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Model of a viscous layer deformation by thermocapillary forces

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Three-dimensional nonstationary flow of viscous incompressible liquid is investigated in a layer generated by nonuniform distribution of temperature on its free boundaries. If the temperature given on the layer boundaries is quadratically dependent on homogeneous coordinates, the external mass forces are absent and the motion arises from the rest state then the problem with free boundaries for the Navier–Stokes equations has the "exact" solution determined from system of equations with two independent variables. Here the free boundaries of the layer remain parallel planes and the distance between them must be also determined. Formulated in present paper are the conditions of both the unique solvability of the reduced problem on the whole in time and the collapse of the solution in finite time. Moreover studied in this paper are qualitative properties of the solution such as its behaviour at large time in the case of global solvability of the problem and the asymptotics of the solution near the collapse moment in the opposite case.

1 Statement of problem.

Considered below is thermocapillary motion of viscous incompressible liquid bounded entirely with free surface. The domain occupied with liquid is denoted by Ω_t, and its boundary is denoted by Γ_t. Liquid density ρ and kinematic coefficient of viscosity ν are supposed to be constant and the coefficient of surface tension σ to be linear function of temperature θ

$$\sigma = \sigma_0 - \kappa(\theta - \theta_0) \tag{1.1}$$

where σ_0, κ and θ_0 are positive constants. We will suppose further that motion arises from the rest state and the external mass forces do not act on liquid. Moreover we will assume that the temperature of free surface $\theta_\Gamma(\vec{x}, t)$ is known function of coordinates $\vec{x} = (x, y, z)$ and time t. Hence the mathematical statement of problem on thermocapillary motion is reduced to determination of the domain Ω_t, $0 < t < T$ and the solution $\vec{v}(\vec{x}, t) = (u, v, w)$, $p(\vec{x}, t)$ of system of the Navier–Stokes equations

$$\vec{v}_t + \vec{v} \cdot \nabla \vec{v} = -\rho^{-1} \nabla p + \nu \Delta \vec{v}, \ \nabla \cdot \vec{v} = 0 \tag{1.2}$$

in this domain satisfying the initial conditions

$$\Omega_0 \text{ is given}, \ \vec{v}(\vec{x}, 0) = 0, \ \vec{x} \in \Omega_0 \tag{1.3}$$

and the conditions on free surface

$$-p\vec{n} + 2\rho\nu D \cdot \vec{n} = -2K\sigma\vec{n} + \nabla_\Gamma \sigma, \tag{1.4}$$

$$\vec{v}\cdot\vec{n} = V_n,\ \vec{x}\in\Gamma_t,\ 0<t<T. \tag{1.5}$$

The following notations are used in (1.4), (1.5): $\vec{n}$ is unit vector of the external normal to the surface Γ_t, $D = [\nabla\vec{v} + (\nabla\vec{v})^*]/2$ is strain velocity tensor, K is mean curvature of the surface Γ_t, $\nabla_\Gamma = \nabla - \vec{n}(\vec{n}\cdot\nabla)$ is surface gradient, V_n is velocity of displacement of surface Γ_t in the direction of $\vec{n}$. After substitution of the expression for σ in the form of (1.1) with $\theta = \theta_\Gamma(\vec{x}, t)$ into (1.4) we obtain the closed statement of the problem with free boundary for the Navier–Stokes equations.

The solvability conditions for the initial boundary value problem (1.2)–(1.5) are derived in [1]. Investigated in [2] are the invariance properties of this problem; the group classification of this problem relative to an "arbitrary element" $\theta_\Gamma(\vec{x}, t)$ is fulfilled there too. The examples of the exact solutions of equations of thermocapillary motion are presented in [3-6] and in chapter 7 of monograph [7] (see also references there). It must be noted that the majority of found exact solutions describe stationary flows determined from system of ordinary differential equations. Solution of plane nonstationary problem for system (1.2) describing thermocapillary flow in a strip is built in [2]. It corresponds to the dependence $\theta_\Gamma = \theta^* + l(t)x^2$, where $\theta^* = const$ and l is an arbitrary function of t and it is defined from system of equations with two independent variables. Possibility of the decrease of order of the problem considered in [2] is caused by the fact that its solution is partially invariant solution [8] of the plane analog of system (1.2). Solution studied in present paper is natural generalization of previous solution for the case of thermocapillary motion in a layer. It corresponds to the temperature distribution on the boundaries of the layer

$$\theta_\Gamma = \theta^* + l(t)x^2/2 + m(t)y^2/2, \tag{1.6}$$

where l and m are arbitrary functions of t.

The further considerations are grounded on the following statement that can be checked by the direct test. If

$$u = (f+g)x,\ v = (f-g)y,\ w = -2\int_0^z f(\varsigma, t)d\varsigma, \tag{1.7}$$

$$p/\rho = \nu w_z(z,t) - \int_0^z w_t(\varsigma, t)d\varsigma - \frac{1}{2}w^2(z,t) + \chi(t),$$

where $f(z,t)$, $g(z,t)$ are the solutions of system of equations

$$f_t + f^2 + g^2 - 2f_z\int_0^z f(\varsigma,t)d\varsigma = \nu f_{zz}, \tag{1.8}$$

$$g_t + 2fg - 2g_z\int_0^z f(\varsigma,t)d\varsigma = \nu g_{zz}$$

and χ is an arbitrary function of t, then functions $\vec{v} = (u, v, w)$, p satisfy the Navier–Stokes equations (1.2). (Note that solution (1.7) of system (1.2) can be determined by regular way as a partially invariant solution with rank two and defect two relative to the four–parameter Lie group generated by translations and Galilei translations along axes x and y [9].)

Let us show that solution (1.7) can be interpreted as a solution describing thermocapillary motion in layer $|z| < s(t)$ where the temperature distribution is prescribed on its boundaries (1.6). In fact in this case $K = 0$, $\nabla_\Gamma \sigma = (-\kappa x l(t), -\kappa y m(t))$ and the condition (1.4) will be satisfied at $z = s(t)$ if functions f and g are subjected to the equalities

$$f_z(s(t),t) = -k[l(t) + m(t)],\ 0 < t < T, \tag{1.9}$$
$$g_z(s(t),t) = -k[l(t) - m(t)],\ 0 < t < T,$$

where $k = \kappa/\rho\nu = const > 0$ and function $\chi(t)$ is chosen in the form

$$\chi = \nu w_z(s(t),t) + \int_0^{s(t)} w_t(\varsigma,t)d\varsigma + \frac{1}{2}w^2(s(t),t).$$

Further we assume that

$$f_z(0,t) = g_z(0,t) = 0,\ 0 < t < T \tag{1.10}$$

and continue functions f, g (determined initially at $0 < z < s(t)$, $0 < t < T$) to the domain $-s(t) < z < 0$ in even way. Then condition (1.4) will be fulfilled at the lower boundary of the layer $z = -s(t)$ too. If we demand the fulfillment of the relation

$$\frac{ds}{dt} = -2\int_0^{s(t)} f(z,t)dz,\ 0 < t < T \tag{1.11}$$

then we can satisfy the condition (1.5) on both boundaries of the layer. Finally we assume that

$$s(0) = a > 0 \tag{1.12}$$

(it corresponds to definition of the initial position of the layer) and

$$f(z,0) = g(z,0) = 0,\ 0 \leq z \leq a. \tag{1.13}$$

Then the initial conditions (1.3) will be satisfied.

