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The principle of the fermionic projector I, introduction and continuum limit

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Abstract

Combining a local gauge principle, the Pauli Exclusion Principle, and the principles of General Relativity in a particular way, we obtain the mathematical framework for the formulation of a new type of variational principle in space-time. The postulate that physics can be formulated within this framework is called the "principle of the fermionic projector."

The principle of the fermionic projector is introduced and discussed. We describe a limiting process with which our variational principles can be analyzed in the setting of relativistic quantum mechanics.

The principle of the fermionic projector was first introduced in [1]. In the subsequent works [2]–[6], the foundations of this principle were clarified, and necessary mathematical tools were developed. Based on these papers, the principle of the fermionic projector can now be stated in its mature form. The present work is the first of a series of forthcoming papers in which the principle of the fermionic projector and its consequences will be worked out in detail.

1 Formulation of the Principle

Although relativistic quantum field theory has been very successful, its present mathematical formulation (in the canonical formalism or with path integrals) is far from being convincing. Partly, the involved problems seem unavoidable: the interaction of quantized fields is simply highly complicated. However, there are also a number of inconsistencies and conceptual difficulties in the underlying mathematical formalism. The situation becomes even more problematic if one tries to include gravitation. Therefore, it seems tempting to look for a mathematical framework which is more appropriate for the formulation of physics than present QFT. The principle of the fermionic projector is a promising attempt in this direction.

Since we want to avoid second quantized fields, our starting point is relativistic quantum mechanics combined with classical field theory. This means more precisely that we consider Dirac particles, described by their quantum mechanical wave functions, which interact via classical fields (e.g. an electromagnetic field, a Yang-Mills field, or a gravitational field). We use the concept of the Dirac sea to reinterpret the negative-energy solutions of the Dirac equation as anti-particle states. Although it is not really essential for what follows, one should keep in mind that the Feynman diagrams of perturbative

quantum field theory can already be derived in the setting of relativistic quantum mechanics (as e.g. explained in [7]); this yields consistency of our approach with the high precision tests of quantum field theory (like the Lamb shift or the anomalous magnetic moment). But clearly, relativistic quantum mechanics does not take into account the particular effects of quantized fields (like the Planck radiation and the photo electric effect); thus our constructions given below should involve some kind of "quantization procedure" for the fields.

Relativistic quantum mechanics and classical field theory incorporate several physical principles: According to the concepts of General Relativity, gravitation is understood in terms of the Lorentzian geometry of space-time. The gauge principle states that the theory should be invariant under local gauge transformations of the classical potentials. In our quantum mechanical context, the Pauli Exclusion Principle says that many-particle wave functions must be anti-symmetric; this implies that each quantum mechanical state may be occupied by at most one Dirac particle.

In this section, we will generalize the notions of relativistic quantum mechanics and classical field theory in several construction steps. This will be done in a very intuitive way. The aim is to work out the essence of the just mentioned physical principles by dropping all additional and less important structures. This will lead us to a quite abstract mathematical framework, in which we shall then formulate the principle of the fermionic projector. We shall end this section with a brief physical overview and discussion.

1.1 Connection between Local Gauge Freedom and the Measurability of Position and Time

Let us first recall some basic notions of gauge theories in Minkowski space. The local gauge principle has its origin in the observation that the electromagnetic potential $A(x)$ of classical electrodynamics is determined only up to gauge transformations of the form

$$A_j \;\to\; A_j + \partial_j \Lambda \tag{1.1}$$

with a real function $\Lambda(x)$. This transformation property was generalized in Yang-Mills theories, from which physical gauge theories like the standard model evolved. In these theories, the potential $A(x)$ takes values in a Lie algebra. The corresponding Lie group $\mathcal{G}$ is called the *gauge group*. Local gauge freedom means that we can choose any section $U(x) \in \mathcal{G}$ and transform the potential according to[1]

$$A_j \;\to\; U A_j U^{-1} \;+\; iU(\partial_j U^{-1}) \quad . \tag{1.2}$$

The gauge transformations of electrodynamics (1.1) are recovered from this formula in the special case of a $U(1)$ gauge group and $U = \exp(i\Lambda)$. The transformation rule (1.2) can be understood more easily if one introduces the gauge-covariant derivative D by

$$D_j \;=\; \partial_j \;-\; iA_j \quad . \tag{1.3}$$

Namely, (1.2) is equivalent to demanding that the gauge-covariant derivative transforms according to the adjoint representation of the gauge group,

$$D_j \;\to\; U D_j U^{-1} \quad . \tag{1.4}$$

[1]Notice that the coupling constant e of the gauge fields was omitted ((1.2) is often written in the form $A_j \to U A_j U^{-1} + ie^{-1} U(\partial_j U^{-1})$). This can be arranged by rescaling the gauge potentials according to $A_j \to e^{-1} A_j$. One should keep in mind, however, that the coupling constant still appears in the field equations, e.g. the Maxwell equations for a Dirac particle in our convention read $\partial_l F^{kl} = 4\pi\, e^2\, \overline{\Psi} \gamma^k \Psi$.

Following the minimal coupling procedure, the gauge potentials are introduced into the physical theory by replacing all partial derivatives by the corresponding gauge-covariant derivatives. Using the requirement that the so-obtained theory should be independent of the choice of the gauge, one can deduce the behavior of all objects of the theory under gauge transformations. For example, a quantum mechanical wave function Ψ which is coupled to the gauge fields must behave under gauge transformations like

$$\Psi(x) \;\to\; U(x)\, \Psi(x) \quad , \tag{1.5}$$

so that its gauge-covariant derivative transforms again according to the fundamental representation, $D_j \Psi \to U\, D_j \Psi$.

In this subsection, we shall give a possible explanation as to why local gauge freedom occurs in physics. Apart from giving some physical insight, this consideration will provide a formalism which will be the starting point for the constructions leading to the principle of the fermionic projector.

We begin with the simple example of the $U(1)$ gauge transformations of the magnetic field for a Schrödinger wave function Ψ in nonrelativistic quantum mechanics. Since it will be sufficient to consider the situation for fixed time, we only write out the spatial dependence of the wave function, $\Psi = \Psi(\vec{x})$ with $\vec{x} \in \mathbb{R}^3$. Taking the nonrelativistic limit for a $U(1)$ gauge group, the gauge freedom (1.5) states that the local phase of the wave function $\Psi(\vec{x})$ is undetermined. This is consistent with the quantum mechanical interpretation of the wave function, according to which the phase of a wave function is not an observable quantity, only its absolute square $|\Psi(\vec{x})|^2$ has a physical meaning as the probability density of the particle. One can even go one step further and take the point of view that the inability to determine the local phase of a quantum mechanical wave function is the physical reason for the local gauge freedom (1.5). Then the $U(1)$ gauge transformations of the magnetic field become a consequence of the principles of quantum mechanics. This argument becomes clearer when stated in more mathematical terms as follows. We consider the usual scalar product on the Schrödinger wave functions,

$$<\Psi \,|\, \Phi> \;=\; \int_{\mathbb{R}^3} \overline{\Psi(\vec{x})}\, \Phi(\vec{x})\, d\vec{x} \quad ,$$

and denote the corresponding Hilbert space by H. On H, the position operators $\vec{X}$ are given as the multiplication operators with the coordinate functions,

$$\vec{X}\, \Psi(\vec{x}) \;=\; \vec{x}\, \Psi(\vec{x}) \quad .$$

As it is common in quantum mechanics, we consider H as an abstract Hilbert space (i.e. we forget about the fact that H was introduced as a space of functions). Then the wave function $\Psi(\vec{x})$ corresponding to a vector $\Psi \in H$ is obtained by constructing a "position representation" of the Hilbert space. In bra/ket notation[2], this is done by choosing an "eigenvector basis" $|\vec{x}>$ of the position operators, i.e.

$$\vec{X}\, |\vec{x}> \;=\; \vec{x}\, |\vec{x}> \quad , \qquad\qquad <\vec{x} \,|\, \vec{y}> \;=\; \delta^3(\vec{x}-\vec{y}) \quad ; \tag{1.6}$$

the wave function is then introduced by

$$\Psi(\vec{x}) \;=\; <\vec{x} \,|\, \Psi> \quad . \tag{1.7}$$

[2]We note that the formal bra/ket notation can be made mathematically precise using spectral measures [3].

The important point for us is that the "eigenvectors" $|\vec{x}>$ of the position operators are only determined up to a phase. Namely, the transformation

$$|\vec{x}> \;\to\; \exp\left(-i\Lambda(\vec{x})\right)|\vec{x}> \tag{1.8}$$

leaves invariant the conditions (1.6) for the "eigenvector basis." If we substitute (1.8) into (1.7), we obtain precisely the transformation (1.5) of the wave function with $U = \exp(i\Lambda)$. The transformation properties of the gauge-covariant derivative (1.4) and of the gauge potentials (1.1) follow from (1.5) if one assumes that $(D_j)_{j=1,2,3}$ are operators on H (and thus do not depend on the representation of H as functions in position space). In physics, the operators $\vec{\pi} = -i\vec{D}$ are called the "canonical momentum operators."

The relation just described between the position representation of quantum mechanical states and the $U(1)$ gauge transformations of the magnetic field was noticed a long time ago. However, the idea of explaining local gauge freedom from quantum mechanical principles was not recognized as being of general significance. In particular, it was never extended to the relativistic setting or to more general gauge groups. The probable reason for this is that these generalizations are not quite straightforward; they make it necessary to formulate relativistic quantum mechanics in a particular way, slightly modifying the usual physical concepts. We shall now outline how this is done for the Dirac theory, sketching those constructions of [2] which are essential for our purpose.

We consider on the four-component Dirac spinors $(\Psi^\alpha(x))_{\alpha=1,\ldots,4}$ in Minkowski space a scalar product of signature $(2,2)$,

$$\prec\!\Psi \,|\, \Phi\!\succ(x) \;=\; \sum_{\alpha=1}^{4} s_\alpha \; \Psi^\alpha(x)^* \; \Phi^\alpha(x) \quad , \qquad\qquad s_1 = s_2 = 1, \quad s_3 = s_4 = -1, \tag{1.9}$$

called *spin scalar product*[3] (Ψ^* is the complex conjugated wave function). We denote the vector space of all Dirac wave functions by H. Integrating the spin scalar product over space-time, we obtain an indefinite scalar product on H,

$$<\Psi \,|\, \Phi> \;=\; \int_{\mathbb{R}^4} \prec\!\Psi \,|\, \Phi\!\succ(x)\, d^4x \quad . \tag{1.10}$$

Furthermore, we introduce on H time/position operators $(X^i)_{i=0,\ldots,3}$ by multiplication with the coordinate functions,

$$X^i \; \Psi(x) \;=\; x^i \; \Psi(x) \quad .$$

We now consider $(H,\, <.|.>)$ as an abstract scalar product space. In order to construct a time/position representation of H, we must choose an "eigenvector basis" of the time/position operators. Since the wave functions have four components, an "eigenvector basis" has in bra/ket notation the form $|x\alpha>$, $x \in \mathbb{R}^4$, $\alpha = 1, \ldots, 4$; it is characterized by the conditions

$$X^i \,|x\alpha> \;=\; x^i \,|x\alpha> \quad , \qquad\qquad <x\alpha \,|\, y\beta> \;=\; s_\alpha \, \delta_{\alpha\beta} \, \delta^4(x-y) \tag{1.11}$$

with s_α as in (1.9). The wave function corresponding to a vector $\Psi \in H$ is defined by

$$\Psi^\alpha(x) \;=\; <x\alpha \,|\, \Psi> \quad . \tag{1.12}$$

[3] We remark for clarity that the spin scalar product is in physics usually denoted by $\overline{\Psi}\Phi$ with the adjoint spinor $\overline{\Psi} = \Psi^*\gamma^0$. Our definition without using the Dirac matrices avoids possible confusion in the generalization to curved space-time.

The conditions (1.11) determine the basis $|x\alpha>$ only up to local isometries of a scalar product of signature $(2,2)$, i.e. up to transformations of the form

$$|x\alpha> \;\to\; \sum_{\beta=1}^{4} (U(x)^{-1})^{\alpha}_{\beta}\, |x\beta> \quad \text{with} \quad U(x) \in U(2,2) \quad . \tag{1.13}$$

If we identify these transformations with gauge transformations and substitute into (1.12), we obtain local gauge freedom of the form (1.5) with gauge group $\mathcal{G} = U(2,2)$. Since gauge transformations correspond to changes of the "eigenvector basis" $|x\alpha>$, we also call $|x\alpha>$ a *gauge*.

From the mathematical point of view, (1.11)–(1.13) is a straightforward generalization of (1.6)–(1.8) to the four-dimensional setting and four-component wave functions, taking into account that the scalar product on the Dirac spinors has signature $(2,2)$. However, our construction departs from the usual description of physics. Namely, the time operator X^0 is not commonly used in relativistic quantum mechanics, and the scalar product (1.10) is unconventional. But these differences are not problematic, as we shall see in the following (see [2] for a more detailed discussion). To avoid confusion, we remark that the scalar product (1.10) could be infinite for physical states, because the time integral might diverge. This problem could be removed for example by considering the system in finite 4-volume and taking a suitable limit. Generally speaking, we do not worry too much about the normalization of quantum mechanical states in this introduction, knowing that this issue can easily be made mathematically precise.

In order to describe the physical interactions with the just obtained $U(2,2)$ gauge freedom, we must incorporate the principles of General Relativity, taking the Dirac operator as the basic object in space-time. This yields a unified description of gravitation and electrodynamics as a classical gauge theory [2]. We briefly outline the underlying construction. According to the equivalence principle, a gravitational field makes it necessary to consider general "curved" coordinate systems in space-time; in other words, space-time is a manifold. Similar to (1.9),(1.10), we consider on this manifold the scalar product space of four-component wave functions $(H, <.|.>)$ and introduce for every coordinate system x^i the multiplication operators X^i. The arbitrariness of the time/position representation of H again yields local $U(2,2)$ gauge freedom. The Dirac operator G is a differential operator of first order on H; i.e., in a chart and a gauge, it takes the form

$$G \;=\; iG^j(x)\frac{\partial}{\partial x^j} \;+\; B(x) \tag{1.14}$$

with 4×4 matrices $G^j(x)$ and $B(x)$. It is a basic fact in General Relativity and gauge theories that the gravitational field, and the gauge potentials, can locally be made zero by choosing a "freely falling" reference frame and a suitable gauge, respectively. Since we are here working exclusively with the Dirac operator, we consider instead the condition that the Dirac operator should coincide locally with the free Dirac operator. More precisely, we demand that for any space-time point p, there is a coordinate system and a gauge such that $G^j(p) = \gamma^j$, $\partial_l G^j(p) = 0$, and $B(p) = 0$, where γ^j are the Dirac matrices of Minkowski space in the Dirac representation. It turns out that, with this local condition, we have introduced both a gravitational and an electromagnetic field. The Lorentzian metric g_{jk} is given by the anti-commutator

$$g^{jk}(x)\, \mathbb{1} \;=\; \frac{1}{2}\left\{G^j(x),\, G^k(x)\right\} \;\equiv\; \frac{1}{2}\left(G^j(x)\, G^k(x) + G^k(x)\, G^j(x)\right) \quad . \tag{1.15}$$

Moreover, the Dirac operator uniquely induces a $U(2,2)$ gauge covariant derivative D, which we call *spin derivative*. The spin connection contains both metric connection coefficients and the electromagnetic potential A. The curvature R_{jk} of the spin connection,

$$R_{jk} \;=\; \frac{i}{2}\,[D_j, D_k] \;\equiv\; \frac{i}{2}\,(D_j D_k - D_k D_j) \quad ,$$

is composed of the Riemannian curvature tensor R_{ijkl} and of the electromagnetic field tensor $F_{jk} = \partial_j A_k - \partial_k A_j$,

$$R_{jk} \;=\; -\frac{i}{8}\, R_{mnjk}\, G^m\, G^n \;+\; \frac{1}{2}\, F_{jk}\, \mathbb{1} \quad .$$

With these tensor fields, one can formulate classical field theory. Finally, the Dirac equation in the gravitational and electromagnetic field takes the form

$$(G - m)\, \Psi \;=\; 0 \quad , \tag{1.16}$$

where m is the mass of the Dirac particle. Notice that, in our description, the Dirac operator determines both the Dirac wave functions and the classical potentials; namely, the wave functions via the Dirac equation (1.16), and the classical potentials via the construction of the metric and the spin derivative. The minimal coupling procedure is no longer used.

The $U(2,2)$ gauge symmetry [2] describes gravitation and electrodynamics, but it does not include the weak and strong interactions. In order to build in additional gauge fields, we must extend the gauge group. Since our gauge group is the isometry group of the spin scalar product, this can be accomplished only by increasing the number of components of the wave functions. In general, one can take wave functions with $p+q$ components and a spin scalar product of signature (p,q),

$$\begin{aligned} \prec \Psi \,|\, \Phi \succ (x) \;&=\; \sum_{\alpha=1}^{p+q} s_\alpha\; \Psi^\alpha(x)^*\; \Phi^\alpha(x) \qquad\qquad \text{with} \\ s_1 = \cdots = s_p = 1 \quad &, \qquad s_{p+1} = \cdots = s_{p+q} = -1 \quad . \end{aligned} \tag{1.17}$$

We call (p,q) the *spin dimension*. Repeating the above construction (1.10)–(1.12) for this spin scalar product yields local gauge freedom with gauge group $\mathcal{G} = U(p,q)$. However, it is not possible to introduce the Dirac operator in this generality. Therefore, we will in what follows always assume that the spin dimension is $(2N, 2N)$ with $N \geq 1$. In this case, one can regard the $4N$ component wave functions as the direct sum of N Dirac spinors, and the above construction of the Dirac operator can be generalized in a straightforward manner. This leads to a Dirac operator of the form (1.14) with $4N \times 4N$ matrices $G^j(x)$ and $B(x)$ satisfying the anti-commutation relations (1.15). The Dirac equation again has the form (1.16). The direct sum of Dirac spinors can be used to describe different types of fermions (e.g. leptons and quarks). Our concept is that the $U(2N, 2N)$ gauge symmetry should, in the correct model, give rise to the gravitational, strong, and electroweak forces.

For clarity, we finally mention some differences of our approach to standard gauge theories. Usually, the gauge groups (e.g. the $SU(2)_{\mathrm{W}}$ or $SU(3)_{\mathrm{S}}$ in the standard model) act on separate indices of the wave functions (namely, on the isospin and color indices, respectively). In contrast to this, our $U(2,2)$ gauge transformations simply act on the spinor index. In our generalization to higher spin dimension (1.17), we make no distinction

between the spinor index and the index of the gauge fields, and they are both combined in one index $\alpha = 1, \ldots, 4N$. Furthermore, we point out that the gauge group and the coupling of the gauge fields to the Dirac particles are, in our setting, completely determined by the spin dimension. Compared to standard gauge theories, where the gauge groups and their couplings can be chosen arbitrarily, this is a strong restriction for the formulation of physical models.

1.2 Projection on Fermionic States

Our setting so far is that of one-particle quantum mechanics based on the Dirac equation (1.16). We will now extend our ideas to many-fermion systems.

A single Dirac particle is described by its wave function $\Psi^\alpha(x) = <x\alpha|\Psi>$, or, in a gauge-independent way, by a vector $\Psi \in H$. Since the phase and normalization of Ψ have no physical significance, we prefer to describe the Dirac particle by the one-dimensional subspace $<\Psi> \equiv \{\lambda\Psi,\ \lambda \in \mathbb{C}\} \subset H$. Now consider the system of n Dirac particles, which occupy the one-particle states $\Psi_1, \ldots, \Psi_n \in H$. Generalizing the subspace $<\Psi>$ of the one-particle system, we will here describe the many-particle state by the subspace $<\Psi_1, \ldots, \Psi_n> \subset H$ spanned by $\Psi_1, \ldots, \Psi_n$. Let us consider for simplicity only the generic case that this subspace is non-degenerate (i.e. there should be no vectors $0 \neq \Psi \in Y$ with $<\Psi|\Phi> = 0$ for all $\Phi \in Y$). Just as in positive definite scalar product spaces, every non-degenerate subspace $Y \subset H$ uniquely determines a projector P_Y on this subspace, characterized by the conditions $P_Y^* = P_Y = P_Y^2$ and $\mathrm{Im}(P_Y) = Y$, where "$*$" denotes the adjoint with respect to the scalar product $<.|.>$. Instead of working directly with the subspace $<\Psi_1, \ldots, \Psi_n> \subset H$, it is more convenient for us to consider the corresponding projector P,

$$P \;=\; P_{<\Psi_1, \ldots, \Psi_n>} \quad .$$

We call P the *fermionic projector*. In this work, we will always describe the Dirac particles of our physical system by a fermionic projector.

The concept of the fermionic projector departs from the usual description of a many-particle state by an anti-symmetric wave function or a vector of the fermionic Fock space. Let us discuss this difference in detail. In many-particle quantum mechanics, the system of Dirac particles $\Psi_1, \ldots, \Psi_n$ is described by the anti-symmetric product wave function

$$\begin{aligned} \Psi^{\alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n) &= (\Psi_1 \wedge \cdots \wedge \Psi_n)^{\alpha_1 \cdots \alpha_n}(x_1, \ldots, x_n) \\ &\equiv \frac{1}{n!} \sum_{\sigma \in S(n)} (-1)^{|\sigma|}\, \Psi_{\sigma(1)}^{\alpha_1}(x_1) \cdots \Psi_{\sigma(n)}^{\alpha_n}(x_n) \quad , \end{aligned} \tag{1.18}$$

where $S(n)$ is the set of all permutations of $\{1, \ldots, n\}$. The wave functions of the form (1.18) are called n-particle Hartree-Fock states. They span the n-particle Fock space $F^n = \bigwedge^n H$. In the fermionic Fock space formalism of QFT, a quantum state is a linear combination of Hartree-Fock states, i.e. a vector of the Fock space $F = \bigoplus_{n=0}^\infty F^n$. In order to connect the fermionic projector with the Fock space formalism, we associate to a projector P_Y on a subspace $Y = <\Psi_1, \ldots, \Psi_n> \subset H$ the wave function (1.18). Because of the anti-symmetrization in (1.18), this mapping is (up to a complex factor) independent of the choice of the basis vectors $\Psi_1, \ldots, \Psi_n$, and gives a one-to-one correspondence between the projectors P_Y on n-dimensional subspaces $Y \subset H$ and n-particle Hartree-Fock states. In this way, one sees that the description of a many-particle state with the fermionic projector is equivalent to using a Hartree-Fock state. With this correspondence, the

formalism of the fermionic projector becomes a special case of the Fock space formalism, obtained by restricting to Hartree-Fock states. In particular, we conclude that the physical concepts behind fermionic Fock spaces, namely the Pauli Exclusion Principle and the fact that quantum mechanical particles are indistinguishable from each other, are also respected by the fermionic projector. However, we point out that the the fermionic projector is not mathematically equivalent to a state of the Fock space, since a vector of the Fock space can in general only be represented by a linear combination of Hartree-Fock states.

Let us analyze what this mathematical difference means physically. If nature is described by a fermionic projector, the joint wave function of all fermions of the Universe must be a Hartree-Fock state. However, this condition cannot be immediately verified in experiments, because measurements can never take into account all existing fermions. In all realistic situations, one must restrict the observations to a small subsystem of the Universe. As is worked out in Appendix A, the effective wave function of a subsystem need *not* be a Hartree-Fock state; it corresponds to an arbitrary vector of the Fock space of the subsystem, assuming that the number of particles of the whole system is sufficiently large. From this, we conclude that the description of the many-particle system with the fermionic projector is indeed physically equivalent to the Fock space formalism. For theoretical considerations, it must be taken into account that the fermionic projector merely corresponds to a Hartree-Fock state; for all practical purposes, however, one can just as well work with the whole Fock space.

We showed after (1.18) that the description of a many-particle state with the fermionic projector implies the Pauli Exclusion Principle. This can also be understood directly in a non-technical way as follows. For a given state $\Psi \in H$, we can form the projector $P_{<\Psi>}$ describing the one-particle state, but there is no projector which would correspond to a two-particle state (notice that the naive generalization $2P_{<\Psi>}$ is not a projector). More generally, every vector $\Psi \in H$ either lies in the image of P, $\Psi \in P(H)$, or it does not. Via these two conditions, the fermionic projector encodes for every state $\Psi \in H$ the occupation numbers 1 and 0, respectively, but it is not possible to describe higher occupation numbers. In this way, the fermionic projector naturally incorporates the Pauli Exclusion Principle in its formulation that each quantum mechanical state may be occupied by at most one fermion.

Let us now describe the form of the fermionic projector P more concretely. Since the fermionic projector is composed of one-particle states which should all satisfy the Dirac equation (1.16), we demand that it be a solution of the operator equation

$$(G - m)\, P \;=\; 0 \quad . \tag{1.19}$$

It is a well-known fact that the Dirac equation admits unphysical solutions of negative energy (see e.g. [7]). Following Dirac's original concept, we here remove this problem of relativistic quantum mechanics by the assumption that all of the negative-energy states are occupied in the vacuum. We thus describe the fermionic vacuum by the fermionic projector

$$P \;=\; P^{\text{sea}} \quad , \tag{1.20}$$

where P^{sea}, the so-called Dirac sea, is the projector on the space spanned by all negative-energy states. The main difficulty in introducing the Dirac sea is to give a meaningful definition of "negative-energy" states in the presence of time-dependent classical fields (e.g. an electromagnetic or gravitational field). A further difficulty is to handle the infinite number of states of the Dirac sea in an indefinite scalar product space. These problems

are resolved in [4, 5, 6]. Moreover, the light-cone expansion [5, 6] yields explicit formulas for P^{sea}, which show in detail how the fermionic projector depends on the classical fields. These results are the mathematical basis of our approach, and we will frequently use them throughout this work.

According to Dirac, systems of particles and anti-particles are obtained from the vacuum by occupying positive-energy states and removing particles from negative-energy states, respectively. Thus we describe a many-particle system by the fermionic projector

$$P \;=\; P^{\text{sea}} \;+\; P^{\text{p}} \;-\; P^{\text{a}} \quad , \tag{1.21}$$

where P^{p} and P^{a} are projectors on the spaces spanned by the particle and anti-particle states of the system, respectively. Since both P^{p} and P^{a} should be composed of solutions of the Dirac equation (1.16), it is clear they must satisfy the Dirac equation

$$(G - m)\, P^{\text{p}} \;=\; 0 \;=\; (G - m)\, P^{\text{a}} \quad . \tag{1.22}$$

Furthermore, we want that P^{p} and P^{a} are composed only of states of positive and negative energy, respectively; since P^{sea} projects on all negative-energy states, this can be written as

$$P^{\text{p}}\, P^{\text{sea}} \;=\; 0 \;=\; P^{\text{sea}}\, P^{\text{p}} \qquad \text{and} \qquad P^{\text{a}}\, P^{\text{sea}} \;=\; P^{\text{a}} \;=\; P^{\text{sea}}\, P^{\text{a}} \quad . \tag{1.23}$$

Actually, the results of [5, 6] imply that the decomposition (1.21)–(1.23) of the fermionic projector into the Dirac sea and the particle/anti-particle states is canonical; i.e. it can be uniquely constructed for any fermionic projector satisfying the Dirac equation (1.19).

For simplicity, the Dirac particles just considered all had the same mass m. For modelling a realistic physical system, the concept of the fermionic projector must be extended to systems of Dirac particles with different masses (like e, μ, τ, and quarks) and massless chiral particles (like neutrinos). These generalizations are quite straightforward; see [4, 6].

We finally mention one general result in [5, 6] which will be important for what follows. Since the classical force fields act on all particles of the Dirac field, the fermionic projector clearly depends on the classical bosonic fields. According to our method of constructing the classical fields from the Dirac operator (outlined in Subsection 1.1) we can say equivalently that the fermionic projector depends on the Dirac operator. This dependence can also be understood more directly via the Dirac equation (1.19). As is obvious from the formulas of the light-cone expansion [5, 6], the fermionic projector contains all the information about the classical fields. In other words, one can uniquely reconstruct the Dirac operator from a given fermionic projector. Therefore it is consistent to consider the fermionic projector as the basic object in space-time and to regard the Dirac operator merely as an auxiliary object which is useful in describing the interaction of the fermions via classical fields.

1.3 Discretization of Space-Time

The ultraviolet divergences of perturbative QFT indicate that the current description of physics should break down at very small distances. It is generally believed that the length scale where yet unknown physical effects should become important is given by the Planck length. Here we will assume that space-time consists, on the Planck scale, of discrete space-time points. The simplest way to discretize space-time would be to replace the space-time continuum by a four-dimensional lattice (as it is e.g. done in lattice gauge theories). In the following construction, we will go much further and discretize space-time

in a way where notions like "lattice spacing" and "neighboring lattice points" are given up. On the other hand, we will retain the principles of General Relativity and our local gauge freedom.

We first consider the situation in a given coordinate system x^i in space-time[4]. For the discretization, we replace the time/position operators X^i by mutually commuting operators with a *purely discrete spectrum*. We take the joint spectrum of these operators, i.e. the set

$$M \;=\; \{x \in \mathbb{R}^4 \mid \text{there is } u \in H \text{ with } X^i u = x^i u \text{ for all } i = 0, \ldots, 3\} \quad ,$$

as our discrete space-time points. We assume that the joint eigenspaces e_x of the X^i,

$$e_x \;=\; \{u \mid X^i u = x^i u \text{ for all } i = 0, \ldots, 3\} \quad , \qquad x \in M,$$

are $4N$-dimensional subspaces of H, on which the scalar product $<.|.>$ has the signature $(2N, 2N)$. Then we can choose a basis $|x\alpha>$, $x \in M$, $\alpha = 1, \ldots, 4N$ satisfying

$$\begin{aligned} X^i\,|x\alpha> \;&=\; x^i\,|x\alpha> \quad , \qquad <x\alpha \mid y\beta> = s_\alpha\,\delta_{\alpha\beta}\,\delta_{xy} \qquad \text{with} \\ s_1 = \cdots = s_{2N} = 1 \quad &, \quad s_{2N+1} = \cdots = s_{4N} = -1 \quad . \end{aligned} \tag{1.24}$$

These relations differ from (1.11) only by the replacement $\delta^4(x-y) \to \delta_{xy}$. It is useful to introduce the projectors E_x on the eigenspaces e_x by

$$E_x \;=\; \sum_{\alpha=1}^{p+q} s_\alpha\,|x\alpha><x\alpha| \quad ; \tag{1.25}$$

they satisfy the relations

$$X^i\,E_x \;=\; x^i\,E_x \qquad \text{and} \tag{1.26}$$

$$E_x^* \;=\; E_x \quad , \qquad E_x\,E_y \;=\; \delta_{xy}\,E_x \quad , \qquad \sum_{x \in M} E_x \;=\; \mathbb{1} \quad , \tag{1.27}$$

where "*" denotes the adjoint with respect to the scalar product $<.|.>$ (these relations immediately follow from (1.24) and the fact that $|x\alpha>$ is a basis). Actually, the operators E_x are independent of the choice of the basis $|x\alpha>$; they are uniquely characterized by (1.26) and (1.27) as the spectral projectors of the operators X^i.

If we change the coordinate system to $\tilde{x}^i = \tilde{x}^i(x)$, the discrete space-time points $M \subset \mathbb{R}^4$ are mapped to different points in $\mathbb{R}^4$, more precisely

$$\tilde{M} = \tilde{x}(M) \quad , \qquad \tilde{E}_{\tilde{x}(x)} = E_x \quad . \tag{1.28}$$

With such coordinate transformations, the relative position of the discrete space-time points in $\mathbb{R}^4$ can be arbitrarily changed. Taking general coordinate invariance seriously on the Planck scale, this is consistent only if we forget about the fact that M and $\tilde{M}$ are subsets of $\mathbb{R}^4$ and consider them merely as index sets for the spectral projectors. In other words, we give up the ordering of the discrete space-time points, which is inherited from the ambient vector space $\mathbb{R}^4$, and consider M and $\tilde{M}$ only as point sets. After this generalization, we can identify M with $\tilde{M}$ (via the equivalence relation $\tilde{x}(x) \simeq x$).

[4]We assume for simplicity that the chart x^i describes all of space-time. The generalization to a non-trivial space-time topology is done in a straightforward way by gluing together different charts.

According to (1.28), the spectral projectors $(E_p)_{p \in M}$ are then independent of the choice of coordinates.

We regard the projectors $(E_p)_{p \in M}$ as the basic objects describing space-time. The time/position operators can be deduced from them. Namely, every coordinate system yields an injection of the discrete space-time points

$$x \;:\; M \hookrightarrow \mathbb{R}^4 \quad , \tag{1.29}$$

and the corresponding time/position operators X^i can be written as

$$X^i \;=\; \sum_{p \in M} x^i(p)\, E_p \quad . \tag{1.30}$$

Since every injection of the discrete space-time points into $\mathbb{R}^4$ can be realized by a suitable choice of coordinates (i.e. for every injection $\iota : M \hookrightarrow \mathbb{R}^4$ there is a chart x^i such that $x(M) = \iota(M)$), we can drop the condition that x is induced by a coordinate system. We can thus take for x in (1.29),(1.30) any embedding of M into $\mathbb{R}^4$.

Let us summarize the result of our construction. We shall describe space-time by an indefinite scalar product space $(H,\, <.|.>)$ and projectors $(E_p)_{p \in M}$ on H, where M is a (finite or countable) index set. The projectors E_p are characterized by the conditions (1.27). Furthermore, we assume that the *spin dimension* is $(2N, 2N)$, i.e. $E_p(H) \subset H$ is, for all $p \in M$, a subspace of signature $(2N, 2N)$. We call $(H,\, <.|.>,\, (E_p)_{p \in M})$ *discrete space-time*. The equivalence principle is taken into account via the freedom in choosing the embeddings (1.29),(1.30) of the discrete space-time points. Moreover, one can choose a basis $|p\alpha>$, $p \in M$, $\alpha = 1, \ldots, 4N$, of H satisfying the conditions

$$E_p\, |q\alpha> \;=\; \delta_{pq}\, |p\alpha> \quad , \qquad\qquad <p\alpha \,|\, q\beta> \;=\; s_\alpha\, \delta_{\alpha\beta}\, \delta_{pq}$$

with s_α as in (1.24); such a basis is called a *gauge*. It is determined only up to transformations of the form

$$|p\alpha> \;\to\; \sum_{\beta=1}^{2N} (U(p)^{-1})^\alpha_\beta\, |p\beta> \quad \text{with} \quad U(p) \in U(2N, 2N) \quad . \tag{1.31}$$

These are the local gauge transformations of discrete space-time.

1.4 The Principle of the Fermionic Projector

For the complete description of a physical system, we must introduce additional objects in discrete space-time $(H,\, <.|.>,\, (E_p)_{p \in M})$. As described at the end of Subsection 1.2, one can, in the space-time continuum, regard the fermionic projector as the basic physical object; namely, it yields via (1.21) the quantum mechanical states, and moreover determines the Dirac operator and thus the classical fields. Therefore, it seems promising to carry over the fermionic projector to discrete space-time. We introduce the *fermionic projector of discrete space-time* P as a projector acting on the vector space H of discrete space-time.

