a simplified theory, in which the in-plane displacements are neglected. By setting $u = 0$, the problem is reduced to the search for a scalar, two-dimensional field describing vertical displacement from the reference plane. The resulting functional, which for isotropic compression has the form

$$\int_\Omega 1 + (1 - |\nabla w|^2)^2 + \sigma^2 |\nabla^2 w| \, d^2 r \,, \tag{1.2}$$

had previously appeared in a different context [4], and has been extensively studied in the mathematical literature partly due to its similarity with the Modica-Mortola functionals in the theory of Gamma-convergence. Deriving the Gamma-limit of (1.2) is still an open problem, even if considerable progress has been recently achieved [5, 2, 11, 8]. The energy of the minimizers scales as $|\Omega| + c\sigma$. A natural approximate solution for small σ, which is generally conjectured to be the correct limit, is the distance function from the boundary. The analogue of (1.2) for the case of nonisotropic compression has also been studied [9].

In parallel with the mathematical progress on the restricted functional (1.2), Jin and Sternberg have relaxed the constraint of zero in-plane displacements [13]. Within a larger (but still restricted) class of functions they have been able to prove that the optimal energy vanishes as $\sigma^{2/3}$ for $\sigma \to 0$, adapting - for the upper bound - a self-similar branching construction used by Kohn and Müller [14, 15] to describe twin refinement in a model of martensitic microstructure. In a different linear stability framework, Audoly [3] also emphasized the importance of the in-plane components.

In this work, we consider the full linearized FvK energy under compressive isotropic strain (1.1), and prove that the actual minimum energy scales linearly in σ. Hence we show that the Jin-Sternberg prediction of vanishing energy for $\sigma \to 0$ is correct, but the scaling in the full problem (1.1) is different from the one in the restricted functional. Some of our candidate minimizers are reminiscent of the distance function, but - at the present level of understanding - this does not appear to be a necessary feature in order to have small energy. In particular, our only structural finding is that, as σ goes to zero, a finite fraction of the total energy concentrates in a thin strip (of width σ) along the boundary.

The arguments for the lower and upper bounds are distinct, and are presented in Section 2 and 3 respectively. In both cases, for the sake of clarity we found it useful to perform first the proof for a simple rectangular geometry, and then show the generalization to a smooth curved boundary. For the lower bound Lipschitz regularity of the boundary suffices, while the upper bound is established for a piecewise C^4 boundary. The latter condition can be relaxed, but we don't pursue this here in order to minimize technicalities. Our approach can be extended to other, nonlinear plate theories and to the full three-dimensional problem. This will be treated in a forthcoming publication.

While writing the present paper, we have become aware of related but independent work of Jin and Sternberg [12], which reaches the same conclusions for the case of a square domain with clamped boundary conditions on two sides and periodic ones on the other two.

2 Lower bound

Let $\Psi = (u, w)$ denote the displacement field. In this Section we prove the lower bound on $I^{(\sigma)}_{\rm FvK}[\Psi, \Omega]$, which gives the following

Theorem 1 (Lower bound) *Let Ω be an open, bounded subset of $\boldsymbol{R}^2$, with Lipschitz boundary. Then, there is a positive constant c_Ω such that for sufficiently small σ one has*

$$I^{(\sigma)}_{\rm FvK}[u, w, \Omega] \geq c_\Omega \sigma \tag{2.1}$$

for any displacement field $\Psi = (u, w)$ which satisfies the boundary conditions $u = w = \nabla w = 0$ on $\partial\Omega$.

For the sake of clarity we first consider the case of a piecewise straight boundary, and then show how the argument is generalized to the case of Lipschitz boundaries. In both cases, we actually prove that any region of size $\sigma \times \sigma$ adjacent to the boundary contains an energy density of order 1. In view of the upper bound presented in Section 3, this implies that for $\sigma \to 0$ a finite fraction of the total energy concentrates on the boundary.

Before starting the proof, we mention some general properties of the functional $I_{\rm FvK}$ defined in Eq. (1.1), which can be more explicitly written as

$$I^{(\sigma)}_{\rm FvK}[u_x, u_y, w] = \int_\Omega (w_{,x}^2 + 2u_{x,x} - 1)^2 + 2(w_{,x}w_{,y} + u_{x,y} + u_{y,x})^2 + (w_{,y}^2 + 2u_{y,y} - 1)^2 + \sigma^2 |\nabla^2 w|^2 \,. \tag{2.2}$$

We shall call strain energy ($I_{\rm strain}$) the first three terms, which depend only on $\nabla\Psi$, and bending energy ($I_{\rm bending}$) the singular perturbation proportional to $|\nabla^2 w|^2$. The standard elasticity scaling gives

$$I^{(\sigma)}_{\rm FvK}[\Psi, \Omega] = \frac{1}{\lambda^2} I^{(\lambda\sigma)}_{\rm FvK}[\Psi^{(\lambda)}, \lambda\Omega] \tag{2.3}$$

where $\Psi(x)^{(\lambda)} = \lambda\Psi(x/\lambda)$. In closing, we notice that $I_{\rm FvK}$ is invariant under changes of coordinates, as is evident from the vectorial representation in Eq. (1.1) (this is not to be confused with invariance under composition of Ψ with a rotation, which does not hold for this geometrically linear model).

If the boundary has a flat part, then for small enough σ we can choose coordinates such that the rectangle $R = (0, \sigma) \times (0, L)$ is contained in Ω and $\{0\} \times (0, L) \subset \partial\Omega$. The rectangle R contains $\lfloor L/\sigma \rfloor$ disjoint squares, $(0, \sigma) \times ((k-1)\sigma, k\sigma)$, for $k = 1, \ldots, \lfloor L/\sigma \rfloor$. We intend to prove that each of them has energy density at least of order 1, which implies the thesis. More precisely, we have

Lemma 1 *Let $Q_\sigma = (0, \sigma)^2$. There is a positive constant c^* such that if $u = w = \nabla w = 0$ for $x = 0$, then*

$$I^{(\sigma)}_{\rm FvK}[u, w, Q_\sigma] \geq c^* \sigma^2 \,. \tag{2.4}$$

Proof. The statement is invariant under rescaling in σ, hence we can assume $\sigma = 1$. We reason by contradiction, and assume that there is a sequence (u^j, w^j) such that $I^{(1)}_{\mathrm{FvK}}[u^j, w^j, Q_1] \to 0$. Then $\nabla^2 w^j \to 0$ in L^2 and since $w^j = \nabla w^j = 0$ on one side of the square, this implies $w^j \to 0$ in $W^{2,2}$. Thus $|\nabla w^j|^2 \to 0$ in $L^1(Q_1)$ and therefore

$$\int_{Q_1} \left|2u^j_{x,x} - 1\right| + 2\left|u^j_{x,y} + u^j_{y,x}\right| + \left|2u^j_{y,y} - 1\right| \to 0 \,. \tag{2.5}$$

For $\beta \in [0, 1]$ consider

$$\chi^j_\beta(s,t) = 2(u^j_x + \beta u^j_y)(s, \beta s + t) - (1+\beta^2)s \,. \tag{2.6}$$

Then

$$\partial_s \chi^j_\beta(s,t) = \left[(2u^j_{x,x} - 1) + 2\beta(u^j_{x,y} + u^j_{y,x}) + \beta^2(2u^j_{y,y} - 1)\right](s, \beta s + t) \to 0 \quad \text{in } L^1(Q_{1/2}) \tag{2.7}$$

by (2.5). Since $\chi^j_\beta(0,t) = 0$ the Poincaré inequality implies that for $s \le 1/4$

$$\int_{1/4}^{1/2} \left|2(u^j_x + \beta u^j_y)(s,y) - (1+\beta^2)s\right| dy \le \int_0^{1/2} \left|\chi^j_\beta(s,t)\right| dt \to 0 \tag{2.8}$$

for all $\beta \in [0, 1]$. This easily leads to a contradiction (take e.g. $\beta = 0$, $\beta = 1$ and $\beta = 1/2$), hence proves the thesis. •