2 Conditions for existence and non-existence of solution.

Here the solvability conditions for the problem (1.8)–(1.13) are formulated and the qualitative properties of its solution are determined. Note that we are interested only in classical solutions of mentioned problem. The input data of the problem (i.e. functions $l(t)$ and $m(t)$) must be subjected to some conditions of smoothness and compatibility in order to ensure the existence of such solutions. Further we assume that these functions are defined for all $t > 0$, moreover

$$l(t), m(t) \in C^{(1+\alpha)/2}[0,\infty),\ 0 < \alpha < 1, \tag{2.1}$$

$$l(0) = m(0) = 0, \tag{2.2}$$

where $C^{(1+\alpha)/2}[0,\infty)$ denotes the space of functions continuous at semiaxis $t \geq 0$ and satisfying the Hölder conditions with exponent $(1+\alpha)/2$ at any compact set. The following notations are used below: S_T is the domain $\{z, t : 0 < z < s(t), 0 < t < T\}$, $C^{2+\alpha,1+\alpha/2}(\bar{S}_T)$ is the Hölder class used in the theory of parabolic equations (its definition can be found in [10]).

Proposition 1. Let the conditions (2.1), (2.2) to be fulfilled. Then one can find such $T > 0$ that the problem (1.8)–(1.13) will have the unique solution $f(z,t)$, $g(z,t)$, $s(t)$, moreover $f, g \in C^{2+\alpha,1+\alpha/2}(\bar{S}_T)$, $s \in C^{2+\alpha/2}[0,T]$.

The proof of this proposition has purely technical character. It is grounded on transition from the Eulerian coordinate z to the Lagrangian coordinate ς in the problem (1.8)–(1.13). The connection between the Lagrangian and Eulerian coordinates is determined in terms of the solution of the Cauchy problem

$$z_t = -2\int_0^z f(\xi,t)d\xi \quad \text{when} \quad t > 0,$$

$$z = \varsigma \quad \text{when} \quad t = 0.$$

Here the domain S_T turns into the rectangle $\Pi = \{\varsigma, t : 0 < \varsigma < a, 0 < t < T\}$ and the equations (1.8) turn into the following equations for the functions $F(\varsigma,t) = f[z(\varsigma,t),t]$, $G(\varsigma,t) = g[z(\varsigma,t),t]$:

$$F_t + F^2 + G^2 = \nu \exp[2\int_0^t F(\varsigma,\tau)d\tau]\{\exp[2\int_0^t F(\varsigma,\tau)d\tau]F_\varsigma\}_\varsigma, \tag{2.3}$$
$$G_t + 2FG = \nu \exp[2\int_0^t F(\varsigma,\tau)d\tau]\{\exp[2\int_0^t F(\varsigma,\tau)d\tau]G_\varsigma\}_\varsigma.$$

The equality

$$z_\varsigma = \exp[-2\int_0^t F(\varsigma,\tau)d\tau]$$

was used for derivation of (2.3). The mentioned solution of the Cauchy problem satisfies this equality. Then the boundary conditions (1.9) are rewritten in the form

$$F_\varsigma(a,t) = -k[l(t)+m(t)]\exp[2\int_0^z F(a,\tau)d\tau],\ 0 < t < T, \tag{2.4}$$
$$G_\varsigma(a,t) = -k[l(t)-m(t)]\exp[2\int_0^z F(a,\tau)d\tau],\ 0 < t < T.$$

The conditions (1.10), (1.13) give the following conditions for the functions F and G

$$F_\varsigma(0,t) = G_\varsigma(0,t) = 0,\ 0 < t < T, \tag{2.5}$$

$$F(\varsigma,0) = G(\varsigma,0) = 0,\ 0 \leq \varsigma \leq a. \tag{2.6}$$

As a result we obtain the initial boundary value problem in the fixed domain (2.4)–(2.6) for the system of quasilinear integro-differential parabolic equations of the second order (2.3). Its local unique solvability in Hölder classes follows from general results of theory of parabolic equations [10] and can be determined for example by method of successive approximations; the convergence of this method is guaranteed for sufficiently small T. If function $F(\varsigma, t)$ is known then function $s(t)$ defining the position of free boundary on the plane z, t is given by formula

$$s(t) = \int_0^a \exp\,[-2\int_0^t F(\varsigma, \tau)d\tau]d\varsigma.$$

Hence the kinematic condition on the free boundary (1.11) is fulfilled automatically.

So solvability of problem (1.8)–(1.13) at small time interval demands from functions $l(t)$, $m(t)$ only the fulfillment of smoothness condition (2.1) and compatibility condition (2.2). As it will be shown below these conditions are insufficient for solvability of the problem "in the whole":

Proposition 2. Assume that

$$l(t) + m(t) \geq 0 \;\text{ for }\; t \geq 0. \tag{2.7}$$

Moreover the inequality (2.7) is strict at some interval $(0, \tau)$. Then the "life span" t_* of solution of problem (1.8)-(1.13) is finite.

Proof. Let us consider the function

$$\bar{f}(t) = \frac{1}{s(t)}\int_0^{s(t)} f(z,t)dz,\;\; h = f - \bar{f}$$

so that quantity $\bar{f}$ is the mean value of function $f(z,t)$ for any fixed t at the interval $[0, s(t)]$ and the mean value of function $h(z,t)$ is equal to zero at this interval for any $t > 0$. Relation (1.11) will take the form

$$\frac{ds}{dt} = -2\bar{f}s. \tag{2.8}$$

So that knowledge of function $\bar{f}$ determines completely the evolution of free boundary in the problem (1.8)-(1.13).

We obtain the identity

$$\frac{d\bar{f}}{dt} = -\bar{f}^2 - \frac{1}{s}\int_0^s (g^2 + 3h^2)dz - \frac{\nu k(l+m)}{s} \tag{2.9}$$

after integration of the first equation (1.8) by z at the interval $[0, s(t)]$ and taking into account the conditions (1.9)–(1.11). Further we may suppose without loss of generality that number τ used in formulation of Proposition 2 is less then the life span t_* of solution of studied problem. As it follows from (2.9) and the conditions of Proposition 2 function $\bar{f}$ decreases monotonically at the interval $[0, t_*)$ and formula (1.13) gives $\bar{f}(0) = 0$. So using (2.8) one can conclude that $s(t) \geq a$ when $0 \leq t < t_*$.

Integration of the identity (2.9) at the interval $(0, \tau)$ and elimination of necessarily nonpositive terms from the right part of the resulting equality lead to the chain of inequalities

$$0 > \bar{f}(\tau) \geq -\nu k\int_0^\tau \frac{l(t)+m(t)}{s(t)}dt \geq -\frac{\nu k}{a}\int_0^\tau [l(t)+m(t)]dt = -\gamma,$$

where $\gamma = \mathit{const} > 0$ in accordance with the condition of Proposition 2. (The rigor of the left inequality is guaranteed by this condition too.) The estimate $\bar{f}(t) \leq (1+\gamma\tau-\gamma t)^{-1}\bar{f}(\tau)$

follows from this fact and inequality $d\bar{f}/dt \le -\bar{f}^2$ arising from (2.9). So far as $\bar{f}(\tau) < 0$ the shown estimate means that the solution of the problem (1.8)-(1.13) is destroyed at finite period of time $t_* \le \gamma^{-1} + \tau$. Proposition 2 is proved.