In analogy to the situation for the continuum, we expect that a physical system can be completely characterized by a fermionic projector in discrete space-time. At this stage, however, it is not at all clear whether this description makes any physical sense. In particular, it seems problematic that neither the Dirac equation nor the classical field equations can be formulated in or extended to discrete space-time; thus it becomes necessary to replace them by equations of completely different type. We take it as an ad-hoc postulate that this can actually be done; namely we assert

The Principle of the Fermionic Projector: A physical system is completely described by the fermionic projector in discrete space-time. The physical equations should be formulated exclusively with the fermionic projector in discrete space-time, i.e. they must be stated in terms of the operators P and $(E_p)_{p \in M}$ on H.

Clearly, the validity and consequences of this postulate still need to be investigated; this is precisely the aim of the present work. The physical equations formulated with P and $(E_p)_{p \in M}$ are called the *equations of discrete space-time*.

1.5 A Variational Principle

Before coming to the general discussion of the principle of the fermionic projector, we give in this subsection an example of a variational principle in discrete space-time. This is done to give the reader an idea of how one can formulate equations in discrete space-time. This example will serve as our model variational principle, and we will often come back to it.

Let us first discuss the general mathematical form of possible equations in discrete space-time. The operators P and $(E_p)_{p \in M}$ all have a very simple structure in that they are projectors acting on H. Therefore, it is certainly not worth studying these operators separately; for physically promising equations, we must combine the projectors P and $(E_p)_{p \in M}$ in a mathematically interesting way. Composite expressions in these operators can be manipulated using the idempotence of P and the relations (1.27) between the projectors $(E_p)_{p \in M}$. First of all, the identities $\sum_{p \in M} E_p = \mathbb{1}$ and $E_p^2 = E_p$ allow us to insert factors E_p into the formulas; e.g.

$$E_x \, P \, \Psi \;=\; E_x \, P \left(\sum_{y \in M} E_y \right) \Psi \;=\; \sum_{y \in M} (E_x \, P \, E_y) \, E_y \, \Psi \quad .$$

Writing

$$P(x,y) \;\equiv\; E_x \, P \, E_y \quad ,$$

we obtain the identity

$$E_x \, (P \, \Psi) \;=\; \sum_{y \in M} P(x,y) \, E_y \, \Psi \quad .$$

This representation of P by a sum over the discrete space-time points resembles the integral representation of an operator in the continuum with an integral kernel. Therefore, we call $P(x,y)$ the *discrete kernel* of the fermionic projector. The discrete kernel can be regarded as a canonical representation of the fermionic projector of discrete space-time, induced by the projectors $(E_p)_{p \in M}$. Now consider a general product of the operators P and $(E_p)_{p \in M}$. Using the relations $P^2 = P$ and $E_x \, E_y = \delta_{xy} \, E_x$, every operator product can be simplified to one with alternating factors P and E_p, i.e. to an operator product of the form

$$E_{x_1} \, P \, E_{x_2} \, P \, E_{x_3} \cdots E_{x_{n-1}} \, P \, E_{x_n} \qquad \text{with } x_j \in M. \tag{1.32}$$

Again using $E_p^2 = E_p$, we can rewrite this product with the discrete kernel as

$$P(x_1, x_2) \, P(x_2, x_3) \, \cdots \, P(x_{n-1}, x_n) \quad . \tag{1.33}$$

We conclude that the equations of discrete space-time should be formed of products of the discrete kernel, where the second argument of each factor must coincide with the first argument of the following factor. We refer to (1.33) as a *chain*.

In analogy to the Lagrangian formulation of classical field theory, we want to set up a variational principle. Our "action" should be a scalar functional depending on the operators P and E_p. Most scalar functionals on operators (e.g. the trace or the determinant) can only be applied to endomorphisms (i.e. to operators which map a vector space into itself). The chain (1.33) is a mapping from the subspace $E_{x_n}(H) \subset H$ to $E_{x_1}(H)$. This makes it difficult to form a scalar, unless $x_1 = x_n$. Therefore, we will only consider *closed chains*

$$P(x,y_1)\, P(y_1,y_2) \cdots P(y_k,x) \;:\; E_x(H) \to E_x(H) \quad .$$

In the simplest case $k=0$, the closed chain degenerates to a single factor $P(x,x)$. This turns out to be too simple for the formulation of a physically interesting action, because the off-diagonal elements of the discrete kernel $P(x,y)$, $x \neq y$, should enter the variational principle. Thus we are led to considering closed chains of two factors, i.e. to the operator product $P(x,y)\, P(y,x)$. Suppose that we are given a real-valued functional $\mathcal{L}$ on the endomorphisms of $E_x(H) \subset H$ (this will be discussed and specified below). Then $\mathcal{L}[P(x,y)\, P(y,x)]$ is a real function depending on two space-time arguments, and we get a scalar by summing over x and y. Therefore, we take for our action S the ansatz

$$S \;=\; \sum_{x,y \in M} \mathcal{L}[P(x,y)\, P(y,x)] \quad . \tag{1.34}$$

This ansatz is called a *two-point action*, and in analogy to classical field theory, we call $\mathcal{L}$ the corresponding *Lagrangian*.

We shall now introduce a particular Lagrangian $\mathcal{L}$. The requirement which will lead us quite naturally to this Lagrangian is that $\mathcal{L}$ should be *positive*. Positivity of the action is desirable because it is a more convincing concept to look for a local minimum of the action than merely for a critical point of an action which is unbounded below. Also, our positive action is simpler and will turn out to work better than for example the action presented in [1]. Let us first consider how one can form a positive functional on $P(x,y)\, P(y,x)$. The closed chain $P(x,y)\, P(y,x)$ is an endomorphism of $E_x(H)$; we abbreviate it in what follows by A. In a given gauge, A is represented by a $4N \times 4N$ matrix. Under gauge transformations (1.31), this matrix transforms according to the adjoint representation,

$$A \;\to\; U(x)\, A\, U(x)^{-1} \quad .$$

Furthermore, A is Hermitian on $E_x(H)$, i.e.

$$<A\,\Psi \,|\, \Phi> \;=\; <\Psi \,|\, A\,\Phi> \qquad\qquad \text{for } \Psi, \Phi \in E_x(H), \tag{1.35}$$

or simply $A^* = A$. In positive definite scalar product spaces, the natural positive functional on operators is an operator norm, e.g. the Hilbert-Schmidt norm $\|B\|_2 = \mathrm{tr}(B^* B)^{\frac{1}{2}}$. In our setting, the situation is more difficult because our scalar product $<.|.>$ is indefinite on $E_x(H)$ (of signature $(2N, 2N)$). As a consequence, Hermitian matrices do not have the same nice properties as in positive definite scalar product spaces; in particular, the matrix A might have complex eigenvalues, and it is in general not even diagonalizable. Also, the operator product A^*A need not be positive, so that we cannot introduce a Hilbert-Schmidt norm. In order to analyze the situation more systematically, we decompose the characteristic polynomial of A into linear factors

$$\det(\lambda - A) \;=\; \prod_{k=1}^{K} (\lambda - \lambda_k)^{n_k} \quad . \tag{1.36}$$

This decomposition is useful because every functional on A can be expressed in terms of the roots and multiplicities of the characteristic polynomial; thus it is sufficient to consider the λ_k's and n_k's in what follows. Each root λ_k corresponds to an n_k-dimensional A-invariant subspace of $E_x(H)$, as one sees immediately from a Jordan representation of A. The roots λ_k may be complex. But since A is Hermitian, (1.35), we know at least that the characteristic polynomial of A is real,

$$\overline{\det(\lambda - A)} \;=\; \det(\lambda - A) \qquad\qquad \text{for } \lambda \in \mathbb{R}.$$

This means that the complex conjugate of every root is also a root with the same multiplicity (i.e. for every λ_k there is a λ_l with $\overline{\lambda_k} = \lambda_l$ and $n_k = n_l$). The reality of the characteristic polynomial is verified in detail as follows. In a given gauge, we can form the transposed, complex conjugated matrix of A, denoted by $A^\dagger$. For clarity, we point out that $A^\dagger$ is *not* an endomorphism of $E_x(H)$, because it has the wrong behavior under gauge transformations (in particular, the trace $\mathrm{tr}(A^\dagger A)$ depends on the gauge and is thus ill-defined). Nevertheless, the matrix $A^\dagger$ is useful because we can write the adjoint of A in the form $A^* = SA^\dagger S$, where S is the spin signature matrix, $S = \mathrm{diag}((s_\alpha)_{\alpha=1,\ldots,4N})$. Since $S^2 = \mathbb{1}$, and since the determinant is multiplicative, we conclude that for any real λ,

$$\begin{aligned}\overline{\det(\lambda - A)} \;&=\; \det(\lambda - A^\dagger) \;=\; \det(\lambda - S^2\, A^\dagger) \\ &=\; \det(\lambda - SA^\dagger S) \;=\; \det(\lambda - A^*) \;=\; \det(\lambda - A) \quad .\end{aligned}$$

An obvious way to form a positive functional is to add up the absolute squares of the roots, taking into account their multiplicities. We thus define the *spectral weight* $|A|$ of A by

$$|A| \;=\; \left(\sum_{k=1}^K n_k \, |\lambda_k|^2 \right)^{\frac{1}{2}} \quad . \tag{1.37}$$

This functional depends continuously on the λ_k; furthermore, it behaves continuously when the roots of the characteristic polynomial degenerate and the multiplicities n_k change. Thus the spectral weight $|\,.\,|$ is a continuous functional. Furthermore, the spectral weight is zero if and only if the characteristic polynomial is trivial, $\det(\lambda - A) = \lambda^{4N}$. This is equivalent to A being nilpotent (i.e. $A^k = 0$ for some k). Thus, in contrast to an operator norm, the vanishing of the spectral weight does not imply that the operator is zero. On the other hand, it does not seem possible to define an operator norm in indefinite scalar product spaces. For our purpose, nilpotent operators are a sufficiently small class of operators, so that the spectral weight is a reasonable concept.

Using the spectral weight, one can write down many positive Lagrangians. The easiest choice would be $\mathcal{L}[A] = |A|^2$. Minimizing the corresponding action (1.34) yields a variational principle which attempts to make the absolute values of the roots $|\lambda_k|$ as small as possible. This turns out to be a too strong minimizing principle. It makes more sense to formulate a variational principle which aspires to equalize the absolute values of all roots. This can be accomplished by combining the expressions $|A^2|^2$ and $|A|^4$. Namely, using that the sum of the multiplicities equals the dimension of the vector space, $\sum_{k=1}^K n_k = 4N$, the Schwarz inequality yields that

$$|A^2|^2 \;=\; \sum_{k=1}^K n_k \, |\lambda_k|^4 \;\geq\; \frac{1}{4N} \left(\sum_{k=1}^K n_k \, |\lambda_k|^2 \right)^2 \;=\; \frac{1}{4N}\, |A|^4 \quad ,$$

and equality holds only if the absolute values of all roots are equal. Thus it is reasonable to minimize $|A^2|^2$, keeping $|A|^4$ fixed. This is our motivation for considering the two-point action:

$$\text{minimize} \quad S \;=\; \sum_{x,y\in M} \Big|(P(x,y)\,P(y,x))^2\Big|^2 \tag{1.38}$$

with the constraint

$$T \;:=\; \sum_{x,y\in M} |P(x,y)\,P(y,x)|^4 \;=\; \text{const} \quad . \tag{1.39}$$

This is our model variational principle.

We next derive the corresponding "Euler-Lagrange equations." For simplicity, we only consider the case that $P(x,y)\,P(y,x)$ can be diagonalized. This is the generic situation; the case of a non-diagonalizable matrix can be obtained from it by an approximation procedure[5]. Thus we assume that the endomorphism $A = P(x,y)\,P(y,x)$ has a spectral decomposition

$$A \;=\; \sum_{k=1}^{K} \lambda_k\, F_k \quad , \tag{1.40}$$

where λ_k are the roots in (1.36), and F_k are operators mapping onto the corresponding eigenspaces (A, K, the λ_k, and the F_k clearly depend on x and y, but we will, for ease in notation, usually not write out this dependence). Since the underlying scalar product space is indefinite, the spectral decomposition (1.40) requires a brief explanation. Suppose that we choose a basis where A is diagonal. In this basis, the operators F_k are simply the diagonal matrices with diagonal entries 1 if the corresponding diagonal elements of A are λ_k, and 0 otherwise. Clearly, these operators map onto the eigenspaces and are orthonormal and complete, i.e.

$$A\,F_k \;=\; \lambda_k\,F_k \quad , \qquad F_k\,F_l \;=\; \delta_{kl}\,F_k \quad \text{, and} \qquad \sum_{k=1}^{K} F_k \;=\; \mathbb{1}_{E_x(H)} \quad .$$

However, the F_k are in general *not* Hermitian (with respect to the spin scalar product). More precisely, taking the adjoint swaps the operators corresponding to complex conjugated eigenvalues,

$$F_k^* \;=\; F_l \qquad \text{when} \qquad \overline{\lambda_k} = \lambda_l \quad . \tag{1.41}$$

These relations can be understood immediately because they ensure that the spectral decomposition (1.40) is Hermitian,

$$\left(\sum_{k=1}^{K} \lambda_k\, F_k\right)^* \;=\; \sum_{k=1}^{K} \overline{\lambda_k}\, F_k^* \;\overset{(1.41)}{=}\; \sum_{k=1}^{K} \lambda_k\, F_k \quad .$$

We now consider continuous variations $P(\tau)$ and $(E_p(\tau))_{p\in M}$, $-\varepsilon < \tau < \varepsilon$, of our operators. The structure of the operators must be respected by the variations, i.e. $P(\tau)$ should be a projector, and the relations (1.27) between the operators $(E_p)_{p\in M}$ should hold

[5]We mention that the case when $P(x,y)P(y,x)$ is not diagonalizable is a bit subtle because our action in this case is continuous, but not differentiable. In other words, the corresponding Euler-Lagrange equations have discontinuities. This effect can be described by taking one-sided limits of the generic equations (1.46)–(1.47); this is the method we will describe and use later on.

for all τ. Continuity of the variation implies that the rank of P and the signature of its image do not change. Thus the variation of P can be realized by a unitary transformation

$$P(\tau) \;=\; U(\tau)\, P\, U(\tau)^{-1} \quad , \tag{1.42}$$

where $U(\tau)$ is a unitary operator on H with $U(0) = \mathbb{1}$. Similarly, the variations of the projectors $(E_p)_{p \in M}$ are also unitary. From (1.27), we can conclude the stronger statement that the variations of all operators $(E_p)_{p \in M}$ can be realized by one unitary transformation, i.e.

$$E_p(\tau) \;=\; V(\tau)\, E_p\, V(\tau)^{-1}$$

with a unitary operator $V(\tau)$ and $V(0) = \mathbb{1}$. Since our action is invariant under unitary transformations of the vector space H, we can, instead of unitarily transforming both P and $(E_p)_{p \in M}$, just as well keep the $(E_p)_{p \in M}$ fixed and *vary only the fermionic projector* by (1.42). To first order in τ, this variation becomes

$$\delta P \;\equiv\; \frac{d}{d\tau} P(\tau)_{|\tau=0} \;=\; i\,[B,\, P] \quad , \tag{1.43}$$

where $B = -iU'(0)$ is a Hermitian operator on H. We will only consider variations where B has *finite support*, i.e. where the kernel $B(x,y) \equiv E_x\, B\, E_y$ of B satisfies the condition

$$B(x,y) \;=\; 0 \quad \text{except for } x, y \in N \subset M \text{ with } N \text{ finite.}$$

This condition can be regarded as the analogue of the assumption in the classical calculus of variations that the variation should have compact support.

Let us compute the variation of the action (1.38) (the constraint (1.39) will be considered afterwards). Writing out the action with the eigenvalues λ_k and multiplicities n_k, we obtain

$$S \;=\; \sum_{x,y \in M} \sum_{k=1}^{K} n_k\, |\lambda_k|^4 \quad .$$

The variation can be computed in perturbation theory to first order,

$$\begin{aligned} \delta S \;&=\; 4\,\mathrm{Re} \sum_{x,y \in M} \sum_{k=1}^{K} n_k\, \overline{\lambda_k}^2\, \lambda_k\, \mathrm{tr}(F_k\, \delta A) \\ &=\; 4\,\mathrm{Re} \sum_{x,y \in M} \sum_{k=1}^{K} n_k\, \overline{\lambda_k}^2\, \lambda_k\, \mathrm{tr}\left(F_k\, (\delta P(x,y)\, P(y,x) \;+\; P(x,y)\, \delta P(y,x))\right) \quad . \end{aligned}$$

Exchanging the names of x and y in the first summand in the trace and using cyclicity of the trace, this expression can be written as an operator product,

$$\delta S \;=\; 4\,\mathrm{Re}\; \mathrm{tr}(Q_1\, \delta P) \quad , \tag{1.44}$$

where the kernel $Q_1(x,y) \equiv E_x\, Q_1\, E_y$ of Q_1 has the form

$$Q_1(x,y) \;=\; \left[\sum_{k=1}^{K} n_k\, \overline{\lambda_k}^2\, \lambda_k\, F_k\right]_{xy} P(x,y) \;+\; P(x,y) \left[\sum_{k=1}^{K} n_k\, \overline{\lambda_k}^2\, \lambda_k\, F_k\right]_{yx} , \tag{1.45}$$

and the subscripts "$_{xy}$" and "$_{yx}$" indicate that the corresponding brackets contain the spectral decomposition of the operators $P(x,y)\, P(y,x)$ and $P(y,x)\, P(x,y)$, respectively.

We note that the trace in (1.44) is well-defined because the trace is actually taken only over a finite-dimensional subspace of H. A short straightforward computation using (1.41) shows that the operator Q_1 is Hermitian. Thus the trace in (1.44) is real, and we conclude that

$$\delta S \;=\; 4\,\mathrm{tr}(Q_1\,\delta P) \quad .$$

The variation of our constraint (1.39) can be computed similarly, and one gets

$$\begin{aligned} \delta T &= 4\,\mathrm{tr}(Q_2\,\delta P) \qquad \text{with} \\ Q_2(x,y) &= \left[(\sum_{l=1}^K n_l\,|\lambda_l|^2)\sum_{k=1}^K n_k\,\overline{\lambda_k}\,F_k\right] P(x,y) \;+\; P(x,y)\left[(\sum_{l=1}^K n_l\,|\lambda_l|^2)\sum_{k=1}^K n_k\,\overline{\lambda_k}\,F_k\right]_{yx} . \end{aligned}$$

Now consider a local minimum of the action. Handling the constraint with a Lagrange multiplier μ, we obtain the condition

$$0 \;=\; \delta S - \mu\,\delta T \;=\; 4\,\mathrm{tr}\left((Q_1-\mu Q_2)\,\delta P\right) \overset{(1.43)}{=} 4i\,\mathrm{tr}\left((Q_1-\mu Q_2)\,[B,P]\right) \quad .$$

Assume that the products $(Q_1-\mu Q_2)\,P$ and $P\,(Q_1-\mu Q_2)$ are well-defined operators. Since B has finite support, we can then cyclically commute the operators in the trace and obtain

$$0 \;=\; 4i\,\mathrm{tr}\left(B\,[P,\,Q_1-\mu Q_2]\right) \quad .$$

Since B is arbitrary, we conclude that $[P,\,Q_1-\mu Q_2]=0$, where our notation with the commutator implicitly contains the condition that the involved operator products are well-defined. Thus our *Euler-Lagrange equations* are the commutator equations

$$[P,\,Q] \;=\; 0 \qquad \text{with} \qquad Q(x,y) \;=\; C_{xy}\,P(x,y) \;+\; P(x,y)\,C_{yx} \quad , \tag{1.46}$$

$$C_{xy} \;=\; \sum_{k=1}^K n_k \left[\overline{\lambda_k}^2\,\lambda_k \;-\; \mu\,\overline{\lambda_k}\sum_{l=0}^K n_l\,|\lambda_l|^2\right]_{xy} F_k \quad . \tag{1.47}$$

In the formula (1.47) for C_{xy}, we consider the spectral decomposition (1.36),(1.40) of the closed chain $P(x,y)\,P(y,x)$ (similarly, C_{yx} refers to the spectral decomposition of $P(y,x)\,P(x,y)$). The equations (1.46)–(1.47) are the equations of discrete space-time corresponding to the variational principle (1.38),(1.39).

1.6 Discussion

In the previous subsections, the principle of the fermionic projector was introduced in a rather abstract mathematical way. Our constructions departed radically from the conventional formulation of physics, so much so, that the precise relation between the principle of the fermionic projector and the notions of classical and quantum physics is not obvious. In order to clarify the situation, we shall now describe the general physical concept behind the principle of the fermionic projector and explain in words the connection to classical field theory, relativistic quantum mechanics, and quantum field theory. Since we must anticipate results which will be worked out later in this and the following papers, the description in this subsection is clearly not rigorous, and is intended only to give a brief qualitative overview.

The constructions in Subsections 1.1 and 1.2 are merely a reformulation of classical field theory and relativistic quantum mechanics. Although they are an important preparation for the following construction steps, they do not by themselves have new physical

implications. Therefore, we need not consider them here, and begin by discussing the concept of discrete space-time of Subsection 1.3. With our definition of discrete space-time, the usual space-time continuum is given up and resolved into discrete space-time points. A-priori, the discrete space-time points are merely a point set, i.e. there are no relations (like e.g. the nearest-neighbor relation on a lattice) between them. Thus one may think of discrete space-time as a "disordered accumulation of isolated points." There exists no time parameter, nor does it make sense to speak of the "spatial distance" between the space-time points. Clearly, this concept of a pure point set is too general for a reasonable description of space-time. Namely, we introduced discrete space-time with the intention of discretizing the space-time continuum on the Planck scale. Thus, for systems which are large compared to the Planck length, the discrete nature of space-time should not be apparent. This means that discrete space-time should, in a certain *continuum limit*, go over to a Lorentzian manifold. However, since M is merely a point set, discrete space-time $(H, <.|.>, (E_p)_{p \in M})$ is symmetric under permutations of the space-time points. Taking a naive continuum limit would imply that the points of space-time could be arbitrarily exchanged, in clear contradiction to the topological and causal structure of a Lorentzian manifold.

In order to avoid this seeming inconsistency, one must keep in mind that we introduced an additional object in discrete space-time: the fermionic projector P. Via its discrete kernel $P(x, y)$, the fermionic projector yields relations between the discrete space-time points. Our idea is that the discrete kernel should provide all structures needed for a reasonable continuum limit. In more detail, our concept is as follows. In the space-time continuum, the fermionic projector is built up of all quantum mechanical states of the fermionic particles of the system. Closely following Dirac's original concept, we describe the vacuum by the "sea" of all negative-energy states; systems with particles and anti-particles are obtained by occupying positive-energy states and removing states from the Dirac sea, respectively. The fermionic projector of the continuum completely characterizes the physical system. In particular, it is shown in [5, 6] that its integral kernel $P(x, y)$ is singular if and only if y lies on the light cone centered at x. In this way, the fermionic projector of the continuum encodes the causal, and thus also topological, structure of the underlying space-time. We have in mind that the fermionic projector of discrete space-time should, similar to a regularization on the Planck scale, approximate the fermionic projector of the continuum. This means that on a macroscopic scale (i.e. for systems comprising a very large number of space-time points), the fermionic projector of discrete space-time can, to good approximation, be identified with a fermionic projector of the continuum. Using the just-mentioned properties of the continuum kernel, we thus conclude that the discrete kernel induces on discrete space-time a structure which is well-approximated by a Lorentzian manifold. However, on the Planck scale (i.e. for systems involving only few space-time points), the discrete nature of space-time becomes manifest, and the notions of space, time, and causality cease to exit.

The critical step for making this concept precise is the formulation of the physical equations intrinsically in discrete space-time. Let us describe in principle how this is supposed to work. In the continuum description, the fermionic projector satisfies the Dirac equation (1.19); furthermore, the classical potentials entering the Dirac equation obey classical field equations. As a consequence of these equations, the fermionic projector of the continuum is an object with very specific properties; this is worked out in detail in [4, 5, 6]. Our idea is that, using the special form of the fermionic projector, it should be possible to restate the Dirac equation and classical field equations directly in terms of the

fermionic projector. Thus we wish to formulate equations in which the fermionic projector enters as the basic object, and which are equivalent to, or a generalization of, both the Dirac equation and the classical field equations. It turns out that it is impossible to state equations of this type in the space-time continuum, because composite expressions in the fermionic projector are mathematically ill-defined. But one can formulate mathematically meaningful equations in discrete space-time, removing at the same time the ultraviolet problems of the continuum theory. The variational principle (1.38),(1.39) leading to the Euler-Lagrange equations (1.46),(1.47) is an example for such equations. Note that this variational principle and the corresponding Euler-Lagrange equations in discrete space-time are clearly not causal, but, for consistency with relativistic quantum mechanics and classical field theory, we demand that they should, in the continuum limit, reduce to local and causal equations (namely, to the Dirac and classical field equations). Since the fermionic projector is not an object which is commonly considered in physics, it is difficult to give an immediate physical interpretation for the equations of discrete space-time; only a detailed mathematical analysis can provide an understanding of the variational principle. If one wishes, one can regard the equations of discrete space-time as describing a direct particle-particle interaction between all the states of the fermionic projector. The collective interaction of the fermions of the Dirac sea with the additional particles and holes should, in the continuum limit, give rise to an effective interaction of fermions and anti-fermions via classical bosonic fields. Eventually, the collective particle-particle interaction should even give a microscopic justification for the appearance of a continuous space-time structure.

Let us now describe the relation to quantum field theory. Since the coupled Dirac and classical field equations, combined with the pair creation/annihilation of Dirac's hole theory, yield precisely the Feynman diagrams of QFT (see e.g. [7]), it is clear that all results of perturbative quantum field theory, in particular the high precision tests of QFT, are also respected by our ansatz (assuming that the equations of discrete space-time have the correct continuum limit). Thus the only question is if the particular effects of quantized fields, namely the Planck radiation and the photo electric effect, can be explained in our framework. The basic physical assumption behind Planck's radiation law is that the energy levels of an electromagnetic radiation mode do not take continuous values, but are quantized in steps of $E = \hbar\omega$. While the quantitative value $\hbar\omega$ of the energy quanta can be understood via the quantum mechanical identification of energy and frequency (which is already used in classical Dirac theory), the crucial point of Planck's assumption lies in the occurrence of discrete energy levels. The photo electric effect, on the other hand, can be explained by a "discreteness" of the electromagnetic interaction: the electromagnetic wave tends not to transmit its energy continuously, but prefers to excite few atoms of the photographic material. We have the conception that these different manifestations of "discreteness" should follow from the equations of discrete space-time if one goes beyond the approximation of an interaction via classical bosonic fields.

If this concept of explaining the effects of quantized fields from the equations of discrete space-time were correct, it would even have consequences for the interpretation of quantum mechanics. Namely, according to the statistical interpretation, quantum mechanical particles are point-like; the absolute value $|\Psi(\vec{x})|^2$ of the wave function gives the probability density for the particle to be at the position $\vec{x}$. Here, we could regard the wave function itself as the physical object; the particle character would come about merely as a consequence of the "discreteness" of the interaction of the wave function with e.g. the atoms of a photographic material. The loss of determinism could be naturally explained by the non-causality of the equations of discrete space-time.

We conclude that the principle of the fermionic projector raises quite fundamental questions on the structure of space-time, the nature of field quantization, and the interpretation of quantum mechanics. Before entering the study of these general questions, however, it is most essential to establish a quantitative connection between the equations of discrete space-time and the Dirac and classical field equations. Namely, the principle of the fermionic projector can make physical sense only if it is consistent with classical field theory and relativistic quantum mechanics; thus it is of importance to first check this consistency. Even this comparatively simple limiting case is of highest physical interest. Indeed, the principle of the fermionic projector provides a very restricted framework for the formulation of physical models; e.g., there is no freedom in choosing the gauge groups, the coupling of the gauge fields to the fermions, or the masses of the gauge bosons. This means that, if a connection could be established to relativistic quantum mechanics and classical field theory, the principle of the fermionic projector would give an explanation for the interactions observed in nature, and would yield theoretical predictions for particle masses and coupling constants. We begin with this study in the following section.

2 The Continuum Limit

According to the principle of the fermionic projector, a physical system is described by the fermionic projector P in discrete space-time $(H, <.|.>, (E_p)_{p \in M})$. In this section, we shall establish a mathematically sound connection between this description and the usual formulation of physics in a space-time continuum. More precisely, we will develop a method with which equations in discrete space-time (like e.g. the Euler-Lagrange equations (1.46),(1.47)) can be analyzed within the framework of relativistic quantum mechanics and classical field theory. Our approach is based on the assumption that the fermionic projector of discrete space-time can be obtained from the well-known fermionic projector of the continuum [4, 5, 6] by a suitable regularization process on the Planck scale. The basic difficulty is that composite expressions in the fermionic projector (like e.g. in (1.46)) depend essentially on how the regularization is carried out; our task is to analyze this dependence in detail. We will show that, if we study the behavior close to the light cone, the dependence on the regularization simplifies considerably and can be described by a finite number of parameters. Taking these parameters as free parameters, we will end up with a meaningful effective continuum theory.

We point out that, since we deduce the fermionic projector of discrete space-time from the fermionic projector of the continuum, the causal and topological structure of the space-time continuum, as well as the Dirac equation and Dirac's hole theory, enter our construction from the very beginning. Thus our procedure cannot give a justification or even derivation of these structures from the equations of discrete space-time. The reason why our method is nevertheless interesting is that we do not need to specify the classical potentials which enter the Dirac equation; in particular, we do not assume that they satisfy the classical field equations. Thus we can hope that an analysis of the equations of discrete space-time should give constraints for the classical potentials; this means physically that the equations of discrete space-time should in the continuum limit yield a quantitative description of the interaction of the Dirac particles via classical fields. This is the first important step in the analysis of the principle of the fermionic projector.

For clarity, we will mainly restrict attention to the case of one type of fermions ($N = 1$) which all have the same mass m. The generalizations to systems of fermions with different masses and to chiral fermions (as considered in [4, 6]) are given in the last Subsection 2.6.

2.1 The Continuum Description

We begin by reviewing some basic facts about the fermionic projector in the space-time continuum (see [4, 5, 6] for details). The fermionic projector was introduced in Subsection 1.2 as the projector on all occupied one-particle states of the physical system, (1.20), (1.21). For the detailed study of the fermionic projector, it is convenient to consider its integral kernel $P(x,y)$, which is defined in a gauge and coordinate system by

$$(P\Psi)^\alpha(x) \;=\; \int \sum_{\beta=1}^4 P^\alpha_\beta(x,y)\, \Psi^\beta(y)\, \sqrt{-g}\, d^4y \quad .$$

If we were in a Hilbert space, we could choose an orthonormal basis (Ψ_a) in the image of P and represent the kernel in the form $P(x,y) = \sum_a \Psi_a(x)\, \overline{\Psi_a(y)}$. On a formal level, such a representation is also true in an indefinite scalar product space, more precisely

$$P(x,y) \;\stackrel{\text{formally}}{=}\; \sum\!\!\!\!\!\!\int_a |\Psi_a(x)\succ\!\!\prec \Psi_a(y)| \quad , \tag{2.1}$$

where a runs over all (discrete or continuous) quantum numbers of the occupied one-particle states. By a careful analysis of the infinite sum and of the normalization of the states Ψ_a, the decomposition (2.1) can be made mathematically precise (cf. [4, 5, 6]).

In the vacuum, the fermionic projector is composed of all plane-wave solutions $(\Psi_{\vec{k}s})$ of the Dirac equation of negative energy, where $\vec{k} \in \mathbb{R}^3$ is momentum and $s = 1, 2$ denotes the two spin orientations. The formula (2.1) yields, if we integrate over $\vec{k}$ and sum over s, the integral over the lower mass shell

$$P(x,y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, (\not{p} + m)\, \delta(p^2 - m^2)\, \Theta(-p^0)\, e^{-ip(x-y)} \quad , \tag{2.2}$$

where $\not{p} \equiv p^j \gamma_j$, and where Θ is the Heaviside function $\Theta(t) = 1$ for $t \geq 0$ and $\Theta(t) = 0$ otherwise. This Fourier integral is a well-defined tempered distribution. In order to compute it, one pulls the Dirac matrices out of the integral by setting

$$P(x,y) \;=\; (i\not{\partial}_x + m)\, T_{m^2}(x,y) \qquad \text{with} \tag{2.3}$$

$$T_a(x,y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \delta(p^2 - a)\, \Theta(-p^0)\, e^{-ip(x-y)} \quad . \tag{2.4}$$

The remaining Fourier integral (2.4) can be calculated explicitly, and one gets an expression involving Bessel functions. The important point for us is that T_a has singularities and poles on the light cone. Namely, using the series representation of the Bessel functions, one obtains

$$\begin{aligned} T_a(x,y) \;=\; & -\frac{1}{8\pi^3} \left(\frac{\mathrm{PP}}{\xi^2} \;+\; i\pi\, \delta(\xi^2)\, \epsilon(\xi^0) \right) \\ & + \frac{a}{32\pi^3} \left(\log(a\,|\xi^2|) \;+\; i\pi\, \Theta(\xi^2)\, \epsilon(\xi^0) \right) \sum_{l=0}^\infty \frac{(-1)^l}{l!\,(l+1)!}\, \frac{(a\xi^2)^l}{4^l} \\ & + (\text{smooth contributions}) \quad , \end{aligned} \tag{2.5}$$

where $\xi \equiv y - x$, ϵ is the step function $\epsilon(t) = 1$ for $t \geq 0$ and $\epsilon(t) = -1$ otherwise, and "PP" denotes the principal value of the pole. Thus the singularities have the form of a

δ-distribution and of a discontinuity on the light cone; the poles are of the order ξ^{-2} or of logarithmic type. By substituting (2.5) into (2.3) and carrying out the derivative $\partial\!\!\!/_x$, one can immediately derive a similar formula for $P(x,y)$. Hence also the fermionic projector of the vacuum has singularities and poles on the light cone; the most singular contributions are of the order $\delta'(\xi^2)$ and ξ^{-4}, respectively.