Proof of Theorem 1 Consider a Lipschitz domain Ω. Then by definition $\partial\Omega$ is locally the graph of a Lipschitz function h and Ω lies locally to one side of $\partial\Omega$. Exploiting again rotational invariance it suffices to show that there is a constant c^*, only depending on the Lipschitz constant L of h, such that

$$I^{(\sigma)}_{\mathrm{FvK}}[u, w, \tilde{Q}_\sigma] \ge c^* \sigma^2 \tag{2.9}$$

for all domains

$$\tilde{Q}_\sigma = \{(x,y) : \ x = h(y) + \tilde{s}, \ \tilde{s} \in (0,\sigma), \ y \in (0,\sigma)\} \tag{2.10}$$

where for simplicity $h(0) = 0$. We may again assume $\sigma = 1$ since scaling leaves the Lipschitz constant of h invariant. Arguing again by contradiction, we assume that there are sequences u^j, w^j and h^j such that

$$I^{(1)}_{\mathrm{FvK}}[u^j, w^j, \tilde{Q}^j_1] \to 0 \tag{2.11}$$

with $\mathrm{Lip}\, h^j \le L$. Taking a subsequence and relabeling we can further assume that $h^j \to h^\infty$ uniformly. Note that the change of variables

$$\tilde{s} = x - h^j(y) \,, \qquad \tilde{t} = y \tag{2.12}$$

is a volume preserving map of $\tilde{Q}^j_1$ onto Q_1. Let $f^j(\tilde{s}, \tilde{t}) = (\nabla w^j)(\tilde{s} + h^j(\tilde{t}), \tilde{t})$. Then $\partial_{\tilde{s}} f^j \to 0$ in $L^2(Q_1)$, and thus

$$||\nabla w^j||_{L^2(\tilde{Q}^j_1)} = ||f^j||_{L^2(Q_1)} \to 0 \tag{2.13}$$

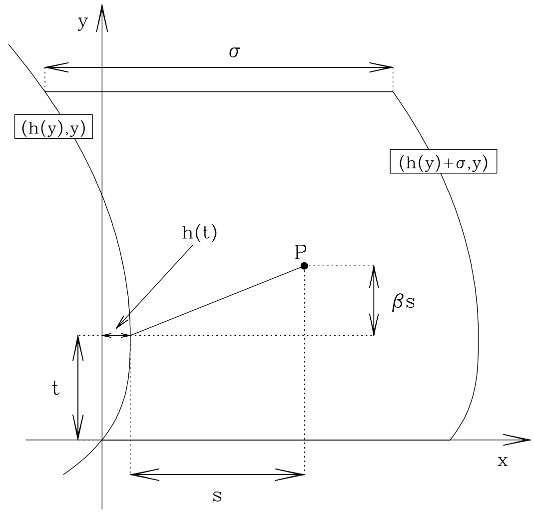

FIGURE 2.1: Domain $\tilde{Q}_\sigma$ used in the proof of Theorem 1, and representation of a point P in the new coordinates, $P = (h(t) + s, \beta s + t)$. The left side is the external boundary. A similar picture applies to Lemma 1, where, by taking $h(t) = 0$, one obtains a square.

by Poincaré inequality. Thus (2.5) holds with Q_1 replaced by $\tilde{Q}_1^j$.

For $0 \leq \beta \leq \min(1/2L, 1)$ the maps

$$\Phi_\beta^j(s,t) = (s + h^j(t), \beta s + t) \tag{2.14}$$

(see Figure 2.1) are bilipschitz (Lipschitz with Lipschitz inverse) with Lipschitz constants bounded independently of j. Moreover there exists a nonempty open set U with

$$U \subset \Phi_\beta^j\left(Q_{1/2}\right) \subset \tilde{Q}_1^j \tag{2.15}$$

for all j and β as above. Let

$$\chi_\beta^j(s,t) = 2(u_x^j + \beta u_y^j)[\Phi_\beta^j(s,t)] - (1+\beta^2)s\,. \tag{2.16}$$

Then, as before (2.5) yields

$$\partial_s \chi_\beta^j \to 0 \qquad \text{in } L^1(Q_{1/2})\,. \tag{2.17}$$

Taking first $\beta = 0$ and then $\beta \neq 0$ we deduce that

$$u_x^j \to u_x^\infty\,, \quad u_y^j \to u_y^\infty \qquad \text{in } L^1(U)\,, \tag{2.18}$$

where

$$2(u_x^\infty + \beta u_y^\infty) \circ \Phi_\beta^\infty(s,t) = (1+\beta^2)s \tag{2.19}$$

Since Φ_β^∞ is again bilipschitz both u_x^∞ (take $\beta = 0$) and u_y^∞ (take $\beta \neq 0$) are Lipschitz. Differentiating (2.19) with respect to s we obtain

$$\begin{pmatrix} 1 \\ \beta \end{pmatrix}^T \nabla u^\infty \begin{pmatrix} 1 \\ \beta \end{pmatrix} = (1+\beta^2) \qquad \text{a.e. in } U\,, \tag{2.20}$$

for all β with $0 \leq \beta \leq \min(1/2L, 1)$. Thus

$$(\nabla u^\infty)^T + \nabla u^\infty = Id \qquad \text{a.e. in } U\,. \tag{2.21}$$

On the other hand differentiating (2.19) with respect to t and taking $\beta = 0$ and $\beta \neq 0$ we obtain

$$\nabla u^\infty \begin{pmatrix} (h^\infty)' \\ 1 \end{pmatrix} = 0\,. \tag{2.22}$$

Multiplication by $((h^\infty)', 1)$ from the left yields a contradiction with (2.21). This finishes the proof of Theorem 1. •

3 Upper bound

This Section is devoted to the construction of a displacement field Ψ with energy bounded by $c\sigma$, where c denotes a generic constant which depends only on the domain. The main idea behind our construction is that the distance function is able to relax compression only in one direction (which, locally, is the one orthogonal to the boundary), whereas compression in the direction parallel to the boundary is relaxed by small-scale in-plane oscillations. Such oscillations, which are the analogue in the present context of the twinned microstructures in solid-to-solid phase transitions [14, 15], will be called *folds* (by "fold" we mean one period in the simplest periodic construction, and in its smoothly deformed versions; this corresponds to a pair of twins in the usual martensitic language). The folding scale must be of order σ close to the boundary, but will be much larger in the interior. The change in oscillatory profile with changing distance from the boundary, which corresponds to the disappearance of some folds, leads to a much lower energy if also the third component is nonzero, providing the main difference with the $\sigma^{2/3}$ scaling obtained by Jin and Sternberg [13] in their constrained problem (see Eqs. 3.10-3.13). The main result of this Section is the following

Theorem 2 (Upper bound) *Let Ω be an open, bounded subset of $\boldsymbol{R}^2$ with piecewise C^4 boundary. Then, there is a positive constant $\bar{c}_\Omega$ such that for small enough σ there is a displacement field $\Psi = (u, w)$ with $u = w = \nabla w = 0$ on $\partial\Omega$ and*

$$I_{\rm FvK}^{(\sigma)}[\Psi, \Omega] \leq \bar{c}_\Omega \sigma\,. \tag{3.1}$$

For simplicity we focus first on a rectangle, with boundary conditions enforced only on the y-axis (Section 3.1), and afterwards (Section 3.2) we show how this construction generalizes to an arbitrary smooth domain Ω.

3.1 Construction for a rectangle

In this Section we construct a deformation field Ψ on the rectangle $R = [0, L_x] \times [0, L_y]$ which obeys the boundary condition $u = w = \nabla w = 0$ on $\{0\} \times [0, L_y]$ and has energy bounded by $c\sigma$. In a first approximation, one can seek the optimal displacement field which is invariant under translations in y, which is easily seen to be $\Psi = \Psi(x) = (0, 0, x)$ (this corresponds to the solution of the simplified functional (1.2) in this geometry). To construct the y-dependent oscillations it is natural to consider the deviation from this solution, hence we define the modified functional

$$\begin{aligned} J[z, v, w, \Omega] &= I_{\text{FvK}}[z - w, v, w + x, \Omega] \\ &= \int_\Omega (w_{,x}^2 + 2z_{,x})^2 + 2(w_{,x}w_{,y} + z_{,y} + v_{,x})^2 + (w_{,y}^2 + 2v_{,y} - 1)^2 + \sigma^2|\nabla^2 w|^2 \,. \end{aligned} \tag{3.2}$$

We first relax the third term of (3.2) by constructing a natural oscillatory profile as a function of y, and then discuss how it changes with changing x. We shall indicate by $\Xi = (z, v, w)$ the modified displacement field which enters J, to distinguish it from the corresponding $\Psi = (z - w, v, w + x)$. We remark that J recovers invariance under translations (but not under rotations). Hence when considering rectangles contained in R we can translate them to have a corner in the origin.