Actually Proposition 2 contains the necessary condition of global solvability of the problem (1.8)-(1.13). Determination of sufficient conditions for the existence of its solution for all $t > 0$ demands essential efforts. Obtaining of the estimate of the module maximum of functions f and g in the domain S_T for all $T > 0$ is the main point here. In the case when such an estimate is obtained the proof of solvability of the problem (1.8)–(1.13) "in the whole" can be fulfilled by scheme of the proof of Theorem 1 of chapter 7 of the book [7] on the ground of the method developed in [10].

The specific character of the considered problem with free boundary consists of the fact that its solution can cease its existence with growth of t by two reasons. The first reason is demonstrated in Proposition 2. Existence of function $\bar{f}(t)$ obtained in process of its proof and equations (2.8) imply that $s \to \infty$ when $t \nearrow t_*$. Vanishing of function $s(t)$ at finite time t^* is the other reason. This possibility explains the conditional character of Proposition 3 formulated below. Further positive quantities (generally speaking, depending on T) are denoted by C_k $(k = 1, 2, \ldots)$.

Proposition 3. Let the following inequalities to be fulfilled

$$l(t) \le 0,\ m(t) \le 0 \ \text{ for } \ t \ge 0 \tag{2.10}$$

Then the alternative written below takes place.

a). One can find such $t^* < \infty$ that $s(t) > 0$ for $0 \le t < t^*$ and $s \to 0$ when $t \nearrow t^*$. In this case the estimates

$$|f(z,t)| \le C_1,\ |g(z,t)| \le C_2 \ \text{ when } \ (z,t) \in \bar{S}_T \tag{2.11}$$

are valid ($T > 0$ – an arbitrary number less then t^*).

b). Inequality $s(t) > 0$ is fulfilled for any finite $t > 0$. Then estimates (2.11) are valid in the domain $\bar{S}_T$ for any $T > 0$.

Proof. Let us introduce into consideration functions $\lambda = f + g$, $\mu = f - g$. As it follows from (1.8) these functions satisfy the equations

$$\lambda_t + \lambda^2 - 2\lambda_z \int_0^z f(\varsigma, t)d\varsigma = \nu\lambda_{zz}, \tag{2.12}$$

$$\mu_t + \mu^2 - 2\mu_z \int_0^z f(\varsigma, t)d\varsigma = \nu\mu_{zz}$$

in the domain S_T. Initial and boundary conditions for the system (2.12) are obatined from (1.9),(1.10),(1.13) and have the form

$$\lambda_z(s(t), t) = -2kl(t),\ \mu_z(s(t), t) = -2km(t), \tag{2.13}$$

$$\lambda_z(0, t) = \mu_z(0, t) = 0,\ 0 < t < T, \tag{2.14}$$

$$\lambda(z, 0) = \mu(z, 0) = 0. \tag{2.15}$$

If we consider the first and the second equations (2.12) as linear ones relative to the functions λ and μ we may apply the maximum principle [11] to the solution of the initial boundary value problems (2.13)–(2.15) for these equations. In accordance with this principle the nonnegativity of the right parts of the conditions (2.13) provided by the inequalities (2.10) and the homogeneity of the conditions (2.14), (2.15) imply the nonnegativity of functions λ and μ in the domain S_T where $T < t^*$ in the case a) and T is an arbitrary positive number in the case b). It means that

$$f \geq 0 \text{ and } |g| \leq f \text{ for } (z,t) \in \bar{S}_T. \tag{2.16}$$

So that the proof of the first inequality (2.11) will imply the proof of the second one. Moreover function $s(t)$ decreases monotonically for $t > 0$ by virtue of (2.8), (2.16), so that this fact together with (2.12) implies the estimate

$$s(t) \leq a \text{ if } t \in [0,T]. \tag{2.17}$$

Obtaining of uniform pointwise estimates of functions f_z, g_z in the domain $\bar{S}_T$ is the next step of the proof. It is evident that it is sufficient for this purpose to obtain the similar estimates for the functions $\xi = \lambda_z$, $\eta = \mu_z$. As it follows from (2.12)–(2.15) these functions are the solutions of the first initial boundary value problems for the linear parabolic equations

$$\xi_t - 2\xi_z \int_0^z f(\varsigma,t)d\varsigma + 2g\xi = \nu\xi_{zz}, \tag{2.18}$$

$$\eta_t - 2\eta_z \int_0^z f(\varsigma,t)d\varsigma + 2g\eta = \nu\eta_{zz},$$

$$\xi(s(t),t) = -2kl(t), \;\; \nu(s(t),t) = -2km(t), \tag{2.19}$$

$$\xi(0,t) = \nu(0,t) = 0, \;\; 0 < t < T, \tag{2.20}$$

$$\xi(z,0) = \nu(z,0) = 0, \;\; 0 \leq z \leq a. \tag{2.21}$$

Estimates written below are valid on the ground of the maximum principle applied to the solutions of the problems (2.18)–(2.21) and inequalities (2.10)

$$0 \leq \xi = \lambda_z \leq C_3 = \max_{0 \leq t \leq T} [-2kl(t)],$$

$$0 \leq \eta = \mu_z \leq C_4 = \max_{0 \leq t \leq T} [-2km(t)].$$

So we conclude from these estimates and definition of λ and μ that

$$0 \leq f_z \leq C_5 \text{ and } |g_z| \leq f_z \text{ for } (z,t) \in S_T \tag{2.22}$$

with $C_5 = C_3 + C_4$.

It arises from inequalities (2.22) that the maximal value of function $f(z,t)$ at some fixed t is achieved in the point $z = s(t)$ belonging to the free boundary of the domain S_T.

So one must obtain the estimate from above of function $f(s(t),t)$ for the completion of the proof of Proposition 3. With this aim let us consider the obvious representation

$$f(s(t),t) = \bar{f}(t) + \frac{1}{s(t)} \int_0^{s(t)} z f_z(z,t) dz. \tag{2.23}$$

The second term of the right part is estimated on the base of inequalities (2.17),(2.22)

$$\frac{1}{s(t)} \int_0^{s(t)} z f_z(z,t) dz \le \frac{aC_5}{2}. \tag{2.24}$$

The estimate from above of function $\bar{f}(t)$ is based on inequality

$$\frac{d\bar{f}}{dt} \le -\frac{\nu k(l+m)}{s}$$

following from (2.9). Integration of this inequality from zero to $t \le T$ with account of the condition $\bar{f}(0) = 0$ and replacement of functions $-l(t)$, $-m(t)$ with their maximal values on the interval $[0,T]$ imply

$$\bar{f}(t) \le \frac{\nu C_5}{2} \int_0^t \frac{d\tau}{s(\tau)} \quad \text{for} \quad t \in [0,T]. \tag{2.25}$$

Using estimates (2.24), (2.25) and representations (2.23) we conclude that

$$f(s(t),t) \le \frac{C_5}{2} \left[\int_0^t \frac{d\tau}{s(\tau)} + a \right] \quad \text{if} \quad 0 \le t \le T.$$

Here: a) $T < t^*$; b) $T > 0$ is arbitrary. The proof of Proposition 3 is complete.