In the case involving interactions, the physical system contains particles and anti-particles; furthermore, all the Dirac particles are moving in a (classical) bosonic field. This is described mathematically as follows. According to (1.21) and (2.1), the integral kernel of the fermionic projector can be written in the form

$$P(x,y) \;=\; P^{\text{sea}}(x,y) \;+\; \sum_{a=1}^{n_f} |\Psi_a(x)\succ\!\prec\Psi_a(y)| \;-\; \sum_{a=1}^{n_a} |\Phi_a(x)\succ\!\prec\Phi_a(y)| \quad , \tag{2.6}$$

where $(\Psi_a)_{a=1,\ldots,n_f}$ and $(\Phi_a)_{a=1,\ldots,n_a}$ are an orthonormal basis for the particle and anti-particle states, respectively (for simplicity, we assume in what follows that the number of particles of our system is finite, i.e. $n_f, n_a < \infty$). Similar to the situation in the vacuum, the Dirac sea P^{sea} has singularities and poles on the light cone; on the other hand, we can in most situations assume that the wave functions Ψ_a and Φ_a are smooth. The motion of the Dirac particles in the bosonic fields is described by the Dirac equation (1.19). As a consequence, both P^{sea} and the wave functions Ψ_a, Φ_a depend on the bosonic fields. In particular, the singularities and poles of P^{sea} on the light cone are influenced by the bosonic fields. This effect is described quantitatively by the formulas of the light-cone expansion [5, 6], which give a representation of $P^{\text{sea}}(x,y)$ of the following general form,

$$\begin{aligned} P^{\text{sea}}(x,y) \;&=\; \sum_n (\text{iterated line integrals over bosonic potentials and fields})\, T^{\text{reg}\,(n)}(x,y) \\ &\quad + (\text{smooth contributions}) \quad , \end{aligned} \tag{2.7}$$

where

$$T^{\text{reg}\,(n)} \;=\; \left(\frac{d}{da}\right)^n T^{\text{reg}}_{a\,|\,a=0} \qquad \text{and} \tag{2.8}$$

$$\begin{aligned} T^{\text{reg}}_a(x,y) \;&=\; -\frac{1}{8\pi^3}\left(\frac{\text{PP}}{\xi^2} \;+\; i\pi\,\delta(\xi^2)\,\epsilon(\xi^0)\right) \\ &\quad + \frac{a}{32\pi^3}\left(\log(|\xi^2|) \;+\; i\pi\,\Theta(\xi^2)\,\epsilon(\xi^0)\right) \sum_{l=0}^{\infty} \frac{(-1)^l}{l!\,(l+1)!}\,\frac{(a\xi^2)^l}{4^l} \\ &\quad + (\text{smooth contributions}) \quad . \end{aligned} \tag{2.9}$$

The detailed expressions for the line integrals in (2.7) are not needed at this point; it suffices to note that they are smooth functions in the bosonic field. Note that T^{reg}_a differs from T_a in that the term $\log(a|\xi^2|)$ in (2.5) was replaced by $\log(|\xi^2|)$. This "regularization" is important because otherwise the a-derivatives in (2.8) would not exist (but clearly, it has nothing to do with the regularizations of the fermionic projector which we will consider later on). The expression (2.9) is a power series in a. The higher powers in a contain more factors ξ^2 and thus have a weaker singularity on the light cone. Thus (2.7) is an expansion in the order of the singularity on the light cone.

2.2 The Method of Variable Regularization

Let us consider how one can get a relation between the continuum fermionic projector and the description of physics in discrete space-time. As discussed in Section 1, discrete space-time should, for macroscopic systems, be equivalent to the usual space-time continuum.

For consistency with relativistic quantum mechanics, the fermionic projector of discrete space-time should in this limit go over to the continuum fermionic projector. Using furthermore that the discretization length should be of the order of the Planck length, we conclude that the fermionic projector of discrete space-time should correspond to a certain "regularization" of the continuum fermionic projector on the Planck scale. Thus it is a physically reasonable method to construct the fermionic projector of discrete space-time from the fermionic projector of the continuum by a suitable regularization process on the Planck scale.

Regularizations of the continuum theory are also used in perturbative QFT in order to make the divergent Feynman diagrams finite. However, there is the following major difference between the regularizations used in QFT and our regularization of the fermionic projector. In contrast to QFT, where the regularization is merely a mathematical technique within the renormalization procedure, we here consider the regularized fermionic projector as the object describing the physical reality. More precisely, the regularized fermionic projector should be a model for the fermionic projector of discrete space-time, which we consider as the basic physical object. As an important consequence, it is not inconsistent for us if the effective continuum theory depends on how the regularization is carried out. Namely in this case, we must regularize in such a way that the regularized fermionic projector is a good microscopic approximation to the "physical" fermionic projector of discrete space-time; only such a regularization can yield the correct effective continuum theory. This concept of giving the regularization a physical significance clearly suffers from the shortcoming that we have no detailed information about the microscopic structure of the fermionic projector in discrete space-time, and thus do not know how the correct regularization should look like. In order to deal with this problem, we shall consider a general class of regularizations. We will analyze in detail how the effective continuum theory depends on the regularization. Many quantities will depend sensitively on the regularization, so much so, that they are undetermined and thus ill-defined in the continuum limit. However, certain quantities will be independent of the regularization and have a simple correspondence in the continuum theory; we call these quantities *macroscopic*. We will try to express the effective continuum theory purely in terms of macroscopic quantities. We cannot expect that the effective continuum theory will be completely independent of the regularization. But for a meaningful continuum limit, it must be possible to describe the dependence on the regularization by a small number of parameters, which we consider as empiric parameters modelling the unknown microscopic structure of discrete space-time. We refer to this general procedure for constructing the effective continuum theory as the *method of variable regularization*.

In order to illustrate the method of variable regularization, we mention an analogy to solid state physics. On the microscopic scale, a solid is composed of atoms, which interact with each other quantum mechanically. On the macroscopic scale, however, a solid can be regarded as a continuous material, described by macroscopic quantities like the density, the pressure, the conductivity, etc. The macroscopic quantities satisfy macroscopic physical equations like the equations of continuum mechanics, Ohm's law, etc. Both the macroscopic characteristics of the solid and the macroscopic physical laws can, at least in principle, be derived microscopically from many-particle quantum mechanics. However, since the details of the microscopic system (e.g. the precise form of the electron wave functions) are usually not known, this derivation often does not completely determine the macroscopic physical equations. For example, it may happen that a macroscopic equation can be derived only up to a proportionality factor, which depends on unknown

microscopic properties of the solid and is thus treated in the macroscopic theory as an empirical parameter. The physical picture behind the method of variable regularization is very similar to the physics of a solid, if one considers on the microscopic scale our description of physics in discrete space-time and takes as the macroscopic theory both relativistic quantum mechanics and classical field theory. Clearly, the concept of discrete space-time is more hypothetical than atomic physics because it cannot at the moment be verified directly in experiments. But we can nevertheless get indirect physical evidence for the principle of the fermionic projector by studying whether or not the method of variable regularization leads to interesting results for the continuum theory.

In the remainder of this subsection, we specify for which class of regularizations we shall apply the method of variable regularization. Our choice of the regularization scheme is an attempt to combine two different requirements. On one hand, we must ensure that the class of regularizations is large enough to clarify the dependence of the effective continuum theory on the regularization in sufficient detail; on the other hand, we must keep the technical effort on a reasonable level. Consider the integral kernel of the continuum fermionic projector (2.6),(2.7). Under the reasonable assumption that the fermionic wave functions Ψ_a, Φ_a and the bosonic potentials are smooth, both the projectors on the particle/anti-particle states in (2.6) and the iterated line integrals in (2.7) are smooth in x and y. The factors $T^{\text{reg}\,(n)}$, however, have singularities and poles on the light cone, (2.8),(2.9). Let us consider what would happen if we tried to formulate a variational principle similar to that in Subsection 1.5 with the continuum kernel (instead of the discrete kernel). The just-mentioned smooth terms in the kernel would not lead to any difficulties; we could just multiply them with each other when forming the closed chain $P(x,y)\,P(y,x)$, the resulting smooth functions would influence the eigenvalues $\lambda_k(x,y)$ in (1.36) in a continuous way. However, the singularities of $T^{\text{reg}\,(n)}$ would cause severe mathematical problems because the multiplication of $T^{\text{reg}\,(n)}(x,y)$ with $T^{\text{reg}\,(n)}(y,x)$ leads to singularities which are ill-defined even in the distributional sense. For example, the naive product $P(x,y)\,P(y,x)$ would involve singularities of the form $\sim \delta'((y-x)^2)\,\delta((y-x)^2)$ and $\sim \delta((y-x)^2)^2$. This simple consideration shows why composite expressions in the fermionic projector make mathematical sense only after regularization. Furthermore, one sees that the regularization is merely needed to remove the singularities of $T^{\text{reg}\,(n)}$. Hence, it seems reasonable to regularize only the factors $T^{\text{reg}\,(n)}$ in (2.7), but to leave the fermionic wave functions Ψ_a, Φ_a as well as the bosonic potentials unchanged. This regularization method implies that the fermionic wave functions and the bosonic potentials are well-defined also for the regularized fermionic projector; using the notation of page 23, they are macroscopic quantities. Therefore, we call our method of only regularizing $T^{\text{reg}\,(n)}$ the assumption of *macroscopic potentials and wave functions*.

The assumption of macroscopic potentials and wave functions means physically that energy and momentum of all bosonic fields, and of each particle/anti-particle of the physical system, should be small compared to the Planck energy. In other words, we exclude the case that the physical potentials and wave functions have oscillations or fluctuations on the Planck scale. Namely, such microscopic inhomogeneities could not be described by smooth functions in the continuum limit, and are thus not taken into account by our regularization method. If, conversely, the potentials and wave functions are nearly constant on the Planck scale, the unregularized and the (no matter by which method) regularized quantities almost coincide, and it is a good approximation to work in the regularized fermionic projector with the unregularized potentials and wave functions.

According to the assumption of macroscopic potentials and wave functions, it remains

to regularize the factors $T^{\text{reg}\,(n)}$ in (2.7). Recall that we constructed the distributions $T^{\text{reg}\,(n)}$ from the continuum kernel of the fermionic projector of the vacuum (2.2) via (2.3) and the expansion in the mass parameter (2.8). An essential step for getting a meaningful regularization scheme is to extend this construction to the case with regularization. Namely, this extension makes it sufficient to specify the regularization of the fermionic projector of the vacuum; we can then deduce the regularized $T^{\text{reg}\,(n)}$ and obtain, by substitution into (2.7), the regularized fermionic projector with interaction (if it were, on the contrary, impossible to derive the regularized $T^{\text{reg}\,(n)}$ from the regularized fermionic projector of the vacuum, the independent regularizations of all functions $T^{\text{reg}\,(n)}$, $n = 0, 1, 2, \ldots$, would involve so many free parameters that the effective continuum theory would be under-determined). Having in mind the extension of (2.3) and (2.8) to the case with regularization (which will be carried out in Subsection 2.6 and Appendix B), we now proceed to describe our regularization method for the fermionic projector of the vacuum. In the vacuum, the kernel of the continuum fermionic projector $P(x, y)$ is given by the Fourier integral (2.2). $P(x, y)$ is invariant under translations in space-time, i.e. it depends only on the difference vector $y - x$. It seems natural and is most convenient to preserve the translation symmetry in the regularization. We thus assume that the kernel of the regularized fermionic projector of the vacuum, which we denote for simplicity again by $P(x, y)$, is translation invariant,

$$P(x, y) \;=\; P(y - x) \quad \text{for} \quad x, y \in M \subset \mathbb{R}^4 \quad . \tag{2.10}$$

We refer to (2.10) as a *homogeneous regularization of the vacuum*. Notice that the assumption (2.10) allows for both discrete and continuum regularizations. In the first case, the set M is taken to be a discrete subset of $\mathbb{R}^4$ (e.g. a lattice), whereas in the latter case, $M = \mathbb{R}^4$. According to our concept of discrete space-time, it seems preferable to work with discrete regularizations. But since continuous regularizations give the same results and are a bit easier to handle, it is worth considering them, too. The assumption of a homogeneous regularization of the vacuum means physically that the inhomogeneities of the fermionic projector on the Planck scale should be irrelevant for the effective continuum theory. Since such microscopic inhomogeneities can, at least in special cases, be described by microscopic gravitational or gauge fields, this assumption is closely related to the assumption of macroscopic potentials and wave functions discussed above.

Taking the Fourier transform in the variable $y - x$, we write (2.10) as the Fourier integral

$$P(x, y) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \tilde{P}(p)\, e^{-ip(x-y)} \tag{2.11}$$

with a distribution $\tilde{P}$. If one considers a discrete regularization, $\tilde{P}$ may be defined only in a bounded region of $\mathbb{R}^4$ (for a lattice regularization with lattice spacing d, for example, one can restrict the momenta to the "first Brillouin zone" $p \in (-\frac{\pi}{d}, \frac{\pi}{d})^4$). In this case, we extend $\tilde{P}$ to all of $\mathbb{R}^4$ by setting it to zero outside this bounded region. Although it is of no relevance for what follows, one should clearly keep in mind that for a discrete regularization, x and y take values only in the discrete set M. Let us briefly discuss the distribution $\tilde{P}$. First of all, $P(x, y)$ should be the kernel of a Hermitian operator; this implies that $P(x, y)^* = P(y, x)$ and thus

$$\tilde{P}(p)^* \;=\; \tilde{P}(p) \qquad \text{for all } p \quad , \tag{2.12}$$

where "$*$" again denotes the adjoint with respect to the spin scalar product (i.e. $\tilde{P}(p)^* = \gamma^0\, \tilde{P}(p)^\dagger\, \gamma^0$). For consistency with the continuum theory, the regularized kernel (2.11)

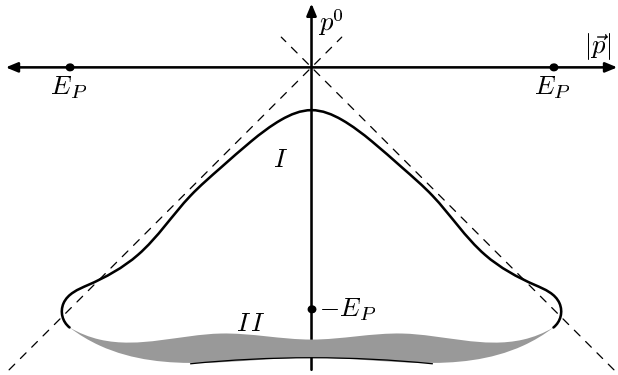

Figure 1: Example for $\tilde{P}$, the regularized fermionic projector of the vacuum in momentum space.

should, for macroscopic systems, go over to the continuum kernel (2.2). Thus we know that $\tilde{P}(p)$ should, for small energy-momentum p (i.e. when both the energy p^0 and the momentum $|\vec{p}|$ are small compared to the Planck energy), coincide with the distribution $(\not{p} + m)\, \delta(p^2 - m^2)\, \Theta(-p^0)$. This is illustrated in the example of Figure 1. In the region I close to the origin, $\tilde{P}$ looks similar to a hyperbola on the lower mass shell. Furthermore, we know that $\tilde{P}$ is a regularization on the Planck scale. This means that, in contrast to the integrand in (2.2), $\tilde{P}$ should decay at infinity, at least so rapidly that the integral (2.11) is finite for all x and y. The length scale for this decay in momentum space should be of the order of the Planck energy E_P. However, the precise form of $\tilde{P}$ for large energy or momentum is completely arbitrary, as is indicated in Figure 1 by the "high energy region" II. This arbitrariness reflects our freedom in choosing the regularization.

We finally make an ansatz for $\tilde{P}$ which seems general enough to include all relevant regularization effects, and which will considerably simplify our analysis in what follows. According to (2.12), $\tilde{P}(p)$ is a Hermitian 4×4 matrix, and can thus be written as a real linear combination of the basis of the Dirac algebra $\mathbb{1}$, $i\gamma^5$, $\gamma^5\gamma^j$, and σ^{jk} (with the pseudoscalar matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and the bilinear covariants $\sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k]$). The integrand of the continuum kernel (2.2) contains only vector and scalar components. It is reasonable to assume that the regularized kernel also contains no pseudoscalar and pseudovector components, because the regularization would otherwise break the symmetry under parity transformations. The inclusion of a bilinear component in $\tilde{P}$, on the other hand, would cause technical complications without giving anything essentially new. Thus we make an ansatz where $\tilde{P}$ is composed only of a vector and a scalar component, more precisely

$$\tilde{P}(p) \;=\; (v_j(p)\, \gamma^j \;+\; \phi(p)\, \mathbb{1})\, f(p) \tag{2.13}$$

with a vector field v and a scalar field ϕ; f is a distribution. We also need to assume that $\tilde{P}$ is reasonably regular and well-behaved; this will be specified in the following subsections. We refer to the ansatz (2.13) as the assumption of a *vector-scalar structure* for the fermionic projector of the vacuum.

2.3 The Regularized Product $P(x,y)\, P(y,x)$ in the Vacuum

According to the method of variable regularization, we must analyze how the effective continuum theory depends on the choice of the regularization. We shall now consider this

problem for the simplest composite expression in the fermionic projector, the closed chain $P(x,y)\,P(y,x)$ in the vacuum. The discussion of this example will explain why we need to analyze the fermionic projector on the light cone. Working out this concept mathematically will eventually lead us to the general formalism described in Subsection 2.6.

Using the Fourier representation (2.11), we can calculate the closed chain to be

$$\begin{aligned} P(x,y)\,P(y,x) &= \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4}\, \tilde{P}(k_1)\, \tilde{P}(k_2)\, e^{-i(k_1-k_2)(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4} \left[\int \frac{d^4q}{(2\pi)^4}\, \tilde{P}(p+q)\, \tilde{P}(q) \right] e^{-ip(x-y)} \quad , \qquad (2.14) \end{aligned}$$

where we introduced new integration variables $p = k_1 - k_2$ and $q = k_2$. Thus the Fourier transform of the closed chain is given by the convolution in the square brackets. This reveals the following basic problem. The convolution in the square bracket involves $\tilde{P}$ for small and for large energy-momentum. Even when p is small, a large q leads to a contribution where both factors $\tilde{P}(p+q)$ and $\tilde{P}(q)$ are evaluated for large energy-momenta. If we look at the example of Figure 1, this means that (2.14) depends essentially on the behavior of $\tilde{P}$ in the high-energy region II and can thus have an arbitrary value. More generally, we conclude that, since the form of $\tilde{P}$ for large energy or momentum is unknown, the value of (2.14) is undetermined.

At first sight, it might seem confusing that the pointwise product $P(x,y)\,P(y,x)$ of the regularized fermionic projector should be undetermined, although the unregularized kernel (2.2) is, for $y-x$ away from the light cone, a smooth function, so that pointwise multiplication causes no difficulties. In order to explain the situation in a simple example, we briefly discuss the fermionic projector $\hat{P}$ obtained by adding to P a plane wave,

$$\hat{P}(x,y) \;=\; P(x,y) \;+\; e^{-ik(x-y)}\, \mathbb{1} \quad .$$

If the energy or the momentum of the plane wave is of the order of the Planck energy, the plane wave is highly oscillating in space-time. Such an oscillating term is irrelevant on the macroscopic scale. Namely, if $\hat{P}$ acts on a macroscopic function η, the oscillating term is evaluated in the weak sense, and the resulting integral $\int \exp(iky)\, \eta(y)\, d^4y$ gives almost zero because the contributions with opposite signs compensate each other. This "oscillation argument" can be made mathematically precise using integration by parts, e.g. in the case of high energy $k^0 \sim E_P$,

$$\int e^{iky}\, f(y)\, d^4y \;=\; -\frac{1}{ik^0} \int e^{iky}\, (\partial_t f)\, d^4y \;\sim\; \frac{1}{E_P} \quad .$$

In the corresponding closed chain

$$\hat{P}(x,y)\, \hat{P}(y,x) \;=\; P(x,y)\, P(y,x) \;+\; P(x,y)\, e^{-ik(y-x)} \;+\; e^{-ik(x-y)}\, P(y,x) \;+\; \mathbb{1} \quad ,$$

the second and third summands are also oscillating. In the last summand, however, the oscillations have dropped out, so that this term affects the macroscopic behavior of the closed chain. This elementary consideration illustrates why the unknown high-energy contribution to the fermionic projector makes it impossible to determine the closed chain pointwise. We remark that for very special regularizations, for example the regularization by convolution with a smooth "mollifier" function having compact support, the pointwise product makes sense away from the light cone and coincides approximately with the

product of the unregularized kernels. But such regularizations are much too restrictive. Namely, we must allow for the possibility that the fermionic projector describes non-trivial (yet unknown) high-energy effects. Thus the high-energy behavior of the fermionic projector must not be constrained by a too simple regularization method.

The fact that the product $P(x,y)\,P(y,x)$ is undetermined for fixed x and y does not imply that a pointwise analysis of the closed chain is mathematically or physically meaningless. But it means that a pointwise analysis would essentially involve the unknown high-energy behavior of $\tilde{P}$; at present this is a problem completely out of reach. Therefore, our strategy is to find a method for evaluating the closed chain in a way where the high-energy behavior of $\tilde{P}$ becomes so unimportant that the dependence on the regularization can be described in a simple way. We hope that this method will lead us to a certain limiting case in which the equations of discrete space-time become manageable.

The simplest method to avoid the pointwise analysis is to evaluate the closed chain in the weak sense. The Fourier representation (2.14) yields that

$$\int P(x,y)\,P(y,x)\,\eta(x)\,d^4x \;=\; \int \frac{d^4p}{(2\pi)^4}\,\tilde{\eta}(p)\left[\int \frac{d^4q}{(2\pi)^4}\,\tilde{P}(p+q)\,\tilde{P}(q)\right] \quad , \tag{2.15}$$

where $\tilde{\eta}$ is the Fourier transform of a smooth function η. For macroscopic η (i.e. a function which is almost constant on the Planck scale), the function $\tilde{\eta}(p)$ is localized in a small neighborhood of $p=0$ and has rapid decay. Thus exactly as (2.14), the integral (2.15) depends on the form of $\tilde{P}$ for large energy-momentum. Hence this type of weak analysis is not helpful. In order to find a better method, we consider again the Fourier integral (2.11) in the example of Figure 1. We want to find a regime for $y-x$ where the "low energy" region I plays an important role, whereas the region II is irrelevant. This can be accomplished only by exploiting the special form of $\tilde{P}$ in the low-energy region as follows. The hyperbola of the lower mass shell in region I comes asymptotically close to the cone $\mathcal{C} = \{p^2 = 0\}$. If we choose a vector $(y-x) \neq 0$ on the light cone $L = \{(y-x)^2 = 0\}$, then the hypersurface $\mathcal{H} = \{p \,|\, p(y-x) = 0\}$ is null and thus tangential to the cone $\mathcal{C}$. This means that for all states on the hyperbola which are close to the straight line $\mathcal{C} \cap \mathcal{H}$, the exponential in (2.11) is approximately one. Hence all these states are "in phase," and thus yield a large contribution to the Fourier integral (2.11). The states in the high-energy region II, however, are not in phase; they will give only a small contribution to (2.11), at least when the vector $(y-x) \in L$ is large, so that the exponential in (2.11) is highly oscillating on the scale $p \sim E_P$. This qualitative argument shows that by considering the fermionic projector on the light cone, one can filter out information on the behavior of $\tilde{P}$ in the neighborhood of a straight line along the cone $\mathcal{C}$. This should enable us to analyze the states on the lower mass shell without being affected too much by the unknown high-energy behavior of $\tilde{P}$. We point out that if $P(x,y)$ depends only on the behavior of $\tilde{P}$ close to the cone $\mathcal{C}$, then the same is immediately true for composite expressions like the product $P(x,y)\,P(y,x)$. Thus restricting our analysis to the light cone should simplify the dependence on the regularization considerably, also for composite expressions like the closed chain. Our program for the remainder of this paper is to make this qualitative argument mathematically precise and to quantify it in increasing generality.

2.4 The Regularized Vacuum on the Light Cone, Scalar Component

For clarity, we begin the analysis on the light cone for the scalar component of (2.13), i.e. we consider the case

$$\tilde{P}(p) \;=\; \phi(p)\, f(p) \quad ; \tag{2.16}$$

the vector component will be treated in the next subsection. We can assume that the spatial component of the vector $y-x$ in (2.11) points in the direction of the x-axis of our Cartesian coordinate system, i.e. $y-x = (t, r, 0, 0)$ with $r > 0$. Choosing cylindrical coordinates ω, k, ρ, and φ in momentum space, defined by $p = (\omega, \vec{p})$ and $\vec{p} = (k, \rho\cos\varphi, \rho\sin\varphi)$, the Fourier integral (2.11) takes the form

$$P(x,y) \;=\; \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\; \tilde{P}(\omega, k, \rho, \varphi)\; e^{i\omega t - ikr} \quad . \tag{2.17}$$

Since the exponential factor in this formula is independent of ρ and φ, we can write the fermionic projector as the two-dimensional Fourier transform

$$P(x,y) \;=\; 2 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk\; h(\omega, k)\; e^{i\omega t - ikr} \tag{2.18}$$

of a function h defined by

$$h(\omega, k) \;=\; \frac{1}{2\,(2\pi)^4} \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\; (\phi\, f)(\omega, k, \rho, \varphi) \quad . \tag{2.19}$$

We want to analyze $P(x,y)$ close to the light cone $(y-x)^2 = 0$ away from the origin $y = x$. Without loss of generality, we can restrict our attention to the upper light cone $t = r$. Thus we are interested in the region $t \approx r > 0$. The "light-cone coordinates"

$$s \;=\; \frac{1}{2}\,(t-r) \quad , \qquad\qquad l \;=\; \frac{1}{2}\,(t+r) \tag{2.20}$$

are well-suited to this region, because the "small" variable s vanishes for $t = r$, whereas the "large" variable l is positive and non-zero. Introducing also the associated momenta

$$u \;=\; -k - \omega \quad , \qquad\qquad v \;=\; k - \omega \quad , \tag{2.21}$$

we can write the fermionic projector as

$$P(s,l) \;=\; \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h(u,v)\, e^{-i(us+vl)} \quad . \tag{2.22}$$

Let us briefly discuss the qualitative form of the function h, (2.19). According to the continuum kernel (2.2), the scalar component (2.16) should, for energy and momentum small compared to the Planck energy E_P, go over to the δ-distribution on the lower mass shell $\tilde{P} = m\,\delta(p^2 - m^2)\,\Theta(-p^0)$. In this limit, the integral (2.19) can be evaluated to be

$$\begin{aligned} h \;&=\; \frac{m}{2\,(2\pi)^4} \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\varphi\; \delta(\omega^2 - k^2 - \rho^2 - m^2)\; \Theta(-\omega) \\ &=\; \frac{m}{4\,(2\pi)^3}\, \Theta(\omega^2 - k^2 - m^2)\, \Theta(-\omega) \;=\; \frac{m}{32\pi^3}\, \Theta(uv - m^2)\, \Theta(u) \quad ; \end{aligned} \tag{2.23}$$

thus integrating out ρ and φ yields a constant function in the interior of the two-dimensional "lower mass shell" $\omega^2 - k^2 = m^2$, $\omega < 0$. From this we conclude that for $u, v \ll E_P$,

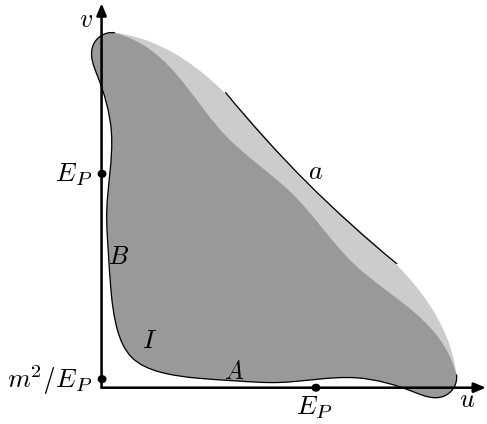

Figure 2: Example for $h(u,v)$, the reduced two-dimensional distribution.

$h(u,v)$ should have a discontinuity along the hyperbola $\{uv = m^2,\ u > 0\}$, be zero below (i.e. for $uv < m^2$) and be nearly constant above. Furthermore, we know that h decays at infinity on the scale of the Planck energy. Similar to our discussion of $\tilde{P}$ (after (2.12)), the precise form of h for large energy or momentum is completely arbitrary. The function $h(u,v)$ corresponding to the example of Figure 1 is shown in Figure 2. The two branches of the hyperbola asymptotic to the u and v axes are labelled by "A" and "B," respectively.

It is instructive to consider the energy scales of our system. The scale for high energies is clearly given by the Planck energy E_P. The relevant low-energy scale, on the other hand, is m^2/E_P (it is zero for massless fermions). Namely, the hyperbola $uv = m^2$ comes as close to the v-axis as as $v \sim m^2/E_P$ before leaving the low-energy region. These two energy scales are also marked in Figure 2. Since we want to analyze the situation close to the light cone, we choose the "small" light-cone parameter s on the Planck scale, i.e.

$$s \sim E_P^{-1} \qquad \text{or} \qquad s < E_P^{-1} \quad . \tag{2.24}$$

The "large" light-cone parameter l, on the other hand, is non-zero and thus yields a third energy scale. We shall always choose this scale between the two extremal energy scales, more precisely

$$\frac{1}{E_P} \ll l < l_{\max} \ll \frac{E_P}{m^2} \quad . \tag{2.25}$$

The parameter $l_{\max}$ was introduced here in order to avoid l being chosen too large. Namely, we will always regard l as being small compared to the length scales of macroscopic physics (a reasonable value for $l_{\max}$ would e.g. be the Fermi length). One should keep in mind that the quotient of the two fundamental energy scales is in all physical situations extremely large; namely $E_P^2/m^2 \gg 10^{35}$. Thus the constraints (2.25) can be easily satisfied and still leave us the freedom to vary l on many orders of magnitude.

In the remainder of this subsection, we will evaluate the Fourier integral (2.22) using the scales (2.24) and (2.25). In preparation, we discuss and specify the function $h(u,v)$ for fixed u, also denoted by $h_u(v)$. As one sees in Figure 2, h_u will in general not be continuous. More precisely in the example of Figure 2, h_u has a discontinuous "jump" from zero to a finite value on the hyperbola (and its extension to the high-energy region), and maybe has a second jump to zero for large v (e.g. on line "a"). For simplicity, we

assume that h_u is always of this general form, i.e.

$$h_u(v) \;=\; \begin{cases} 0 & \text{for } v < \alpha_u \text{ or } v > \beta_u \\ \text{smooth} & \text{for } \alpha_u \leq v \leq \beta_u \end{cases} \tag{2.26}$$

with parameters $\alpha_u < \beta_u$. The case of less than two discontinuities can be obtained from (2.26) by setting $h_u(\alpha_u)$ or $h_u(\beta_u)$ equal to zero, or alternatively by moving the position of the discontinuities α_u or β_u to infinity. We remark that the discontinuity at $v = \beta_u$ will become irrelevant later; it is here included only to illustrate why the behavior of the fermionic projector on the light cone is independent of many regularization details. Without regularization, $h_u(v)$ is for $v \geq \alpha_u$ a constant function, (2.23). Thus the v-dependence of $h_u(v)$ for $\alpha_u \leq v \leq \beta_u$ is merely a consequence of the regularization, and it is therefore reasonable to assume that the v-derivatives of $h_u(v)$ scale in inverse powers of the regularization length E_P. More precisely, we demand that there is a constant $c_1 \ll l E_P$ with

$$|h_u^{(n)}(v)| \;\leq\; \left(\frac{c_1}{E_P}\right)^n \max |h_u| \qquad \text{for } \alpha_u \leq v \leq \beta_u \quad , \tag{2.27}$$

where the derivatives at $v = \alpha_u$ and β_u are understood as the right- and left-sided limits, respectively. This regularity condition is typically satisfied for polynomial, exponential, and trigonometric functions, but it excludes small-scale fluctuations of h_u. Clearly, we could also consider a more general ansatz for h_u with more than two discontinuities or weaker regularity assumptions. But this does not seem to be the point at the moment, because all interesting effects, namely the influence of discontinuities for small and large v, as well as of smooth regions, can already be studied in the setting (2.26),(2.27).

Let us analyze the v-integral of the Fourier transform (2.22),

$$P_u(l) \;:=\; \int_{-\infty}^{\infty} h_u(v)\, e^{-ivl}\, dv \quad . \tag{2.28}$$

According to the first part of (2.25), the exponential factor in (2.28) is highly oscillating on the scale $v \sim E_P$. Thus we can expect that the smooth component of h_u gives only a small contribution to the integral (2.28), so that the discontinuities at α_u and β_u play the dominant role. In order to make this picture mathematically precise, we iteratively integrate in (2.28) K times by parts,

$$\begin{aligned} P_u(l) \;&=\; \int_{\alpha_u}^{\beta_u} h_u(v)\, e^{-ivl}\, dv \;=\; -\frac{1}{il} \int_{\alpha_u}^{\beta_u} dv\; h_u(v)\, \frac{d}{dv} e^{-ivl} \\ &=\; -\frac{1}{il}\, h_u(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} \;+\; \frac{1}{il} \int_{\alpha_u}^{\beta_u} h_u'(v)\, e^{-ivl}\, dl \\ &=\; \cdots \;=\; -\frac{1}{il} \sum_{n=0}^{K-1} \left(\frac{1}{il}\right)^n \; h_u^{(n)}(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} \;+\; \left(\frac{1}{il}\right)^K \int_{\alpha_u}^{\beta_u} h_u^{(K)}(v)\, e^{-ivl}\, dl \quad . \end{aligned} \tag{2.29}$$

If we bound all summands in (2.29) using the first part of (2.25) and the regularity condition (2.27), each v-derivative appears in combination with a power of l^{-1}, and this gives a factor $c_1/(lE_P) \ll 1$. Thus we can in the limit $K \to \infty$ drop the integral term in (2.29) and obtain

$$P_u(l) \;=\; -\frac{1}{il} \sum_{n=0}^{\infty} \left(\frac{1}{il}\right)^n \; h_u^{(n)}(v)\, e^{-ivl}\Big|_{\alpha_u}^{\beta_u} \quad . \tag{2.30}$$

This expansion converges fast, as its summands decay like $(c_1/(lE_P))^n$.