For definiteness, we focus on a rectangle $A = [0, l] \times [0, h]$, which can be thought of as a small piece of our domain R. If no dependence on x is present, we can set $z = 0$ so that the first two strain terms are identically zero. The third one can be made to vanish by choosing w and v which satisfy the differential equation

$$w_{,y}^2 + 2v_{,y} = 1 \,. \tag{3.3}$$

One possible choice is

$$\tilde{w}^h(y) = \frac{h}{\pi\sqrt{2}} \sin \frac{2\pi y}{h} \tag{3.4}$$

$$\tilde{v}^h(y) = -\frac{h}{8\pi} \sin \frac{4\pi y}{h} \,. \tag{3.5}$$

Note that there is considerable freedom in this choice, any other smooth solution with the suitable scaling and boundary conditions would only cause changes in the constants appearing below. For later reference, we note that $\tilde{v}$, $\tilde{w}$ and their derivatives are bounded:

$$\sup_{y \in [0,h]} \left| \partial_h^n \partial_y^m \tilde{w}^h(y) \right| \le c h^{1-n-m} \,, \qquad \text{for } n \ge 0 \,, \;\; m \ge 0 \,, \tag{3.6}$$

and the same bounds hold for $\tilde{v}^h$. These bounds will be needed up to third derivatives (here and below, c denotes a generic positive constant which can depend only on the domain).

The energetic cost of (3.4-3.5) is purely from bending, and equals $4\pi^2\sigma^2 l/h$. We further note that the boundary values on the horizontal sides of A do not depend on h, i.e.

$$\tilde{w}_{,y}^h(0) = \tilde{w}_{,y}^h(h) = \sqrt{2} \,, \qquad \tilde{w}^h(0) = \tilde{w}^h(h) = \tilde{v}^h(0) = \tilde{v}^h(h) = 0 \,. \tag{3.7}$$

This shows that different pieces with different h can be easily put together across an horizontal boundary, whereas some additional construction is needed to match different h's along vertical boundaries. The boundary conditions (3.7) can hence be used to divide our global construction into independent pieces along directions parallel to the coordinate axes. From the form of the functional it is clear that we can safely join different pieces of our test function provided that z, v, w and ∇w are continuous across internal boundaries. We now formalize these boundary conditions, both for horizontal and vertical segments.

Definition 1 *We say that a displacement field $\Xi = (z, v, w)$ satisfies standard boundary conditions along a horizontal segment $[x_0, x_0 + l] \times \{y_0\}$ if on that segment it obeys $z = v = w = 0$ and $w_{,y} = \sqrt{2}$.*

Definition 2 *We say that a displacement field $\Xi = (z, v, w)$ satisfies standard boundary conditions along a vertical segment $\{x_0\} \times [y_0, y_0 + h]$ if along that segment $w(x_0, y) = \tilde{w}^h(y - y_0)$, $v(x_0, y) = \tilde{v}^h(y - y_0)$, with $z = w_{,x} = 0$.*

We now consider a region where the oscillation period h changes as a function of x. It is more convenient to use a domain symmetric with respect to the x axis. Since $\tilde{w}$ and $\tilde{v}$ are periodic in y they do not need to be changed (or translated), and all the results above are also valid in a symmetric region $A' = [0, l] \times [-h/2, h/2]$, with the only difference that the sign of the boundary values changes. Indeed, we get

$$\tilde{w}^h_{,y}(\pm h/2) = -\sqrt{2}\,, \qquad \tilde{w}^h(\pm h/2) = \tilde{v}^h(\pm h/2) = 0\,. \tag{3.8}$$

Let $a(x)$ be the oscillation period, which will be of order h and change on a distance of order $l \gg h$, and consider the region $[0, l] \times [-a(x)/2, a(x)/2]$. Define w and v as in (3.4-3.5), with h replaced by the local $a(x)$:

$$w(x, y) = \tilde{w}^{a(x)}(y)\,, \qquad v(x, y) = \tilde{v}^{a(x)}(y)\,, \tag{3.9}$$

and consider the energy (3.2). The third strain term is still zero, but the x derivatives are no longer zero. Since $w_{,y}$ is of order 1, whereas $v_{,x}$ and $w_{,x}$ are of order h/l, the first strain term is $O(h/l)^4$, the second one $O(h/l)^2$. Hence for $h \ll l$ the latter is the most dangerous, and we choose to cancel it by a suitable choice of z. This can be done in such a way that the scaling of the first term is unchanged. Indeed, the second strain term vanishes provided that

$$-z_{,y} = w_{,x} w_{,y} + v_{,x} = \frac{da(x)}{dx}\left[\frac{\partial \tilde{w}^a(y)}{\partial y}\frac{\partial \tilde{w}^a(y)}{\partial a} + \frac{\partial \tilde{v}^a(y)}{\partial a}\right]. \tag{3.10}$$

Solving for z we get

$$z(x, y) = \frac{da(x)}{dx}\left[\tilde{z}^{a(x)}(y) + \zeta(x)\right] \tag{3.11}$$

where $\zeta(x)$ is a still undetermined function of x, and

$$\tilde{z}^h(y) = \frac{y^2}{2h} + \frac{h}{8\pi^2}\left(\cos\frac{4\pi y}{h} - 1\right) + \frac{y}{8\pi}\sin\frac{4\pi y}{h} \tag{3.12}$$

is a solution of

$$-\frac{\partial \tilde{z}^h(y)}{\partial y} = \frac{\partial \tilde{w}^h(y)}{\partial y}\frac{\partial \tilde{w}^h(y)}{\partial h} + \frac{\partial \tilde{v}^h(y)}{\partial h} \tag{3.13}$$

which obeys the same estimates as $\tilde{w}^h$ and $\tilde{v}^h$ in Eq. (3.6). This implies $|z| \le |a'|h$, $|z_{,x}| \le |a''|h + |a'|h/l$, and $|z_{,y}| \le |a'|$, provided that $|\zeta| \le h$ and $|\zeta_{,x}| \le h/l$. Note that (3.13) has been derived without assuming an explicit form for $a(x)$, and only the explicit form (3.12) for $\tilde{z}^h$ depends on the choice made in (3.4-3.5) for $\tilde{w}^h$ and $\tilde{v}^h$.

The displacement field so constructed reduces to $\Xi_0 = (0, \tilde{v}^{a(x)}, \tilde{w}^{a(x)})$ on the two vertical boundaries (i.e. for $x = 0$ and for $x = l$), provided that da/dx vanishes both for $x = 0$ and for $x = l$. Along the oblique boundaries $[0,l] \times \{\pm a(x)\}$ instead we have $v = w = 0$, and $w_{,y} = -\sqrt{2}$, but z is nonzero. Indeed, $\tilde{z}^h(0) = 0$, but $\tilde{z}^h(\pm h/2) = h/8$. Thus in the construction below we shall need to exploit our freedom to choose $\zeta(x)$ to ensure continuity of z.

At this point, it would be tempting to simply let $a(x)$ tend to zero at the points where a fold has to disappear (branching points). However, this would lead to infinite energy concentrating at the point where a vanishes, because the bending term would be of order

$$\sigma^2 \int |\nabla^2 w| dx dy \sim \sigma^2 \int \frac{1}{a(x)} dx\,, \tag{3.14}$$

which diverges if $a(0) = a'(0) = 0$. This problem can be avoided by stopping a at some minimum value (called η below) not smaller than σ, and then joining with smooth interpolation in a final thin layer. In particular, we set

$$a(x) = \frac{h}{2}\left[1 - \phi\left(\frac{x}{l}\right)\right] + \eta\phi\left(\frac{x}{l}\right)\,, \tag{3.15}$$

where $\phi : [0,1] \to [0,1]$ is a smooth function such that $\phi(0) = 0$, $\phi(1) = 1$, $\phi'(0) = \phi'(1) = 0$, with bounded derivatives, $0 \le \phi' \le 2$ and $|\phi''| \le 4$, which further satisfies $\phi(t) + \phi(1-t) = 1$ and $\phi(t) \ge t^2$. For example, $\phi(x) = 2x^2$ for $x \in [0, 1/2]$, $\phi(x) = 1 - 2(1-x)^2$ for $x \in [1/2, 1]$. The "dangerous" bending term (3.14) can then be estimated using, for $\lambda > 0$,

$$\int_0^1 \frac{dt}{\lambda + \phi(t)} \le \int_0^\infty \frac{dt}{\lambda + t^2} = \frac{\pi}{2\lambda^{1/2}}\,, \tag{3.16}$$

and the strain energy will be of order 1 only in the small region of height η where the sign of $w_{,y}$ has to change.