3 Qualitative properties of solutions.

It will be shown below that all hypothetical possibilities considered in Proposition 3 can be realized. In order not to overload the paper we consider two simple cases of the behaviour of functions $l(t)$ and $m(t)$ defining the "destiny" of the solution of our problem.

Proposition 4. Let the solution of the problem (1.8)–(1.13) to be determined in some domain S_T. Suppose that conditions (2.10) are fulfilled and moreover

$$l(t) = m(t) = 0 \quad \text{when} \quad t \ge \tau \tag{3.1}$$

and $l + m \not\equiv 0$ when $0 \le t \le \tau$. Then the problem (1.8)–(1.13) is solvable in the domain S_T for any $T > 0$ and the following estimates are valid: either

$$s = C_6 t^{-2} + \mathrm{O}(t^{-3}) \quad \text{when} \quad t \to \infty \tag{3.2}$$

$$f = t^{-1} + \mathrm{O}(t^{-2}), \; g = \mathrm{O}(t^{-2}) \quad \text{when} \quad 0 \le z \le s(t) \tag{3.3}$$

or

$$s = C_7 t^{-1} + \mathrm{O}(t^{-2}) \quad \text{when} \quad t \to \infty \tag{3.4}$$

$$f = g = t^{-1}/2 + O(t^{-2}) \text{ when } 0 \leq z \leq s(t).$$

The last situation is possible only in the case $l = 0$ or $m = 0$ for all $t \geq 0$.

Proof. First of all note that Proposition 1 implies that in any case one can find the existance time τ of the solution of the problem (1.8)–(1.13). So far as the conditions of Proposition 3 are fulfilled here the inequalities

$$\lambda = f + g \geq 0,\ \mu = f - g \geq 0 \tag{3.5}$$

are valid in the domain $\bar{S_T}$, moreover at least one of the functions λ, μ is not identically equal to zero on the upper boundary of this domain, i.e. at $t = \tau$, $0 \leq z \leq s(\tau)$. In other case we arrive to contradiction with the condition $l + m \not\equiv 0$, $0 \leq t \leq \tau$ (functions λ and μ satisfy this condition as the solutions of the problem (2.12)–(2.15) by virtue of the strict maximum principle [11]). So one can conclude from here and (3.5) that $\bar{f}(\tau) > 0$.

Now we can use the identity (2.9) where the last right term is absent at $t \geq \tau$ as it follows from (3.1). The inequality $d\bar{f}/dt \leq -\bar{f}^2$ follows from mentioned identity and integation of this inequality from $t = \tau$ with account of positiveness of $\bar{f}(\tau)$ implies the estimate

$$\bar{f}(t) \leq \frac{\bar{f}(\tau)}{1 + (t - \tau)\bar{f}(\tau)} \text{ when } t \geq \tau. \tag{3.6}$$

In accordance with (2.8) the estimate from above of function $\bar{f}(t)$ implies the estimate from below of function $s(t)$. So that (2.8), (3.6) give

$$s(t) \geq \frac{s(\tau)}{[1 + (t - \tau)\bar{f}(\tau)]^2} \text{ when } t \geq \tau. \tag{3.7}$$

The global existence theorem is valid for the problem (1.8)–(1.13) on the ground of Proposition 3 and inequality (3.7).

Now let us obtain the asymptotic representations (3.2)–(3.4). With this aim we use formulation of the problem (1.8)–(1.13) in the Lagrangian coordinates (2.3)–(2.6) where the boundary condition (2.4) is homogeneous for $t \geq \tau$ by virtue of the assumption (3.1). We introduce into consideration functions $\Lambda(\varsigma, t) = \lambda(z, t)$, $\mathrm{M}(\varsigma, t) = \mu(z, t)$ and obtain the initial boundary value problem for them

$$\Lambda_t + \Lambda^2 = \nu \exp[\int_0^t (\Lambda + \mathrm{M})dt]\{exp\Big[\int_0^t (\Lambda + \mathrm{M})dt\Big]\Lambda_\varsigma\}_\varsigma, \tag{3.8}$$

$$\mathrm{M}_t + \mathrm{M}^2 = \nu \exp\Big[\int_0^t (\Lambda + \mathrm{M})dt\Big]\{exp\Big[\int_0^t (\Lambda + \mathrm{M})dt\Big]\mathrm{M}_\varsigma\}_\varsigma$$

in semistrip $\Sigma_\tau = \{\varsigma, t : 0 < \varsigma < a,\ t > \tau\}$,

$$\Lambda_\varsigma(a, t) = \mathrm{M}_\varsigma(a, t) = 0,\ t > \tau, \tag{3.9}$$

$$\Lambda_\varsigma(0, t) = \mathrm{M}_\varsigma(0, t) = 0,\ t > \tau, \tag{3.10}$$

$$\Lambda(\varsigma, \tau) = \Lambda_0(\varsigma),\ \mathrm{M}(\varsigma, \tau) = \mathrm{M}_0(\varsigma),\ 0 \leq \varsigma \leq a, \tag{3.11}$$

where the functions Λ_0, M_0 are defined by the equalities

$$\Lambda_0(\varsigma) = \lambda[z(\varsigma,\tau),\tau], \ \mathrm{M}_0(\varsigma) = \mu[z(\varsigma,\tau),\tau], \tag{3.12}$$

where τ is a parameter and the connection between the Lagrangian coordinate ς and the Eulerian coordinate z is given by formula

$$z(\varsigma,\tau) = \int_0^\varsigma \exp\,[-2\int_0^t F(\rho,\sigma)d\sigma]d\rho, \ 0 \le \varsigma \le a, \ t \ge 0$$

Boundedness of function $F(\varsigma,t) = f(z,t)$ for $\varsigma \in [0,a]$ and any finite $t \ge 0$ guaranteed by the proved fact of solvability of the problem (1.8)–(1.13) "in the whole" provides the mutual uniqueness of correspondence between the variables ς and z.

The existence of solutions of system (3.8) not depending on ς is the remarkable peculiarity of this system. Such solutions are compatible with the boundary conditions (3.9), (3.10). This circumstance permits to use them as the barrier functions for the solution of the problem (3.8)–(3.10). We choose these functions in the following form

$$\Lambda^-(t) = \frac{\lambda_{min}}{1+\lambda_{min}(t-\tau)}, \ \Lambda^+(t) = \frac{\lambda_{max}}{1+\lambda_{max}(t-\tau)},$$

$$\mathrm{M}^-(t) = \frac{\mu_{min}}{1+\mu_{min}(t-\tau)}, \ \mathrm{M}^+(t) = \frac{\mu_{max}}{1+\mu_{max}(t-\tau)},$$

where λ_{min} (λ_{max}) and μ_{min} (μ_{max}) are the minimal (maximal) values of functions $\lambda(z,\tau)$ and $\mu(z,\tau)$ at the interval $0 \le z \le s(\tau)$.

Here we use the condition of Proposition 4, $l(t) + m(t) \not\equiv 0$ for $0 \le \ t \le \tau$. We may assume without loss of generality that one can find such an interval (t_1,t_2), $0 \le t_1 < t_2 \le \tau$ that the strict inequality is fulfilled there

$$l(t) < 0 \text{ when } t_1 < t < t_2. \tag{3.13}$$

At this time function $m(t)$ can vanish identically (note that both functions l and m are nonpositive for $t \ge 0$ in accordance with the condition (2.10)). The case when $l = 0$ for all $t \ge 0$ and the inequality analogous with (3.13) is fulfilled for the function $m(t)$ is considered in a similar way.