Using (2.28), we can write the Fourier transform (2.22) as

$$P(s,l) \;=\; \int_{-\infty}^{\infty} P_u(l)\, e^{-ius}\, du \quad . \tag{2.31}$$

Notice that, apart from the constraints (2.25), the "large" variable l can be freely chosen. We want to study the functional dependence of (2.31) on the parameter l. In preparation, we consider an integral of the general form

$$\int_a^b f(u)\, e^{-i\gamma(u)\, l}\, du \quad , \tag{2.32}$$

where we assume that $(u, \gamma(u))$ is a curve in the high-energy region, more precisely $\gamma \sim E_P$. Assume furthermore that γ is monotone with $|\gamma'| \sim 1$ and that $(b-a) \sim E_P$. By transforming the integration variable, we can then write (2.32) as the Fourier integral

$$\int_{\gamma(a)}^{\gamma(b)} f\, |\gamma'|^{-1}\, e^{-i\gamma l}\, d\gamma \quad . \tag{2.33}$$

If the function $f\, |\gamma'|^{-1}$ is smooth, its Fourier transform (2.33) has rapid decay in the variable l. Under the stronger assumption that $f\, |\gamma'|^{-1}$ varies on the scale E_P, we conclude that the length scale for this rapid decay is of the order $l \sim E_P^{-1}$. As a consequence, the rapid decay can be detected even under the constraint $l < l_{\max}$ imposed by (2.25), and we say that (2.33) has *rapid decay in* l. The reader who feels uncomfortable with this informal definition can immediately make this notion mathematically precise by an integration by parts argument similar to (2.29) imposing for $f\, |\gamma'|^{-1}$ a condition of type (2.27). The precise mathematical meaning of rapid decay in l for the integral (2.32) is that for every integer k there should be constants $c \sim 1$ and $l_{\min} \ll l_{\max}$ such that for all $l \in (l_{\min}, l_{\max})$,

$$\int_a^b f(u)\, e^{-i\gamma(u)\, l}\, du \;\leq\; c\, (lE_P)^{-k} \int_a^b |f(u)|\, du \quad .$$

We return to the analysis of the integral (2.31). The boundary terms of (2.30) at β_u yield contributions to $P(s,l)$ of the form

$$-\left(\frac{1}{il}\right)^{n+1} \int_{-\infty}^{\infty} h_u^{(n)}(\beta_u)\, e^{-i\beta_u l - ius}\, du \quad . \tag{2.34}$$

Recall that the points (u, β_u) are in the high-energy region (in the example of Figure 2, these points lie on curve "a"). According to (2.24), the length scale for the oscillations of the factor $\exp(-ius)$ is $u \sim E_P$. Under the reasonable assumption that β_u is monotone and that the functions $|\beta'(u)|^{-1}$ and $h_u^{(n)}(\beta_u)$ vary on the scale E_P, the integral (2.34) is of the form (2.33), and the above consideration yields that (2.34) has rapid decay in l. We remark that this argument could be extended to the case where β_u has extremal points (basically because the extrema give contributions only for isolated momenta u and thus can be shown to be negligible), but we will not go into this here. Having established rapid decay in l for (2.34), it remains to consider the boundary terms in (2.34) at α_u, more precisely

$$P(s,l) \;=\; \sum_{n=0}^{\infty} \left(\frac{1}{il}\right)^{n+1} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, e^{-i\alpha_u l - ius}\, du \;+\; (\text{rapid decay in } l) \quad . \tag{2.35}$$

We cannot again apply our "oscillation argument" after (2.32), because α_u tends asymptotically to zero on branch "A" of the hyperbola (see Figure 2), so that the factor $\exp(-i\alpha_u l)$ is non-oscillating in this region. We expand this factor in a Taylor series,

$$P(s,l) \;=\; \sum_{n,k=0}^{\infty} \frac{1}{k!}\, (il)^{k-n-1} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, (-\alpha_u)^k\, e^{-ius}\, du \quad . \tag{2.36}$$

In the region where $l\alpha_u \not\ll 1$, this expansion might seem problematic and requires a brief explanation. First of all, α_u becomes large near $u = 0$ (on branch "B" of the hyperbola in Figure 2). In the case without regularization, the power expansion of the factor $\exp(-i\alpha_u\, l)$ corresponds to an expansion in the mass parameter (recall that in this case, $\alpha_u = m^2/u$ according to (2.23)), and it would lead in (2.36) to a singularity of the integrand at the origin. Indeed, this difficulty is a special case of the logarithmic mass problem, which is discussed in [5, 6] and finally resolved by working with the "regularized" distribution T_a^{reg}, (2.9). Using these results, the behavior of the unregularized $P(s,l)$ for small momenta $u \ll E_P$ is well understood. Our oscillation argument after (2.32) yields that the regularization for $u \ll E_P$ (i.e. the form of the extension of branch "B" of the hyperbola to the high-energy region) affects $P(s,l)$ merely by rapidly decaying terms. Thus it is sufficient to consider here the integrand in (2.36) away from the origin $u = 0$. When combined with the results in [5, 6], our analysis immediately yields a complete description of the regularized fermionic projector near the light cone. Furthermore, the function α_u might become large for $u \sim E_P$, and this is a more subtle point. One way of justifying (2.36) would be to simply assume that $l_{\max}\alpha_u \ll 1$ along the whole extension of branch "A" to the high-energy region. A more general method would be to split up the curve (u, α_u) in the high-energy region $u \sim E_P$ into one branch where the expansion (2.36) is justified and another branch where our oscillation argument after (2.32) applies. The intermediate region $l\alpha_u \sim 1$, where none of the two methods can be used, is generically so small that it can be neglected. In order to keep our analysis reasonably simple, we here assume that α_u is sufficiently small away from the origin, more precisely

$$\alpha_u \;<\; \alpha_{\max} \;\ll\; l_{\max}^{-1} \qquad\qquad \text{for } u \sim E_P. \tag{2.37}$$

For a fixed value of $k - n$, all summands in (2.36) have the same l-dependence. Let us compare the relative size of these terms. According to our regularity assumption (2.27), the derivatives of h scale like $h_u^{(n)} \sim E_P^{-n}$. Using the bound (2.37), we conclude that, for a fixed power of l, the summands in (2.36) decrease like $(\alpha_{\max}/E_P)^n$. Thus it is a very good approximation to drop the summands for large n. At first sight, it might seem admissible to take into account only the first summand $n = 0$. Unfortunately, the situation is not quite so simple. For example, it may happen that, when restricted to the curve (u, α_u), the function $h(u,v)$ is so small that the summands for $n = 0$ in (2.36) are indeed not dominant. More generally, we need to know that for some $n_0 \geq 0$, the function $h_u^{(n_0)}(\alpha_u)$ is really of the order given in (2.27), i.e.

$$|h_u^{(n_0)}(\alpha_u)| \;\geq\; c \left(\frac{c_1}{E_P}\right)^{n_0} \max |h_u| \qquad\qquad \text{for } u \sim E_P \tag{2.38}$$

and a positive constant c. If this condition is satisfied, we may neglect all summands for $n > n_0$, and collecting the terms in powers of l, we conclude that

$$P(s,l) \;=\; \frac{1}{(il)^{n_0+1}} \sum_{k=0}^{\infty} (-il)^k \sum_{n=\max(n_0-k,0)}^{n_0} \frac{(-1)^{n_0-n}}{(k-n_0+n)!} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, \alpha_u^{k-n_0+n}\, e^{-ius}\, du$$

$$+ \sum_{n=n_0+1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} h_u^{(n)}(\alpha_u)\, e^{-ius}\, du$$
$$+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (\alpha_{\max}/E_P) \ll (l_{\max} E_P)^{-1}) \quad . \tag{2.39}$$

In our case, the function h_u has in the low-energy region according to (2.23) the form $h_u(\alpha_u) = m/(32\pi^3)\,\Theta(u)$. Hence it is natural to assume that (2.38) is satisfied for $n_0 = 0$. Introducing the shorter notation

$$h(u) \;:=\; h_u(\alpha(u)) \quad , \qquad h^{[n]}(u) \;:=\; h_u^{(n)}(\alpha_u) \quad , \qquad \alpha(u) \;:=\; \alpha_u \quad , \tag{2.40}$$

we have thus derived the following result.

Expansion of the scalar component: *Close to the light cone (2.24),(2.25), the scalar component (2.16) of the fermionic projector of the vacuum has the expansion*

$$P(s,l) \;=\; \frac{1}{il} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} h\, \alpha^k\, e^{-ius}\, du \tag{2.41}$$
$$+ \sum_{n=1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} h^{[n]}\, e^{-ius}\, du \tag{2.42}$$
$$+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (\alpha_{\max}/E_P) \ll (l_{\max} E_P)^{-1}) \tag{2.43}$$

with suitable regularization functions h, $h^{[n]}$, and α. In the low-energy region $u \ll E_P$, the regularization functions are

$$h(u) \;=\; \frac{m}{32\pi^3}\,\Theta(u) \quad , \qquad h^{[n]}(u) \;=\; 0 \quad , \qquad \alpha(u) \;=\; \alpha_u \;=\; \frac{m^2}{u} \quad . \tag{2.44}$$

In this expansion, the l-dependence is written out similar to a Laurent expansion. The main simplification compared to our earlier Fourier representation (2.11) is that the dependence on the regularization is now described by functions of only one variable, denoted by h, $h^{[n]}$, and α. In composite expressions in $P(s,l)$, we will typically get convolutions of these functions; such one-dimensional convolutions are convenient and can be easily analyzed. The simplification to one-dimensional regularization functions became possible because many details of the regularization affect only the contribution with rapid decay in l, which we do not consider here. Notice that the summands in (2.41) and (2.42) decay like $(l\,\alpha_{\max})^k/k! \ll (l/l_{\max})^k/k!$ and $(lE_P)^{-n}$, respectively. In the low-energy limit (2.44), the expansion (2.41) goes over to a power series in m^2, and we thus refer to (2.41) as the *mass expansion*. In the mass expansion, the regularization is described by only two functions h and α. The series (2.42), on the other hand, is a pure regularization effect and is thus called the *regularization expansion*. It involves an infinite number of regularization functions $h^{[n]}$. Accordingly, we will use the notions of mass and regularization expansions also for other expansions of type (2.39).

In the expansion (2.39), the fermionic projector is described exclusively in terms of the function $h(u,v)$ in a neighborhood of the discontinuity along the curve (u, α_u). Let us go back to the definition of h, (2.19), and consider what this result means for the regularized fermionic projector in momentum space (2.16). In the case without regularization (2.23), we saw that integrating out the cylindrical coordinates ρ and φ yields a discontinuity of h whenever the 2-plane $\omega, k = \text{const}$ meets and is tangential to the hyperboloid $\omega^2 - k^2 - \rho^2 = m^2$. Indeed, this picture is true in the general case, i.e. the discontinuity of h can always

be associated to a contribution to $\tilde{P}$ which describes a hypersurface in four-dimensional momentum space. The simplest way to recover the discontinuity of h when integrating out the cylindrical coordinates would be to choose $\tilde{P}$ of the form (2.16) with a function ϕ and the spherically symmetric distribution $f = \delta(|\vec{p}| - \omega - \alpha(-|\vec{p}| - \omega))$. Since spherically symmetric regularizations seem too restrictive, it is preferable to describe the discontinuity of h more generally by a contribution to $\tilde{P}$ of the form

$$\phi(\vec{p})\, \delta(\omega - \Omega(\vec{p})) \quad , \tag{2.45}$$

which is singular on the hypersurface $\omega = \Omega(\vec{p})$. For small momentum $|\vec{p}| \ll E_P$, the surface should clearly go over to the mass shell given by $\Omega = -\sqrt{|\vec{p}|^2 + m^2}$ and $\phi = m/|2\Omega|$; also, it is reasonable to assume that ϕ and Ω are smooth and sufficiently regular. This consideration shows that that for the behavior of the fermionic projector on the light cone (2.39), the essential role is played by states lying on a hypersurface. We refer to these one-particle states as the *surface states* of the fermionic projector of the vacuum. This result seems physically convincing because the surface states naturally generalize the states on the lower mass shell known from relativistic quantum mechanics. By integrating out the cylindrical coordinates for the ansatz (2.45), one can express the regularization functions $h_u^{(n)}$ in (2.39) in terms of ϕ and the geometry of the hypersurface. But we point out that in contrast to the just discussed discontinuity of h, the partial derivatives of h depend also on states other than surface states. For example, a contribution to $\tilde{P}$ of the form $b(\omega, \vec{p})\, \Theta(\omega - \Omega(\vec{p}))$ with Ω as in (2.45) and a smooth function b has a discontinuity on the surface Ω and affects all of the regularization functions $h_u^{(n)}$ for $n \geq 1$ (as one verifies by a short computation). Thinking of the decomposition of the fermionic projector into the one-particle states, such non-surface contributions would consist of a large number of states, and would thus make it necessary to introduce many additional fermions into our system. It does not seem quite reasonable or appropriate to considerably increase the number of particles of the system with the only purpose of having more freedom for the derivative terms of h in (2.39). It seems easiest and is physically most convincing to assume that all of the regularization functions in (2.39) come about as a consequence of surface states. We refer to this assumption as the *restriction to surface states*. It is of no relevance for the scalar component (2.41),(2.43), but will yield an important relation between the regularization functions for the vector component in the next subsection. To avoid confusion, we point out that the restriction to surface states clearly does not imply that $\tilde{P}$ is of the form (2.45). It imposes a condition only on the behavior of $\tilde{P}$ in a neighborhood of our hypersurface; namely that the only distributional or non-regular contribution to $\tilde{P}$ in this neighborhood should be the hypersurface itself.

For clarity, we finally review our assumptions on the regularization. Our first assumption was that the function $h(u, v)$ has, for every fixed u, at most two discontinuities at $\alpha(u)$ and $\beta(u)$, and is sufficiently regular otherwise, (2.27). Furthermore, the function $\beta(u)$ had to be monotone and again sufficiently regular. For the function $\alpha(u)$, we assumed that (2.37) holds. Since h is obtained from $\tilde{P}$, (2.16), by integrating out the cylindrical coordinates (2.19), these assumptions implicitly pose conditions on the fermionic projector of the vacuum. Although they could clearly be weakened with more mathematical effort, these conditions seem sufficiently general for the moment. In order to understand this better, one should realize that integrating out the cylindrical coordinates does in the generic case (i.e. unless if there are singularities parallel to the plane $\omega, k = \text{const}$) improve the regularity. The restriction to the generic case is in most situations justified by the fact that the direction $y - x$ and the coordinate system in (2.17) can be freely chosen. Using the above

assumptions on $h(u,v)$, we showed that the dominant contribution to the fermionic projector on the light cone is made by states on a hypersurface in four-dimensional momentum space. With the "restriction to surface states," we assumed finally that the behavior on the light cone (2.39) is completely characterized by these states.

2.5 The Regularized Vacuum on the Light Cone, Vector Component

We shall now extend the previous analysis to the vector component in (2.13). More precisely, we will analyze the Fourier integral (2.11) for

$$\tilde{P}(p) \;=\; v_j(p)\, \gamma^j\, f(p) \tag{2.46}$$

close to the light cone. We again choose light-cone coordinates (s, l, x_2, x_3) with $y - x = (s, l, 0, 0)$ (s and l are given by (2.20), while x_2 and x_3 are Cartesian coordinates in the orthogonal complement of the sl-plane). The associated momenta are denoted by $p = (u, v, p_2, p_3)$ with u and v according to (2.21). As in (2.18), we integrate out the coordinates perpendicular to u and v,

$$h_j(u,v) \;:=\; \frac{1}{2\,(2\pi)^4} \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3\; (v_j\, f)(u, v, p_2, p_3) \quad , \tag{2.47}$$

and obtain a representation of the fermionic projector involving two-dimensional Fourier integrals,

$$P(s,l) \;=\; \gamma^j\, P_j(s,l) \quad \text{with} \quad P_j(s,l) \;:=\; \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h_j(u,v)\, e^{-i(us+vl)} \quad . \tag{2.48}$$

The tensor indices in (2.47) and (2.48) refer to the coordinate system (s, l, x_2, x_3). For clarity, we denote the range of the indices by $j = s, l, 2, 3$; thus

$$\gamma^s \;=\; \frac{1}{2}\,(\gamma^0 - \gamma^1) \quad , \qquad \gamma^l \;=\; \frac{1}{2}\,(\gamma^0 + \gamma^1) \quad , \tag{2.49}$$

where $\gamma^0, \ldots, \gamma^3$ are the usual Dirac matrices of Minkowski space. According to the continuum kernel (2.2), $\tilde{P}$ has in the case without regularization the form $\tilde{P} = \not{p}\, \delta(p^2 - m^2)\, \Theta(-p^0)$, and h_j can be computed similar to (2.23) to be

$$\gamma^j\, h_j(u,v) \;=\; \frac{1}{32\pi^3} (-u\gamma^s - v\gamma^l)\, \Theta(uv - m^2)\, \Theta(u) \quad . \tag{2.50}$$

This limiting case specifies the regularized $h_j(u,v)$ for small energy-momentum $u, v \ll E_P$. In order to keep the form of the functions h_j in the high-energy region sufficiently general, we merely assume in what follows that the h_j satisfy all the conditions we considered for the function h in the previous subsection (see the summary in the last paragraph of Subsection 2.4).

Our main result is the following.

Expansion of the vector component: *Close to the light cone (2.24),(2.25), the vector component (2.46) of the fermionic projector of the vacuum has the expansion $P = \gamma^j P_j$ with*

$$P_s(s,l) \quad = \quad \frac{1}{il} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} -u\, g_s\;\; \alpha^k\, e^{-ius}\, du$$

$$
\begin{aligned}
&+ \sum_{n=1}^{\infty} \frac{1}{(il)^{n+1}} \int_{-\infty}^{\infty} -u\, g_s^{[n]}\, e^{-ius}\, du \\
&+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (\alpha_{\max}/E_P) \ll (l_{\max} E_P)^{-1}) \qquad (2.51) \\
P_l(s,l) \;&=\; \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left[(k-1)\, \alpha^k \;+\; k\, \frac{b}{u}\, \alpha^{k-1} \right] g_l\, e^{-ius}\, du \\
&+ \sum_{n=1}^{\infty} \frac{1}{(il)^{n+2}} \int_{-\infty}^{\infty} -(n+1)\, g_l^{[n]}\, e^{-ius}\, du \\
&+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (\alpha_{\max}/E_P) \ll (l_{\max} E_P)^{-1}) \qquad (2.52) \\
P_{2/3}(s,l) \;&=\; \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left[\alpha^k \;+\; k\, \frac{b_{2/3}}{u}\, \alpha^{k-1} \right] g_{2/3}\, e^{-ius}\, du \\
&+ \sum_{n=1}^{\infty} \frac{1}{(il)^{n+2}} \int_{-\infty}^{\infty} g_{2/3}^{[n]}\, e^{-ius}\, du \\
&+ (\text{rapid decay in } l) \;+\; (\text{higher orders in } (\alpha_{\max}/E_P) \ll (l_{\max} E_P)^{-1}) \qquad (2.53)
\end{aligned}
$$

and suitable regularization functions g_j, $g_j^{[n]}$, b, $b_{2/3}$, and the mass regularization function α as in (2.42). In the low energy region $u \ll E_P$, the regularization functions have the form

$$
\begin{aligned}
g_s(u) \;&=\; \frac{1}{32\pi^3}\, \Theta(u) \quad , \qquad g_s^{[n]}(u) \;=\; 0 && (2.54) \\
g_l(u) \;&=\; \frac{1}{32\pi^3}\, \Theta(u) \quad , \qquad g_l^{[n]}(u) \;=\; b(u) \;=\; 0 && (2.55) \\
g_{2/3}(u) \;&=\; g_{2/3}(u) \;=\; b_{2/3}(u) \;=\; 0 \quad . && (2.56)
\end{aligned}
$$

Before entering the derivation, we briefly discuss these formulas. For this, we consider the situation where, like in the case without regularization, the vector $v(p)$ in (2.46) points into the direction p. In this case, we can write the vector component as

$$
\tilde{P}(p) \;=\; p_j \gamma^j\, (\phi f)(p) \quad , \qquad (2.57)
$$

where (ϕf) has the form of the scalar component considered in Subsection 2.4. Since multiplication in momentum space corresponds to differentiation in position space, we obtain for (2.48)

$$
P(s,l) \;=\; -i \left(\gamma^s \frac{\partial}{\partial s} + \gamma^l \frac{\partial}{\partial l} + \gamma^2 \frac{\partial}{\partial x^2} + \gamma^3 \frac{\partial}{\partial x^3} \right) P_{\text{scalar}}(s,l) \quad ,
$$

where P_{scalar} is the scalar component (2.22) with h as in (2.19). We now substitute for P_{scalar} the expansion on the light cone (2.41)–(2.43) and carry out the partial derivatives. For the s- and l-components, this gives exactly the expansions (2.51) and (2.52) with

$$
g_s \;=\; g_l \;=\; h \quad , \qquad g_s^{[n]} \;=\; g_l^{[n]} \;=\; h^{[n]} \quad , \qquad b \;=\; 0 \quad . \qquad (2.58)
$$

For the components $j = 2, 3$, the calculation of the partial derivatives is not quite so straightforward because the expansion of the scalar component (2.41)–(2.43) was carried out for fixed x_2 and x_3. Nevertheless, one can deduce also the expansion (2.53) from (2.41)–(2.43) if one considers x_2 and x_3 as parameters of the regularization functions h, $h^{[n]}$, and α, and differentiates through, keeping in mind that differentiation yields a

factor $1/l$ (to get the scaling dimensions right). In this way, the simple example (2.57) explains the general structure of the expansions (2.51) and (2.52). We point out that the regularization function b vanishes identically in (2.58). This means that b is non-zero only when the direction of the vector field v is modified by the regularization. Thinking in terms of the decomposition into the one-particle states, we refer to this regularization effect as the *shear of the surface states*.

We shall now derive the expansions (2.51)–(2.53). Since the Fourier integrals in (2.48) are of the form (2.22), they have the expansion (2.39), valid close to the light cone (2.24),(2.25). It remains to determine the parameter n_0 in (2.39). We consider the components $j = s, l, 2$, and 3 separately. According to (2.50), the function h_s in the low-energy region looks similar to the hyperbola depicted in Figure 2. The main difference to the low-energy behavior of the scalar component (2.23) is the additional factor u in h_s which grows linearly along branch "A" of the hyperbola. Thus in the low-energy region away from the origin,

$$(h_s)_u(\alpha_u) \;\sim\; E_P \qquad \text{and} \qquad \max_{v \in (0, E_P)} |(h_s)_u(v)| \;\sim\; E_P \quad . \tag{2.59}$$

From this behavior it is natural to assume that h_s satisfies the bound (2.38) with $n_0 = 0$. Because of the linearly growing factor u in the low-energy region, it is convenient to write the regularization functions in the form

$$(h_s)_u(\alpha_u) \;=:\; -u\, g_s(u) \quad , \qquad (h_s)_u^{(n)}(\alpha_u) \;=:\; -u\, g_s^{[n]}(u) \tag{2.60}$$

with suitable functions g_s and $g_s^{[n]}$ (this can be done because, as explained after (2.36), close to the origin $u = 0$, we can work with the unregularized fermionic projector). This yields the expansion (2.51). According to (2.50) and (2.60), the regularization functions have the low-energy limit (2.54). For the l-component, the situation is much different. According to (2.50), the function h_l in the low-energy limit has the form

$$h_l(u, v) \;=\; -\frac{1}{32\pi^3}\, v\, \Theta(uv - m^2) \quad . \tag{2.61}$$

The factor v decreases like m^2/u along branch "A" of the hyperbola. Thus in the low-energy region away from the origin,

$$(h_l)_u(\alpha_u) \;\sim\; m^2/E_P \qquad \text{whereas} \qquad \max_{v \in (0, E_P)} |(h_l)_u(v)| \;\sim\; E_P \quad . \tag{2.62}$$

Therefore, we cannot assume that h_l satisfies the bound (2.38) with $n_0 = 0$. But $(h_l)_u^{(1)}(\alpha_u) \sim 1$ in the low-energy region, and thus we may choose $n_0 = 1$. We conclude that it is necessary to take into account two inner summands in (2.39), more precisely

$$P_l(s, l) \;=\; \frac{1}{(il)^2} \sum_{k=0}^{\infty} \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \Big[(h_l)'_u(\alpha_u)\, \alpha_u^k \;-\; k\, (h_l)_u(\alpha_u)\, \alpha_u^{k-1} \Big]\, e^{-ius}\, du + \cdots, \tag{2.63}$$

where "$\cdots$" stands for the regularization expansion and all terms neglected in (2.39). In the low-energy region, we have according to (2.61) and (2.44),

$$(h_l)_u(\alpha_u) \;=\; -\frac{1}{32\pi^3}\, \frac{m^2}{u} \;=\; (h_l)'_u(\alpha_u)\, \alpha_u \quad .$$

Thus in this region, the two summands in the square brackets of (2.63) are of the same order of magnitude, and none of them can be neglected. In view of the low-energy limit, we introduce the regularization functions as

$$\begin{aligned} (h_l)'_u(\alpha_u) &=: -g_l(u) \quad , \qquad (h_l)_u^{[1+n]}(\alpha_u) =: -(n+1)\, g_l^{[n]}(u) \\ (h_l)'_u(\alpha_u)\, \alpha_u - (h_l)_u(\alpha_u) &=: \frac{b(u)}{u}\, g_l(u) \quad ; \end{aligned} \tag{2.64}$$

this yields the expansion (2.52). According to (2.50), the regularization functions have the low-energy limit (2.55). We finally consider the components $j = 2$ and 3. According to (2.50), these components are identically equal to zero in the low-energy limit. But for $u \sim E_P$, they might behave similar to P_s or P_l. To be on the safe side, we choose $n_0 = 1$. Denoting the regularization functions by

$$\begin{aligned} (h_{2\!/\!3})'_u(\alpha_u) &=: g_{2\!/\!3}(u) \quad , \qquad (h_{2\!/\!3})_u^{[1+n]}(\alpha_u) =: g_{2\!/\!3}^{[n]}(u) \\ -(h_{2\!/\!3})_u(\alpha_u) &=: \frac{b_{2\!/\!3}(u)}{u}\, g_{2\!/\!3}(u) \quad , \end{aligned} \tag{2.65}$$

we obtain the expansion (2.53). According to (2.50), the regularization functions $g_{2\!/\!3}$, $g_{2\!/\!3}^{[h]}$, and $b_{2\!/\!3}$ vanish in the low-energy region, (2.56).

For clarity, we point out that choosing $n_0 = 1$ (as in (2.52) and (2.53)) is an extension of setting $n_0 = 0$ (as in (2.51)), obtained by taking into account more summands of the general expansion (2.36). Nevertheless, the different behavior in the low-energy region (2.59),(2.62) suggests that (2.52) and (2.53) should not be merely more general formulas than (2.51), but that the behavior of $P_j(s,l)$, $j = l, 2, 3$, should be really different from that of $P_s(s,l)$. We shall now make this difference precise. Comparing (2.59) and (2.62) (and using that $h_{2\!/\!3}$ vanishes in the low-energy region), it is reasonable to impose that there should be a constant $\varepsilon_{\text{shear}} > 0$ with

$$|(h_j)_u(\alpha_u)| \;<\; \varepsilon_{\text{shear}}\, |(h_s)_u(\alpha_u)| \qquad \text{for } u \sim E_P \text{ and } j = l, 2, \text{ or } 3. \tag{2.66}$$

In view of (2.59) and (2.62), $\varepsilon_{\text{shear}}$ should be as small as

$$\varepsilon_{\text{shear}} \;\sim\; \frac{m^2}{E_P^2} \quad . \tag{2.67}$$

However, if the surface states have shear (as defined earlier in this subsection), the constant $\varepsilon_{\text{shear}}$ must in general be chosen larger. In order to keep our analysis as general as possible, we will not specify here how $\varepsilon_{\text{shear}}$ scales in the Planck energy, but merely assume that $m^2/E_P^2 < \varepsilon_{\text{shear}} \ll 1$. Using (2.60), (2.64), and (2.65), the condition (2.66) can be expressed in terms of the regularization functions g_j and b_j as

$$\left(\frac{b}{u} + \alpha_u\right) g_l,\; \frac{b_{2\!/\!3}}{u}\, g_{2\!/\!3} \;<\; \varepsilon_{\text{shear}}\, u\, g_s \qquad \text{for } u \sim E_P. \tag{2.68}$$

It is interesting to analyze what the condition (2.66) means for the functions P_j. We begin with the case without regularization. In this case, the vector component of $P(x,y)$ points into the direction $y - x$, more precisely $P(x,y) = i(y-x)_j \gamma^j\, S(x,y)$ with a scalar distribution S. In a composite expression like the closed chain $P(x,y)\, P(y,x)$, one can contract the tensor indices and obtains in a formal calculation $P(x,y)\, P(y,x) = (y -$

$x)^2\ S(x,y)\ S(y,x)$ with a scalar factor $(y-x)^2$ which vanishes on the light cone. Let us consider this contraction in our light-cone coordinates. Before the contraction, each factor $(y-x)_j \gamma^j = 2l\ \gamma^s + 2s\ \gamma^l \approx 2l\gamma^s$ is, if we take only the leading contribution on the light cone (i.e. the lowest order in s/l), proportional to l. After the contraction, however, the product $(y-x)^2 = 4ls$ is proportional to both l and s. Thus the contraction yields, to leading order on the light cone, a dimensionless factor s/l. While the factor l^{-1} changes the scaling behavior in the "large" variable, the factor s tends to make the composite expression "small" near the light cone. The analysis of the scaling behavior in l can immediately be extended to the case with regularization by looking at the expansions (2.51) and (2.52). Let us consider as an example the leading term of the mass expansion. For the expansion (2.51), this is the summand $k=0$, and it scales like $P_s(s,l) \sim 1/l$. If we assume that (2.66) holds with $\varepsilon_{\text{shear}}$ according to (2.67), then (2.68) shows that $b(u) \sim 1$, and the summands in the square bracket are in (2.52) are of comparable size. Hence the leading term of the expansion (2.52) is also the summand $k=0$, and it scales in l like $P_l(s,l) \sim 1/l^2$. Hence the leading term of the sum $\gamma^l P_l + \gamma^s P_s$ behaves like $P \sim 1/l + \mathcal{O}(1/l^2)$. Since s and l are null directions, a contraction of the tensor indices in the closed chain leads only to mixed products of the form $P_s\ P_l$, and this mixed product scales in l like $P_s\ P_l \sim 1/l^3$. Thus, exactly as in the case without regularization, the contraction of the tensor indices yields an additional factor l^{-1}. If on the other hand, the condition (2.66) were violated, the regularization function b could be chosen arbitrarily large. But if b becomes large enough, the cominant contribution to (2.52) is the summand $k=1$ (notice that b does not appear in the summand $k=0$), and hence $P_l(s,l) \sim 1/l$. This implies that $P_s\ P_l \sim 1/l^2$, and the contraction does no longer yield an additional factor l^{-1}. This consideration is immediately extended to the components $P_{2\beta}$ by considering the l-dependence of the summands in (2.53). We conclude that the condition (2.66) with $\varepsilon_{\text{shear}} \ll 1$ means that the contraction of the tensor indices yields a scalar factor which vanishes on the light cone. Therefore, we refer to this condition by saying that the *vector component is null on the light cone*. If one wishes, one can simply take this condition as an additional assumption on the fermionic projector of the vacuum. However, the property of the vector component being null on the light cone also arises in the study of composite expressions in the fermionic projector as a compatibility condition, and can thus be derived from the equations of discrete space-time (we shall come back to this derivation in our analysis of the equations of discrete space-time in forthcoming papers).

The next question is if our regularization functions α, g_j, $g_j^{[n]}$, and b, which appear in our expansions (2.51)–(2.53), are all independent of each other, or whether there are some relations between them. Recall that the regularization functions are derived from the boundary values of the functions $\partial_v^n h_j(u,v)$, $n \geq 0$, on the curve (u, α_u) (see (2.60), (2.64), and (2.65)). Since the $(h_j)_{j=s,l,2,3}$ were treated in our two-dimensional Fourier analysis as four independent and (apart from our regularity assumptions) arbitrary functions, we can certainly not get relations between the regularization functions by looking at the situation in the uv-plane. But we can hope that when we consider the surface states in four-dimensional momentum space (as introduced in Subsection 2.4), the geometry of the hypersurface defined by these states might yield interesting restrictions for the regularization functions. First of all, we mention that our discussion of surface states of the previous subsection applies without changes also to the vector component; we will in what follows make use of the restriction to surface states. Since in the low-energy region the regularization is irrelevant and the results of [5, 6] apply, we can furthermore restrict our attention to large energy and momentum $\omega, |\vec{k}| \sim E_P$. We choose polar coordinates

$(\omega, k = |\vec{k}|, \vartheta, \varphi)$ in momentum space and introduce the "mass shell coordinates"

$$U \;=\; -|\vec{k}| - \omega \quad , \qquad\qquad V \;=\; |\vec{k}| - \omega \quad . \tag{2.69}$$

Notice that, in contrast to the coordinates u and v, (2.21), the variables (2.69) are the spherically symmetric part of a four-dimensional coordinate system $(U, V, \vartheta, \varphi)$. Extending also the notation (2.49) in a spherically symmetric way, we introduce the Dirac matrix

$$\gamma^S \;=\; \frac{1}{2}\left(\gamma^0 \;-\; \frac{\vec{\gamma}\vec{k}}{k}\right) \quad .$$

Let us consider what the expansions (2.51), (2.52), and (2.53) tell us about the surface states. As explained before (2.45), the discontinuities of h_j come about in (2.47) when the plane $u, v = \text{const}$ meets and is tangential to the hypersurface of the surface states. We denote the tangential intersection point of the surface $u, v = \text{const}$ with the hypersurface by $Q = (U, V, \vartheta, \varphi)$. In the high-energy region under consideration, the variable U is of the order E_P. The variable V, on the other hand, will be of order $\alpha(U) < \alpha_{\max}$. Thus our hypersurface is close to the mass cone in the sense that $V/U \sim \alpha_{\max}/E_P \ll 1$. As a consequence, the angle ϑ is small (more precisely, $\vartheta \leq \sqrt{\alpha_{\max}/E_P}$), and we conclude that, to leading order in $\alpha_{\max}/E_P$, $V = \alpha(U)$. Hence we can write the hypersurface as a graph $V = A(U, \vartheta, \varphi)$ with a function A satisfying the condition

$$A(U, \vartheta = 0) \;=\; \alpha(U) \;+\; (\text{higher orders in } \alpha_{\max}/E_P) \quad .$$

One can think of the functions $A(u, \vartheta, \varphi)$ as the extension of α to the four-dimensional setting. In order to determine the structure of the Dirac matrices, we first recall that the assumption that the vector component is null on the light cone implied in our consideration after (2.51) that the parameter n_0 corresponding to P_l, P_2, and P_3 was equal to one. This means that to leading order in $\alpha_{\max}/E_P$, only the function $h_s(u, v)$ is discontinuous on the curve (u, α_u), and we conclude that the distribution $\tilde{P}$ is on the hypersurface at the point Q a scalar multiple of γ^s; we use the short notation $\tilde{P}(Q) \sim \gamma^s$. Using again that ϑ is small, we obtain that to leading order in $\alpha_{\max}/E_P$, $\tilde{P}(U, A(U, \vartheta = 0), \vartheta = 0) \sim \gamma^s$. Since the spatial direction of the vector $y - x$ in (2.11) can be chosen arbitrarily, we can by rotating our coordinate system, immediately extend this result to general ϑ and φ, namely $\tilde{P}(U, \alpha(U, \vartheta, \varphi), \vartheta, \varphi) \sim \gamma^S$. Hence the surface states are described by a contribution to $\tilde{P}$ of the form

$$-32\pi^3\, g(U, \vartheta, \varphi)\, \gamma^S\, \delta(V - A(U, \vartheta, \varphi)) \;+\; (\text{higher orders in } \alpha_{\max}/E_P) \tag{2.70}$$

with some function g. It is reasonable to assume that the functions in (2.70) are sufficiently regular. Similar to our regularity condition (2.27) for h, we here assume that the derivatives of A and g_S have the natural scaling behavior in E_P. More precisely, for all $n_1, n_2, n_3 \geq 0$ there should exist a constant $c \ll l E_P$ with

$$|\partial_U^{n_1} \partial_\vartheta^{n_2} \partial_\varphi^{n_3} A(U, \vartheta, \varphi)| \;+\; |\partial_U^{n_1} \partial_\vartheta^{n_2} \partial_\varphi^{n_3} g(U, \vartheta, \varphi)| \;\leq\; c\, E_P^{-n_1}\, \max(|A| + |g|) \tag{2.71}$$

for all $U \sim E_P$.