We come therefore to the following

Lemma 2 *Given a rectangle $B = (0,l) \times (-h/2, h/2)$ with $h \le l$ there is a displacement field $\Xi = (z, v, w)$ which satisfies the standard horizontal boundary conditions on the top and bottom sides of B (Def. 1), and the vertical ones (Def. 2) on the three segments $\{0\} \times (-h/2, 0)$, $\{0\} \times (0, h/2)$, and $\{l\} \times (-h/2, h/2)$, with energy bounded by*

$$J[\Xi, B] \le c\left[\frac{h^5}{l^3} + \frac{\sigma^{3/2} l}{h^{1/2}} + \sigma^{3/2} h^{1/2}\right]\,. \tag{3.17}$$

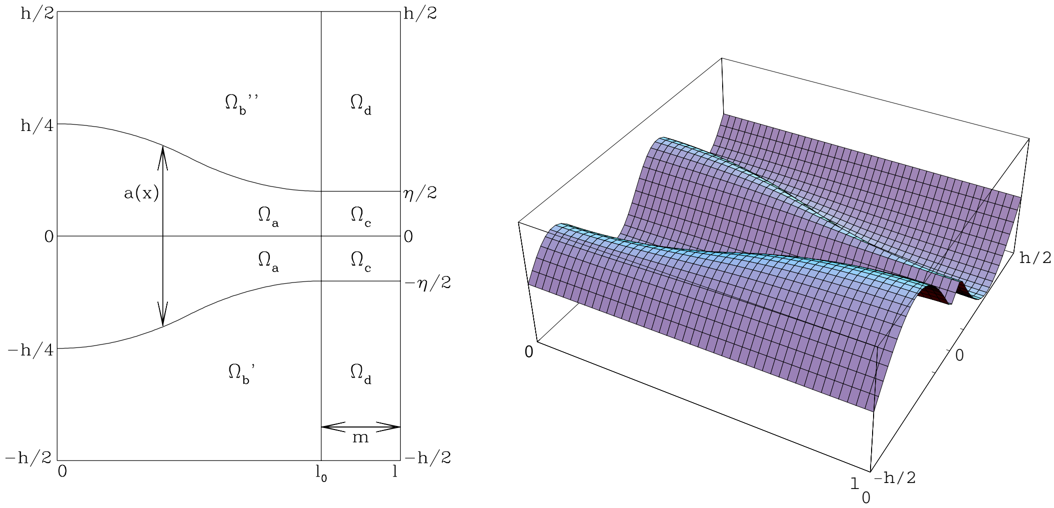

FIGURE 3.1: Subdivision of the domain B used in the construction of Lemma 2 (left panel) and representation of the constructed w in Ω_a and Ω_b (right panel).

Proof. We first decompose the domain into the part of length l_0 where the inner twin smoothly decreases its width from $h/2$ to η, and the one where it disappears by interpolation, of length $m = l - l_0$ (see Figure 3.1). The values of m and η will be chosen below, now we merely assume the ordering $\eta \le m \le h \le l$ which allows us to simplify many terms.

For $x \in [0, l_0]$ the width of the inner twin is given by

$$a(x) = \frac{h}{2}\left[1 - \phi\left(\frac{x}{l_0}\right)\right] + \eta\phi\left(\frac{x}{l_0}\right) , \tag{3.18}$$

which smoothly decreases from $h/2$ to η. The domain B is then naturally divided into three parts: Ω_a is the region $|y| \le a(x)/2$ occupied by the "small" twin, Ω_b' is the region $-h/2 \le y \le -a(x)/2$ occupied by the first half of the "large" fold, and Ω_b'' is the one occupied by the other half (see Figure 3.1). In Ω_b', we set

$$w = \tilde{w}^{h-a(x)}\left(y + \frac{h}{2}\right) , \qquad v = \tilde{v}^{h-a(x)}\left(y + \frac{h}{2}\right) , \qquad z = a'(x)\tilde{z}^{h-a(x)}\left(y + \frac{h}{2}\right) \tag{3.19}$$

and the same in Ω_b'', with the arguments replaced by $y - h/2$ [e.g. $w = \tilde{w}^{h-a}(y - h/2)$, etc.]. Using Eq. (3.7) and $\tilde{z}^h(0) = 0$ it is easy to see that on the external boundary this choice of w, v, z satisfies Definition 1. Along the internal boundaries $[0, l_0] \times \{\pm a(x)/2\}$ we get instead $w = v = 0$, $w_{,y} = -\sqrt{2}$ [from (3.8)], and $z = a'(x)(h - a(x))/8$ [from $\tilde{z}^h(\pm h/2) = h/8$].

In Ω_a we set

$$w = \tilde{w}^{a(x)}(y) , \quad v = \tilde{v}^{a(x)}(y) , \quad z = a'(x)\left[\tilde{z}^{a(x)}(y) + \zeta(x)\right] \tag{3.20}$$

where ζ is still to be determined. On the internal boundaries we get $z = a'(x)(\zeta + a/8)$, hence $\zeta(x) = (h - 2a(x))/8$ leads to a continuous z. For w and v Eq. (3.8) holds again. This shows that the Ξ so constructed has enough smoothness and matches all required boundary conditions.

We now come to the energy estimate. The only nonzero strain term is the first one, which is bounded using $|w_{,x}| \le |a_{,x}||\tilde{w}^a_{,a}| \le ch/l$ and $|z_{,x}| \le |a_{,x}|^2|\tilde{z}^a_{,a}| + |a_{,xx}||\tilde{z}^a + \zeta| \le ch^2/l^2$, leading to

$$J_{\text{strain}}[\Xi, \Omega_a \cup \Omega_b] \le c\frac{h^5}{l^3} \tag{3.21}$$

(where $\Omega_b = \Omega_b' \cup \Omega_b''$). We now compute the bending term. We need to estimate

$$w_{,xx} = a_{,xx}\tilde{w}^a_{,a} + a_{,x}^2\tilde{w}^a_{,aa}\,, \qquad w_{,xy} = a_{,x}\tilde{w}^a_{,ay}\,, \qquad w_{,yy} = \tilde{w}^a_{,yy}\,. \tag{3.22}$$

Given the bounds on a and on $\tilde{w}^h$, we get $|\nabla^2 w| \le c/a(x)$ in Ω_a, and $|\nabla^2 w| \le c/h$ in Ω_b. Performing the y integration first we get

$$J_{\text{bending}}[\Xi, \Omega_a \cup \Omega_b] = \sigma^2 \int_{\Omega_a\cup\Omega_b} \left|\nabla^2 w\right|^2 \le \sigma^2 \int_0^{l_0} \frac{c}{h} + \frac{c}{a(x)} \le \frac{\sigma^2 l}{\sqrt{\eta h}} \tag{3.23}$$

where in the last step we have used the definition of $a(x)$ and Eq. (3.16). This concludes the construction in the region $[0, l_0] \times [0, h]$.

In $[l_0, l]$ we define Ξ as a smooth interpolation between the values at $x = l_0$ and and $x = l$,

$$\Xi(x, y) = \Xi(l_0, y)\left[1 - \phi\left(\frac{x}{m}\right)\right] + \Xi(l, y)\phi\left(\frac{x}{m}\right)\,. \tag{3.24}$$

This has small energy because two boundary values differ significantly only in the small set $\Omega_c = [l_0, l] \times [-\eta/2, \eta/2]$, which has relative size $|\Omega_c|/|\Omega_c \cup \Omega_d| = \eta/h$. In the large set $\Omega_d = [l_0, l] \times \{\eta \le 2|y| \le h\}$ instead the boundary conditions are similar (the relative difference $|\Xi(l_0) - \Xi(l)|/(|\Xi(l_0)| + |\Xi(l)|)$ is of order η/h). It is also clear that, since $\eta \le m$, the y-derivatives are the most dangerous, hence we focus on them in the estimates.

In Ω_d the difference from the zero-strain boundary condition, $\Phi(x, y) = \Xi(x, y) - \Xi(l, y)$, is small. For example, for $y < -\eta/2$,

$$|w_{,y}(l_0, y) - w_{,y}(l, y)| = \left|\tilde{w}^{h-\eta}_{,y}\left(y + \frac{h}{2}\right) - \tilde{w}^h_{,y}\left(y + \frac{h}{2}\right)\right| \le c\frac{\eta}{h} \tag{3.25}$$

because $\tilde{w}^h$ has bounded second derivatives, and the same for the other components. By direct integration we get $|\Phi(l_0, \cdot)| \le c\eta$. From (3.24) we then estimate the full gradient of Φ in Ω_d, obtaining $|\nabla\Phi| \le \eta(1/m + 1/h)$. This in turn gives $J_{\text{strain}}[\Xi, \Omega_d] \le \int (\nabla\Phi)^2 + (\nabla\Phi)^4 \le \eta^2 h/m$ (some terms have disappeared because $\eta \le m \le h$).