At first let us consider the special case $m = 0$ for all $t \ge 0$. Then the second condition (2.13) is homogeneous and implies the equality $\mu = 0$ in the domain $\bar{S}_T$ for any $T > 0$ on the ground of the uniqueness theorem for the solution of the initial boundary value problem (2.12)–(2.15) for the function μ. It means that functions f and g coincide for all $z \in [0,s(t)]$, $t \ge 0$.

On the other hand values of the function λ are strictly positive on the upper boundary $t = \tau$, $0 \le z \le s(\tau)$ of the domain S_T in consequence of inequality (3.13) and strict maximum principle applied to solution of the problem (2.12)–(2.15) for the function λ. So $\lambda_{min} = min\lambda(z,\tau) > 0$. Now let us consider the function $P^- = \Lambda - \Lambda^-$. By virtue of (3.8)–(3.10) it is the solution of the following problem

$$P^-_t + (\Lambda + \Lambda^-)P^- = \nu \exp\,(\int_0^t \Lambda dt')[\exp\,(\int_0^t \Lambda dt')P^-_\varsigma]_\varsigma, \ (\varsigma,t) \in \Sigma_T,$$

$$P^-_\varsigma(0,t) = P^-_\varsigma(a,t) = 0, \ t > \tau,$$

$$P^{-}(\varsigma,\tau)=\Lambda_0(\varsigma)-\lambda_{min},\ 0\le\varsigma\le a$$

(here we take into account that $\lambda[z(\varsigma,\tau),\tau]=\Lambda_0(\varsigma)$ in accordance with (3.12)). It follows from the maximum principle that $P^{-}(\varsigma,\tau)\ge 0$ in semistrip $\bar{\Sigma}_T$, this fact implies the estimate

$$\Lambda(\varsigma,\tau)\ge\frac{\lambda_{min}}{1+\lambda_{min}(t-\tau)}\ \text{for}\ (\varsigma,t)\in\bar{\Sigma}_T$$

by virtue of definition of this function.

The inequality

$$\Lambda(\varsigma,t)\le\frac{\lambda_{max}}{1+\lambda_{max}(t-\tau)}\ \text{for}\ (\varsigma,t)\in\bar{\Sigma}_T$$

is obtained in a similar way. Then we rewrite the obtained inequality in terms of function $\lambda(z,t)$ and take into account that $\lambda=2f,\ f=g$ by virtue of $\mu=0$, so that

$$\frac{\lambda_{min}}{1+\lambda_{min}(t-\tau)}\le 2f(z,t)\le\frac{\lambda_{max}}{1+\lambda_{max}(t-\tau)}\ \text{when}\ 0\le z\le s(t),\ t\ge\tau.$$

The correctness of asymptotics (3.4) for the functions f and g is obtained, consequently the asymptotics for the function s follows immediatly from (2.8).

Let us pass to the analysis of general case where the inequality written (3.13) is fulfilled parallel with

$$m(t)<0\ \text{when}\ t_3<t<t_4, \tag{3.14}$$

where t_3 and t_4 are some numbers from interval $[0,\tau]$. First of all it is obtained here that the inequality (3.14) implies $\mu_{min}=\min\mu(z,\tau)>0$. This fact petrmits to prove the nonnegativity of functions $Q^{-}=\mathrm{M}-\mathrm{M}^{-}$, $Q^{+}=\mathrm{M}^{+}-\mathrm{M}$ in the domain $\bar{\Sigma}_T$ and to obtain the estimates

$$\frac{\mu_{min}}{1+\mu_{min}(t-\tau)}\le\mathrm{M}(\varsigma,t)\le\frac{\mu_{max}}{1+\mu_{max}(t-\tau)}\ \text{for}\ (\varsigma,t)\in\bar{\Sigma}_T.$$

The last inequalities take the following form in terms of functions f and g with account of the relation $\mu=f-g$

$$\frac{\mu_{min}}{1+\mu_{min}(t-\tau)}\le f(z,t)-g(z,t)\le\frac{\mu_{max}}{1+\mu_{max}(t-\tau)}\ \text{when}\ 0\le z\le s(t),\ t\ge\tau.$$

The obtained estimates from above and below of function $\Lambda(\varsigma,t)$ imply the inequalities for the function $\lambda=f+g$

$$\frac{\lambda_{min}}{1+\lambda_{min}(t-\tau)}\le f(z,t)+g(z,t)\le\frac{\lambda_{max}}{1+\lambda_{max}(t-\tau)}\ \text{when}\ 0\le z\le s(t),\ t\ge\tau.$$

As a result we come to the relations $f+g=t^{-1}+\mathrm{O}(t^{-2})$, $f-g=t^{-1}+\mathrm{O}(t^{-2})$ when $t\to\infty$, $0\le z\le s(t)$. This fact proves the correctness of asymptotic representations (3.3) for the general case, when the both functions $l(t)$ and $m(t)$ take negative values even at some part of the interval $(0,\tau)$. The use of (2.8) and (3.3) gives the demanded asymptotics (3.2) of function $s(t)$. The proof of Proposition 4 is complete.

Proposition 5. Let us suppose that solution of the problem (1.8)–(1.13) is defined in the domain S_T. If inequality (2.10) and condition

$$l + m = -A/\nu k = const < 0 \text{ when } t \geq \tau \tag{3.15}$$

are fulfilled then one can find such finite $t^* > 0$ that $s(t) > 0$ for $0 \leq t < t^*$ and $s \to 0$ when $t \nearrow t^*$.

Proof. Let us use the identity (2.9) and rewrite it in terms of functions f and g

$$\frac{d}{dt}\int_0^s f dz + \int_0^s (3f^2 + g^2)dz = -\nu k(l+m) \tag{3.16}$$

We introduce into consideration functions

$$U(t) = \int_0^{s(t)} f(z,t)dz, \; V(t) = \int_0^{s(t)} g(z,t)dz. \tag{3.17}$$

Using the Cauchy–Bunyakovsky inequality and condition (3.15) we obtain from (3.16) the differential inequality for the function U

$$\frac{dU}{dt} \leq -\frac{3U^2}{s} + A \text{ when } t \geq \tau. \tag{3.18}$$

Note that inequality $U(\tau) = U_0 > 0$ is valid by virtue of the conditions of Proposition 5. Moreover function $U(t)$ nonnegative at $t \geq \tau$ can't vanish as long as $s(t) > 0$. These statements follow from strict maximum principle applied to the functions $f+g \quad f-g$ (see the beginning of the Proof of Proposition 4). Then it follows from (2.8), (3.17) that $ds/dt < 0$ for mentioned values of t. This fact permits to rewrite the inequality (3.18) in more comforatble form by introducing the function $U^2 = W(s)$

$$\frac{dW}{ds} - \frac{3W}{s} \geq -A \quad s \leq s(\tau) = s_0.$$

Integartion of the last inequality leads to the result

$$W(s) \geq \frac{sA}{2}\left(1 - \frac{s^2}{s_0^2} + \frac{2U_0^2 s^3}{As_0^3}\right) \equiv \frac{sA}{2}R^2(s),$$

moreover $R(s) \geq C_8 > 0$ for $s \in [0, s_0]$. Estimate written below is obtained from this fact and relation $ds/dt = -2W^{1/2}(s)$ following from (2.8), (3.17) and definition of function W

$$\int_s^{s_0} \frac{dr}{R(r)\sqrt{2Ar}} \geq t - \tau.$$

Integral entering this estimate converges in zero. This fact guarantees finiteness of the value of t^* corresponding to the moment of vanishing of function s and we obtain the estimate of t^*

$$t^* \leq \frac{1}{\sqrt{2A}}\int_0^{s_0} \frac{ds}{R(s)\sqrt{s}} + \tau.$$

Proposition 5 is proved.