The form of the surface states (2.70) allows us to calculate the regularization functions g_j, $g_j^{[n]}$, and b_j. For this, we first represent the matrix γ^S in (2.70) in the Dirac basis $(\gamma^j)_{j=s,l,2,3}$; this yields the contributions of the surface states to the distributions $(v_j\, f)$.

By substituting into (2.47) and carrying out the integrals over p_2 and p_3, one obtains the functions h_j. The regularization functions can finally be computed via (2.60), (2.64), and (2.65). This whole calculation is quite straightforward, and we only state the main results. To leading order in v/u, we can take A and g as constant functions in the calculation of the s- and l-components , and thus the calculation of $\gamma^s h_s + \gamma^l h_l$ reduces to the integral

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp_3 \; \left(\gamma^s + \frac{v}{u}\, \gamma^l \right) g(u, \vartheta = 0)\, \delta \left(v - \alpha_u - \frac{p_2^2 + p_3^2}{u} \right)$$
$$+(\text{higher orders in } v/u,\, \alpha_{\max}/E_P) \quad .$$

An evaluation in cylindrical coordinates yields that both $g_s(u)$ and $g_l(u)$ are equal to $g(u, \vartheta = 0)$, and we thus have the important relation

$$g_s(u) \;=\; g_l(u) \;=:\; g(u) \quad . \tag{2.72}$$

In the case without shear of the surface states, this relation was already found in (2.58); we now see that it holds in a much more general setting. The calculation of the angular components $j = 2, 3$ gives for $g_{2/3}$ contributions proportional to $u\, \partial_{2/3} A$ and $u\, \partial_{2/3} g$. Unfortunately, this is not very helpful to us because we have no information on the derivatives of A and g. The computation of the regularization functions $g_j^{[n]}$ involves higher derivatives of the functions in (2.70) and becomes quite complicated. We remark that the above analysis of the surface states can be carried out similarly for the scalar component of the previous subsection and gives relations between the regularization functions h and $h^{[n]}$, (2.40), but these relations all depend on unknown details of the geometry of the hypersurface. We thus conclude that (2.72) is the only relation between the regularization functions which can be derived with our present knowledge on the surface states,

We finally mention one assumption on the regularization which, although we will not use it in the present work, might be worth considering later. The 4×4 matrix $(\not{p} + m)$ in the integrand of the unregularized fermionic projector (2.2) has the special property of being singular of rank two. This means that the fermionic projector is composed of only two occupied fermionic states, for every momentum p on the mass shell. The natural extension of this property to the case with regularization is that for every p on the hypersurface defined by the surface states, the matrix $\tilde{P}(p)$ corresponding to the vector-scalar structure (2.13) should be of rank two. We refer to this property as the assumption of *half occupied surface states*. In terms of the functions $h(u, v)$ and $h_j(u, v)$, it means that $h_s(u, \alpha(u))\, h_l(u, \alpha_u) = h(u, \alpha_u)^2$. Using (2.40), (2.60), (2.64), and (2.72), the assumption of half occupied surface states yields the following relation between the regularization functions of the scalar and vector components,

$$(\alpha(u)\, u + b(u))\, g(u)^2 \;=\; h(u)^2 \quad . \tag{2.73}$$

2.6 The General Formalism

In this subsection, we shall extend our previous analysis on the light cone in three ways: to the case with interaction, to systems of Dirac seas as introduced in [4], and to composite expressions in the fermionic projector. Our first step is to develop a method which allows us to introduce a regularization into the formulas of the light-cone expansion (2.7). We will here only motivate and describe this method, the rigorous justification is given in Appendix B. Since the formulas of the light-cone expansion involve the factors $T^{\text{reg}\,(n)}$,

(2.8),(2.9), we begin by bringing these distributions into a form similar to our expansion of the regularized scalar component (2.41). By partly carrying out the Fourier integral (2.4) in the light-cone coordinates introduced in Subsection 2.4 (see (2.20) and (2.21)), we can write the distribution T_a as

$$T_a(s,l) \;=\; \frac{1}{32\pi^3}\,\frac{1}{il}\int_0^\infty e^{-\frac{ial}{u}-ius}\,du \quad . \tag{2.74}$$

This formula can be regarded as a special case of the expansion (2.35) (notice that the function $h(u,v)$ corresponding to T_a is computed similar to (2.23)), but (2.74) holds also away from the light cone. The distribution T_a is not differentiable in a at $a=0$, as one sees either directly in position space (2.5) or equivalently in (2.74), where formal differentiation leads to a singularity of the integrand at $u=0$. This problem is bypassed in [5, 6] by working instead of T_a with the distribution T_a^{reg}, (2.9). Let us briefly consider what this "regularization" means in the integral representation (2.74). The formal a-derivative of (2.74),

$$\frac{d}{da}T_a(s,l) \;=\; -\frac{1}{32\pi^3}\int_0^\infty \frac{1}{u}\,e^{-\frac{ial}{u}-ius}\,du \quad ,$$

is well-defined and finite for $a\neq 0$ because of the oscillating factor $\exp(-ial/u)$. However, the limit $a\to 0$ leads to a logarithmic divergence. Thus one must subtract a logarithmic counterterm before taking the limit; more precisely,

$$T^{\text{reg}\,(1)}(s,l) \;=\; -\frac{1}{32\pi^3}\lim_{a\to 0}\int_{-\infty}^\infty \left[\frac{1}{u}\,e^{-\frac{ial}{u}}\,\Theta(u) \;-\; (1+\log a)\,\delta(u)\right] e^{-ius}\,du \quad .$$

The higher a-derivatives $T^{\text{reg}\,(n)}$, $n>1$, are defined similarly using suitable counterterms which are localized at $u=0$. Since we do not need the details in what follows, we simply write

$$T^{\text{reg}\,(n)}(s,l) \;=\; -\frac{1}{32\pi^3}\,(-il)^{n-1}\int_0^\infty \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius}\,du \quad . \tag{2.75}$$

Consider a summand of the light-cone expansion (2.7),

$$\text{(iterated line integrals in bosonic potentials and fields)}\; T^{\text{reg}\,(n)}(s,l) \quad . \tag{2.76}$$

According to our assumption of macroscopic potentials and wave functions described in Subsection 2.2, we shall regularize only the distribution $T^{\text{reg}\,(n)}$, keeping the iterated line integral unchanged. Let us briefly analyze what this assumption means quantitatively. Not regularizing the iterated line integral in (2.76), denoted in what follows by F, will be a good approximation if and only if F is almost constant on the Planck scale. In other words, not regularizing F is admissible if we keep in mind that this method can describe the regularized fermionic projector only modulo contributions of the order $\partial_j F/E_P$. In the case that this last derivative acts on the bosonic potentials and fields contained in F, we obtain the limitation already mentioned in Subsection 2.2 that energy and momentum of the bosonic fields should be small compared to the Planck energy. More precisely, we can describe the fermionic projector only to leading order in $(l_{\text{macro}}E_P)^{-1}$, where l_{macro} is a typical length scale of macroscopic physics. A point we did not pay attention to earlier is that the iterated line integrals also involve factors $(y-x)$ which are contracted with the bosonic potentials and fields (see [5, 6] for many examples). Thus in light-cone coordinates, F will in general contain factors of l. If the derivative in $\partial_j F$ acts on a factor l, this factor is annihilated. Hence keeping the iterated line integrals in (2.76) unchanged

can describe only the leading order in $(lE_P)^{-1}$ of the fermionic projector. We conclude that the assumption of macroscopic potentials and wave functions is justified if and only if we restrict our analysis to the *leading order in* $(lE_P)^{-1}$ *and* $(l_{macro} E_P)^{-1}$. We remark that going beyond the leading order in $(lE_P)^{-1}$ or $(l_{\text{macro}} E_P)^{-1}$ would make it impossible to describe the interaction by classical fields, and is thus at present out of reach.

The restriction to the leading order in $(lE_P)^{-1}$ is a considerable simplification for what follows. First of all, we can neglect all regularization expansions (which are just expansions in powers of $(lE_P)^{-1}$; see e.g. (2.42) and the discussion thereafter), and thus we do not need to consider the regularization functions $h^{[n]}$ and $g_j^{[n]}$. Next we compare for given k the summands in (2.51)–(2.53) (the analysis for fixed k is justified assuming that the vector component is null on the light cone; see (2.66) and the discussion thereafter). One sees that the tensor index $j = s$ gives the leading contribution in $(lE_P)^{-1}$ to the vector component. This is a great simplification when tensor indices are contracted in composite expressions. Namely, when the vector component is contracted with the bosonic potentials or fields, it suffices to consider the contribution P_s, (2.51). If vector components are contracted with each other, the products of type $P_{2/3}\, P_{2/3}$ are according to (2.51)–(2.53) of higher order in $(lE_P)^{-1}$ or $\varepsilon_{\text{shear}}$ than corresponding products of type $P_s\, P_l$. Hence in such contractions, we must take into account both P_s and P_l, but we can again neglect the components P_2 and P_3. We conclude that the only regularization functions which should be of relevance here are those appearing in (2.41) and in the mass expansions of (2.51) and (2.52), i.e. the four functions

$$\alpha(u) \quad , \qquad g(u) \quad , \qquad h(u) \quad , \text{and} \qquad b(u) \tag{2.77}$$

with g given by (2.72).

Under the assumption of macroscopic potentials and wave functions, it suffices to regularize the factor $T^{\text{reg}\,(n)}$ in (2.76). Our method for regularizing $T^{\text{reg}\,(n)}$ is to go over to the integral representation (2.75) and to insert the regularization functions (2.77) into the integrand. The procedure depends on whether the contribution to the light-cone expansion is of even or odd order in the mass parameter m. Furthermore, we must treat the factors $(y-x)_j \gamma^j$ in the light-cone expansion separately. The precise regularization method is the following.

Regularization of the light-cone expansion: *A summand of the light-cone expansion (2.7) which is proportional to* m^p,

$$m^p \,(\text{iterated line integrals in bosonic potentials and fields})\, T^{\text{reg}\,(n)}(s,l) \quad , \tag{2.78}$$

has the regularization

$$\begin{aligned}
&(-1)\,(\text{iterated line integrals in bosonic potentials and fields}) \\
&\quad \times (-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} \\
&\quad + (\text{rapid decay in } l) \;+\; (\text{higher orders in } (lE_P)^{-1}, (l_{\text{macro}} E_P)^{-1}, \varepsilon_{\text{shear}}) \;. \qquad (2.79)
\end{aligned}$$

A contribution to the light-cone expansion (2.7) which is proportional to m^p *and contains a factor* $(y-x)_j \gamma^j$,

$$\begin{aligned}
&m^p \,(\text{iterated line integrals in bosonic potentials and fields}) \\
&\quad \times (y-x)_j \gamma^j\, T^{\text{reg}\,(n)}(s,l) \quad , \qquad (2.80)
\end{aligned}$$

is properly regularized according to

$$
\begin{aligned}
&(-1)\ \text{(iterated line integrals in bosonic potentials and fields)} \\
&\times (-il)^{n-1} \int_{-\infty}^{\infty} du \left[2l\, \gamma^s \left(\frac{1}{u^n} \right)^{\text{reg}} + 2in\, \gamma^l \left(\frac{1}{u^{n+1}} \right)^{\text{reg}} + 2l\, b(u)\, \gamma^l \left(\frac{1}{u^{n+2}} \right)^{\text{reg}} \right] \\
&\times e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} + (\text{contributions} \sim \gamma^2, \gamma^3) \\
&+ (\text{rapid decay in } l) + (\text{higher orders in } (lE_P)^{-1}, (l_{\text{macro}} E_P)^{-1}, \varepsilon_{\text{shear}})\ . \qquad (2.81)
\end{aligned}
$$

In these formulas, the regularization function a is given by

$$a(u) \;=\; u\, \alpha(u) \quad , \qquad (2.82)$$

ε_{shear} is defined via (2.66), and l_{macro} is a macroscopic length scale.

Let us briefly explain and motivate this regularization method (see Appendix B for the derivation). First of all, we note that, after writing the factor $(y-x)_j \gamma^j$ together with the iterated line integrals, the expression (2.80) is of the form (2.78), and the regularization rule (2.79) applies. Thus (2.81) is an extension of (2.79) giving additional information on the l-component of the factor $(y-x)_j \gamma^j$. As we shall see later, this information is essential when the factor $(y-x)_j$ in (2.80) is to be contracted with another factor $(y-x)^j$ in a composite expression. To explain the formula (2.79), we first point out that the expansions of the scalar and vector components (2.41)–(2.43), (2.51), and (2.52) do not involve the mass parameter m. The reason is that m was absorbed into the regularization functions g, h, and α, as one sees by considering the low-energy limit; see (2.44), (2.54), and (2.55). Furthermore, we note that each contribution to the mass expansions of the scalar or vector components contains either a factor h or g (see (2.41), (2.51), and (2.52)), and it is therefore reasonable that we should also use exactly one of these factors here. As a consequence, the power m^p in (2.78) uniquely determines how many factors of each regularization function we should take. Namely for even p, we must take one factor g and $p/2$ factors α, whereas the case of odd p gives rise to one factor h and $(p-1)/2$ factors α. On the other hand, we know that the insertion of the regularization functions into (2.75) should modify the behavior of the integrand only for large $u \sim E_P$; in particular, we should for small u have a behavior $\sim u^{-n}$. In order to comply with all of these conditions, one must insert the regularization functions precisely as in (2.79). In order to motivate (2.81), we consider the expansion of the vector component (2.51),(2.52). Recall that the regularization function b vanishes in the low-energy region (2.55) and describes the shear of the surface states (as explained after (2.58)). Since this effect is not related to the mass of the Dirac particle, it is plausible that we should not associate to b a power of m. For the mass expansion of the vector component, we should thus collect all terms to a given power of α. The contribution $\sim \alpha^k$ to $\gamma^s P_s + \gamma^l P_l$ takes, according to (2.51) and (2.52)), the form

$$\frac{1}{il}\, \frac{(-il)^k}{k!} \int_{-\infty}^{\infty} \left(-u\, \gamma^s + \frac{k-1}{il}\, \gamma^l - \frac{b}{u}\, \gamma^l \right) g\, \alpha^k\, e^{-ius}\, du \quad .$$

In order to obtain the correct behavior in the low-energy region, we must multiply this formula by $-2l$ and choose $k = n+1$. This explains the form of the square bracket in (2.81). The combination of the regularization functions g, h, and a in (2.81) can be understood exactly as in (2.79) using power counting in m.

Our constructions so far were carried out for the case $N = 1$ of one Dirac sea. We shall now generalize our regularization method to systems of Dirac seas as defined in [4, Section 4] and will also introduce a compact notation for the regularization. We first outline how chiral particles (e.g. neutrinos) can be described. Without regularization [4], a chiral Dirac sea is obtained by multiplying the Dirac sea of massless particles with the chiral projectors $\chi_{L/R} = \frac{1}{2}(\mathbb{1} \mp \gamma^5)$; for example in the vacuum and left/right handed particles,

$$\tilde{P}(p) \;=\; \chi_{L/R}\, \not{p}\, \delta(p^2)\, \Theta(-p^0) \quad . \tag{2.83}$$

The most obvious regularization method is to deduce the regularized chiral Dirac sea from a Dirac sea regularized with our above methods again by multiplying from the left with a chiral projector. This simple method indeed works, under the following assumptions. First of all, we must ensure that the regularized fermionic projector of the vacuum is a Hermitian operator. To this end, we must assume that the scalar component ϕ in (2.13) be identically equal to zero (this generalizes the requirement of massless particles needed in the case without regularization). Hence we regularize (2.83) by setting

$$\tilde{P}(p) \;=\; \chi_{L/R}\, v_j(p) \gamma^j\, f(p) \quad .$$

The expansions near the light cone are then obtained from (2.42),(2.43) and (2.51)–(2.53) by setting the scalar regularization functions h and $h^{[n]}$ to zero and by multiplying with $\chi_{L/R}$. Assuming furthermore that the external field is causality compatible [4], the formulas of the light-cone expansion are regularized likewise by taking the regularizations (2.79) and (2.81) with h set identically equal to zero, and by multiplying from the left with a chiral projector $\chi_{L/R}$. We next consider the generalization to systems of Dirac seas. In the vacuum, we can describe a system of Dirac seas by taking, similar to the construction in [4, Section 4], a direct sum of regularized Dirac seas and by using instead of the chiral projectors $\chi_{L/R}$ the chiral asymmetry matrix X. Since we may choose the regularization functions for each Dirac sea independently, this procedure clearly increases the total number of regularization functions. However, it is natural to impose that the regularization should respect all symmetries among the Dirac seas. More precisely, if the fermionic projector of the vacuum contains identical Dirac seas (e.g. corresponding to an underlying color $SU(3)$ symmetry), then we will always use the same regularization functions for all of these Dirac seas. Once the regularization has been specified for the vacuum, we can again apply the rules (2.78)–(2.81) to regularize the light-cone expansion. Namely, in the special case that the external field is diagonal in the Dirac sea index, we can simply take the direct sum of the contributions (2.79),(2.81), using in each summand the regularization functions of the corresponding vacuum Dirac sea. In the general case of a non-diagonal external field, the regularization functions can be inserted uniquely if one uses that, according to the assumption of macroscopic potentials and wave functions of Subsection 2.2, the fermionic projector is modified by the external field only on the macroscopic scale, so that its microscopic structure is the same as in the vacuum. For example, one can in the case of a gravitational and YM field make the external field locally to zero by transforming to a suitable coordinate sytem and gauge, can in this system insert the regularization functions as in the vacuum, and can finally transform back to the original system. We conclude that the generalization of our regularization method to systems of Dirac seas is quite straightforward and canonical. Therefore, we can introduce a short notation for the regularizations of the factors $T^{\text{reg}\,(n)}$ in the light-cone expansion by simply adding a label for the order in the mass parameter. More precisely, we introduce

in the case $N = 1$ of one Dirac sea the following abbreviations for the Fourier integrals in (2.79) and (2.81),

$$T^{(n)}_{[p]} \equiv -(-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} \tag{2.84}$$

$$\begin{aligned} (\not\xi\, T^{(n)}_{[p]}) \equiv\; & -(-il)^{n-1} \int_{-\infty}^{\infty} du\, e^{-ius} \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases} \\ & \times \left[2l\, \gamma^s \left(\frac{1}{u^n}\right)^{\text{reg}} + 2in\, \gamma^l \left(\frac{1}{u^{n+1}}\right)^{\text{reg}} + 2l\, b(u)\, \gamma^l \left(\frac{1}{u^{n+2}}\right)^{\text{reg}} \right] \end{aligned} \tag{2.85}$$

$$T^{(n)}_{\{p\}} \equiv -(-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius}\, b(u) \times \begin{cases} h(u)\, a(u)^{\frac{p-1}{2}} & \text{for } p \text{ odd} \\ g(u)\, a(u)^{\frac{p}{2}} & \text{for } p \text{ even} \end{cases}. \tag{2.86}$$

In the case of a system of Dirac seas, we use the same notation for the corresponding direct sum. With this notation, the regularization of the light-cone expansion is carried out (modulo all the contributions neglected in (2.79) and (2.81)) merely by the replacement $m^p\, T^{(n)}(x,y) \to T^{(n)}_{[p]}$, and by marking with brackets that the factors $(y-x)_j \gamma^j$ and $T^{(n)}_{[p]}$ belong together, where we use as in [5, 6] the abbreviation $\xi \equiv y - x$. We call a factor $\not\xi$ inside the brackets $(\not\xi T^{(n)}_{[p]})$ an *inner factor* $\not\xi$. Notice that the functions $T^{(n)}_{\{p\}}$ in (2.86) involve the regularization function b; they will be needed below to handle contractions between the inner factors.

We finally come to the analysis of composite expressions in the fermionic projector. In Subsection 2.3, we already discussed the simplest composite expression, the closed chain $P(x,y)\, P(y,x)$ in the vacuum. In order to analyze the closed chain near the light cone, we substitute for $P(x,y)$ and $P(y,x)$ the regularized formulas of the light-cone expansion and multiply out. It is convenient to use that the fermionic projector is Hermitian and thus $P(y,x) = P(x,y)^*$ (where "$*$" denotes the adjoint with respect to the spin scalar product); hence the light-cone expansion of $P(y,x)$ is obtained from that for $P(x,y)$ by taking the adjoint (with respect to the spin scalar product). The iterated line integrals can be multiplied with each other giving a smooth function; also we can simplify the resulting product of Dirac matrices using their anti-commutation relations. Denoting the adjoints of (2.84) and (2.85) by $\overline{T^{(n)}_{[p]}}$ and $(\overline{\not\xi T^{(n)}_{[p]}})$, respectively, we thus obtain for the closed chain a sum of terms of the following forms,

$$F\, T^{(n_1)}_{[r_1]}\, \overline{T^{(n_2)}_{[r_2]}}, \quad F\, (\xi_{j_1} T^{(n_1)}_{[r_1]})\, \overline{T^{(n_2)}_{[r_2]}}, \quad F\, T^{(n_1)}_{[r_1]}\, (\overline{\xi_{j_2} T^{(n_2)}_{[r_2]}}), \quad F\, (\xi_{j_1} T^{(n_1)}_{[r_1]})\, (\overline{\xi_{j_2} T^{(n_2)}_{[r_2]}}) \tag{2.87}$$

where F is a smooth function in x and y, and n_j, r_j are integer parameters. Here the tensor indices of the inner factors ξ are contracted either with each other or with tensor indices in the smooth prefactor F. Actually, the closed chain is a too simple example for us; what we really have in mind are Euler-Lagrange equations like (1.46),(1.47). Anticipating a result in a forthcoming paper on the principle of the fermionic projector, we mention that the analysis of such Euler-Lagrange equations can be reduced to products of the form (2.87), if we allow for more than two factors $T^{(.)}_{[.]}$. Hence our key problem is to mathematically handle products of the form

$$\begin{aligned} \text{(smooth function)} \quad & (\xi_{j_1} T^{(n_1)}_{[r_1]}) \cdots (\xi_{j_a} T^{(n_a)}_{[r_a]})\, T^{(n_{a+1})}_{[r_{a+1}]} \cdots T^{(n_b)}_{[r_b]} \\ & \times (\overline{\xi_{j_{b+1}} T^{(n_{b+1})}_{[r_{b+1}]}}) \cdots (\overline{\xi_{j_c} T^{(n_c)}_{[r_c]}})\, \overline{T^{(n_{c+1})}_{[r_{c+1}]}} \cdots \overline{T^{(n_d)}_{[r_d]}} \end{aligned} \tag{2.88}$$

with $1 \le a < b < c < d$, parameters n_i, p_i, and tensor indices j_i. Here the tensor indices of the inner factors ξ_i are again contracted either with other inner factors or with tensor indices in the smooth prefactor. We mention for clarity that, since the factors in (2.88) are complex functions or, in the case $N > 1$ of systems of Dirac seas, direct sums of complex functions, the product (2.88) clearly is commutative.

The inner factors in (2.88) can be simplified using the particular form (2.84),(2.85) of $T^{(n)}_{[r]}$ and $(\xi_j T^{(n)}_{[r]})$. We begin with the case of an inner factor which is contracted with a tensor index in the smooth prefactor, i.e. with products of the form

$$\cdots F^j\,(\xi_j T^{(n)}_{[r]}) \cdots \qquad \text{or} \qquad \cdots F^j\,\overline{(\xi_j T^{(n)}_{[r]})} \cdots$$

and a smooth vector field F, where "$\cdots$" stands for any other factors of the form as in (2.88). According to (2.85), to leading order in $(lE_P)^{-1}$ it suffices to take into account the s-component, and thus (2.84) yields that $(\not{\xi} T^{(n)}_{[r]}) \approx 2l\,\gamma^s\, T^{(n)}_{[r]}$. Since $2l\,\gamma^s$ coincides on the light cone with $\not{\xi}$, we conclude that, to leading order in $(lE_P)^{-1}$,

$$F^j\,(\xi_j T^{(n)}_{[r]}) \;=\; F^j \xi_j\, T^{(n)}_{[r]} \qquad \text{and} \qquad F^j\,\overline{(\xi_j\, T^{(n)}_{[r]})} \;=\; F^j \xi_j \overline{T^{(n)}_{[r]}} \quad . \tag{2.89}$$

These relations coincide with what one would have expected naively. We next consider the case of two inner factors which are contracted with each other, i.e. products of the form

$$\cdots (\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \cdots \;, \quad \cdots (\xi_j T^{(n_1)}_{[r_1]})\overline{(\xi^j T^{(n_2)}_{[r_2]})} \cdots \;, \text{ or } \quad \cdots \overline{(\xi_j T^{(n_1)}_{[r_1]})}\overline{(\xi^j T^{(n_2)}_{[r_2]})} \cdots \;. \tag{2.90}$$

In this case, the product cannot be calculated naively because the factor $\xi_j \xi^j = \xi^2$ vanishes on the light cone. But we can still compute the product using the Fourier representation (2.85). Since the s- and l-directions are null, only the mixed products of the s- and l-components in (2.85) contribute, and we obtain

$$\begin{aligned}
(\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \;=\; & (-il)^{n_1-1}\, l \int_{-\infty}^{\infty} du_1\, \frac{1}{u_1^{n_1}}\, e^{-iu_1 s} \times \begin{cases} h(u_1)\, a(u_1)^{\frac{p_1-1}{2}} & \text{for } p_1 \text{ odd} \\ g(u_1)\, a(u_1)^{\frac{p_1}{2}} & \text{for } p_1 \text{ even} \end{cases} \\
& \times (-il)^{n_2-1} \int_{-\infty}^{\infty} du_2 \left[\frac{2in_2}{u_2^{n_2+1}} + \frac{2l\, b(u_2)}{u_2^{n_2+2}} \right] e^{-iu_2 s} \times \begin{cases} h(u_2)\, a(u_2)^{\frac{p_2-1}{2}} & \text{for } p_2 \text{ odd} \\ g(u_2)\, a(u_2)^{\frac{p_2}{2}} & \text{for } p_2 \text{ even} \end{cases} \\
& + (-il)^{n_1-1} \int_{-\infty}^{\infty} du_1 \left[\frac{2in_1}{u_1^{n_1+1}} + \frac{2l\, b(u_1)}{u_1^{n_1+2}} \right] e^{-iu_1 s} \times \begin{cases} h(u_1)\, a(u_1)^{\frac{p_1-1}{2}} & \text{for } p_1 \text{ odd} \\ g(u_1)\, a(u_1)^{\frac{p_1}{2}} & \text{for } p_1 \text{ even} \end{cases} \\
& \times (-il)^{n_2-1}\, l \int_{-\infty}^{\infty} du_2\, \frac{1}{u_2^{n_2}}\, e^{-iu_2 s} \times \begin{cases} h(u_2)\, a(u_2)^{\frac{p_2-1}{2}} & \text{for } p_2 \text{ odd} \\ g(u_2)\, a(u_2)^{\frac{p_2}{2}} & \text{for } p_2 \text{ even} \end{cases}
\end{aligned}$$

(where we omitted for ease in notation the indices "reg"), and similarly for the two other products in (2.90). In the case of systems of Dirac seas, this calculation can be done for each summand of the direct sum separately. Rewriting the Fourier integrals using the notation (2.84) and (2.86), we get the following result.

Contraction rules: *To leading order in* $(lE_P)^{-1}$,

$$\begin{aligned}
& (\xi_j T^{(n_1)}_{[r_1]})(\xi^j T^{(n_2)}_{[r_2]}) \\
& \quad = \; -2\, T^{(n_1)}_{[r_1]}\, (n_2\, T^{(n_2+1)}_{[r_2]} + T^{(n_2+2)}_{\{r_2\}}) \;-\; 2\, (n_1\, T^{(n_1+1)}_{[r_1]} + T^{(n_1+2)}_{\{r_1\}})\, T^{(n_2)}_{[r_2]}
\end{aligned} \tag{2.91}$$

$$(\xi_j T^{(n_1)}_{[r_1]})(\overline{\xi^j T^{(n_2)}_{[r_2]}})$$
$$= -2\, T^{(n_1)}_{[r_1]}\, (n_2\, \overline{T^{(n_2+1)}_{[r_2]}} + \overline{T^{(n_2+2)}_{\{r_2\}}}) \;-\; 2\, (n_1\, T^{(n_1+1)}_{[r_1]} + T^{(n_1+2)}_{\{r_1\}})\, \overline{T^{(n_2)}_{[r_2]}} \tag{2.92}$$
$$(\overline{\xi_j T^{(n_1)}_{[r_1]}})(\overline{\xi^j T^{(n_2)}_{[r_2]}})$$
$$= -2\, \overline{T^{(n_1)}_{[r_1]}}\, (n_2\, \overline{T^{(n_2+1)}_{[r_2]}} + \overline{T^{(n_2+2)}_{\{r_2\}}}) \;-\; 2\, (n_1\, \overline{T^{(n_1+1)}_{[r_1]}} + \overline{T^{(n_1+2)}_{\{r_1\}}})\, \overline{T^{(n_2)}_{[r_2]}} \quad . \tag{2.93}$$

By iteratively applying (2.89) and the contraction rules (2.91)–(2.93), we can in (2.88) eliminate all inner factors ξ, and end up with products of the form

$$(\text{smooth function})\, T^{(a_1)}_\bullet \cdots T^{(a_p)}_\bullet\, \overline{T^{(b_1)}_\bullet} \cdots \overline{T^{(b_q)}_\bullet} \tag{2.94}$$

with parameters a_i, b_i and $p, q \geq 1$, where each subscript "$\bullet$" stands for an index $[r]$ or $\{r\}$. We call the product $T^{(a_1)}_\bullet \cdots T^{(a_p)}_\bullet\, \overline{T^{(b_1)}_\bullet} \cdots \overline{T^{(b_q)}_\bullet}$ a *monomial*.

We point out that the above transformation rules for the inner factors (2.89) and (2.91)–(2.93) are identities valid pointwise (i.e. for fixed x and y) close to the light cone. We anticipate that Euler-Lagrange equations like (1.46),(1.47) do not lead us to evaluate the products of the form (2.88) pointwise, but merely in the weak sense (this will be explained in detail in the forthcoming papers on the principle of the fermionic projector). Therefore, we now go over to a weak analysis of the monomials. In the case of a continuous regularization, we thus consider the integral

$$\int d^4x\, \eta(x)\, T^{(a_1)}_\bullet \cdots T^{(a_p)}_\bullet\, \overline{T^{(b_1)}_\bullet} \cdots \overline{T^{(b_q)}_\bullet} \tag{2.95}$$

with a test function η. Before coming to the derivation of calculation rules for the integrand in (2.95), we must think about how the test function η is to be chosen. As explained in Subsection 2.3 in the example of the closed chain (2.15), a weak integral in general depends essentially on the unknown high-energy behavior of the fermionic projector and is therefore undetermined. To avoid this problem, we must evaluate (2.95) in such a way that our expansions near the light cone become applicable. To this end, we assume that η has its *support near the light cone*, meaning that in light-cone coordinates (s, l, x_2, x_3), the "large" variable l satisfies on the support of η the conditions (2.25). For clarity, we remark that this definition does not state that the support of η should be in a small neighborhood of the light cone, but merely in a strip away from the origin. This is sufficient because we shall extract information on the behavior near the light cone by considering the singularities of the integral for $E_P \to \infty$ (see 2.101 below). Furthermore, we assume that η is *macroscopic* in the sense that its partial derivatives scale in powers of l^{-1} or l^{-1}_{macro}. Under these assumptions, the integrand in (2.95) is macroscopic in l, and carrying out the s- and l-integrals in (2.95) gives a function which is macroscopic in the "transversal" variables x_2 and x_3. Therefore in the three variables (l, x_2, x_3), a weak analysis is equivalent to a pointwise analysis, and thus it suffices to consider the s-integral in (2.95), i.e. the expression

$$\int_{-\infty}^{\infty} ds\; \eta\, T^{(a_1)}_\bullet \cdots T^{(a_p)}_\bullet\, \overline{T^{(b_1)}_\bullet} \cdots \overline{T^{(b_q)}_\bullet} \tag{2.96}$$

for fixed l, x_2, and x_3. In the case of a discrete regularization, the integral in (2.95) must be replaced by a sum over all space-time points, i.e. we must consider instead of (2.95) the weak sum

$$\sum_{x \in M} \eta(x)\, T^{(a_1)}_\bullet \cdots T^{(a_p)}_\bullet\, \overline{T^{(b_1)}_\bullet} \cdots \overline{T^{(b_q)}_\bullet} \quad , \tag{2.97}$$

where $M \subset \mathbb{R}^4$ are the discrete space-time points, and η is a macroscopic function in $\mathbb{R}^4$ with support near the light cone. Up to a normalization factor, (2.97) can be regarded as a Riemann sum which approximates the integral (2.95). Assuming that the space-time points have a generic position in $\mathbb{R}^4$ and keeping in mind that the function inside the sum (2.97) is macroscopic in the variables l, x_2, and x_3, the Riemann sum and the integral indeed coincide to leading order in $(lE_P)^{-1}$ and $(l_{\text{macro}} E_P)^{-1}$. Hence it is reasonable to work also in the discrete case with the one-dimensional integral (2.96).

Let us analyze the integral (2.96) in more detail. We first consider how (2.96) scales in the Planck energy. In the limit $E_P \to \infty$, the factors $T_\bullet^{(n)}$ go over to distributions which are in general singular on the light cone. Hence their product in (2.96) becomes ill-defined for $E_P \to \infty$ even in the distributional sense, and thus we expect that the integral (2.96) should diverge for $E_P \to \infty$. The order of this divergence can be determined using the following power counting argument. Keeping in mind that the regularization functions decay on the Planck scale $u \sim E_P$, the Fourier integrals (2.84) and (2.86) behave on the light cone (i.e. for $s = 0$) like

$$T^{(n)} \;\sim\; \log^g(E_P)\, E_P^{-n+1}$$

with $g = 1$ in the case $n = 1$ and $g = 0$ otherwise. Hence the product in the integrand of (2.96) scales on the light cone as

$$T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}} \;\sim\; \log^g(E_P)\, E_P^L \tag{2.98}$$

with an integer $g \geq 0$ and

$$L \;=\; p + q \;-\; \sum_{j=1}^{p} a_j \;-\; \sum_{k=1}^{q} b_k \quad . \tag{2.99}$$

We call L the *degree* of the monomial. We restrict attention to the case $L > 1$. In this case, the product (2.98) diverges in the limit $E_P \to \infty$ at least quadratically. If s is not zero, the oscillations of the factor $\exp(-ius)$ in (2.84) and (2.86) lead to a decay of $T_\bullet^{(n)}$ on the scale $s \sim E_P^{-1}$. This consideration shows that the dominant contribution to the integral (2.96) when $E_P \to \infty$ is obtained by evaluating η on the light cone, and the scaling behavior of this contribution is computed by multiplying (2.98) with a factor E_P^{-1}. We conclude that (2.96) diverges in the limit $E_P \to \infty$, and its leading divergence is proportional to

$$\eta(s=0)\, \log^g(E_P)\, E_P^{L-1} \qquad \text{with} \qquad g \geq 0 \quad . \tag{2.100}$$

By substituting the Fourier representations (2.84) and (2.86) into (2.96), one can rewrite the products in (2.96) as convolutions of the regularization functions (this is explained in detail in Appendix C for a particular choice of regularization functions). This calculation, which is straightforward and shall not be given here, shows that, to leading order in $(lE_P)^{-1}$ and $(l_{\text{macro}} E_P)^{-1}$, the integral (2.96) is indeed proportional to (2.100); this calculation also allows one to compute the parameter g. Collecting the factors of l in (2.84) and (2.86), we obtain the following result.