In Ω_c instead we just use the uniform bound on the gradient $|\nabla\Xi| \le c$, which gives for the strain energy $J_{\text{strain}}[\Xi, \Omega_c] \le c|\Omega_c| = cm\eta$.

The bending term is again computed separately in the two subregions. In Ω_d the same argument as in (3.25) allows to control $|\nabla^2 w| \le \eta/m^2$ (this result depends on the ordering $m \le h$ and on control on the third derivative of $\tilde{w}^h$).

In Ω_c the dominant contribution is instead the $w_{,yy}$ derivative of order $1/\eta$ (because of the boundary condition at $x = l_0$), and this leads to a total $|\nabla^2 w| \leq 1/\eta^2$ (using the ordering $\eta \leq m$). The total bending energy is then

$$J_{\text{bending}}[\Xi, \Omega_c \cup \Omega_d] \leq \frac{c\sigma^2\eta^2 h}{m^3} + \frac{cm\sigma^2}{\eta} . \tag{3.26}$$

Collecting the various terms we get

$$J[\Xi, B] \leq c \left[\frac{h^5}{l^3} + \frac{\sigma^2 l}{(\eta h)^{1/2}} + \frac{\eta^2 h}{m} + \frac{m\sigma^2}{\eta} \right] \tag{3.27}$$

where the two irrelevant terms $m\eta$ and $\sigma^2\eta h m^{-3}$ have been dropped. We finally fix $\eta = \sigma$ and $m = (\sigma h)^{1/2}$ and get the final result of Lemma 2. •

We now show how the basic building block constructed in Lemma 2 delivers a test function with energy scaling linearly in σ in a simple geometry.

Lemma 3 *Given $R = (0, L_x) \times (0, L_y)$, for small enough σ there is a Ψ with $u = w = \nabla w = 0$ for $x = 0$, and with energy bounded by $c\sigma$. Further, this result can be achieved with a Ψ which for $x > c\sigma^{1/3}$ does not depend on x, and which obeys $|\Psi - (0, 0, x)| \leq c\sigma^{1/2}$, $|\nabla\Psi| \leq c$, $|\nabla^2\Psi| \leq c/\sigma$.*

Proof. The main part of our construction is based on a geometric subdivision of the domain. For large x, one can take the profiles of (3.4-3.5) with h of order $\sigma^{1/2}$ and $z = 0$, reaching an energy of the correct order of magnitude (the strain part vanishes, and $|\nabla^2 w| \leq c\sigma^{-1/2}$). For small x, we need to refine, down to scale σ. Since from Lemma 2 we know how to double the period of oscillation, it is natural to fix the widths at the various stages to $h_i = \sigma 2^i$, for $1 \leq i \leq N$, where N is defined by $2^N \simeq \sigma^{-1/2}$. The error arising from taking an integer approximation to the solution of this equation is negligible for small enough σ, and will not be explicitly considered in the following. Clearly it is always possible to construct Ψ in a larger domain $[0, L_x] \times [0, \sigma 2^N]$ which contains an integer number of twins, and then restrict.

We seek a sequence of spacings l_i which constitute the widths of the regions where branching takes place. We apply Lemma 2 on all rectangles of size $l_i \times h_i$, and then from the profiles Ξ_i in the rectangles we obtain $\tilde{\Psi} = (z - w, v, w + x)$. Finally, for $x \in [0, \sigma]$ we modify $\tilde{\Psi}$ using a smooth interpolation between $\tilde{\Psi}$ and the boundary condition,

$$\Psi(x, y) = \tilde{\Psi}(x, y)\phi\left(\frac{x}{\sigma}\right) . \tag{3.28}$$

Since $|\tilde{\Psi}| \leq c\sigma$ for $x \leq \sigma$, we have $\nabla\Psi$ of order 1 and $\nabla^2 w$ of order $1/\sigma$, hence the energy in the strip where $\phi \neq 1$ is controlled by the area, i.e. $L_y\sigma$. The total energy is then bounded by

$$I_{\text{FvK}}[\Psi, R] \leq c\sigma L_y \left[1 + \sum_{i=1}^{N} \left[\frac{h_i^4}{l_i^3} + \frac{\sigma^{3/2} l_i}{h_i^{3/2}} + \sigma^{3/2} h_i^{-1/2} \right] \right] . \tag{3.29}$$

The first term corresponds to both the boundary layer and the x-independent oscillations at large-x, while the series comes from the region of branching (Lemma 2). Since the third term in the series directly sums to σ, we focus on the remaining two.

A natural criterion to choose the spacings l_i is to minimize the energy bound, which amounts to choose for each i the l_i which minimizes

$$\frac{h_i^4}{l_i^3} + \frac{\sigma^{3/2}}{h_i^{3/2}} l_i \,, \tag{3.30}$$

which is $l_i = h_i(h_i/\sigma)^{3/8}$. With this choice, both series converge as $2^{-i/8}$, and the energy estimate is proven. With this construction the branching process covers a region $\sum_i l_i \simeq \sigma^{5/16}$.

It is also interesting, even if not needed for the following discussion, to try to constrain the branching process to a smaller region close to the boundary. In order to do this, one should optimize not only the energy contribution per branching step in (3.29), but also the consumption in horizontal distance, hence minimize

$$\frac{h_i^4}{l_i^3} + \frac{\sigma^{3/2}}{h_i^{3/2}} l_i + \mu l_i \tag{3.31}$$

for l_i, where μ is a suitable penalization parameter which will be fixed later. This gives

$$l_i = h_i \left(\frac{4}{(\sigma/h_i)^{3/2} + \mu} \right)^{1/4} \tag{3.32}$$

which has a crossover from $l_i \propto h_i$ (at large i) to $l_i \propto h_i(h_i/\sigma)^{3/8}$ (at small i), (the second scaling clearly coincides with the one obtained above). In practice, it is simpler to take

$$l_i = \begin{cases} \sigma 2^{11i/8} & 1 \le i \le k \\ \sigma 2^i 2^{3k/8} & i > k \end{cases} \tag{3.33}$$

where the constant has been chosen to ensure continuity at k, and the variable k replaces μ, which is now given by $\mu = 2^{-3k/2}$. The resulting energy contribution from the series is

$$\sum_{i=1}^{N} \frac{h_i^4}{l_i^3} + \frac{\sigma^{3/2}}{h_i^{3/2}} l_i = \sigma \sum_{i=1}^{k} 2^{-i/8} + 2^{-i/8} + \sigma \sum_{i=k}^{N} 2^{i-\frac{9}{8}k} + 2^{-\frac{1}{2}i+\frac{3}{8}k} \le \sigma \left(1 + 2^{N-\frac{9}{8}k} + 2^{\frac{3}{8}k-\frac{1}{2}}\right) \tag{3.34}$$

which is bounded by $c\sigma$ provided that $8N/9 \le k \le N$. The total length,

$$\sum_i l_i = \sum_{i=1}^{k} \sigma 2^{11i/8} + \sum_{i=k}^{N} \sigma 2^i 2^{3k/8} = \sigma 2^{3k/8} (2^N + 2^k) \,, \tag{3.35}$$

is clearly optimized by choosing the smallest value, $k = 8N/9$, which gives $\sum_i l_i = \sigma 2^{4N/3} = \sigma^{1/3} < \sigma^{5/16}$. This concludes the proof of Lemma 3. •

3.2 Construction for a generic domain

The construction in a generic domain is considerably simplified if the domain is first triangulated on a scale smaller than the minimum radius of curvature of the boundary. Such a triangulation depends only on the domain, and not on σ, hence it only affects the constant in (3.1), not the scaling. It is clear that any domain with C^4 boundary can be subdivided into a finite number of pieces, each of which has two straight sides and a curved (C^4 regular) one, with radius of curvature bounded from below by a given multiple of the side length. The three angles can be further assumed to be less than $\pi/2$. Then, if one can construct a function with energy $c\sigma$ in such pieces, with the usual $u = w = \nabla w = 0$ boundary conditions, by putting them together one gets the result for the full domain. Hence we can focus on curvilinear triangles, which satisfy the following

Definition 3 *A simply connected, bounded set $\Sigma \subset \boldsymbol{R}^2$ is said to be if type A if its boundary is the union of three curves of class C^4, which join at angles less than $\pi/2$, and whose radius of curvature is always larger than 10 times the diameter of Σ.*

As done above, it is natural to start the construction with the distance function, which is singular along three smooth curves in the interior (see Figure 3.2). It is simple to see that they divide Σ into three parts, ω_1, ω_2 and ω_3, which obey the following

Definition 4 *A simply connected, bounded set $\omega \subset \boldsymbol{R}^2$ is said to be if type B if its boundary is the union of three curves of class C^4, which join three points X, Y and P, such that the angles at X and Y are less than $\pi/4$, and all radii of curvature are larger than 10 times the diameter of ω. The XY side is called external side.*

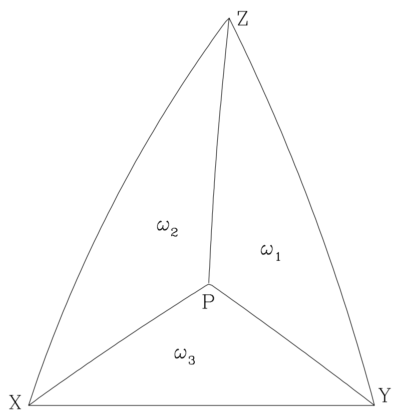

FIGURE 3.2: Each curvilinear triangle Σ (of type A) is subdivided into three pieces ω_i of type B along the singular set of the distance function to the boundary, which bisects the angles.