The interest in investigating of the behaviour of solution of the problem (1.8)–(1.13) near the moment t^* follows from this Proposition. The simplest solution of this question can be found in the case when both functions $l(t)$ and $m(t)$ take constant values beginning with some τ.

Proposition 6. Let the conditions of Proposition 5 to be fulfilled. Moreover

$$l - m = -B/\nu k = const \quad t \geq \tau \tag{3.19}$$

with $|B| \leq A$. Then the following relations are valid

$$\frac{s}{(t^* - t)^2} \to 4A(1 + \sqrt{1 - 8\beta^2/9}), \tag{3.20}$$

$$\frac{U}{t^* - t} \to A(1 + \sqrt{1 - 8\beta^2/9}), \tag{3.21}$$

$$\frac{V}{t^* - t} \to \frac{4B}{3} \quad t \nearrow t^*. \tag{3.22}$$

where $\beta = B/A$ ($|\beta| \leq 1$) and functions $U(t)$, $V(t)$ are defined by equalities (3.17).

Proof. Identity (3.15) and the analogous identity

$$\frac{d}{dt}\int_0^s g dz + 4\int_0^s f g dz = -\nu k(l - m), \tag{3.23}$$

obtained by integration of the second equation (1.8) by z on the interval $[0, s(t)]$ with the use of relations (1.9)–(1.11) are the base of the proof. Both identities are considered for the values $t \geq \tau$ when there right parts are constant by virtue of the conditions (3.15), (3.19).

Let us denote the mean value of function $g(z,t)$ on the interval $0 \leq z \leq s(t)$ as $\bar{g}(t)$ and put $j(z,t) = g - \bar{g}$. The identities (3.16), (3.23) can be rewritten in the form

$$\frac{dU}{dt} + \frac{3U^2 + V^2}{s} + \int_0^s (3h^2 + j^2) dz = A, \tag{3.24}$$

$$\frac{dV}{dt} + \frac{4UV}{s} + \int_0^s hj dz = B \text{ when } t \in [\tau, t^*)$$

with the help of these functions and functions $\bar{f}(t)$, $h(z,t) = f - \bar{f}$ introduced before.(Here the evident equalities $U = s\bar{f}$, $V = s\bar{g}$ following from (3.17) and definition of functions $\bar{f}$ and $\bar{g}$ were taken into account.)

System (3.24) is not closed relative to the functions U and V, however this fact does not prevent to find the asymptotics of its solution near the moment t^* when $s(t^*) = 0$. The point is that the integral terms of (3.24) tend to zero quickly when $t \nearrow t^*$. The proof is grounded on representations

$$f(z,t) = \bar{f}(t) + \int_{b(t)}^z f_z(\varsigma, t) d\varsigma, \tag{3.25}$$

$$g(z,t) = \bar{g}(t) + \int_{c(t)}^z g_z(\varsigma, t) d\varsigma,$$

where $b(t)$ ($c(t)$) is a point from the interval $[0, s(t)]$ where function $f(z,t)$ ($g(z,t)$) takes its mean value as a function of z on mentioned interval. Using the uniform estimates

(2.22) of functions $|f_z|$, $|g_z|$ valid by virtue of (2.10) and remembering the definitions of functions h, j we arrive to inequalities

$$|h| \leq C_5 s,\ |j| \leq C_5 s \text{ when } 0 \leq z \leq s(t),\ \tau \leq t \leq t^*.$$

Now if system (3.24) is rewritten in the form

$$\frac{dU}{dt} + \frac{3U^2 + V^2}{s} + \Phi(t) = A, \qquad (3.26)$$
$$\frac{dV}{dt} + \frac{4UV}{s} + \Psi(t) = B,$$

where

$$\Phi = \int_0^{s(t)} [3h^2(z,t) + j^2(z,t)]dz,\ \Psi = 4\int_0^{s(t)} h(z,t)j(z,t)dz$$

then the estimates

$$|\Phi| \leq C_9 s^3,\ |\Psi| \leq C_9 s^3 \text{ when } \tau \leq t \leq t^* \qquad (3.27)$$

with $C_9 = 4C_5^2/3 = const$ will be valid for the functions $\Phi(t)$, $\Psi(t)$.

It follows from strict monotonicity of function s on the interval $[\tau, t^*)$ obtained in Proposition 5 that we may convert the dependence of s on t and consider Φ and Ψ as functions of variable s $\Phi[t(s)] = \phi(s)$, $\Psi[t(s)] = \psi(s)$, where $0 \leq s \leq s_0 = s(\tau)$. The further reasoning is grounded on transformation of (3.26) into the dynamic system of the third order with the help of change of variables

$$U = (As)^{1/2} q(\rho),\ V = (As)^{1/2} r(\rho),\ \rho = \ln(1/s). \qquad (3.28)$$

Substitution of (3.28) into (3.26) with regard to the relation $ds/dt = -2U$ leads to the system of equations

$$2q\frac{dq}{d\rho} + 2q^2 + r^2 = 1 - \frac{\varphi(s)}{A},$$
$$2q\frac{dr}{d\rho} + 3qr = \beta - \frac{\psi(s)}{A}, \qquad (3.29)$$
$$\frac{ds}{d\rho} = -s,\ \rho \geq \rho_0,$$

where $\rho_0 = \ln(1/s_0)$, $\beta = B/A = \text{const}$, $|\beta| \leq 1$ by virtue of conditions of Proposition 6. Our aim is to investigate the behaviour of solution of the Cauchy problem

$$q(\rho_0) = q_0,\ r(\rho_0) = r_0,\ s = \exp(-\rho_0) = s_0 \qquad (3.30)$$

for the system (3.29) when $\rho \to \infty$ where

$$q_0 = (As_0)^{-1/2} \int_0^{s_0} f(z,t)dz,\ r_0 = (As_0)^{-1/2} \int_0^{s_0} g(z,\tau)dz.$$

Moreover it is assumed that functions f and g are already defined in the domain S_τ so that $|r_0| \leq q_0$, $q_0 > 0$ on the base of (2.16), (3.17) and (3.28) (note that conditions of

Proposition 6 guarantee the fulfillment of inequalities (2.10) providing estimates (2.16)). The inequalities

$$q(\rho) > 0,\ |r(\rho)| \leq q(\rho) \tag{3.31}$$

for any finite $\rho \geq \rho_0$ follow also from mentioned relations but here they play the role of a priori estimates for solution of the Cauchy problem (3.29), (3.30).