Weak evaluation of the monomials near the light cone: *Consider the integral (2.96) for a monomial of degree $L > 1$. Then there is an integer $g \geq 0$ and a real coefficient c_{reg} independent of s and l such that for every macroscopic test function η,*

$$\begin{aligned} \int_{-\infty}^{\infty} ds\, \eta\, T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}} \;&=\; \frac{c_{\text{reg}}}{(il)^L}\, \eta(s=0)\, \log^g(E_P)\, E_P^{L-1} \\ &+ (\text{higher orders in } (lE_P)^{-1} \text{ and } (l_{\text{macro}} E_P)^{-1}) \quad . \end{aligned} \tag{2.101}$$

The coefficient c_{reg} clearly depends on the indices of the monomial and on the details of the regularization. We call c_{reg} a *regularization parameter*.

Integrals of type (2.96) can be transformed using integration by parts. More precisely,

$$\int_{-\infty}^{\infty} ds \left(\frac{d}{ds}\eta\right) T_\bullet^{(a_1)} \cdots \overline{T_\bullet^{(b_q)}} = -\int_{-\infty}^{\infty} ds\, \eta \frac{d}{ds}\left(T_\bullet^{(a_1)} \cdots \overline{T_\bullet^{(b_q)}}\right) \tag{2.102}$$

$$= -\int_{-\infty}^{\infty} ds\, \eta \left[\left(\frac{d}{ds} T_\bullet^{(a_1)}\right) T_\bullet^{(a_2)} \cdots \overline{T_\bullet^{(b_q)}} + \cdots + T_\bullet^{(a_1)} \cdots \overline{T_\bullet^{(b_{q-1})}} \left(\frac{d}{ds}\overline{T_\bullet^{(b_q)}}\right)\right] \tag{2.103}$$

where we applied the Leibniz rule in the last step. Differentiating (2.84) and (2.86) with respect to s yields that

$$\frac{d}{ds} T_\bullet^{(n)} = -l\, T_\bullet^{(n-1)} \qquad \text{and} \qquad \frac{d}{ds}\overline{T_\bullet^{(n)}} = -l\, \overline{T_\bullet^{(n-1)}} \quad . \tag{2.104}$$

With these relations, we can carry out the derivatives in (2.103). Notice that the differentiation rules (2.104) decrease the index n by one. According to (2.99) and (2.101), decrementing the upper index of a factor $T_\bullet^{(a_j)}$ or $\overline{T_\bullet^{(b_k)}}$ increments the degree of the monomial and yields in the weak integral a factor of the order E_P/l. Using furthermore that η is macroscopic (as defined after (2.95)), we conclude that each summand in (2.103) dominates the left side of (2.102) by one order in (lE_P) or $(l_{\text{macro}}E_P)$. We have thus derived the following result.

Integration-by-parts rule: *Consider a monomial of degree $L > 0$. In a weak analysis near the light cone, we have to leading order in $(lE_P)^{-1}$ and $(l_{macro}E_P)^{-1}$,*

$$T_\bullet^{(a_1-1)} \cdots T_\bullet^{(a_p)} \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}} + \cdots + T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p-1)} \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}}$$
$$T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)} \overline{T_\bullet^{(b_1-1)}} \cdots \overline{T_\bullet^{(b_q)}} + \cdots + T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)} \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q-1)}} = 0 \quad . \tag{2.105}$$

This calculation rule yields relations between the monomials. In the case $p = 1$ of only one factor $T^{(a_1)}$, one can by iteratively applying the integration-by-parts rule arrange that a_1 is zero; thus it suffices to consider monomials of the form

$$T^{(0)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}} \qquad \text{with} \qquad b_1 \leq \cdots \leq b_q. \tag{2.106}$$

In the case $p > 1$, one can likewise express every monomial as a unique linear combination of monomials of the following form,

$$T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}} \quad \text{with} \quad a_1 = a_2 \leq \cdots \leq a_p \text{ and } b_1 \leq \cdots \leq b_q \quad , \tag{2.107}$$

as is verified by simple induction. Notice that in (2.106) and (2.107), we ordered the factors $T_\bullet^{(a_j)}$ and $\overline{T_\bullet^{(b_k)}}$ for increasing values of a_j and b_k. This does not completely fix the ordering; namely one can arbitrarily reorder those factors for which the indices a_j or b_k coincide. Since the products in (2.106) and (2.107) are commutative, configurations which can be transformed into each other by such reorderings clearly describe the same monomial and shall be identified. We call the monomials of the form (2.106) and (2.107) the *basic monomials* and the corresponding regularization parameters c_{reg} the *basic regularization parameters*.

With the above constructions, we have developed the mathematical framework for analyzing composite expressions in the fermionic projector in the continuum. Our procedure

is outlined as follows. We first substitute for the fermionic projector the regularized formulas of the light-cone expansion; this yields sums of products of the form (2.88), where the smooth prefactor involves the bosonic potentials and fields as well as the wave functions of the Dirac particles and anti-particles of the system. Applying our contraction rules, we then eliminate all inner factors and obtain terms of the form (2.94). When evaluated in the weak sense (2.101), the l-dependence determines the degree L of the monomial, and the dependence on the regularization is described for each monomial by the corresponding regularization parameters c_{reg}. Using our integration-by-parts rule, we can furthermore restrict attention to the basic monomials (2.106),(2.107) and the corresponding basic regularization parameters. As is shown in Appendix C for the physically most relevant case $p \neq q$, the basic monomials are linearly independent in the sense that there are no further identities between them. Therefore, it is a reasonable method to take the basic regularization parameters as empirical parameters modelling the unknown microscopic structure of space-time. If this is done, the composite expressions in the fermionic projector reduce to expressions in the bosonic fields and fermionic wave functions, involving a small number of free parameters. This procedure for analyzing composite expressions in the fermionic projector is called the *continuum limit*. In the forthcoming papers on the principle of the fermionic projector, we shall apply these methods to variational principles like the example described in Subsection 1.5.

A Connection to the Fock Space Formalism

In this appendix it is shown that for an observer who is making measurements only in a subsystem of the whole physical system, the concept of the fermionic projector is equivalent to the fermionic Fock space formalism, assuming that the number of fermions of the whole system is infinite. The following consideration applies in the same way to either a space-time continuum or to discrete space-time.

Let P be a fermionic projector acting on the vector space H. The observables correspond to operators $\mathcal{O}$ on H (for simplicity, we only consider one-particle observables; the generalization to many-particle observables is straightforward by repeating the following argument on a finite tensor product over H). Our subsystem is described by a non-degenerate subspace $K \subset H$; we decompose H as a direct sum $H = K \oplus L$ with $L = K^\perp$. We assume that the observables are localized in N; i.e. they are trivial on L,

$$\mathcal{O}_{|L} \;=\; \mathbb{1}_{|L} \quad . \tag{A.1}$$

We choose a (properly normalized) basis $\Psi_1, \ldots, \Psi_n$ of the subspace $P(H) \subset H$, and decompose the states Ψ_j in the form

$$\Psi_j \;=\; \Psi_j^K + \Psi_j^L \qquad \text{with} \qquad \Psi_j^K \in K\,, \quad \Psi_j^L \in L \quad .$$

If we substitute into (1.18), we obtain for the many-particle wave function the expression

$$\Psi \;=\; \sum_{\pi \in \mathcal{P}(n)} (-1)^{|\pi|} \left(\bigwedge_{j \in \pi} \Psi_j^K \right) \wedge \left(\bigwedge_{j \notin \pi} \Psi_j^L \right) \quad , \tag{A.2}$$

where $\mathcal{P}(n)$ denotes the set of all subsets of $\{1, \ldots, n\}$. For measurements in our subsystem, we must calculate the expectation value $<\Psi|\mathcal{O}|\Psi>_F$[6], where the operators $\mathcal{O}$ act on the Fock space according to

$$\mathcal{O}(\Psi_1 \wedge \cdots \wedge \Psi_n) \;=\; (\mathcal{O}\Psi_1) \wedge \cdots \wedge \Psi_n \;+\; \Psi_1 \wedge (\mathcal{O}\Psi_2) \cdots \wedge \Psi_n \;+\; \Psi_1 \wedge \cdots \wedge (\mathcal{O}\Psi_n) \quad ,$$

and where $<.|.>_F$ is the scalar product on the Fock space, induced by the scalar product $<.|.>$ on H. It is useful to rewrite the expectation value with the statistical operator S, i.e.

$$<\Psi|\mathcal{O}|\Psi>_F \;=\; \mathrm{tr}_F(S\,\mathcal{O}) \qquad \text{with} \qquad S \;=\; |\Psi><\Psi|_F \quad ,$$

where tr_F denotes the trace over the Fock space. Using (A.1), we can take the partial trace over L and obtain, using (A.2),

$$<\Psi|\mathcal{O}|\Psi>_F \;=\; \mathrm{tr}_{F_K}(S^K\,\mathcal{O}) \qquad \text{with} \tag{A.3}$$

$$S^K \;=\; \sum_{k=0}^{n} \sum_{\substack{\pi,\,\pi' \in \mathcal{P}(n),\\ \#\pi = \#\pi' = k}} c_{\pi,\pi'}\; |\wedge_{i\in\pi}\, \Psi_i^K><\wedge_{j\in\pi'}\, \Psi_j^K|_{F_K} \tag{A.4}$$

$$c_{\pi,\pi'} \;=\; (-1)^{|\pi|+|\pi'|} <\wedge_{i\notin\pi}\, \Psi_i^L|\, \wedge_{j\notin\pi'}\, \Psi_j^L>_F \quad ,$$

[6] We remark for clarity that this expectation value does not coincide with that of a measurement in nonrelativistic quantum mechanics. Namely, in the continuum, the scalar product $<.|.>$ involves a time integration. But one can get a connection to nonrelativistic measurements by considering operators $\mathcal{O}$ with a special time dependence (which, for example, act on the wave functions only in a short time interval $[t, t+\Delta t]$).

where tr_{F_K} is the trace over the Fock space $F_K = \oplus_{k=0}^\infty \wedge^k K$ generated by K. Thus our subsystem is described by a statistical operator S^K on F_K, which is composed of mixed states consisting of different numbers of particles. Since the constants $c_{\pi,\pi'}$ depend on the wave functions Ψ^L outside our subsystem, we can consider them as arbitrary numbers.

In the limit when the number n of particles of the whole system tends to infinity, (A.4) goes over to a statistical operator of the form

$$S^K = \sum_{k=0}^\infty \sum_{\alpha,\beta=0}^\infty c_{\alpha\beta}^{(k)} \, |\Psi_\alpha^{(k)}><\Psi_\beta^{(k)}|_{F_K} \tag{A.5}$$

with arbitrary complex coefficients $c_{\alpha\beta}^{(k)}$ and k-particle states $\Psi_\alpha^{(k)} \in F_K^k$. This statistical operator differs from a general statistical operator S_{gen}^K in that it is diagonal on the k-particle subspaces (i.e. that the wave functions in the "bra" and in the "ket" of (A.5) are both k-particle states); more precisely, S_{gen}^K has, compared to (A.5), the more general form

$$S_{\text{gen}}^K = \sum_{k,l=0}^\infty \sum_{\alpha,\beta=0}^\infty c_{\alpha\beta}^{(k,l)} \, |\Psi_\alpha^{(k)}><\Psi_\beta^{(l)}|_{F_K} \quad . \tag{A.6}$$

We remark for clarity that a pure state of the Fock space $\Psi \in F_K$ has a decomposition $\Psi = \sum_{k=0}^\infty \lambda_k \Psi^{(k)}$, and thus the corresponding statistical operator is

$$S = |\Psi><\Psi|_{F_K} = \sum_{k,l=0}^\infty \lambda_k \, \overline{\lambda_l} \, |\Psi^{(k)}><\Psi^{(l)}|_{F_K} \quad .$$

This statistical operator is a special case of (A.6), but it is *not* of the form (A.5).

The difference between (A.5) and (A.6) becomes irrelevant if we keep in mind that all physically relevant observables commute with the particle number operator. Namely in this case, every expectation value reduces to the sum of the expectation values in the k-particle Fock spaces,

$$\begin{aligned} \mathrm{tr}_{F_K}(S_{\text{gen}}^K \, \mathcal{O}) &= \sum_{k,l=0}^\infty \sum_{\alpha,\beta=0}^\infty c_{\alpha\beta}^{(k,l)} <\Psi_\beta^{(l)} \,|\, \mathcal{O} \,|\, \Psi_\alpha^{(k)}>_{F_K} \\ &= \sum_{k=0}^\infty \sum_{\alpha,\beta=0}^\infty c_{\alpha\beta}^{(k,k)} <\Psi_\beta^{(k)} \,|\, \mathcal{O} \,|\, \Psi_\alpha^{(k)}>_{F_K} \quad . \end{aligned}$$

If we choose the coefficients $c_{\alpha\beta}^{(k)}$ in (A.5) to be $c_{\alpha\beta}^{(k)} = c_{\alpha\beta}^{(k,k)}$, these expectation values are also obtained from the statistical operator S^K,

$$\mathrm{tr}_{F_K}(S_{\text{gen}}^K \, \mathcal{O}) = \mathrm{tr}_{F_K}(S^K \, \mathcal{O}) \quad .$$

We conclude that it is no loss of generality to describe the subsystem by the statistical operator S^K.

B The Regularized Causal Perturbation Theory

In Subsection 2.6, we gave a procedure for regularizing the formulas of the light-cone expansion (2.78)–(2.82). We shall now derive this regularization procedure. The basic idea

is to extend the causal perturbation expansion [4] to the case with regularization, in such a way that the causality and gauge symmetry are preserved for macroscopic perturbations. Using the methods of [5, 6], one can then analyze the behavior of the so-regularized Feynman diagrams near the light cone. For simplicity, we will restrict attention to the first order in perturbation theory. But our methods could also be applied to higher order Feynman diagrams, and the required gauge symmetry suggests that our main result, Theorem B.2, should hold to higher order in perturbation theory as well. Our analysis is based on [4, 5], and we will use also the notation introduced in these papers.

We first state our assumptions on the fermionic projector of the vacuum. As in Section 2, we describe the vacuum by a fermionic projector $P(x,y)$ of the form (2.11) with vector-scalar structure (2.13). For small energy-momentum, $\tilde{P}$ should coincide with the unregularized fermionic projector of the vacuum, i.e.

$$\tilde{P}(k) \;=\; (\not{k} + m)\, \delta(k^2 - m^2)\, \Theta(-k^0) \qquad \text{if } |k^0| \ll E_P \text{ and } |\vec{k}| \ll E_P. \tag{B.1}$$

Furthermore, we assume that the vector component is null on the light cone (i.e. that (2.66) holds with $\varepsilon_{\text{shear}} \ll 1$), and that P satisfies all the regularity assumptions considered in Subsections 2.4 and 2.5. For simplicity, we finally assume that the support of $\tilde{P}$ lies in the interior of the lower mass cone,

$$\text{supp } \tilde{P} \;\subset\; \{k \,|\, k^2 \geq 0 \text{ and } k^0 \leq 0\} \quad . \tag{B.2}$$

This last condition is quite strong, but nevertheless reasonable. In particular, it is satisfied when P is composed of one-particle states wich are small perturbations of the Dirac eigenstates on the lower mass shell.

In this appendix, we shall address the question of how one can introduce a classical external field into the system. For clarity, we will develop our methods mainly in the example of an external electromagnetic field. As described in Subsection 2.2, we consider the regularized fermionic projector as a model for the fermionic projector of discrete space-time. In this sense, the regularization specifies the microscopic structure of space-time. Following the concept of macroscopic potentials and wave functions introduced in Subsection 2.2, the electromagnetic field should modify the fermionic projector only on length scales which are large compared to the Planck length, but should leave the microscopic structure of space-time unchanged. In order to fulfill this requirement, we impose the following conditions. First of all, we assume that the electromagnetic field be "macroscopic" in the sense that it can be described by an electromagnetic potential A which vanishes outside the low-energy region, i.e.

$$\tilde{A}(k) \;=\; 0 \qquad \text{unless } |k^0| \ll E_P \text{ and } |\vec{k}| \ll E_P, \tag{B.3}$$

where $\tilde{A}$ is the Fourier transform of A. We denote the fermionic projector in the presence of the electromagnetic field by $P[\not{A}]$. In order to prevent that the electromagnetic potential might influence the microscopic structure of space-time locally, we demand that A can locally be made to zero by a gauge transformation. Thus we impose that the usual behavior under $U(1)$ gauge transformations

$$P[\not{A} + (\not{\partial}\Lambda)](x,y) \;=\; e^{i\Lambda(x)}\, P[\not{A}](x,y)\, e^{-i\Lambda(y)} \tag{B.4}$$

(with a real function Λ) should hold also for the regularized fermionic projector, assuming that the involved potentials A and $(A + \partial\Lambda)$ are both macroscopic (B.3). We point out

that, because of the gauge symmetry in discrete space-time (following from the freedom in choosing the gauge (1.31)), the local phase transformations in (B.4) are irrelevant in the equations of discrete space-time, and thus the transformation law (B.4) implies the freedom to transform the electromagnetic potential according to $\mathcal{A} \to \mathcal{A} + \partial\!\!\!/ \Lambda$. Finally, we must rule out the possibility that the electromagnetic potential might influence the microscopic structure of space-time in a nonlocal way. For this purpose, we impose that the perturbation expansion for the regularized fermionic projector be causal, in the sense introduced in [4].

Let us consider how these conditions can be implemented in the perturbation theory to first order. We first recall the standard perturbation theory for Dirac eigenstates. For a solution Ψ of the free Dirac equation $(i\partial\!\!\!/ - m)\Psi = 0$, the perturbation to first order in A, which we denote by $\Delta\Psi[\mathcal{A}]$, is given by

$$\Delta\Psi[\mathcal{A}](x) \;=\; -\int d^4y\, s_m(x,y)\, \mathcal{A}(y)\, \Psi(y) \quad , \tag{B.5}$$

where $s_m(x,y)$ is the Dirac Green's function

$$s_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \frac{\mathrm{PP}}{k^2 - m^2}\, (k\!\!\!/ + m)\, e^{-ik(x-y)} \quad , \tag{B.6}$$

and "PP" denotes the principal value (see [4, 5] for details). If we consider $s_m(x,y)$ as the integral kernel of an operator s_m and the potentials as multiplication operators, we can calculate $\Delta\Psi$ in the case $\mathcal{A} = \partial\!\!\!/\Lambda$ to be

$$\begin{aligned} \Delta\Psi[\partial\!\!\!/\Lambda] \;&=\; -s_m\, (\partial\!\!\!/\Lambda)\, \Psi \;=\; i s_m\, [i\partial\!\!\!/ - m,\, \Lambda]\, \Psi \\ &=\; i((i\partial\!\!\!/ - m) s_m)\, \Lambda\, \Psi \;-\; i s_m\, \Lambda\, ((i\partial\!\!\!/ - m)\Psi) \;=\; i\Lambda\, \Psi \quad . \end{aligned} \tag{B.7}$$

Thus in this case, $\Delta\Psi(x) = i\Lambda(x)\, \Psi(x)$ is simply the contribution linear in Λ to the phase transformed wave function $\exp(i\Lambda(x))\, \Psi(x)$; this shows explicitly that the perturbation calculation is gauge invariant.

As a consequence of the regularization, the fermionic projector $P(x,y)$ is in general not composed of Dirac eigenstates. Therefore, we next consider a wave function Ψ which is not necessarily a solution of the free Dirac equation. But according to (B.2), we may assume that its Fourier transform $\tilde{\Psi}$ has its support in the interior of the mass cone,

$$\mathrm{supp}\; \tilde{\Psi} \;\subset\; \{k \,|\, k^2 \geq 0\} \quad . \tag{B.8}$$

In this case, we can introduce $\Delta\Psi[\mathcal{A}]$ as follows. The spectral projector p_μ of the free Dirac operator $i\partial\!\!\!/$ corresponding to the eigenvalue $\mu \in \mathbb{R}$ has the form

$$p_\mu(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \epsilon(\mu)\, (k\!\!\!/ + \mu)\, \delta(k^2 - \mu^2)\, e^{-ik(x-y)} \tag{B.9}$$

(see [4]; notice that we added the step function $\epsilon(\mu)$ to allow for the case $\mu < 0$). Since the real axis is only part of the spectrum of the free Dirac operator (namely, the free Dirac operator has also an imaginary spectrum), the spectral projectors $(p_\mu)_{\mu \in \mathbb{R}}$ are clearly not complete, i.e. $\int_{-\infty}^{\infty} p_\mu d\mu \neq \mathbb{1}$. By integrating (B.9) over μ,

$$\int_{-\infty}^{\infty} p_\mu(x,y)\, d\mu \;=\; \int \frac{d^4k}{(2\pi)^4}\, \Theta(k^2)\, e^{-ik(x-y)} \quad , \tag{B.10}$$

one sees more precisely that the operator $\int_{-\infty}^{\infty} p_\mu d\mu$ is the projector on all the momenta in the mass cone $\{k \,|\, k^2 \geq 0\}$. But according to (B.8), Ψ lies in the image of this projector, and we can thus use the spectral projectors p_μ to decompose Ψ into eigenstates of the free Dirac operator. Each eigenstate can then be perturbed using (B.5). This leads us to introduce $\Delta\Psi[\A]$ according to

$$\Delta\Psi[\A] \;=\; -\int_{-\infty}^{\infty} d\mu\, s_\mu\, \A\, p_\mu\, \Psi \quad . \tag{B.11}$$

This definition of $\Delta\Psi$ shows the correct behavior under gauge transformations; namely, we have similar to (B.7),

$$\Delta\Psi[\partial\!\!\!/\Lambda] \;=\; i\int_{-\infty}^{\infty} d\mu\, s_\mu\, [i\partial\!\!\!/ - \mu,\, \Lambda]\, p_\mu\, \Psi \;=\; i\Lambda \left(\int_{-\infty}^{\infty} p_\mu\, d\mu\right) \Psi \;\stackrel{(B.10),(B.8)}{=}\; i\Lambda\, \Psi \quad . \tag{B.12}$$

Thinking in terms of the decomposition (2.1) of the fermionic projector into the one-particle states, it seems natural to introduce the perturbation of the fermionic projector $\Delta P[\A]$ by perturbing each one-particle state according to (B.11). This leads to the formula

$$\Delta P[\A] \;=\; -\int_{-\infty}^{\infty} d\mu\, (s_\mu\, \A\, p_\mu\, P \;+\; P\, p_\mu\, \A\, s_\mu) \quad . \tag{B.13}$$

The gauge symmetry can again be verified explicitly. Namely, a calculation similar to (B.12) using (B.2) yields that

$$\Delta P[\partial\!\!\!/\Lambda](x,y) \;=\; i\Lambda(x)\, P(x,y) - iP(x,y)\, \Lambda(y) \quad ,$$

and this is the contribution linear in Λ to (B.4). The perturbation calculation (B.13) is immediately extended to a general perturbation $\mathcal{B}$ as considered in [4] by setting

$$\Delta P[\mathcal{B}] \;=\; -\int_{-\infty}^{\infty} d\mu\, (s_\mu\, \mathcal{B}\, p_\mu\, P \;+\; P\, p_\mu\, \mathcal{B}\, s_\mu) \quad . \tag{B.14}$$

Let us verify if the perturbation calculation (B.14) is causal in the sense of [4]. Since it seems impossible to write (B.14) in a manifestly causal form (like e.g. [4, equation (3.9)]), we apply here a different method, which allows us to analyze the causality of the perturbation expansion in momentum space. As mentioned in [4, Section 5], the causality of the perturbation expansion can be understood via the causality of the line integrals over the external potentials and fields which appear in the light cone expansion. More precisely, causality means that the light-cone expansion of $\Delta P(x,y)$ should involve only line integrals along the line segment $\overline{xy}$, but no unbounded line integrals like for example $\int_0^\infty d\lambda\, \mathcal{B}(\lambda y + (1-\lambda)x)$. This way of understanding the causality of the perturbation expansion yields a simple condition in momentum space. Namely if $\mathcal{B}$ has the form of a plane wave of momentum q, i.e. $\mathcal{B}(x) = \mathcal{B}_q \exp(-iqx)$, then the unbounded line integrals become infinite when q goes to zero (for $\mathcal{B}_q$ fixed), whereas integrals along the line segment $\overline{xy}$ are clearly bounded in this limit. Hence we can say that the perturbation calculation (B.14) is causal only if it is regular in the limit $q \to 0$. In order to analyze this condition, we substitute the explicit formulas (B.6) and (B.9) into (B.14) and obtain

$$\begin{aligned}
\Delta P[\mathcal{B}](x,y) \;=\; & -\int_{-\infty}^{\infty} d\mu\, \epsilon(\mu) \int \frac{d^4k}{(2\pi)^4} \\
& \times \Big(\frac{\mathrm{PP}}{(k+q)^2 - \mu^2}\, (k\!\!\!/ + q\!\!\!/ + \mu)\, \mathcal{B}_q\, (k\!\!\!/ + \mu)\, \delta(k^2 - \mu^2)\, \tilde{P}(k)\, e^{-i(k+q)x + iky} \\
& \quad + \tilde{P}(k)\, \delta(k^2 - \mu^2)\, (k\!\!\!/ + \mu)\, \mathcal{B}_q\, (k\!\!\!/ - q\!\!\!/ + \mu)\, \frac{\mathrm{PP}}{(k-q)^2 - \mu^2}\, e^{-ikx + i(k-q)y} \Big) \quad .
\end{aligned}$$

We set $q = \varepsilon \hat{q}$ with a fixed vector $\hat{q}$ and consider the behavior for $\epsilon \searrow 0$. Taking only the leading order in ε, one can easily carry out the μ-integration and gets

$$\Delta P[\mathcal{B}](x,y) \;=\; -\frac{1}{\varepsilon} \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)} \\ \times \left(\frac{\mathrm{PP}}{2k\hat{q} + \varepsilon \hat{q}^2} \,(\not{\hat{q}}\, \mathcal{B}_q + \mathcal{B}_q \,\not{\hat{q}})\, \tilde{P}(k) \;+\; \tilde{P}(k)\, (\not{\hat{q}}\, \mathcal{B}_q + \mathcal{B}_q \,\not{\hat{q}})\, \frac{\mathrm{PP}}{-2k\hat{q} + \varepsilon \hat{q}^2} \right) + \mathcal{O}(\varepsilon^0)\,. \quad \text{(B.15)}$$

Since

$$\lim_{\epsilon \searrow 0} \frac{\mathrm{PP}}{2k\hat{q} + \varepsilon \hat{q}^2} \;=\; \lim_{\epsilon \searrow 0} \frac{\mathrm{PP}}{2k\hat{q} - \varepsilon \hat{q}^2} \;=\; \frac{\mathrm{PP}}{2k\hat{q}}$$

in the sense of distributions in the argument $k\hat{q}$ (notice that this kind of convergence is sufficient using the regularity of $\tilde{P}$), the leading singularity of (B.15) for $\varepsilon \searrow 0$ has the form

$$-\frac{1}{\varepsilon} \int \frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \frac{\mathrm{PP}}{2k\hat{q}} \,\Big[\{\mathcal{B}_q,\, \not{\hat{q}}\},\, \tilde{P}(k) \Big] \quad . \qquad \text{(B.16)}$$

Taking the Fourier transform in the variable $(x-y)$, it is clear that (B.16) vanishes only if the commutator/anti-commutator combination $[\{\mathcal{B}_q, \not{\hat{q}}\}, \tilde{P}(k)]$ is zero for all k. Since the perturbation $\mathcal{B}_q$ can be arbitrary, one sees (for example by considering a scalar perturbation, $\mathcal{B}_q \sim \mathbb{1}$) that it is a necessary condition for the perturbation calculation (B.14) to be regular in the limit $q \to 0$ that

$$[\not{\hat{q}},\, \tilde{P}(k)] \;=\; 0 \qquad \text{for all } k. \qquad \text{(B.17)}$$

This commutator vanishes only if the vector field $v(k)$ in (2.13) is a multiple of k, or, using the notation of Subsection 2.5, if the surface states have no shear. We conclude that the perturbation calculation (B.14) is in general not causal.

Before resolving this causality problem, we briefly discuss how this problem comes about. The condition (B.17) can be stated equivalently that the operator P must commute with the free Dirac operator. In other words, the perturbation calculation (B.14) is causal only if the fermionic projector of the vacuum is composed of eigenstates of the free Dirac operator. In this formulation, our causality problem can be understood directly. Namely, since our perturbation method is based on the perturbation calculation (B.5) for Dirac eigenstates, it is not astonishing that the method is inappropriate for non-eigenstates, because the perturbation expansion is then performed around the wrong unperturbed states. It is interesting to see that this shortcoming leads to a breakdown of causality in the perturbation expansion.

In order to comply with causality, we must modify the perturbation calculation (B.14). Our idea is to deduce the perturbation calculation for the fermionic projector from that for a modified fermionic projector, which satisfies the causality condition (B.17). The simplest idea for modifying the fermionic projector would be to introduce a unitary transformation $\tilde{U}(k) \in U(2,2)$ which makes the vector $v(k)$ in (2.13) parallel to k, more precisely

$$\tilde{U}(k)^{-1}\, v_j(k)\, \gamma^j\, \tilde{U}(k) \;=\; \lambda(k)\, \not{\hat{q}} \qquad \text{with } \lambda(k) \in \mathbb{R}.$$

However, a unitary transformation is too restrictive because it keeps the Lorentzian scalar product $v(k)^2$ invariant, and thus cannot be used for example in the case where $v(k)$ is space-like, but k is time-like. Therefore, we shall consider here a linear combination of

unitary transformations. More precisely, we introduce for $L > 1$ and $l = 1, \ldots, L$ unitary operators $\tilde{U}_l(k) \in U(2,2)$ and real coefficients c_l such that

$$\sum_{l=1}^{L} c_l(k) = 1 \qquad \text{and} \qquad v_j(k)\, \gamma^j \;=\; \sum_{l=1}^{L} c_l(k)\, \tilde{U}_l(k)\, \lambda(k)\, \not{k}\, \tilde{U}_l(k)^{-1} \tag{B.18}$$

with $\lambda(k) \in \mathbb{R}$. The existence of $(\tilde{U}_l, c_l)$ is guaranteed by the fact that the $U(2,2)$ transformations comprise Lorentzian transformations [3]. Clearly, the representation (B.18) is not unique. According to (B.1), we can choose the transformation (B.18) to be the identity in the low-energy region, and can thus assume that

$$\tilde{U}_l(k) \;=\; \mathbb{1} \qquad \text{if } |k^0| \ll E_P \text{ and } |\vec{k}| \ll E_P. \tag{B.19}$$

Furthermore, the regularity assumptions and the particular properties of the fermionic projector mentioned before (B.2) give rise to corresponding properties of the operators $\tilde{U}_l$; this will be specified below (see (B.31) and (B.51)). The operators obtained by multiplication with $\tilde{U}_l(k)$ in momentum space are denoted by U_l; they have in position space the kernels

$$U_l(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\, \tilde{U}_l(k)\, e^{-ik(x-y)} \quad . \tag{B.20}$$

Finally, we introduce the "modified fermionic projector" Q by replacing the vector field $v(k)$ in (2.13) by $\lambda(k)\, \not{k}$, i.e.

$$\tilde{Q}(k) \;=\; (\lambda(k)\, \not{k} \;+\; \phi(k)\, \mathbb{1})\, f(k) \quad . \tag{B.21}$$

According to (B.18), the fermionic projector P is obtained from Q by the transformation

$$P \;=\; \sum_{l=1}^{L} c_l\, U_l\, Q\, U_l^{-1} \quad . \tag{B.22}$$

The modified fermionic projector (B.21) satisfies the condition $[\tilde{Q}(k),\, \not{k}] = 0$. Hence the perturbation calculation for Q does not suffer from our above causality problem, and we can introduce $\Delta Q[\mathcal{B}]$ in analogy to (B.14) by

$$\Delta Q[\mathcal{B}] \;:=\; -\int_{-\infty}^{\infty} d\mu\, (s_\mu\, \mathcal{B}\, p_\mu\, Q \;+\; Q\, p_\mu\, \mathcal{B}\, s_\mu) \quad . \tag{B.23}$$

We now deduce the perturbation of P by applying to (B.23) a transformation analogous to that in (B.22), namely

$$\Delta P[\mathcal{B}] \;:=\; \sum_{l=1}^{L} c_l\, U_l\, \Delta Q[\mathcal{B}]\, U_l^{-1} \tag{B.24}$$

$$\;=\; -\sum_{l=1}^{L} c_l \int_{-\infty}^{\infty} d\mu\, U_l\, (s_\mu\, \mathcal{B}\, p_\mu\, Q \;+\; Q\, p_\mu\, \mathcal{B}\, s_\mu)\, U_l^{-1} \quad . \tag{B.25}$$

This last transformation should not affect the causality (in the sense of [4]) because if (B.23) is regular when the momentum q of the bosonic potential goes to zero, then the transformed operator (B.24) will clearly also be regular in this limit. We call (B.25) the *regularized causal perturbation* of the fermionic projector to first order.

The perturbation calculation (B.25) requires a detailed explanation. Qualitatively speaking, the difference between (B.14) and (B.25) is that the spectral projectors p_μ, the Green's functions s_μ, and the perturbation operator $\mathcal{B}$ have been replaced by the unitarily transformed operators

$$U_l\, p_\mu\, U_l^{-1} \quad , \qquad U_l\, s_\mu\, U_l^{-1} \quad \text{, and} \qquad U_l\, \mathcal{B}\, U_l^{-1} \quad , \tag{B.26}$$

and that a linear combination is taken. According to (B.19), the unitary transformations in (B.26) have no influence on the macroscopic properties of these operators, i.e. on the behavior when these operators are applied to wave functions with support in the low-energy region. But the transformation (B.26) changes the operators on the microscopic scale, in such a way that causality is fulfilled in the perturbation expansion. We point out that in the case where $\mathcal{B}$ is the usual operator of multiplication with the external potentials, the transformed operator $U_l \mathcal{B} U_l^{-1}$ is in general no longer a multiplication operator in position space; thus one can say that the classical potentials have become nonlocal on the microscopic scale. In order to better understand why the causality problem of (B.14) has disappeared in (B.25), it is useful to observe that Q commutes with the spectral projectors p_μ. This means that Q is composed of eigenstates of the Dirac operator, so that the perturbation expansion is now performed around the correct unperturbed states.