Let us concentrate on one of the three pieces, say ω_1. Since our construction is based on oscillations superimposed to the distance function, it is natural to use as variables the arc length along the external side and the distance function itself. The construction of Section 3.1 will need to be modified in order to take into account the curvature of the boundary, and care will also be needed to properly enforce all boundary conditions and ensure smooth matching of different pieces.

Having presented the general scheme, we start to discuss the details of the construction in a curvilinear triangle ω of type B. Let t denote arc length along the external side of ω, and s the distance function from that side. The change of coordinates is

$$\phi(s,t) = \alpha(t) + sn(t) \tag{3.36}$$

where $n = (-\alpha')^\perp$ is the inner normal, and $(a_x, a_y)^\perp = (-a_y, a_x)$ (this defines the direction of increasing t). The two new basis vectors are

$$e_s = n(t)\,, \qquad e_t = n^\perp(t) = \alpha'(t)\,; \tag{3.37}$$

they both depend on t but not on s, and have derivatives proportional to the local curvature $\kappa(t)$,

$$n' = -\kappa n^\perp\,, \qquad (n^\perp)' = \kappa n\,. \tag{3.38}$$

We shall denote the basis vectors with (e_s, e_t) when using (s,t) as coordinates, and with $(n, n^\perp)$ when using (x,y). In the new coordinates, the domain ω takes the form

$$\tilde{\omega} = \{0 \le t \le T\,, 0 \le s \le f(t)\} \tag{3.39}$$

where $f(t)$ is C^4 in $[0,T_1]$ and in $[T_1,T]$, $|f'(t)| \le c_1$, and $|f(t)| \le \operatorname{diam}\omega \le 1/(2\sup|\kappa(t)|)$ [the point $(s,t) = (f(T_1), T_1)$ corresponds to the corner P]. We are now ready to express the functional in the new coordinates.

Lemma 4 *Let ω be a domain of type B as in Definition 4, and let (s,t) be distance to the external side and arc length, as in (3.36). Then, $I_{\text{FvK}}[\Psi,\omega] = \tilde{I}[\Psi,\tilde{\omega}]$, which is the sum of*

$$\begin{aligned}\tilde{I}_{\text{strain}}[\Psi,\tilde{\omega}] \quad = \quad & \int_{\tilde{\omega}} \left[\tilde{w}_{,s}^2 + 2\tilde{u}_{s,s} - 1\right]^2 + 2\left[\gamma\tilde{w}_{,s}\tilde{w}_{,t} + \tilde{u}_{t,s} + \gamma\tilde{u}_{s,t} + \gamma\kappa\tilde{u}_t\right]^2 \\ & + \left[\gamma^2\tilde{w}_{,t}^2 + 2\gamma\tilde{u}_{t,t} - 2\gamma\kappa\tilde{u}_s - 1\right]^2 \frac{dsdt}{\gamma(s,t)}\,,\end{aligned} \tag{3.40}$$

and of

$$\tilde{I}_{\text{bending}}[\Psi,\tilde{\omega}] \le c\sigma^2 \int_{\tilde{\omega}} |\nabla^2\tilde{w}|^2 + |\nabla\tilde{w}|^2\,, \tag{3.41}$$

where

$$\gamma(s,t) = \frac{1}{1 - s\kappa(t)}\,, \tag{3.42}$$

$\kappa(t)$ is the curvature of the external side of ω, $\tilde{\omega}$ is the image of ω under the change of variables (3.36), and $u_s = u \cdot e_s$ and $u_t = u \cdot e_t$ are components taken with respect to s and t.

Proof. To express the gradients of u and w in the new coordinates we start from the gradient of the transformation,

$$\frac{\partial(x,y)}{\partial(s,t)} = \nabla\phi = n \otimes e_s + (1 - s\kappa)n^\perp \otimes e_t \,, \tag{3.43}$$

and its inverse,

$$\frac{\partial(s,t)}{\partial(x,y)} = (\nabla\phi)^{-1} = e_s \otimes n + \gamma e_t \otimes n^\perp \,, \tag{3.44}$$

where $\gamma(s,t)$ was defined in (3.42). We are now ready to compute the gradients of $u(x,y)$ and $w(x,y)$ in terms of $\tilde{u}(s,t)$ and $\tilde{w}(s,t)$, which are

$$\nabla w = \tilde{w}_{,s} e_s + \gamma \tilde{w}_{,t} e_t \tag{3.45}$$

and

$$\nabla u = \partial_s \tilde{u} \otimes n + \gamma \partial_t \tilde{u} \otimes n^\perp \,. \tag{3.46}$$

More explicitly, since $\tilde{u} = \tilde{u}_s e_s + \tilde{u}_t e_t$, and considering the dependence of e_s and e_t on t, we get

$$\nabla u = \begin{pmatrix} \tilde{u}_{s,s} & \gamma(\tilde{u}_{s,t} + \kappa \tilde{u}_t) \\ \tilde{u}_{t,s} & \gamma(\tilde{u}_{t,t} - \kappa \tilde{u}_s) \end{pmatrix} . \tag{3.47}$$

The last term we need is the second gradient of w,

$$\nabla^2 w = \begin{pmatrix} \tilde{w}_{s,s} & \gamma \tilde{w}_{,ts} + \kappa\gamma^2 \tilde{w}_{,t} \\ \gamma \tilde{w}_{,st} + \gamma_{,s} \tilde{w}_{,t} & \gamma^2 \tilde{w}_{,tt} + \gamma\gamma_{,t} \tilde{w}_{,t} - \kappa\gamma \tilde{w}_{,s} \end{pmatrix} . \tag{3.48}$$

By a simple substitution we write I_{FvK} in the new variables, and the thesis follows. •

The construction of a deformation with small energy $\tilde{J}$ is analogous to the one performed in Section 3.1. In parallel to Eq. (3.2), we start by subtracting the distance function, and obtain

$$\tilde{J}_{\mathrm{strain}}[\tilde{z}, \tilde{v}, \tilde{w}, \tilde{\omega}] = \tilde{I}_{\mathrm{strain}}[\tilde{z} - \tilde{w}, \tilde{v}, \tilde{w} + s, \tilde{\omega}] \tag{3.49}$$

$$= \int_{\tilde{\omega}} \left[\tilde{w}_{,s}^2 + 2\tilde{z}_{,s}\right]^2 + 2\left[\gamma \tilde{w}_{,s} \tilde{w}_{,t} + \tilde{v}_{,s} + \gamma \tilde{z}_{,t} + \gamma\kappa \tilde{v}\right]^2 + \left[\gamma^2 \tilde{w}_{,t}^2 + 2\gamma \tilde{v}_{,t} - 2\gamma\kappa \tilde{z} + 2\gamma\kappa \tilde{w} - 1\right]^2 \frac{dsdt}{\gamma} \,. \tag{3.50}$$

The bending term is bounded by $\tilde{J}_{\mathrm{bending}} \leq c\sigma^2 \int |\nabla^2 \tilde{w}|^2 + (|\nabla \tilde{w}| + 1)^2$. The second term is negligible in our construction with bounded gradients, and will be dropped from now on. As above, we shall denote by $\tilde{\Xi}$ the rescaled displacement field $(\tilde{z}, \tilde{v}, \tilde{w})$.