First of all note that trajectory of the dynamic system (3.29) outgoing from the point (q_0, r_0, s_0) can't go out the limits of the cylindrical sector $K_N = \{q, r, s : 0 < q^2 + r^2 < N^2,\ |r| < q,\ 0 < s < s_0\}$ of phase space $\mathcal{R}^3$ (where N is sufficiently large) when $\rho \geq \rho_0$. It is sufficient for the proof of this statement to check that no points of output of the dynamic system (3.29) are situated on the boundary of the domain K_N (definition of points of output and points of input find, for example, in [12]). In fact the rectangles $q = r,\ 0 \leq s \leq s_0$ and $q = -r,\ 0 \leq s \leq s_0$ can't contain points of output in view of inequalities (3.31). The upper basis of K_N, i.e. the circular sector $0 \leq q \leq N,\ |r| \leq q,\ s = s_0$, consists of points of input in accordance with the third equation of system (3.29). The lower basis of K_N can't contain points of output so far as it corresponds to the value $\rho = \infty$. Now one must check the absence of points of output on the cylindrical part of the boundary K_N, i.e. on the set $H_N = \{q, r, s : q^2 + r^2 = N^2,\ |r| \leq q,\ 0 \leq s \leq s_0\}$.

The field of directions of the dynamic system (3.29) is characterized by vector $\vec{l}$ with components $(2q)^{-1}[1 - 2q^2 - r^2 - A^{-1}\varphi(s)]$, $(2q)^{-1}[\beta - 3qr - A^{-1}\psi(s)]$, $-s$. The scalar product $\vec{l} \cdot \vec{n}$ of vector $\vec{l}$ and unit vector of the external normal $\vec{n} = (\cos\omega, \sin\omega, 0)$ to the surface H_N where $\omega = \mathrm{arctg}(r/q)$ gives

$$\vec{l} \cdot \vec{n} = -N(1 + \sin^2\omega) - \frac{[A^{-1}\varphi(s) - 1]\cos\omega + [A^{-1}\psi(s) - \beta]\sin\omega}{(2N\cos\omega)}.$$

So far as $|\omega| \leq \pi/4$ on the surface H_N, $|\varphi| \leq C_9 s_0^3$, $|\psi| \leq C_9 s_0^3$ by virtue of inequalities (3.27) and $|\beta| \leq 1$ one can obtain the fulfillment of inequality $\vec{l} \cdot \vec{n} < 0$ on the surface H_N by choosing number N larger then $\max\,[(1 + A^{-1}C_9 s_0^3)^{1/2}, (q_0^2 + r_0^2)^{1/2}]$ and supplying simultaneously the belonging of the point (q_0, r_0, s_0) to the set $\bar{K}_N$. Then both inequalities (3.31) and a priori estimate

$$q \leq N \text{ when } \rho \geq \rho_0 \tag{3.32}$$

are valid for the solution of the Cauchy problem (3.29), (3.30).

Let us return to the dynamic system (3.29). It has the unique rest point with coordinates

$$q = q^* \equiv 0.5(1 + \sqrt{1 - 8\beta^2/9})^{1/2}, \tag{3.33}$$

$$r = r^* \equiv 2^{-1/2}\mathrm{Sgn}\beta(1 + \sqrt{1 - 8\beta^2/9})^{1/2},\ s = 0$$

in the domain K_N. Linearization of (3.29) near the rest point leads to the system

$$\frac{dQ}{d\rho} = -2Q - \frac{r^*}{q^*}R,$$

$$\frac{dR}{d\rho} = -\frac{3r^*}{2q^*}Q - \frac{3}{2}R,\ \frac{dS}{d\rho} = -S.$$

Eigenvalues of the matrix of this system are

$$\lambda_{1,2} = \frac{-7q^* \pm \sqrt{(q^*)^2 + 24(r^*)^2}}{2q^*}, \ \lambda_3 = -1.$$

So far as $q^* > 0$ and $|r^*| \le q^*$ in view of $|\beta| \le 1$ all eigenvalues λ_i $(i = 1, 2, 3)$ are negative. Rest point $(q^*, r^*, 0)$ of system (3.29) is stable in accordance with Lyapunov theorem; it is a three–dimensional analog of knot.

The proof of the fact that trajectory of the dynamic system (3.29) starting from the point (q_0, r_0, s_0) in the "moment" $\rho = \rho_0$ finishes at $\rho \to \infty$ in the rest point $(q^*, r^*, 0)$ will complete the proof of Proposition 6. If this fact is obtained relations (3.20)–(3.22) are derived without any problems. Projection of mentioned trajectory on the plane $s = 0$ approaches asymptoticly (by virtue of the third equation (3.29)) at $\rho \to \infty$ to trajectory of two-dimensional dynamic system

$$\frac{dq}{d\rho} = (2q)^{-1}(1 - 2q^2 - r^2), \qquad (3.34)$$
$$\frac{dr}{d\rho} = (2q)^{-1}(\beta - 3qr)$$

outgoing from the point (q_0, r_0) at $\rho = \rho_0$; denote this trajectory by L. As it was proved befor curve L is contained in a circular sector $\bar{D}_N = \{q, r : q^2 + r^2 \le N^2, |r| \le q\}$.

At first let us suppose that $|\beta| < 1$. Then vector

$$\vec{m}(q, r) = ((2q)^{-1}(1 - 2q^2 - r^2), (2q)^{-1}(\beta - 3qr))$$

does not vanish on the boundary of the domain $D_{N,\epsilon} = \{q, r : \epsilon^2 < q^2 + r^2 < N^2, |r| < q\}$, where the number $\epsilon > 0$ is chosen less then $q_0 > 0$. (Note that for $|\beta| < 1$ and sufficiently small ϵ all points of the arch $q^2 + r^2 = \epsilon^2$, $|r| \le q$ are points of input for system (3.34), so that trajectory L does not fall outside the limits of not only the sector $\bar{D}_N$ but also the domain $\bar{D}_{N,\epsilon}$ for $\rho \ge \rho_0$.) So far as $\vec{m} \ne 0$ on the boundary of the domain $D_{N,\epsilon}$ we may calculate the rotation of vector field $\vec{m}(q, r)$ on mentioned boundary. Simple calculations show that this rotation is eqaul to unit. So far as knot $q = q^*$, $r = r^*$ is the unique singular point of the field $\vec{m}$ in the domain $\bar{D}_{N,\epsilon}$ and the knot index is equal to unit then system (3.34) has no limit cycles in the domain $\bar{D}_{N,\epsilon}$. So it follows from this fact that point (q^*, r^*) is the limit point of curve L when $\rho \to \infty$.

Now let $\beta = 1$ (case $\beta = -1$ is considered in a similar way). Here vector $\vec{m}$ vanishes on the boundary of the domain $D_{N,\epsilon}$: $\vec{m}(1/\sqrt{3}, 1/\sqrt{3}) = 0$. If at the same time $q_0 = r_0 = 1/\sqrt{3}$ then trajectory L consists of one mentioned point. If $q_0 = r_0 \ne 1/\sqrt{3}$, $\epsilon \le q_0 \le N$ then line L is a part of mentioned segment of the straight line $q = r$. In this case the dependence $q(\rho)$ is defined from solution of the Cauchy problem

$$\frac{dq}{d\rho} = (2q)^{-1}(1 - 3q^2) \text{ when } \rho > \rho_0, \ q(\rho_0) = q_0.$$

It is evident that $q \to q^* = 1/\sqrt{3}$ when $\rho \to \infty$. If the point (q_0, r_0) lies strictly inside the domain $\bar{D}_{N,\epsilon}$ then we can narrow down a little the opening angle of this domain and achieve the situation when vector field $\vec{m}$ has no zeros on the boundary of the domain

$\epsilon^2 < q^2+r^2 < N^2$, $|r| < (1-\delta)q$ containing the point (q_0, r_0) (the last fact can be provided for small ehough $\delta > 0$). Now the reasoning written above about the rotation of the field $\vec{m}$ can be repeated almost literally.