Let us consider a gauge transformation. In the case $\mathcal{B} = \partial\!\!\!/ \Lambda$, the perturbation (B.25) is computed to be

$$\begin{aligned}
\Delta P[\partial\!\!\!/ \Lambda] &= i \sum_{l=1}^L c_l \int_{-\infty}^\infty d\mu\, U_l \left(s_\mu \,[i\partial\!\!\!/ - \mu,\, \Lambda]\, p_\mu\, Q \;+\; Q\, p_\mu\, [i\partial\!\!\!/ - \mu,\, \Lambda]\, s_\mu \right) U_l^{-1} \\
&= i \sum_{l=1}^L c_l \int_{-\infty}^\infty d\mu\, U_l \left(\Lambda\, p_\mu\, Q \;-\; Q\, p_\mu\, \Lambda \right) U_l^{-1} \\
&= \sum_{l=1}^L c_l \left(i U_l \Lambda \left(\int_{-\infty}^\infty p_\mu\, d\mu \right) Q U_l^{-1} \;-\; i U_l Q \left(\int_{-\infty}^\infty p_\mu\, d\mu \right) \Lambda U_l^{-1} \right) .
\end{aligned} \tag{B.27}$$

By construction of $\tilde{Q}$, we can assume that the distributions $\tilde{P}$ and $\tilde{Q}$ have the same support, and thus (B.2) holds for $\tilde{Q}$ as well,

$$\text{supp}\, \tilde{Q} \;\subset\; \{k \,|\, k^2 \geq 0 \text{ and } k^0 \leq 0\} \quad . \tag{B.28}$$

Hence, according to (B.10), the projectors $\int_{-\infty}^\infty p_\mu d\mu$ in (B.27) can be omitted, and we conclude that

$$\Delta P[\partial\!\!\!/ \Lambda] \;=\; \sum_{l=1}^L c_l \left(i U_l \Lambda U_l^{-1}\, U_l Q U_l^{-1} \;-\; i U_l Q U_l^{-1}\, U_l \Lambda U_l^{-1} \right) \quad . \tag{B.29}$$

If in this formula we were allowed to replace the factors $U_l \Lambda U_l^{-1}$ by Λ, we could substitute (B.22) and would obtain the contribution linear in Λ to the required transformation law (B.4). Indeed, the difference between Λ and $U_l \Lambda U_l^{-1}$ is irrelevant, as one sees in detail as follows. We consider one summand in (B.29) and set for ease in notation $U = U_l$. According to (B.19), the operators Λ and $U \Lambda U^{-1}$ coincide macroscopically (i.e. when applied to functions with support in the low-energy region), and thus (B.29) yields gauge symmetry on the macroscopic scale. However, such a macroscopic gauge symmetry is not sufficient for us; namely, to ensure that the microscopic structure of space-time is not

influenced by the electromagnetic field, it is essential that (B.4) holds even on the Planck scale. In order to show microscopic gauge invariance, we consider the operator $U\Lambda U^{-1}$ in momentum space,

$$(U\Lambda U^{-1}\, f)(q) \;=\; \int \frac{d^4p}{(2\pi)^4}\, \tilde{U}(q)\, \tilde{\Lambda}(q-p)\, \tilde{U}(p)^{-1}\, f(p) \quad , \tag{B.30}$$

where $\tilde{\Lambda}$ is the Fourier transform of Λ, and f is a test function in momentum space. Since we assume that the electromagnetic potential $\A = \partial\!\!\!/\Lambda$ is macroscopic (B.3), the integrand in (B.30) vanishes unless $q-p$ is in the low-energy region. More precisely, we can say that

$$|q^0 - p^0|,\, |\vec{q} - \vec{p}| \;\sim\; l_{\text{macro}}^{-1} \quad ,$$

where l_{macro} denotes a typical length scale of macroscopic physics. Since the vector $q-p$ is in this sense small, it is reasonable to expand the factor $\tilde{U}(q)$ in (B.30) in a Taylor series around p. As the operators $\tilde{U}_l$ are characterized via (B.18), we can assume that they have similar regularity properties as P. In particular, we may assume that the partial derivatives of $\tilde{U}_l(p)$ scale in powers of E_P^{-1}, in the sense that there should be a constant $c \ll l_{\text{macro}} E_P$ such that

$$|\partial^\kappa \tilde{U}_l(p)| \;\leq\; \left(\frac{c}{E_P}\right)^{|\kappa|} \qquad \text{for any multi-index } \kappa. \tag{B.31}$$

From this we conclude that the Taylor expansion of $\tilde{U}(q)$ around p is an expansion in powers of $(l_{\text{macro}} E_P)^{-1}$, and thus

$$\begin{aligned}(U\Lambda U^{-1}\, f)(q) \;&=\; \int \frac{d^4p}{(2\pi)^4}\, \tilde{U}(p)\, \tilde{\Lambda}(q-p)\, \tilde{U}(p)^{-1}\, f(p) \\ &\quad + (\text{higher orders in } (l_{\text{macro}} E_P)^{-1}).\end{aligned} \tag{B.32}$$

Using that $\tilde{\Lambda}(q-p)$ is a multiple of the identity matrix, the factors $\tilde{U}(p)$ and $\tilde{U}(p)^{-1}$ in (B.32) cancel each other. We conclude that the operators $U\Lambda U^{-1}$ and Λ coincide up to higher order in $(l_{\text{macro}} E_P)^{-1}$. For the integral kernels in position space, we thus have

$$(U\Lambda U^{-1})(x,y) \;=\; \Lambda(x)\, \delta^4(x-y) \;+\; (\text{higher orders in } (l_{\text{macro}} E_P)^{-1}). \tag{B.33}$$

We point out that this statement is much stronger than the equality of the operators $U\Lambda U^{-1}$ and Λ on the macroscopic scale that was mentioned at the beginning of this paragraph. Namely, (B.33) shows that these operators coincide even microscopically, up to a very small error term. Notice that it was essential for the derivation that Λ is a scalar function (for example, (B.33) would in general be false if we replaced Λ by $\A$). Using (B.33) in each summand of (B.29) and applying (B.22), we conclude that

$$\begin{aligned}\Delta P[\partial\!\!\!/\Lambda](x,y) \;&=\; i\Lambda(x)\, P(x,y) \;-\; iP(x,y)\, \Lambda(y) \\ &\quad + (\text{higher orders in } (l_{\text{macro}} E_P)^{-1}).\end{aligned} \tag{B.34}$$

This shows gauge symmetry of the perturbation calculation (B.25).

It is interesting that, according to (B.34), gauge symmetry holds only up to an error term. This is unproblematic as long as the length scales of macroscopic physics are large compared to the Planck length. But (B.34) indicates that the regularized causal perturbation theory fails when energy or momentum of the perturbation $\mathcal{B}$ are of the order of the

Planck energy. In this case, the distinction between the "macroscopic" and "microscopic" length scales, on which our constructions relied from the very beginning (cf. (B.3)), can no longer be made, and it becomes impossible to introduce a causal and gauge invariant perturbation theory.

We conclude the discussion of the regularized causal perturbation expansion by pointing out that our construction was based on condition (B.17), which is only a necessary condition for causality. Hence the causality of (B.25) has not yet been proved. We shall now perform the light-cone expansion of (B.25). This will show explicitly that the light-cone expansion involves, to leading orders in $(l_{\text{macro}} E_P)^{-1}$ and $(l E_P)^{-1}$, no unbounded line integrals, thereby establishing causality in the sense of [4].

In the remainder of this appendix, we will analyze the regularized causal perturbation calculation (B.25) near the light cone. Our method is to first perform the light-cone expansion of ΔQ, and then to transform the resulting formulas according to (B.24) to finally obtain the light-cone expansion of ΔP. In preparation, we describe how a decomposition into Dirac eigenstates can be used for an analysis of the operator Q near the light cone. A short computation using (B.21) and (B.28) yields that $\tilde{Q}$ can be represented in the form

$$\tilde{Q}(k) \;=\; \int_{-\infty}^{\infty} d\mu\, w_\mu(\vec{k})\, \epsilon(\mu)\, (\not{k} + \mu)\, \delta(k^2 - \mu^2)\, \Theta(-k^0) \tag{B.35}$$

with the real-valued distribution

$$w_\mu(\vec{k}) = (\phi(k) + \mu\, \lambda(k))\, f(k) \qquad \text{and} \qquad k(\vec{k}) = (-\sqrt{|\vec{k}|^2 + \mu^2}, \vec{k}). \tag{B.36}$$

This representation can be understood as follows. According to (B.9), the distributions $\epsilon(\mu)\, (\not{k} + \mu)\, \delta(k^2 - \mu^2)$ in the integrand of (B.35) are the spectral projectors of the free Dirac operator in momentum space. The factor $\Theta(-k^0)$ projects out all states on the upper mass cone, and the function $w_\mu(\vec{k})$ multiplies the states on the lower mass shell $k = (-\sqrt{|\vec{k}|^2 + \mu^2}, \vec{k})$ with a scalar weight factor. In this sense, (B.35) can be regarded as the spectral decomposition of the operator Q into Dirac eigenstates. Notice that the factor $\delta(k^2 - \mu^2)\, \Theta(-k^0)$ in (B.35) is the Fourier transform of the distribution T_a, (2.4). Exactly as described for the scalar component in Subsection 2.4, we are here interested only in the regularization effects for large energy or momentum and may thus disregard the logarithmic mass problem (see [5, 6] for details). Therefore, we "regularize" T_a according to (2.9) and consider instead of (B.35) the operator

$$\tilde{Q}^{\text{reg}}(k) \;:=\; \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, w_\mu(\vec{k})\, (\not{k} + \mu)\, T^{\text{reg}}_{\mu^2}(k) \quad ,$$

where $T^{\text{reg}}_a(k)$ is the Fourier transform of (2.9). We expand the distribution $T^{\text{reg}}_{\mu^2}$ in a power series in μ^2,

$$\tilde{Q}^{\text{reg}}(k) \;=\; \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, w_\mu(\vec{k})\, (\not{k} + \mu) \sum_{n=0}^{\infty} \frac{1}{n!}\, T^{\text{reg}\,(n)}(k)\, \mu^{2n} \quad ,$$

where $T^{\text{reg}\,(n)}$ is given by (2.8). Commuting the integral and the sum, we obtain

$$\tilde{Q}^{\text{reg}}(k) \;=\; 32\pi^3 \sum_{n=0}^{\infty} \frac{1}{n!} \Big(g_{[n]}(\vec{k})\, \not{k} \;+\; h_{[n]}(\vec{k}) \Big)\, T^{\text{reg}\,(n)}(k) \tag{B.37}$$

with

$$g_{[n]}(\vec{k}) = \frac{1}{32\pi^3} \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, w_\mu(\vec{k})\, \mu^{2n} \tag{B.38}$$

$$h_{[n]}(\vec{k}) = \frac{1}{32\pi^3} \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, w_\mu(\vec{k})\, \mu^{2n+1} \quad . \tag{B.39}$$

The representation (B.37) is very useful because it reveals the behavior of the operator Q near the light cone. To see this, we consider the Fourier transform of (B.37) in light-cone coordinates (s, l, x_2, x_3). For the Fourier transform of the factor $T^{\text{reg}\,(n)}(k)$, we have the representation (2.75). This representation can immediately be extended to the Fourier transform of $\not{k}\, T^{\text{reg}\,(n)}(k)$ by acting on (2.75) with the differential operator $i\not{\partial}$; more precisely in light-cone coordinates $y - x = (s, l, x_2, x_3)$,

$$\begin{aligned} &\int \frac{d^4k}{(2\pi)^4}\, \not{k}\, T^{\text{reg}\,(n)}(k)\, e^{-ik(x-y)} \\ &= -\frac{1}{32\pi^3}\, (-il)^{n-2} \int_0^\infty \left[il\, \gamma^s \left(\frac{1}{u^{n-1}}\right)^{\text{reg}} - (n-1)\, \gamma^l \left(\frac{1}{u^n}\right)^{\text{reg}} \right] e^{-ius} \quad . \end{aligned} \tag{B.40}$$

In order to treat the factors $g_{[n]}$ and $h_{[n]}$ in (B.37), we note that the Fourier transform of (B.37) can be computed similar as described in Subsection 2.4 by integrating out the transversal momenta according to (2.19) and analyzing the remaining two-dimensional Fourier integral (2.22) with the integration-by-parts method (2.29). If this is done, the functions $g_{[n]}$ and $h_{[n]}$ appear in the integrand of (2.22). Our regularity assumption on the fermionic projector of the vacuum (see Subsections 2.4 and 2.5) imply that $g_{[n]}$ and $h_{[n]}$ are smooth functions, whose partial derivatives scale in powers of E_P^{-1}. Hence all derivative terms of the functions $g_{[n]}$ and $h_{[n]}$ which arise in the integration-by-parts procedure (2.29) are of higher order in $(lE_P)^{-1}$. Taking into account only the leading order in $(lE_P)^{-1}$, we thus obtain a representation of the fermionic projector of the vacuum involving only $g_{[n]}$ and $h_{[n]}$ at the boundary $v = \alpha_u$. Comparing this representation with (2.75) and (B.40), we conclude that the Fourier transform of (B.37) is obtained, to leading order in $(lE_P)^{-1}$, simply by inserting the functions $g_{[n]}$ and $h_{[n]}$ into the integrands of (2.75) and (B.40), evaluated along the line $\vec{k} = (k_x = 2u, k_y = 0, k_z = 0)$. Thus

$$\begin{aligned} Q^{\text{reg}}(s,l) &= -\sum_{n=0}^\infty \frac{1}{n!}\, (-il)^{n-1} \int_0^\infty \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius}\, h_{[n]}(u)\, du \\ &\quad - \sum_{n=0}^\infty \frac{1}{n!}\, (-il)^{n-2} \int_0^\infty \left[il\, \gamma^s \left(\frac{1}{u^{n-1}}\right)^{\text{reg}} - (n-1)\, \gamma^l \left(\frac{1}{u^n}\right)^{\text{reg}} \right] e^{-ius}\, g_{[n]}(u)\, du \\ &\quad + (\text{higher orders in } (lE_P)^{-1}), \end{aligned} \tag{B.41}$$

where $h_{[n]}(u)$ and $g_{[n]}(u)$ are the functions (B.38) and (B.39) with $\vec{k} = (-2u, 0, 0)$.

The decomposition of the operator Q into Dirac eigenstates (B.35) is also useful for analyzing its perturbation ΔQ.

Lemma B.1 *Let $B(x) \in C^2(\mathbb{R}^4) \cap L^1(\mathbb{R}^4)$ be a matrix potential which decays so fast at infinity that the functions $x_i \mathcal{B}(x)$ and $x_i x_j \mathcal{B}(x)$ are also L^1. Then the light-cone expansion of the operator $\Delta Q[\mathcal{B}]$, (B.23), is obtained by regularizing the light-cone expansion of the Dirac sea to first order in the external potential [5] as follows. A summand of the light-cone expansion of the Dirac sea which is proportional to m^p,*

$$m^p \text{ (iterated line integrals in bosonic potentials and fields) } T^{\text{reg}\,(n)}(s,l) \quad ,$$

must be replaced by

$$(-1)\ \textit{(iterated line integrals in bosonic potentials and fields)}$$
$$\times (-il)^{n-1} \int_{-\infty}^{\infty} du \left(\frac{1}{u^n}\right)^{reg} e^{-ius} \times \begin{cases} h_{[\frac{p-1}{2}]} & \textit{for } p \textit{ odd} \\ g_{[\frac{p}{2}]} & \textit{for } p \textit{ even} \end{cases}$$
$$+ \textit{(rapid decay in } l) + \textit{(higher orders in } (lE_P)^{-1}, (l_{macro} E_P)^{-1}). \qquad \text{(B.42)}$$

A contribution $\sim m^p$ *which contains a factor* $(y-x)_j \gamma^j$,

$$m^p \textit{ (iterated line integrals in bosonic potentials and fields) } (y-x)_j \gamma^j\, T^{reg\,(n)}(s,l) \quad ,$$

is to be replaced by

$$(-1)\ \textit{(iterated line integrals in bosonic potentials and fields)}$$
$$\times (-il)^{n-1} \int_{-\infty}^{\infty} du \left[2l\, \gamma^s \left(\frac{1}{u^n}\right)^{reg} + 2in\, \gamma^l \left(\frac{1}{u^{n+1}}\right)^{reg} \right]$$
$$\times e^{-ius} \times \begin{cases} h_{[\frac{p-1}{2}]} & \textit{for } p \textit{ odd} \\ g_{[\frac{p}{2}]} & \textit{for } p \textit{ even} \end{cases} + \textit{(contributions} \sim \gamma^2, \gamma^3)$$
$$+ \textit{(rapid decay in } l) + \textit{(higher orders in } (lE_P)^{-1}, (l_{macro} E_P)^{-1}) \quad . \qquad \text{(B.43)}$$

In these formulas, $g_{[n]}$ *and* $h_{[n]}$ *are the functions (B.38),(B.39) with* $\vec{k} = (-2u, 0, 0)$.

Proof: By substituting (B.6) and (B.35) into (B.23), we obtain the following representation for ΔQ in momentum space,

$$\begin{aligned} \Delta Q[\mathcal{B}] \left(k + \frac{q}{2},\, k - \frac{q}{2}\right) &= -\int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, (\not{k} + \frac{\not{q}}{2} + \mu)\, \mathcal{B}_q\, (\not{k} - \frac{\not{q}}{2} + \mu) \\ &\quad \times \left(\frac{\mathrm{PP}}{(k+\frac{q}{2})^2 - \mu^2}\, w_\mu(\vec{k} - \frac{\vec{q}}{2})\, T_{\mu^2}(k - \frac{q}{2}) + w_\mu(\vec{k} + \frac{\vec{q}}{2})\, T_{\mu^2}(k + \frac{q}{2})\, \frac{\mathrm{PP}}{(k-\frac{q}{2})^2 - \mu^2} \right) \\ &= \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, (\not{k} + \frac{\not{q}}{2} + \mu)\, \mathcal{B}_q\, (\not{k} - \frac{\not{q}}{2} + \mu) \\ &\qquad \times \frac{\mathrm{PP}}{2kq} \left(w_\mu(\vec{k} + \frac{\vec{q}}{2})\, T_{\mu^2}(k + \frac{q}{2}) - w_\mu(\vec{k} - \frac{\vec{q}}{2})\, T_{\mu^2}(k - \frac{q}{2}) \right) \quad . \end{aligned} \qquad \text{(B.44)}$$

Since we are here interested in the regularization effects for large energy or momentum, we may disregard the logarithmic mass problem and work on the level of the formal light-cone expansion of [5, Section 3] (our constructions could be made rigorous using the resummation method of [5, Section 4]). As in [5, Section 3], we expand the distributions T_{μ^2} in a Taylor series in q and rewrite the resulting k-derivatives as derivatives with respect to μ^2. This gives

$$T_{\mu^2}(k \pm \frac{q}{2}) = \sum_{l,r=0}^{\infty} c_{lr}\, (\pm kq)^l \left(\frac{q^2}{4}\right)^r T_{\mu^2}^{(l+r)}(k) \qquad \text{(B.45)}$$

with combinatorial factors c_{lr} whose detailed form is not needed in what follows. Next, we expand (B.45) in a Taylor series in μ^2 and obtain

$$T_{\mu^2}(k \pm \frac{q}{2}) = \sum_{n,l,r=0}^{\infty} c_{nlr}\, \mu^{2n}\, (\pm kq)^l \left(\frac{q^2}{4}\right)^r T^{(n+l+r)}(k) \qquad \text{(B.46)}$$

with new combinatorial factors c_{nlr}. We substitute the expansions (B.46) into (B.44) and write the even and odd terms in kq together,

$$\begin{aligned}\Delta Q[\mathcal{B}]\left(k+\frac{q}{2},\, k-\frac{q}{2}\right) \;&=\; -\int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, (\not k+\frac{\not q}{2}+\mu)\, \mathcal{B}_q\, (\not k-\frac{\not q}{2}+\mu) \\ &\times \left(\frac{\mathrm{PP}}{2kq} \sum_{n,l,r=0,\, l \text{ even}}^{\infty} c_{nlr}\, \mu^{2n}\, (kq)^l \left(\frac{q^2}{4}\right)^r T^{(n+l+r)}(k) \left(w_\mu(\vec k+\frac{\vec q}{2}) - w_\mu(\vec k-\frac{\vec q}{2})\right) \right. \\ &\left. + \frac{\mathrm{PP}}{2kq} \sum_{n,l,r=0,\, l \text{ odd}}^{\infty} c_{nlr}\, \mu^{2n}\, (kq)^l \left(\frac{q^2}{4}\right)^r T^{(n+l+r)}(k) \left(w_\mu(\vec k+\frac{\vec q}{2}) + w_\mu(\vec k-\frac{\vec q}{2})\right) \right). \quad \text{(B.47)}\end{aligned}$$

We first consider the contributions to (B.47) for even l. These terms contain the factor $(w_\mu(\vec k+\frac{\vec q}{2}) - w_\mu(\vec k-\frac{\vec q}{2}))$. If the distribution w_μ were a smooth function and its derivatives had the natural scaling behavior in powers of the Planck length, we could immediately conclude that $|w_\mu(\vec k+\frac{\vec q}{2}) - w_\mu(\vec k-\frac{\vec q}{2})| \sim |\vec q|\, |\partial w_\mu| \sim (l_{\text{macro}} E_P)^{-1}$, and thus all the terms for even l would be negligible. Unfortunately, the situation is more difficult because w_μ is in general not a smooth function (cf. (B.36)), and we obtain the desired regularity in $\vec k$ only after the μ-integration has been carried out. This makes it necessary to use the following argument. Consider one summand in (B.47) for even l. After carrying out the μ-integration, this summand yields a finite number of contributions to $\Delta Q(k+\frac{q}{2}, k-\frac{q}{2})$ of the following form,

$$\frac{\mathrm{PP}}{kq}\, (kq)^l \left(\frac{q^2}{4}\right)^r \cdots\, \mathcal{B}_q\, \cdots T^{(n+l+r)}(k)\, \left(g(\vec k+\frac{\vec q}{2}) - g(\vec k+\frac{\vec q}{2})\right) \quad , \qquad \text{(B.48)}$$

where each symbol "$\cdots$" stands for a possible factor $\not k$ or $\not q$, and where g is a scalar function, which coincides with one of the functions $g_{[n]}$ or $h_{[n]}$ (see (B.38) and (B.39)). As already mentioned after (B.40), our regularity assumptions on the fermionic projector of the vacuum imply that the functions $g_{[n]}$ and $h_{[n]}$, and thus also g, are smooth, and that their derivatives scale in powers of the Planck length. We now transform (B.48) to position space. Our regularity assumptions on $\mathcal{B}$ mean in momentum space that $\mathcal{B}(q) \in C^2 \cap L^\infty$. As a consequence, we can carry out the q-integration in the Fourier integral and obtain a contribution to $\Delta Q(x,y)$ of the form

$$\int \frac{d^4k}{(2\pi)^4}\, T^{(n+l+r)}(k)\, \left[F(k, x+y, \vec q) - F(k, x+y, -\vec q)\right]\, e^{-ik(x-y)} \qquad \text{(B.49)}$$

with a (matrix-valued) function F which is differentiable in $\vec q$ and whose $\vec q$-derivative is of the order E_P^{-1}. In the low-energy region, the function g in (B.48) is constant and thus F is homogeneous in k of degree at most $l+1$. After transforming to light-cone coordinates, this implies that (B.49) is close to the light cone dominated by the fermionic projector of the vacuum (i.e. in light-cone coordinates, $|(B.49)| \leq \text{const}(l)\, |P(s,l)|$). The mean value theorem yields that the square bracket in (B.49) is of the order $(l_{\text{macro}} E_P)^{-1}$, and we conclude that all summands in (B.47) for even l are of higher order in $(l_{\text{macro}} E_P)^{-1}$.

It remains to consider the summands in (B.47) for odd l. In this case, one factor kq cancels the principal value, and we obtain

$$\Delta Q[\mathcal{B}]\left(k+\frac{q}{2},\, k-\frac{q}{2}\right) \;=\; -\int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, (\not k+\frac{\not q}{2}+\mu)\, \mathcal{B}_q\, (\not k-\frac{\not q}{2}+\mu)$$

$$\times \sum_{n,l,r=0}^{\infty} C_{nlr}\, \mu^{2n}\, (kq)^{2l} \left(\frac{q^2}{4}\right)^r T^{(n+2l+1+r)}(k) \left(w_\mu(\vec{k}+\frac{\vec{q}}{2}) + w_\mu(\vec{k}-\frac{\vec{q}}{2})\right)$$
$$+(\text{higher orders in } (l_{\text{macro}} E_P)^{-1}) \tag{B.50}$$

with some combinatorial factors C_{nlr}. This formula has similarities to the light-cone expansion of the Dirac sea in momentum space [5, equation (3.15)]. In [5, Section 3], we proceeded by rewriting the factors kq as k-derivatives acting on $T^{(.)}$. When taking the Fourier transform, these k-derivatives were integrated by parts onto the exponential factor $\exp(-ik(x-y))$ to yield factors $(y-x)$. After collecting and rearranging all resulting terms, we obtained the line-integrals of the light-cone expansion. This method can be applied also to the integrand of (B.50), and we can carry out the μ-integration afterwards. We shall not go through all these constructions steps in detail here, but merely consider what happens in principle. Whenever a k-derivative ∂_{k^j} acts on the factors w_μ in the integration-by-parts procedure, we get instead of a factor $(y-x)_j\, w_\mu$ (which is obtained when the k-derivative acts on the exponential $\exp(-ik(x-y))$) a factor $\partial_j w_\mu$. After carrying out the μ-integration, one sees that the resulting term is of higher order in $(lE_P)^{-1}$. Thus we can, to leading order in $(lE_P)^{-1}$, neglect all derivatives of the factors w_μ. But then, the integration-by-parts procedure reduces to the construction in [5, Section 3], and we thus obtain precisely the line integrals of the light-cone expansion [5]. Furthermore, we can replace the factor $(w_\mu(\vec{k}+\frac{\vec{q}}{2}) + w_\mu(\vec{k}-\frac{\vec{q}}{2}))$ in (B.50) by $2w_\mu(\vec{k})$, because a Taylor expansion of this factor around $\vec{q}=0$ amounts, again after carrying out the μ-integration, to an expansion in powers of $(l_{\text{macro}} E_P)^{-1}$, and it thus suffices to take into account the leading term of this expansion. These considerations show that the light-cone expansion of (B.50) differs from that in [5] merely by the additional μ-integration and the factor $w_\mu(\vec{k})$. Hence the light-cone expansion of (B.50) is obtained from that of the Dirac sea by the following replacements,

$$m^p\, T^{(n)}(x,y) \;\to\; \int \frac{d^4k}{(2\pi)^4} \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, \mu^p\, T^{(n)}(k)\, e^{-ik(x-y)}\, w_\mu(\vec{k})$$
$$m^p\, (y-x)_j \gamma^j\, T^{(n)}(x,y) \;\to\; \int \frac{d^4k}{(2\pi)^4} \int_{-\infty}^{\infty} d\mu\, \epsilon(\mu)\, \mu^p\, (-2i\not{k})\, T^{(n+1)}(k)\, e^{-ik(x-y)}\, w_\mu(\vec{k})$$

(where we used that $(y-x)^j T^{(n)}(x,y) = 2\partial_{x^j} T^{(n+1)}(x,y)$, see [5, equation (3.5)]). The lemma follows by carrying out the μ-integrals applying (B.38),(B.39), and by analyzing the behavior near the light cone as explained before (B.41). ∎

From this lemma, we can deduce the light-cone expansion of the regularized fermionic projector.

Theorem B.2 *The light-cone expansion of the regularized causal perturbation (B.25) is obtained by regularizing the light-cone expansion of the Dirac sea to first order in the external potential [5] as follows. A summand of the light-cone expansion of the Dirac sea which is proportional to m^p,*

$$m^p \text{ (iterated line integrals in bosonic potentials and fields) } T^{reg\,(n)}(s,l) \quad ,$$

must be replaced by (2.79). A contribution $\sim m^p$ which contains a factor $(y-x)_j \gamma^j$,

$$m^p \text{ (iterated line integrals in bosonic potentials and fields) } (y-x)_j \gamma^j\, T^{reg\,(n)}(s,l) \quad ,$$

is to be replaced by (2.81). In these formulas, g, h, a, and b are the regularization functions introduced in Subsections 2.4 and 2.5 (see (2.72), (2.40), (2.82), and (2.64)).

Proof: As mentioned at the beginning of this appendix, we assume here that the vector component is null on the light cone (2.66). Let us consider what this condition tells us about the operators U_l. According to (B.19), the operators $\tilde{U}_l$ are trivial in the low-energy region. Conversely, for large energy or momentum, (2.66) yields that the vector field $v(k)$ is parallel to k, up to a perturbation of the order $\varepsilon_{\text{shear}}$. Hence we can assume that the transformation (B.22) is a small perturbation of the identity, in the sense that

$$c_l \,|\tilde{U}_l(k) - \mathbb{1}| \;\sim\; \varepsilon_{\text{shear}} \qquad \text{for all } k. \tag{B.51}$$

We next derive the light-cone expansion of ΔP by transforming the result of Lemma B.1 according to (B.24). Since the transformation (B.24) is small in the sense of (B.51), it leaves the iterated line integrals in (B.42) and (B.43) unchanged to leading order in $\varepsilon_{\text{shear}}$. Hence it suffices to consider the transformation of the u-integrals in (B.42) and (B.43). The u-integral in (B.42) is as a homogeneous scalar operator invariant under the unitary transformations. In the u-integral in (B.43), on the other hand, only the Dirac matrices γ^l and γ^s are modified. More precisely, we have to leading order in $\varepsilon_{\text{shear}}$,

$$\begin{aligned}
\sum_{l=1}^{L} c_l \,(\tilde{U}_l \gamma^s \tilde{U}_l^{-1})(u, v=\alpha_u) &= \gamma^s + \frac{b_1(u)}{u^2}\,\gamma^l + (\text{contributions} \sim \gamma^2, \gamma^3) \\
\sum_{l=1}^{L} c_l \,(\tilde{U}_l \gamma^l \tilde{U}_l^{-1})(u, v=\alpha_u) &= \gamma^l + \frac{b_2(u)}{u^2}\,\gamma^s + (\text{contributions} \sim \gamma^2, \gamma^3)
\end{aligned}$$

with suitable regularization functions b_s and b_l which are small in the following sense,

$$\frac{b_{1/2}(u)}{u^2} \;\sim\; \varepsilon_{\text{shear}} \quad .$$

Notice that in the high-energy region $u \sim E_P$, the contribution $\sim \gamma^l$ in the integrand of (B.43) is smaller than the contribution $\sim \gamma^s$ by a relative factor of $(lE_P)^{-1}$. Hence we can neglect b_2, whereas b_1 must be taken into account. We conclude that the transformation (B.24) of the contributions (B.42) and (B.43) is carried out simply by the replacement

$$\gamma^s \;\to\; \gamma^s \;+\; \frac{b_1(u)}{u^2}\,\gamma^l \quad . \tag{B.52}$$

It remains to derive relations between the regularization functions $g_{[n]}$, $h_{[n]}$, and b_s, which appear in the transformed contributions (B.42) and (B.43), and the regularization functions g, h, a, and b in (2.79) and (2.81). For this, we apply the transformation (B.21) to Q^{reg}, (B.41). Exactly as described above, this transformation reduces to the replacement (B.52), and we obtain the following expansion of the fermionic projector near the light cone,

$$\begin{aligned}
P^{\text{reg}}(s,l) \;=\; & -\sum_{n=0}^{\infty} \frac{1}{n!}\,(-il)^{n-1} \int_0^\infty \left(\frac{1}{u^n}\right)^{\text{reg}} e^{-ius}\, h_{[n]}(u)\, du \\
& - \sum_{n=0}^{\infty} \frac{1}{n!}\,(-il)^{n-2} \int_0^\infty \left[il\,\gamma^s \left(\frac{1}{u^{n-1}}\right)^{\text{reg}} - (n-1)\,\gamma^l \left(\frac{1}{u^n}\right)^{\text{reg}} + il\,\gamma^s\, b(u) \left(\frac{1}{u^{n+1}}\right)^{\text{reg}} \right] \\
& \qquad \times e^{-ius}\, g_{[n]}(u)\, du \;+\; (\text{higher orders in } \varepsilon_{\text{shear}},\, (lE_P)^{-1}).
\end{aligned}$$

Comparing this result with the formulas for the fermionic projector derived in Subsections 2.4 and 2.5 (see (2.42),(2.43) and (2.51),(2.52)), one gets the following identities between the regularization functions,

$$g_{[n]}(u) \;=\; g(u)\, a(u)^n \quad , \qquad h_{[n]}(u) \;=\; h(u)\, a(u)^n \quad , \qquad b_1(u) \;=\; b(u) \quad . \qquad \blacksquare$$

We finally explain in which sense the regularized causal perturbation theory is unique. In order to ensure regularity of the perturbation theory in the limit when the momentum q of the external field goes to zero, one must satisfy a causality condition similar to (B.17), and to this end has to work with a modified fermionic projector Q. Since we must modify the direction of the vector field v, it is natural to describe the transformation from Q to P by linear combinations of unitary transformations (B.22). Nevertheless, we remark that one could just as well work with a different or more general transformation $Q \to P$. The reason is that the particular form of this transformation enters only in the proof of Theorem B.2, and we use merely that this transformation is close to the identity, in the sense similar to (B.51). Hence the restriction to transformations of type (B.22) is no loss in generality. Furthermore, we point out that the gauge symmetry (B.34) uniquely determines the precise form of how the potential $\mathcal{B}$ enters into the perturbation calculation (e.g. one may not replace $\mathcal{B}$ in (B.25) by $U_l^{-1} \mathcal{B} U_l$). We conclude that our construction of the regularized causal perturbation theory is canonical up to the freedom in choosing the coefficients $c_l(k)$ and the unitary transformations $\tilde{U}_l(k)$. By assuming regularity (B.31) and the bound (B.51), the arbitrariness in choosing $(c_l, \tilde{U}_l)$ was constrained so much that it has no influence on the regularization of the light-cone expansion. Indeed, the c_l and $\tilde{U}_l$ do not enter the statement of Theorem B.2. Thus we can say that the regularized causal perturbation expansion is unique up to contributions of higher order in $(lE_P)^{-1}$, $(l_{\text{macro}} E_P)^{-1}$, and $\varepsilon_{\text{shear}}$.

C Linear Independence of the Basic Monomials

In this appendix, we shall prove the following theorem.

Theorem C.1 *We consider for given p and q, $p \neq q$, the basic monomials (2.106) or (2.107) and evaluate them according to (2.96) weakly near the light cone, to leading order in $(lE_P)^{-1}$ and $(l_{macro} E_P)^{-1}$. If the weak integral over a linear combination of the basic monomials vanishes for every choice of η and the regularization functions, then the linear combination is trivial.*

Before coming to the proof, we make a few remarks. Notice that we consider only the case $p \neq q$ where the number of factors $T_\bullet^{(a_j)}$ and $\overline{T_\bullet^{(b_k)}}$ are different. Indeed, this is the case relevant for our applications. The case $p = q$ is a bit more complicated because the monomials are then (up to a sign) invariant under the transformation $T_\bullet^{(a_j)} \leftrightarrow \overline{T_\bullet^{(b_j)}}$, which exchanges the first p and the last p factors of the monomial, as one sees by taking the complex conjugate of (2.101). But taking into account this additional symmetry, the statement of our theorem and its proof could be extended immediately.