To fully accomodate for the presence of the γ factors in $\tilde{J}_{\mathrm{strain}}$ would require changing the basis functions $\tilde{w}^h$ depending on position. However, if $|\Xi|$ is small a simple rescaling will do. Indeed, we have

Lemma 5 *Let Ξ be a displacement field satisfying $|\Xi| \leq c_1\sigma^{1/2}$ and $|\nabla\Xi| \leq c_2$, in a domain $\tilde{\omega}$ of type (3.39). Then,*

$$\tilde{J}[\tilde{\Xi}, \tilde{\omega}] \leq c\,(J[\Xi, \tilde{\omega}] + \sigma) \tag{3.51}$$

where $\tilde{\Xi} = (\tilde{z}, \tilde{v}, \tilde{w}) = (\gamma^{-2}z, \gamma^{-1}v, \gamma^{-1}w)$, J was defined in (3.2), and the constant c depends on c_1, c_2 and $\tilde{\omega}$.

Proof. First observe that $1/2 \leq \gamma \leq 3/2$, and $|\nabla\gamma|$, $|\nabla^2\gamma|$ are bounded (because the external side of ω is C^4). Then, the jacobian γ^{-1} in (3.50) can be dropped, and the rescaled functions satisfy $\nabla\tilde{w} = \gamma^{-1}\nabla w + O(\sigma^{1/2})$, and analogous estimates hold for $\tilde{z}$ and $\tilde{v}$. We now consider the three strain terms of Eq. (3.50). The first one is bounded using

$$\tilde{w}_{,s}^2 + 2\tilde{z}_{,s} = \gamma^{-2}(w_{,s}^2 + 2z_{,s}) + O(\sigma^{1/2}) \tag{3.52}$$

which then gives

$$\int_{\tilde{\omega}} (\tilde{w}_{,s}^2 + 2\tilde{z}_{,s})^2 \leq (\sup\gamma)^{-4} \int_{\tilde{\omega}} (w_{,s}^2 + 2z_{,s})^2 + c\sigma|\tilde{\omega}|\,. \tag{3.53}$$

The second one,

$$\gamma\tilde{w}_{,s}\tilde{w}_{,t} + \tilde{v}_{,s} + \gamma\tilde{z}_{,t} + \gamma\kappa\tilde{v} = \gamma^{-1}\left(w_{,s}w_{,t} + v_{,s} + z_{,t}\right) + O(\sigma^{1/2}) \tag{3.54}$$

and the third one are treated similarly. Finally, consider the bending term. Since derivatives of γ are bounded, we get $|\nabla^2\tilde{w} - \gamma\nabla^2 w| \leq c$, and the Lemma is proved. •

So far, we have reduced the problem of constructing Ψ on a general domain to that of constructing Ξ on a standard domain $\tilde{\omega}$ of the form (3.39), with $J[\Xi, \tilde{\omega}] \leq c\sigma$. The curvature terms in the functional have disappeared in view of Lemma 5, and only the right boundary $s = f(t)$ of $\tilde{\omega}$ is still not straight. In analogy with Lemma 3 we now (Lemma 6) construct Ξ in $\tilde{\omega}$ with small $J[\Xi, \tilde{\omega}]$ and $|\Xi| \leq c\sigma$ on $\partial\tilde{\omega}$. Then (Lemma 7) we generate the corresponding $\tilde{\Xi}$ with the same boundary condition and small $\tilde{J}[\tilde{\Xi}, \tilde{\omega}]$ (using Lemma 5), and smoothly match three such constructions to obtain Ψ defined on a set of type A.

Lemma 6 *Let $\tilde{\omega}$ be a domain of type (3.39). Then there exists a field Ξ defined on $\tilde{\omega}$ such that*

$$J[\Xi, \tilde{\omega}] \leq c\sigma\,, \tag{3.55}$$

with

$$|\Xi| \leq c\sigma^{1/2}\,, \qquad |\nabla\Xi| \leq c\,, \qquad |\nabla^2 w| \leq c\sigma^{-1} \tag{3.56}$$

on $\tilde{\omega}$, with $|\Xi| \leq c\sigma$ on $\partial\tilde{\omega}$.

Proof. The construction is analogous to the one used in Lemma 3 for the rectangle. Let N be the integer that most closely solves $2^N = \sigma^{-1/2}$, and divide the domain in horizontal stripes of height $2^N\sigma \simeq \sigma^{1/2}$. On these boundaries, we impose the usual horizontal boundary conditions (Definition 1). Each stripe is now filled with pieces of two types:

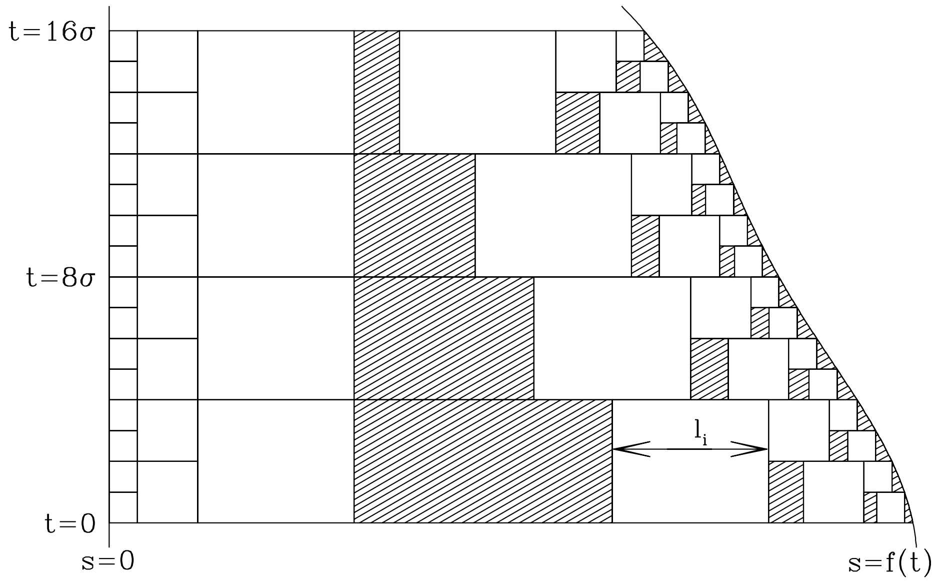

FIGURE 3.3: A slice of the transformed domain $\tilde{\omega}$, filled with pieces of type F (dashed rectangles) or B (empty rectangles).

- branching pieces, which at step i have height $h_i = 2^i\sigma$ and width $l_i = 2^{11i/8}\sigma$, which are called Bi and in which Ξ is given by Lemma 2, and
- flat pieces, which at step i have the same height $h_i = 2^i\sigma$ and variable width (for $i < N$ it will be less than l_i), which are called Fi and in which $\Xi = (0, \tilde{v}^{h_i}(s-s_0), \tilde{w}^{h_i}(s-s_0))$ according to Eqs. (3.4-3.5).

The construction in each stripe is done in three pieces: in the central region there is a unique fold of height $h_N \simeq \sigma^{1/2}$, in the left and right parts branching takes place, down to scale $h_0 = \sigma$ (see Figure 3.3). The left part of the construction (close to the straight boundary $\{s = 0\}$) is identical to the one of Lemma 3, and only uses Bi pieces. The right part is best understood starting from the curved boundary $\{s = f(t)\}$, and using all Bi pieces in order, putting each of them as close as possible to the mentioned boundary. The remaining empty spaces, which at step i (for $i < N$) have width controlled by $h_i \sup|f'(t)|$, are filled with Fi pieces. The two branching regions are finally joined by a (possibly long) FN piece. If there is not enough space to complete both constructions, they are stopped at the largest possible $\bar{i}$, and the joined with a $F\bar{i}$ piece, whose width is bounded by $l_{\bar{i}+1}$ [hence its energy is bounded by the energy of $B(\bar{i}+1)$].

The estimate of the energy goes as follows. The branching pieces are exactly those that would be used in a rectangle with the same height and width, with possibly some excluded, hence their energy is of order σ. The flat pieces have an energy proportional to $\sigma^2 l/h$ [this was computed after Eq. (3.6)], and since for each of them l/h is bounded by the slope of the boundary each of them has energy bounded by σ^2, and their number is bounded by $c\sum_i 2^i \le c2^N \le c\sigma^{-1/2}$. This yields an energy contribution of at most $c\sigma$ (except for the FN

pieces). Finally each FN piece contributes at most $c\sigma^{3/2}$ and there can be at most $c\sigma^{-1/2}$ such pieces. •

We finally come to the full construction for any set Σ of type A.