So now we have shown that the relations

$$q \to q^*, \ r \to r^* \text{ when } \rho \to \infty. \tag{3.35}$$

take place under the conditions of Proposition 6. So far as $\rho = \ln(1/s)$ it follows from (3.35) and definition of q^* and r^* (3.33) with account of equations (3.28) that

$$\frac{U}{(As)^{1/2}} \to 0.5(1+\sqrt{1-8\beta^2/9})^{1/2}, \tag{3.36}$$

$$\frac{V}{(As)^{1/2}} \to [0.5(1-\sqrt{1-8\beta^2/9})]^{1/2} \text{ when } s \to 0.$$

The first of this relations means that $ds/dt \to -[As(1+\sqrt{1-8\beta^2/9})]^{1/2}$ when $t \nearrow t^*$ by virtue of the equation $ds/dt = -2U$. The limit equality (3.20) follows from this fact and then the relations (3.21), (3.22) can be obtained from (3.36). The proof of Proposition 6 is complete.

This Proposition deserves some comments. The reason for introducing into consideration functions $U(t), V(t)$ is the following: these functions remain bounded in the limit $t \nearrow t^*$ in contarst to the functions $\bar{f}(t), \bar{g}(t)$; moreover $U = \mathrm{O}(t^*-t), V = \mathrm{O}(t^*-t)$ when $t \nearrow t^*$. On the other hand the equalities written below follow from the formulae $U = \bar{f}s$, $V = \bar{g}s$, relations (3.20)–(3.22) and representations (3.25) by virtue of the equations (2.22)

$$f(z,t) = \frac{1}{4(t^*-t)} + \mathrm{O}(t^*-t)^2, \tag{3.37}$$

$$g(z,t) = \frac{3}{8\beta(t^*-t)}(1-\sqrt{1-8\beta^2/9}) + \mathrm{O}(t^*-t)^2 \text{ when } t \nearrow t^*, \ 0 \le z \le s(t).$$

Thus formulae (3.20), (3.37) imply that smooth joining of free boundaries of the layer takes place in the moment t^* although the longitudinal components of the liquid velocity grow infinitly when $t \nearrow t^*$.

4 Discussion and Conclusion.

a). System (1.8) permits solutions with $f = g$, $f = -g$ and $g = 0$. In accordance with (1.7) the first two cases describe plane flows. These cases can be realized if one put $m = 0$ and $l = 0$ in (1.6). Case $g = 0$ corresponds to the equality $l = m$ for all $t \ge 0$. In this case solution of the problem (1.8)–(1.13) describes axisymmetric motion.

b). Proposition 2 gives sufficient conditions for blowing up of solution of the problem (1.8)–(1.19) in finite time t_*. Note that this phenomenon has a purely inertial character; viscous forces can't prevent it although these forces guarantee the space regularity of the solution.

The question about the structure of the solution singularity near the moment t_* is still open. This question is studied in detail in [13] for the plane analog of the discussed problem. Speaking more precisely, considered in [13] is equation (1.8) where $g = f$; the modified (for this case) problem with free boundary (1.9)–(1.13) is investigated – one put $l = m = 0$ in the condition (1.9) and substitute condition (1.13) for f by the following:

$$f(z, 0) = f_0(z), \quad 0 \leq z \leq a.$$

(We shall call this problem as problem P.)

Let function f_0 to satisfy the natural smoothness and compatibility conditions and also the inequality $f_0 \leq 0$ for $z \in [0, a]$ and some "steepness condition" [13]. Then solution f, s of problem P has the asymptotics

$$s(t) \sim \frac{\pi}{2\sqrt{\alpha(t_* - t)}}, \quad f(z, t) \sim -\frac{\cos^2(z\sqrt{\alpha(t_* - t)})}{t_* - t}$$

when $t \nearrow t_*$, $0 \leq z < s(t)$, where $\alpha = \text{const} > 0$.

c). The sufficient conditions of solvability of problem P for all $t \geq 0$ are determined in [14]. Also constructed in this paper is class of its exact solutions of the form

$$f = a(t) + b(t)\cos[\pi n z / s(t)], \tag{4.1}$$

where n is a natural number and functions a, b, s form the solution of the dynamic system. The value of solutions (4.1) consists of the fact that these solutions represent the main terms of both the blowing up of solutions of problem P when $t \nearrow t_*$ and its regular solutions when $t \to \infty$.

d). Let us consider the problem (1.8)–(1.13) for the case $l + m = 0$ where function $l(t)$ is nonnegative and $l > 0$ at some interval $0 < t_1 < t < t_2 < \infty$. Then the statement of Proposition 2 is valid although its proof requires some small modification. The case $l = -m$ is interesting from physical point of view so far as applied to the free boundary of the layer tangential stress at the direction of axis x has at $x = \pm y$ the same magnitude but opposite sign with tangential stress at the direction of axis y. It is difficult to predict the qualitative behaviour of solution with growth of t following "physical concepts". In this case the analysis of the problem (1.8)–(1.13) shows that the layer thickness $2s(t)$ is a monotonically increasing function of time and there exists some $t_* < \infty$ such that $s \to \infty$ when $t \nearrow t_*$.

e). Here we suggest some commests for Proposition 4. The exeptional case described by formulae (2.29) correspond to the plane motion. If the equality $l = 0$ or $m = 0$ is broken at some arbitrarily small interval of time then the solution of the problem (1.8)–(1.13) is symmetrized with growth of t as it follows from relations (2.28) (note that $g = 0$ for axisymmetric motion). The essential distinction between plane and three-dimensional regimes of thinning of the layer is demonstrated by the asymptotics of function s: in the first case $s \sim t^{-1}$ and in the second case $s \sim t^{-2}$ when $t \to \infty$.

f). Let us consider the problem on thermocapillary motion of viscous liquid in a layer at the linear dependence of the free boundary temperature on space coordinates

$$\theta_\Gamma = A(t)x + B(t)y.$$

It is not difficult to see that its solution can be obtained in the form

$$u = u(z, t), \; v = v(z, t), \; w = 0, \; p = 0, \; s = a = \text{const}.$$

Functions u, v are determined as solutions of the second initial boundary value problem for the linear equation of heat conduction (the details are omitted).

So the linear by coordinates distribution of temperature on plane free boundary of viscous layer does not lead to its thickness change. It is natural to suppose on the basis of this observation that the main change of the layer thickness under the action of thermocapillary forces takes place near the critical points of the temperature field on free surface. This concept can be considered as an additional motivation for investigation of solutions of the Navier–Stokes equations of the form (1.7). The unboundedness of functions u and v when $x, y \to \infty$ is the evident defect of this solution. However we can consider it as a solution describing the local behaviour of liquid in vicinity of critical points of the temperature field on free boundary.

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