We point out that Theorem C.1 does not imply that the basic monomials are independent in the sense that, by choosing suitable regularization functions, the basic regularization parameters can be given arbitrary values. Theorem C.1 states that there are no identities between the basic monomials, but the basic regularization parameters might

nevertheless be constrained by inequalities between them (e.g., certain regularization parameters might be always positive). Furthermore, we remind that the assumption of half occupied surface states (cf. the last paragraph of Subsection 2.5) yields the relation (2.73) between the regularization functions, which might also give constraints for the regularization parameters. For these reasons, one should in applications always verify that the values for the basic regularization parameters obtained in the effective continuum theory can actually be realized by suitable regularization functions.

Proof of Theorem C.1: The proof is organized as follows. We first calculate for a given monomial the weak integral (2.96) to leading order in $(lE_P)^{-1}$ and $(l_{\text{macro}}E_P)^{-1}$, choosing a class of regularization functions which is particularly easy to handle. By analyzing the dependence on the regularization, we shall find a procedure for reconstructing the indices a_j and b_k of the basic monomial from the weak integral. Then we will generalize this construction, with the goal of determining also the lower indices "$\bullet$" of the monomial. We conclude the proof indirectly. Namely, assuming a non-trivial linear combination of basic monomials which vanishes independent of the regularization, we show that at least one coefficient of the linear combination must be zero, giving a contradiction.

Before beginning, we reduce the problem to the case $p > q$ as follows. If $p < q$, we take the complex conjugate of the basic monomials (2.106) or (2.107) and revert the roles of p and q. This yields the monomials

$$T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T^{(0)}} \qquad \text{with} \qquad a_1 \leq \cdots \leq a_p \tag{C.1}$$

and

$$T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}} \quad \text{with} \quad a_1 \leq \cdots \leq a_p \text{ and } b_1 = b_2 \leq \cdots \leq b_q \quad , \tag{C.2}$$

respectively. By iteratively applying the integration-by-parts rule (2.105), the monomials (C.1) and (C.2) can be transformed into linear combinations of the monomials (2.107) with $p > q$. As is easily verified by going through the combinatorics, this transformation yields a one-to-one mapping between linear combinations of (C.1) or (C.2) and linear combinations of the corresponding basic monomials (2.107). Hence the linear independence of the basic monomials (2.107) implies the linear independence of both (C.1) and (C.2).

We consider for given integers $n_1, \ldots, n_p, m_1, \ldots m_q \geq 0$ the following integral over a product,

$$E \;:=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} f^{(n_1)} \cdots f^{(n_p)}\, \overline{f^{(m_1)}} \cdots \overline{f^{(m_q)}}\, ds \quad , \tag{C.3}$$

where the functions $f^{(n)}$ are defined as the Fourier integrals

$$f^{(n)}(s) \;=\; \int_0^\infty u^n\, e^{-\frac{u}{2E_P}}\, e^{-ius}\, du \quad , \tag{C.4}$$

and $\overline{f^{(n)}}$ is the complex conjugate of $f^{(n)}$. Exactly as required for the Fourier integrals (2.84) and (2.86), the integrand in (C.4) decays on the Planck scale $u \sim E_P$. Working with an exponential decay is particularly convenient because, after substituting (C.4) into (C.3) and carrying out the s-integration, all exponential factors can be taken out of the resulting convolution integrals. More precisely, we obtain

$$E \;=\; \int_0^\infty (p^{(n_1)} * \cdots * p^{(n_p)})(u)\; (p^{(m_1)} * \cdots * p^{(m_q)})(u)\, e^{-\frac{u}{E_P}}\, du \quad , \tag{C.5}$$

where $p^{(n)}$ is the polynomial $p^{(n)}(u) = u^n$, and the convolution "$*$" is defined by

$$(f * g)(u) \;=\; \int_0^u f(v)\, g(u-v)\, dv \quad . \tag{C.6}$$

The convolution of two polynomials is computed using integration by parts to be

$$\begin{aligned}(p^{(n_1)} * p^{(n_2)})(u) \;&=\; u^{n_1+n_2+1} \int_0^1 t^{n_1}\,(1-t)^{n_2}\, dt \\ &=\; u^{n_1+n_2+1}\, \frac{n_2}{n_1+1} \int_0^1 t^{n_1+1}\,(1-t)^{n_2-1}\, dt \;=\; \cdots \;=\; \frac{n_1!\, n_2!}{(n_1+n_2+1)!}\, p^{(n_1+n_2+1)}(u) \quad .\end{aligned}$$

By iteratively applying this identity in (C.5), we can calculate all convolutions and obtain

$$E \;=\; \frac{n_1! \cdots n_p!}{(N+p-1)!}\, \frac{m_1! \cdots m_q!}{(M+q-1)!} \int_0^\infty u^{N+M+p+q-2}\, e^{-\frac{u}{E_P}}\, du \quad ,$$

with

$$N \;=\; \sum_{j=1}^{p} n_j \qquad \text{and} \qquad M \;=\; \sum_{k=1}^{q} m_k \quad .$$

We finally carry out the remaining u-integral to conclude that

$$E \;=\; \frac{n_1! \cdots n_p!}{(N+p-1)!}\, \frac{m_1! \cdots m_q!}{(M+q-1)!}\, (N+M+p+q-2)!\, E_P^{N+M+p+q-1} \quad . \tag{C.7}$$

Suppose that the weak integral (2.96) is given to leading order in $(l_{\text{macro}} E_P)^{-1}$ and $(l E_P)^{-1}$, for any choice of η and the regularization functions. We will now show how this information can be used to reconstruct the indices a_j and b_k of the monomial. First, we substitute the Fourier representations (2.84) and (2.86) into (2.96). Collecting the powers of l, one sees that (2.96) $\sim l^{-L}$ with L given by (2.99). We choose the regularization functions as

$$g(u) \;=\; h(u) \;=\; \left(1 + \sum_{j=1}^{p+q} \lambda_j\, u^{\alpha_j}\right) e^{-\frac{u}{2E_P}}\, \Theta(u) \quad , \qquad a(u) \;=\; b(u) \;=\; 1 \tag{C.8}$$

with real parameters λ_j and integers α_j with

$$\alpha_j \;>\; 2(p+q) \;-\; L \quad . \tag{C.9}$$

Furthermore, we choose the test function to be one on the light cone, $\eta(s=0) = 1$. Then, to leading order in $(l E_P)^{-1}$ and $(l_{\text{macro}} E_P)^{-1}$, the integral (2.96) is computed by evaluating η on the light cone and rewriting the products as convolutions in momentum space. More precisely,

$$\begin{aligned}&\int_{-\infty}^{\infty} ds\, \eta\, T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}} \\ &\quad=\; c\,(il)^{-L} \int_0^\infty (e^{(a_1)} * \cdots * e^{(a_p)})(u)\, (e^{(b_1)} * \cdots * e^{(b_q)})(u)\, e^{-\frac{u}{E_P}}\, du\end{aligned} \tag{C.10}$$

with a real constant $c = \pm 2\pi$ and the functions

$$e^{(n)}(u) \;=\; \left(1 + \sum_{j=1}^{p+q} \lambda_j\, u^{\alpha_j}\right) \left(\frac{1}{u^n}\right)^{\text{reg}} \tag{C.11}$$

("$*$" is again the convolution (C.6)). Notice that the expression (C.10) with $e^{(n)}$ according to (C.11) is a polynomial in the variables λ_j. The coefficients of this polynomial are sums of terms again of the form (C.10), whereby each factor $e^{(n)}$ is to be replaced by either $(u^{-n})^{\mathrm{reg}}$ or $u^{\alpha_j - n}$. We want to pick the term which involves no factors $(u^{-n})^{\mathrm{reg}}$ and contains each factor $u^{\alpha_j - n}$, $j = 1, \ldots, p+q$, exactly once, and thus consider the Taylor coefficient

$$\frac{\partial}{\partial\lambda_1} \cdots \frac{\partial}{\partial\lambda_{p+q}} \int_{-\infty}^{\infty} ds\, \eta\, T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}}\, {}_{|\lambda_1 = \cdots = \lambda_{p+q} = 0} \quad . \tag{C.12}$$

According to the bounds (C.9), both $u^{\alpha_i - a_j}$ and $u^{\alpha_i - b_k}$ are positive powers of u (for every $i = 1, \ldots, p+q$, $j = 1, \ldots, p$, and $k = 1, \ldots, q$). Hence the convolution integrals obtained when computing (C.12) are precisely of the form (C.5) and can be calculated according to (C.7). Since we are interested only in the combinatorics of the two fractions in (C.7), it is convenient to divide by the third and fourth factor in (C.7). This leads us to introduce the function F by

$$\begin{aligned} &F(\alpha_1, \ldots, \alpha_{p+q}) \\ &\quad = \frac{(il)^L}{c\, R!\, E_P^{R+1}} \frac{\partial}{\partial\lambda_1} \cdots \frac{\partial}{\partial\lambda_{p+q}} \int_{-\infty}^{\infty} ds\, \eta\, T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}}\, {}_{|\lambda_1 = \cdots = \lambda_{p+q} = 0} \end{aligned} \tag{C.13}$$

where c is the same constant as in (C.9) and

$$R \;=\; L - 2 + \sum_{j=1}^{p+q} \alpha_j \quad .$$

Carrying out the partial derivatives with the product rule and applying (C.7), we obtain

$$\begin{aligned} &F(\alpha_1, \ldots, \alpha_{p+q}) \\ &\quad = \sum_{\sigma \in S(p+q)} \frac{(\alpha_{\sigma(1)} - a_1)! \cdots (\alpha_{\sigma(p)} - a_p)!}{(p - 1 + \sum_{j=1}^{p} (\alpha_{\sigma(j)} - a_j))!} \, \frac{(\alpha_{\sigma(p+1)} - b_1)! \cdots (\alpha_{\sigma(p+q)} - b_q)!}{(q - 1 + \sum_{k=1}^{q} (\alpha_{\sigma(p+k)} - b_k))!} \end{aligned} , \tag{C.14}$$

where $S(p+q)$ is the set of all permutations of $\{1, \ldots, p+q\}$.

In order to reconstruct the parameters α_j and β_k from (C.14), we analyze the asymptotic behavior when the variables α_j tend to infinity, in successive order. We first consider the asymptotics when $\alpha_1 \to \infty$ for fixed $\alpha_2, \ldots, \alpha_{p+q}$. Using the formula

$$\frac{\alpha!}{(\alpha + n)!} \;=\; \frac{1}{\alpha^n} \left(1 + \mathcal{O}(\alpha^{-1})\right) \quad , \tag{C.15}$$

the summands of (C.14) are in the cases $\sigma^{-1}(1) \leq p$ or $\sigma^{-1}(1) > p$ to leading order in α_1 proportional to

$$\alpha_1^{1 - p - \sum_{j,\, \sigma(j) \neq 1} (\alpha_{\sigma(j)} - a_j)} \qquad \text{and} \qquad \alpha_1^{1 - q - \sum_{k,\, \sigma(p+k) \neq 1} (\alpha_{\sigma(p+k)} - b_k)} \quad , \tag{C.16}$$

respectively. Notice that the sum in the exponent on the left runs over $p-1$ values for j, whereas the corresponding sum on the right consists of $q-1$ summands. Since $p > q$ and the α_j are sufficiently large (C.9), we conclude that the power on the left of (C.16) is larger than the power on the right. Furthermore, the power of α_1 becomes maximal when the parameter $a_{\sigma^{-1}(1)}$, which does not appear in the sum on the left of (C.16), is minimal;

in other words when $a_{\sigma^{-1}(1)} = a_1$. Finally, the maximal power of α_1 is obtained for those permutations for which the variables $\alpha_{\sigma(j)}$ in the sum on the left of (C.16) are minimal. Therefore it is convenient to set $\beta_k = \alpha_{p+k}$, $k = 1, \ldots, q$, and to choose our variables like

$$\alpha_2, \ldots, \alpha_p \;<\; \beta_1, \ldots, \beta_q \quad .$$

Then to leading order in α_1, (C.14) simplifies to

$$\begin{aligned} C \sum_{\sigma \in S(\{2,\ldots,p\})} & \alpha_1^{1-p-\sum_{j=2}^{p}(\alpha_{\sigma(j)}-a_j)} \, (\alpha_{\sigma(2)} - a_2)! \cdots (\alpha_{\sigma(p)} - a_p)! \\ \times \sum_{\pi \in S(q)} & \frac{(\beta_{\pi(1)} - b_1)! \cdots (\beta_{\pi(q)} - b_q)!}{(q - 1 + \sum_{k=1}^{q}(\beta_{\pi(k)} - b_k))!} \quad , \end{aligned} \tag{C.17}$$

where $S(\{2, \ldots, p\})$ denotes the set of all permutations of $\{2, \ldots, p\}$, and C is a combinatorial factor which depends only on the parameters a_j and b_k, counting degeneracies among these parameters (the detailed form of C will not be needed here). We next consider the behavior of (C.17) when β_1 becomes large, for all other parameters fixed. Again using (C.15), we obtain to leading order in β_1 the asymptotic formula

$$\begin{aligned} C \sum_{\sigma \in S(\{2,\ldots,p\})} & \alpha_1^{1-p} \prod_{j=2}^{p} \alpha_1^{-(\alpha_{\sigma(j)}-a_j)} \, (\alpha_{\sigma(j)} - a_j)! \\ \times \sum_{\pi \in S(\{2,\ldots,q\})} & \beta_1^{1-q} \prod_{k=2}^{q} \beta_1^{-(\beta_{\pi(k)}-b_k)} \, (\beta_{\pi(k)} - b_k)! \end{aligned} \tag{C.18}$$

with a new combinatorial factor C. We divide (C.18) by the product of factorials

$$\mathcal{P} \;:=\; \left(\prod_{j=2}^{p} \alpha_j! \right) \left(\prod_{k=2}^{q} \beta_k! \right) \tag{C.19}$$

and consider the asymptotic behavior when the variables $\beta_2, \ldots, \beta_q, \alpha_2, \ldots, \alpha_p$ tend to infinity, in the order given. Applying (C.15) in each step and picking those permutations σ and π which yield the largest power of the corresponding variable, we obtain the expression

$$C \, \alpha_1^{1-p} \, \beta_1^{1-q} \left(\prod_{j=2}^{p} \alpha_1^{a_j - \alpha_j} \, \alpha_j^{-a_j} \right) \left(\prod_{k=2}^{q} \beta_1^{b_k - \beta_k} \, \beta_k^{-b_k} \right) \quad . \tag{C.20}$$

According to its construction, this is a formula for $F(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)/\mathcal{P}$ valid asymptotically if

$$\alpha_1 \gg \beta_1 \gg \cdots \gg \beta_q \gg \alpha_2 \gg \cdots \gg \alpha_p \gg a_j, b_k \quad . \tag{C.21}$$

The main simplification compared to (C.14) is that the sum over the permutations has now disappeared. By looking at the dependence of (C.20) on the variables $\alpha_2, \ldots, \alpha_p$ and $\beta_2, \ldots, \beta_q$, one can immediately determine the parameters $a_2, \ldots, a_p$ and $b_2, \ldots, b_q$, respectively. Since we are considering a basic monomial (2.107), the parameter a_1 coincides with a_2. We remind that the degree L of the monomial is known from the l-dependence of the weak integral (see before (C.8)). Thus we can apply (2.99) to determine b_1. This concludes our procedure for reconstructing the indices a_j and b_k of the monomial.

We now turn attention to the lower indices "$\bullet$" of the monomial. In order to distinguish between square brackets, $\bullet = [.]$, and curly brackets, $\bullet = \{.\}$, we modify the above construction as follows. We choose the regularization functions g, h, and a as in (C.8) and set

$$b(u) \;=\; 1 + \mu\, u^{\gamma} \tag{C.22}$$

with real parameters μ and γ, $0 < \gamma < 1$. We introduce the function H by adding to (C.13) a μ-derivative,

$$\begin{aligned} &H(\gamma; \alpha_1, \ldots, \alpha_{p+q}) \\ &\quad = \quad \frac{(il)^L}{c\, R!\, E_P^{R+1}}\, \frac{\partial}{\partial\mu} \frac{\partial}{\partial\lambda_1} \cdots \frac{\partial}{\partial\lambda_{p+q}} \int_{-\infty}^{\infty} ds\; \eta\, T_{\bullet}^{(a_1)} \cdots T_{\bullet}^{(a_p)}\, \overline{T_{\bullet}^{(b_1)}} \cdots \overline{T_{\bullet}^{(b_q)}}\,{}_{|\mu=\lambda_j=0} \end{aligned} \tag{C.23}$$

with

$$R \;=\; L - 2 + \gamma + \sum_{j=1}^{p+q} \alpha_j \quad .$$

Carrying out the μ-derivative with the Leibniz rule, one gets a sum of terms in which the μ-derivative acts always on one of the factors $T_{\{.\}}^{(a_j)}$ or $\overline{T_{\{.\}}^{(b_j)}}$. According to our ansatz (C.22), taking the μ-derivative of $T_{\{.\}}^{(n)}$ yields an additional factor u^{γ} in the Fourier integral (2.86). If combined with $(u^{-n})^{\mathrm{reg}}$, this additional factor amounts to the replacement $n \to n - \gamma$. Hence computing the products in (C.23) leads to convolutions of the form (C.5), with the only difference that the involved powers n_j and m_j are no longer integer, but positive reals. The formula (C.7) is still valid if one replaces the factorials by corresponding Gamma functions. In the asymptotic region (C.21), we can again expand similar to (C.20). More precisely, if the ν-derivative in (C.23) acts on a factor $T_{\{.\}}^{(a_i)}$ with $a_i \neq a_1$, then the resulting contribution to $H(\gamma; \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)/\mathcal{P}$ has in the region (C.21) the asymptotic form

$$n\, C\, \alpha_1^{1-p-\gamma}\, \beta_1^{1-q}\, \alpha_{i_0}^{\gamma} \left(\prod_{j=2}^{p} \alpha_1^{a_j - \alpha_j}\, \alpha_j^{-a_j} \right) \left(\prod_{k=2}^{q} \beta_1^{b_k - \beta_k}\, \beta_k^{-b_k} \right) \quad , \tag{C.24}$$

where i_0 is the smallest index with $a_{i_0} = a_i$, and n is a combinatorial factor counting the number of factors $T_{\{.\}}^{(a_j)}$ with $a_j = a_i$. On the other hand, if the μ-derivative in (C.23) acts on a factor $T_{\{.\}}^{(a_j)}$ with $a_i = a_1$, then $a_i - \gamma$ will be smaller than all a_j, $1 \leq j \leq p$, and thus the permutation leading to the largest power of α_1 in (C.16) will satisfy $\sigma^{-1}(1) = i$. Using furthermore that (C.14) depends on $\alpha_{\sigma(i)}$ and a_i only in the combination $\alpha_{\sigma(i)} - a_i$, the replacement $a_i \to a_i - \gamma$ can be stated equivalently as $\alpha_1 \to \alpha_1 + \gamma$. This explains why the resulting contribution to $H(\gamma; \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)/\mathcal{P}$ has in the region (C.21) asymptotically the form

$$n\, C\, (\alpha_1 + \gamma)^{1-p}\, \beta_1^{1-q} \left(\prod_{j=2}^{p} (\alpha_1 + \gamma)^{a_j - \alpha_j}\, \alpha_j^{-a_j} \right) \left(\prod_{k=2}^{q} \beta_1^{b_k - \beta_k}\, \beta_k^{-b_k} \right) \quad , \tag{C.25}$$

where n again counts the number of factors $T^{(a_j)}$ with $a_j = a_i$. If the μ-derivative acts on the factors $\overline{T_{\{.\}}^{(b_i)}}$, we have similar formulas; namely in the case $b_i \neq b_1$,

$$n\, C\, \alpha_1^{1-p}\, \beta_1^{1-q-\gamma} \left(\prod_{j=2}^{p} \alpha_1^{a_j - \alpha_j}\, \alpha_j^{-a_j} \right) \beta_{i_0}^{\gamma} \left(\prod_{k=2}^{q} \beta_1^{b_k - \beta_k}\, \beta_k^{-b_k} \right) \quad , \tag{C.26}$$

whereas for $a_i = a_1$,

$$n\, C\, \alpha_1^{1-p}\, (\beta_1+\gamma)^{1-q} \left(\prod_{j=2}^{p} \alpha_1^{a_j-\alpha_j}\, \alpha_j^{-a_j}\right) \left(\prod_{k=2}^{q} (\beta_1+\gamma)^{b_k-\beta_k}\, \beta_k^{-b_k}\right) \quad . \tag{C.27}$$

Here i_0 in (C.26) is the smallest index with $b_{i_0} = b_i$, and n again denotes combinatorial factors. The asymptotic formulas (C.24)–(C.27) are promising because the dependence on the parameters $\alpha_2, \ldots, \alpha_p, \beta_2, \ldots, \beta_q$, and γ allows us to distinguish between the terms of types (C.24)–(C.27) and makes it possible to determine the corresponding parameters i and n. For clarity, we point out that that the condition $\gamma < 1$ is needed in (C.24) to ensure the correct ordering of the parameters $a_1 \leq \ldots \leq a_{i_0-1} \leq a_{i_0} - \gamma \leq a_{i_0+1} \leq \ldots \leq a_p$ (choosing $\gamma > 1$ would make it necessary to reorder these parameters, making our construction more complicated). Similarly, (C.26) holds only if $\gamma < 1$. In (C.25) and (C.27), however, the condition $\gamma < 1$ can be dropped, because the required orderings $a_1 - \gamma \leq a_2 \leq \ldots \leq a_p$ and $b_1 - \gamma \leq b_2 \leq \ldots \leq b_q$ hold true even for large γ.

In order to get information also on the values r of the lower indices $\bullet = [r]$ or $\bullet = \{r\}$, we must extend our construction as follows. We choose the function g as in (C.8) and the other regularization functions according to

$$h(u) \;=\; \left(1+\sum_{i=1}^{\infty} \nu_i\, u^{\delta_i}\right) \left(1+\sum_{j=1}^{p+q} \lambda_j\, u^{\alpha_j}\right) e^{-\frac{u}{2E_P}}\, \Theta(u) \tag{C.28}$$

$$a(u) \;=\; \left(1+\sum_{i=1}^{\infty} \nu_i\, u^{\delta_i}\right)^2 \quad , \qquad b(u) \;=\; 1+\sum_{k=1}^{\infty} \mu_k\, u^{\gamma_k} \tag{C.29}$$

with λ_j, α_j as in (C.8), real parameters ν_i, γ_k, and positive parameters δ_i, γ_k. Generalizing both (C.13) and (C.23), we introduce for integers $A, B \geq 0$ the functions

$$\begin{aligned} &K_{AB}(\delta_1, \ldots, \delta_A, \gamma_1, \ldots, \gamma_B; \alpha_1, \ldots, \alpha_{p+q}) \\ &\quad = \frac{(il)^L}{cR!\, E_P^{R+1}} \left(\prod_{i=1}^{A} \frac{\partial}{\partial \nu_i}\right) \left(\prod_{k=1}^{B} \frac{\partial}{\partial \mu_k}\right) \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_{p+q}} \\ &\qquad\qquad \times \int_{-\infty}^{\infty} ds\; \eta\, T_\bullet^{(a_1)} \cdots T_\bullet^{(a_p)}\, \overline{T_\bullet^{(b_1)}} \cdots \overline{T_\bullet^{(b_q)}}\, {}_{|\nu_i = \mu_k = \lambda_j = 0} \end{aligned} \tag{C.30}$$

with

$$R \;=\; L - 2 + \sum_{i=1}^{A} \delta_i \;+\; \sum_{k=1}^{B} \gamma_k \;+\; \sum_{j=1}^{p+q} \alpha_j \quad .$$

After carrying out the derivatives with the Leibniz rule and expanding in the region (C.21), we obtain in generalization of (C.20) and (C.24)–(C.27) for $K_{AB}/\mathcal{P}$ a sum of terms of the asymptotic form

$$\begin{aligned} &C\, (\alpha_1+x_1)^{1-p}\, (\beta_1+y_1)^{1-q} \\ &\quad \times \left(\prod_{j=2}^{p} (\alpha_1+x_1)^{a_j-x_j-\alpha_j}\, \alpha_j^{-a_j+x_j}\right) \left(\prod_{k=2}^{q} (\beta_1+y_1)^{b_k-y_k-\beta_k}\, \beta_k^{-b_k+y_k}\right) \quad , \end{aligned} \tag{C.31}$$

where x_j is the sum of the parameters γ_k and δ_i corresponding to those derivatives ∂_{μ_k} and ∂_{ν_i} which act on the factor $T_\bullet^{(a_j)}$. Similarly, y_j encodes via a sum of the parameters

γ_k and δ_i which derivatives act on $\overline{T_\bullet^{(b_j)}}$. Generalizing our procedure in (C.24), where we introduced the index i_0, we here reorder those factors $T_\bullet^{(a_j)}$, for which the index a_j coincides, in such a way that the corresponding parameters x_j are decreasing. More precisely, after carrying out the derivatives in (C.30) with the Leibniz rule, we permute the factors $T_\bullet^{(a_j)}$ with the same upper index such that if $a_i = a_j$ for $i < j$, then $x_i \geq x_j$. The degeneracies among the parameters b_k are treated similarly. The combinatorics of all these permutations is taken into account by the prefactor C in (C.31). Finally, we point out that, similar as explained after (C.27), the asymptotic formula (C.31) is correct only if

$$x_j < 1 \text{ for } j = 2, \ldots, p \qquad \text{and} \qquad y_k < 1 \text{ for } k = 2, \ldots 1.$$

However, x_1 and y_1 may be chosen arbitrarily large.

Consider a non-trivial linear combination of basic monomimals, i.e. the expression

$$\sum_{\tau=1}^{J} c_\tau \, T_{\bullet^\tau}^{(a_1^\tau)} \cdots T_{\bullet^\tau}^{(a_p^\tau)} \, \overline{T_{\bullet^\tau}^{(b_1^\tau)}} \cdots \overline{T_{\bullet^\tau}^{(b_q^\alpha)}} \tag{C.32}$$

with $1 \leq J < \infty$ and real coefficients $c_\tau \neq 0$, where the notation $\bullet^\tau$ points out that the lower indices may depend on τ. We assume that this linear combination vanishes in a weak evaluation near the light cone to leading order in $(lE_P)^{-1}$ and $(l_{\text{macro}}E_P)^{-1}$, for any choice of the regularization functions. Since the l-dependence of the weak integral determines the degree of the monomial (see before (C.8)), we can assume that all the monomials in (C.32) have the same degree L. We consider the functions K_{AB}, (C.30), for every monomial in (C.32), and extend them by linearity to the linear combination (C.32). For every monomial in (C.32), we have for $K_{AB}/\mathcal{P}$ in the region (C.21) the asymptotic formula (C.31). In order to distinguish in (C.31) between the different monomials, we label the parameters x_j and y_k by an additional index τ. Notice that the derivatives in (C.30) give zero when A or B are chosen sufficiently large, also the number of monomials in (C.32) is finite. We shall in what follows consider the finite number of configurations (A, B) where the derivatives in (C.30) do not give zero, for at least one of the monomials in (C.32). By choosing γ_k and δ_i small enough, we can arrange that

$$x_j^\tau < 1 \text{ and } y_k^\tau < 1 \qquad \text{for all } \tau \in \{1, \ldots, Jb\} \tag{C.33}$$

and $j = 1, \ldots, p$, $k = 1, \ldots, q$. According to (C.21), α_1 is dominant parameter. Thus we may restrict attention to those contributions (C.31) for which the order of α_1 is maximal, i.e. to the monomials with

$$\sum_{j=2}^{p} a_j^\tau - x_j^\tau \;=\; \max_{\lambda \in \{1,\ldots,J\}} \sum_{j=2}^{p} a_j^\lambda - x_j^\lambda \quad . \tag{C.34}$$

We select the monomials satisfying these conditions and denote the corresponding parameters τ by $\tau \in A_1$. Out of the contributions satisfying (C.34), we pick those for which the power of β_1 is maximal,

$$\sum_{k=2}^{q} b_k^\tau - y_k^\tau \;=\; \max_{\lambda \in A_1} \sum_{k=2}^{q} b_k^\lambda - y_k^\lambda \quad ,$$

and denote the parameters τ satisfying these conditions by $\tau \in A_2$. Next we choose the monomials for which the power of β_2 is maximal,

$$b_2^\tau - x_2^\tau \;=\; \min_{\lambda \in A_2} b_2^\lambda - x_2^\lambda \quad ,$$

and denote the corresponding parameters τ by $\tau \in A_3$. Proceeding in this way for the variables $\beta_3, \ldots, \beta_q$ and $\alpha_2, \ldots, \alpha_p$, we end up with a non-empty index set $\Lambda \ni \tau$. We conclude that, in the asymptotic region (C.21) and to leading order in the variables $\alpha_1, \beta_1, \ldots, \beta_q, \alpha_2, \ldots, \alpha_p$, it suffices to consider those monomials with $\tau \in \Lambda$. Using (C.33), all of these monomials satisfy the conditions

$$a_j^\tau \;=\; \max_{\lambda \in \Lambda} a_j^\lambda \;\text{ for } j = 2, \ldots, p \qquad \text{and} \qquad b_k^\tau \;=\; \max_{\lambda \in \Lambda} b_k^\lambda \;\text{ for } k = 2, \ldots, q. \tag{C.35}$$

Furthermore, for the corresponding leading contributions to $K_{AB}/\mathcal{P}$ of the form (C.31),

$$x_j^\tau \;=\; \max_{\lambda \in \Lambda} x_j^\lambda \;\text{ for } j = 2, \ldots, p \qquad \text{and} \qquad y_k^\tau \;=\; \max_{\lambda \in \Lambda} y_k^\lambda \;\text{ for } k = 2, \ldots, q. \tag{C.36}$$

Together with the degree formula (2.99) and the condition for basic monomials $a_1 = a_2$ (see (2.107)), the relations (C.35) yield that the parameters $a_1^\tau, \ldots, a_p^\tau$ and $b_1^\tau, \ldots, b_q^\tau$ are independent of $\tau \in \Lambda$. The relations (C.36) imply that the parameters $x_2^\tau, \ldots, x_p^\tau$ and $y_2^\tau, \ldots y_q^\tau$ do not depend on $\tau \in \Lambda$, too. We remind that the parameters x_j and y_k determine how the ν- and μ-derivatives act on the factors $T_\bullet^{(a_j)}$ and $\overline{T_\bullet^{(b_k)}}$, respectively. Hence to leading order in the asymptotic region (C.21), the linear combination (C.32) reduces to a linear combination of monomials with fixed parameters (a_j, b_k) and a fixed configuration of derivatives acting on the factors $T_\bullet^{(a_2)}, \ldots, T_\bullet^{(a_p)}$ and $\overline{T_\bullet^{(b_2)}}, \ldots, \overline{T_\bullet^{b_q}}$. We write the fact that this linear combination vanishes in the symbolic form

$$\begin{aligned} \sum_{\tau \in \Lambda} c_\tau \; &(\cdots T_\tau^{(a_1)})(\partial_\nu^{\rho_2} \partial_\mu^{\sigma_2} T_\tau^{(a_2)}) \cdots (\partial_\nu^{\rho_p} \partial_\mu^{\sigma_p} T_\tau^{(a_p)}) \\ &\times \overline{(\cdots T_\tau^{(b_1)})(\partial_\nu^{\rho_{p+2}} \partial_\mu^{\sigma_{p+2}} T_\tau^{(b_2)}) \cdots (\partial_\nu^{\rho_{p+q}} \partial_\mu^{\sigma_{p+q}} T_\tau^{(b_q)})} \;=\; 0 \quad , \end{aligned} \tag{C.37}$$

where ρ_j and σ_j are disjoint subsets of $\{1, \ldots, A\}$ and $\{1, \ldots, B\}$, respectively, and

$$\partial_\nu^\rho \partial_\mu^\sigma T_\bullet^{(n)} \;\equiv\; \left(\prod_{i \in \rho} \frac{\partial}{\partial \nu_i} \right) \left(\prod_{k \in \sigma} \frac{\partial}{\partial \mu_k} \right) T_\bullet^{(n)}{}_{|\nu_i = \mu_k = 0} \quad .$$

As indicated by the dots in front of the factors $T_\bullet^{(a_1)}$ and $T_\bullet^{(b_1)}$ in (C.37), the configuration of the derivatives acting on these factors has not yet been determined; this is because (C.36) gives no information on x_1 and y_1. In order to determine x_1 and y_1, we go back to the asymptotic formula (C.31). The difficulty is that the contributions to (C.31) involving the highest powers of α_1 and β_1, which we considered so far, do not depend on x_1 and y_1. But we can use that, as mentioned after (C.31), both x_1 and y_1 can be chosen arbitrarily large, and can in this way influence (C.31) even when we restrict attention to the highest order in α_1 and β_1. More precisely, we increase the parameters γ_k and δ_i corresponding to those partial derivatives, which are not written out in (C.37) and hence act on one of the factors $T_\bullet^{(a_1)}$ or $\overline{T_\bullet^{(b_1)}}$, and choose them of the order β_1. This allows us to distinguish between all of the configurations of the partial derivatives. We conclude that every configuration of the partial derivatives in (C.37) vanishes separately; i.e. again in symbolic notation,

$$\sum_{\tau \in \Lambda} c_\tau \; (\partial_\nu^{\rho_1} \partial_\mu^{\sigma_1} T_\tau^{(a_1)}) \cdots (\partial_\nu^{\rho_p} \partial_\mu^{\sigma_p} T_\tau^{(a_p)}) \, \overline{(\partial_\nu^{\rho_{p+1}} \partial_\mu^{\sigma_{p+1}} T_\tau^{(b_1)})} \cdots \overline{(\partial_\nu^{\rho_{p+q}} \partial_\mu^{\sigma_{p+q}} T_\tau^{(b_q)})} \;=\; 0 \quad , \tag{C.38}$$

where ρ_j and σ_j are now partitions of $\{1, \ldots, A\}$ and $\{1, \ldots, B\}$, respectively. Clearly, we consider in (C.38) a configuration of the derivatives which really appears, i.e. at least one summand in (C.38) should be non-zero.

We choose from the finite number of configurations (A, B) under consideration those configurations where A is maximal, and out of these configurations the one where B is maximal. For this choice of A and B, the ν- and μ-derivatives in (C.38) determine the monomial, in the sense that every possible configuration of the derivatives in (C.38) gives zero except for at most one $\tau \in \Lambda$ (this is verified by elementary combinatorics). As a consequence, for our choice of A and B, the derivatives in (C.38) are non-zero for exactly one summand. Hence in this summand, the coefficient c_τ must vanish, giving a contradiction. ∎

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