Lemma 7 *Let Σ be a set of type A (see Definition 3). There is a constant c'_Σ such that for small enough σ, there is a displacement field $\Psi = (u, w)$ such that $u = w = \nabla w = 0$ on $\partial\Sigma$, and $I^{(\sigma)}_{\mathrm{FvK}}[\Psi, \Sigma] \le c'_\Sigma \sigma$.*

Proof. The displacement field is constructed by interpolating between the one constructed in Lemma 6 for each of the three type-B pieces which compose Σ and a smooth field satisfying the boundary conditions. The smooth field is obtained by joining the distance function smoothly to zero (and zero gradient) along the boundary, and convoluting with a smooth kernel to eliminate the singularity along the internal boundaries. Let η be a smooth mollifier with support in the ball of radius $1/2$ and let $\eta_\sigma(x) = \sigma^{-2}\eta(x/\sigma)$. Define

$$W_d(r) = \mathrm{dist}(r, \partial\Sigma)\phi\left(\frac{\mathrm{dist}(r, \partial\Sigma)}{\sigma} - 1\right), \tag{3.57}$$

and

$$w_d = \eta_\sigma * W_d, \tag{3.58}$$

where ϕ is as defined before Eq. (3.16). It is easy to verify that $|\nabla w_d| \le c$, $|\nabla^2 w_d| \le c/\sigma$, $|w_d - \mathrm{dist}(r, \partial\Sigma)| \le c\sigma$, and $w_d = \nabla w_d = 0$ on $\partial\Sigma$. Hence $\Psi_d = (0, 0, w_d)$ has bounded energy density, and obeys the prescribed boundary conditions. We divide set Σ of type A into three sets of type B along the singular lines of $\mathrm{dist}(r, \partial\Sigma)$, and impose as boundary conditions $\Psi = \Psi_d$ and $\nabla\Psi = \nabla\Psi_d$ along those lines.

Let ω_k be one of the three pieces of type B, and let γ_1^k be its external boundary, and γ_2^k, γ_3^k the other two smooth pieces of $\partial\omega_k$. In ω_k, let $\tilde{\Xi}$ be the displacement field constructed (on its image $\tilde{\omega}_k$) in Lemma 6, rescaled as in Lemma 5, and mapped back into the (x, y) coordinate system. Let $\tilde{\Psi} = (z - w, v, w + \mathrm{dist}(r, \gamma_1^k))$ be the corresponding Ψ-field, and define the interpolation function

$$\phi_k(r) = \prod_{i=1}^{3} \phi\left(\frac{\mathrm{dist}(r, \gamma_i^k)}{\sigma}\right), \tag{3.59}$$

which is smooth, zero on $\partial\omega_k$ and one in the interior. Now let

$$\Psi(r) = \tilde{\Psi}(r)\phi_k(r) + \Psi_d(r)\left[1 - \phi_k(r)\right]. \tag{3.60}$$

It is clear that this field differs from $\tilde{\Psi}$ only in a region of measure $c\sigma$, where it has bounded energy density, i.e. Ψ still has energy bounded by $c\sigma$. Further, Ψ agrees with Ψ_d up to the first gradient along $\partial\omega_k$, hence it satisfies the given boundary conditions and joins smoothly along internal boundaries. This concludes the proof. •

Proof of Theorem 2 Theorem 2 is an immediate consequence of Lemma 7, by triangulation of the domain Ω into sets of type A. •

Acknowledgements

We thank R. V. Kohn for stimulating discussions, and W. Jin and P. Sternberg for sending us a copy of their work prior to publication. S.M. thanks Weimin Jin for interesting discussions at the Northwestern University Centennial PDE Meeting and Northwestern University for its hospitality. Part of this work was done while S.M. enjoied the generous hospitality of the Department of Mathematics at the University of Minnesota as Ordway Visiting Professor. This work was partially supported by the EU TMR network *"Phase Transitions in Crystalline Solids"*, contract. no. FMRX-CT98-0229.

A Materials with nonzero Poisson's ratio

The general form of the linearized FvK functional under isotropic compression is [17, 10]

$$I_{\mathrm{FvK}}^{(\sigma,\nu)}[u,w,\Omega] = \int_\Omega (1-\nu)|\epsilon|^2 + \nu(\mathrm{Tr}\,\epsilon)^2 + \sigma^2\left[(1-\nu)|\nabla^2 w|^2 + \nu(\Delta w)^2\right], \tag{A.1}$$

where $\nu \in [-1, 1/2]$ is the Poisson ratio [16], the rescaled deformation ϵ is defined by

$$\epsilon = \nabla u + (\nabla u)^T + \nabla w \otimes \nabla w - Id, \tag{A.2}$$

and we use $|M|^2 = \mathrm{Tr}\, M^T M$ for the matrix norm. For $\nu = 0$, (A.1) reduces to (1.1).

We now show that for the purpose of proving upper and lower bounds we can restrict to $\nu = 0$ without loss of generality. Indeed, since $(\mathrm{Tr}\, M)^2 \le 2|M|^2$ for all 2×2 matrices M, it follows that

$$(1-|\nu|)|M|^2 \le (1-\nu)|M|^2 + \nu(\mathrm{Tr}\, M)^2 \le (1+|\nu|)|M|^2 \tag{A.3}$$

which implies that $I_{\mathrm{FvK}}^{(\sigma,\nu)}$ is bounded from above and from below by a multiple of $I_{\mathrm{FvK}}^{(\sigma,0)}$ for all values of Poisson's ratio ν in $(-1, 1/2]$.

References

[1] M. Ben Amar and Y. Pomeau, *Crumpled paper*, Proc. Roy. Soc. Lond. A **453** (1997), 729–755.

[2] L. Ambrosio, C. De Lellis, and C. Mantegazza, *Line energies for gradient vector fields in the plane*, Calc. Var. Partial Diff. Eqs. **9** (2000), 327–355.

[3] B. Audoly, *Stability of straight delamination blisters*, Phys. Rev. Lett. **83** (1999), 4124–4127.

[4] P. Aviles and Y. Giga, *A mathematical problem related to the physical theory of liquid crystal configurations*, Proc. Centre Math. Anal. Austr. Nat. Univ. **12** (1987), 1–16.

[5] ———, *On lower semicontinuity of a defect energy obtained by a singular limit of the Ginzburg-Landau type energy for gradient fields*, Proc. Roy. Soc. Edin. A **129** (1999), 1–17.

[6] E. Cerda, S. Chaieb, F. Melo, and L. Mahadevan, *Conical dislocations in crumpling*, Nature **401** (1999), 46–49.

[7] P. G. Ciarlet, *Theory of plates*, Mathematical elasticity, vol. II, Elsevier, Amsterdam, 1997.

[8] A. DeSimone, R. V. Kohn, S. Müller, and F. Otto, *A compactness result in the gradient theory of phase transitions*, Proc. Roy. Soc. Edin. A (2000), in press.

[9] G. Gioia, A. DeSimone, M. Ortiz, and A. M. Cuitino, *Folding energetics in thin-film diaphragms*, preprint cond-mat/0005058 (2000).

[10] G. Gioia and M. Ortiz, *Delamination in compressed thin films*, Adv. Appl. Mech. **33** (1997), 119–192.

[11] W. Jin and R. V. Kohn, *Singular perturbation and the energy of folds*, J. Nonlinear Sci. **10** (2000), 355–390.

[12] W. Jin and P. Sternberg, *Energy estimates of the von Kármán model in thin-film blistering*, preprint, 2000.

[13] ———, *In-plane displacements in thin-film blistering*, preprint, 2000.

[14] R. V. Kohn and S. Müller, *Branching of twins near an austenite-twinned-martensite interface*, Phil. Mag. A **66** (1992), 697–715.

[15] ———, *Surface energy and microstructure in coherent phase transitions*, Comm. Pure Appl. Math. **47** (1994), 405–435.

[16] L. D. Landau and E. Lifshitz, *Theory of elasticity*, 3rd ed., Course of theoretical Physics, vol. 7, Pergamon Press, Oxford, 1986.

[17] A. E. Lobkovsky, *Boundary layer analysis of the ridge singularity in a thin plate*, Phys. Rev. E **53** (1996), 3750–3759.

[18] A. E. Lobkovsky, S. Gentges, Hao Li, D. Morse, and T. A. Witten, *Scaling properties of stretching ridges in a crumpled elastic sheet*, Science **270** (1995), 1482–1485.

[19] M. Ortiz and G. Gioia, *The morphology and folding patterns of buckling-driven thin-film blisters*, J. Mech. Phys. Solids **42** (1994), 531–559.

[20] Y. Pomeau, *Buckling of thin plates in the weakly and strongly nonlinear regimes*, Phil. Mag. B **78** (1998), 235–242.